Toward a minimal renormalizable supersymmetric $SU(5)$ grand unified model with tribimaximal mixing from A_4 flavor symmetry

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(Received 6 October 2009; published 11 January 2010)

We address the problem of rationalizing the pattern of fermion masses and mixings by adding a non-Abelian flavor symmetry in a grand unified framework. With this purpose, we include an A_4 flavor symmetry into a unified renormalizable supersymmetric grand unified theory $SU(5)$ model. With the help of the "type II seesaw" mechanism we are able to obtain the pattern of observed neutrino mixings in a natural way, through the so-called tribimaximal matrix.

DOI: 10.1103/PhysRevD.81.016004 PACS numbers: 11.30.Hv, 12.10.−g, 12.15.Ff, 14.60.Pq

I. INTRODUCTION

The experimental discovery of flavor oscillations of neutrinos, with the consequence that their masses are different from zero, is certainly a clear indication that there is new physics beyond the content of the standard model [1]. One of the most attractive and beautiful scenarios in which we can set this information is represented by the grand unification theories (GUT), that describes the merging of gauge couplings into a single one at a very high energy ($\sim 10^{16}$ GeV), as suggested by the running of the gauge coupling constants. Inside a unification theory, moreover, it is also possible to try to find an answer to some important and unsolved questions in flavor physics: the low energy data described in the quark sector by the Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix as well as the hierarchy between the quark masses. In the leptonic sector the low energy information is far from being as exhaustive as in the quark sector; one possibility is to assume a particular form for the mixing matrix: the so-called *tribimaximal* matrix [2], which is consistent with our information coming from neutrino oscillations on neutrino mass splittings and mixing angles. The most acclaimed possibility in order to explain the hierarchy between the masses comes from the introduction of a continuous flavor symmetry, as elegantly explained in [3–5], while the mixing can be explained by introducing discrete symmetries. For example, in [6–18] several attempts have been made to face the flavor puzzle by introducing discrete flavor symmetries such as S_3, S_4, A_4, T', and so on. Some attempts, as in [19], have been done to embed the A_4 flavor symmetry into a large flavor symmetry in order to explain also the hierarchy among the third and the other two generations; in particular, the authors have shown that the discrete symmetry A_4 can help us in solving both aspects of the flavor problem: lepton-quark mixing hierarchy and family mass hierarchy. The flavor symmetry A_4, as shown, for example, in [20,21], is very promising also in its extension to flavor group compatible with $SO(10)$-like grand unification. For example, by embedding A_4 into a group like $SU(3) \times U(1)$, as in [22], it is possible to explain both large neutrino mixing and fermion mass hierarchy in $SO(10)$ grand unified theory of flavor (GUTF). Considering as underlying unification theory $SU(5)$ instead of $SO(10)$, the situation becomes very different: the standard model ordinary matter for each family is embedded in two distinct $SU(5)$ representations; this peculiarity makes the way in which the matter content of the theory transforms under the action of the A_4 symmetry not obvious, allowing for different combinations (see for instance [23,24]).

In this paper we introduce the flavor symmetry A_4 in the context of a unified $SU(5)$ theory featuring a type II seesaw mechanism for neutrino mass generation. Our starting point is the model described in [25,26], which is a renormalizable model in which no matter fields besides the standard model ones are introduced. To this model we add two ingredients: the flavor symmetry, introduced in order to produce tribimaximal mixing in the neutrino sector, and supersymmetry, which, as we shall see, makes the needed vacuum alignment somehow more natural.

II. FIELD CONTENT AND $SU(5) \otimes A_4$ INVARIANCE

In order to clarify our notation we now open a small window on the A_4 proprieties, referring as an example to [27] for a more detailed discussion. In particular, in this work we use the basis where the A_4 elements S and T act on a **3** multiplet as

$$S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{1}$$

Given two triplets (a_1, a_2, a_3) and (b_1, b_2, b_3), three non-equivalent singlets can be formed from the $\mathbf{3} \otimes \mathbf{3}$ composition:

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$$\mathbf{1} = a_1b_1 + a_2b_2 + a_3b_3,$$
$$\mathbf{1}' = a_1b_1 + \omega^2 a_2b_2 + \omega a_3b_3, \quad (2)$$
$$\mathbf{1}'' = a_1b_1 + \omega a_2b_2 + \omega^2 a_3b_3$$

while the two inequivalent triplets one can form are $\{a_2b_3, a_3b_1, a_1b_2\}$ and $\{a_3b_2, a_1b_3, a_2b_1\}$. Here as usual $\omega = \exp(2\pi i/3)$. From the decomposition of the direct product $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$ we have two different singlets, as follows:

$$(a_2b_3c_1 + a_3b_1c_2 + a_1b_2c_3),$$
$$(a_3b_2c_1 + a_1b_3c_2 + a_2b_1c_3). \quad (3)$$

We also introduce the **4** representation, which is really simply a singlet added to a triplet; this is useful in order to keep our notation compact. For instance the Higgs multiplet belonging to a **5** representation with respect to $SU(5)$ properties behaves as the direct sum $\mathbf{3} \oplus \mathbf{1}$ under A_4:

$$\mathbf{5}_{\mathbf{H}} \sim \mathbf{3} \oplus \mathbf{1} \rightarrow \{\mathbf{5}_{\mathbf{H}}^{k=1,2,3}, \tilde{\mathbf{5}}_{\mathbf{H}}\}. \quad (4)$$

We now give the $SU(5)$ and A_4 field properties we choose in this work, for Higgs (H) and matter (T) representations, as follows:

$SU(5)$	$\mathbf{10_T}$	$\bar{\mathbf{5}}_\mathbf{T}$	$\bar{\mathbf{5}}_\mathbf{H}$	$\mathbf{5_H}$	$\overline{\mathbf{45}}_\mathbf{H}$	$\mathbf{45_H}$	$\overline{\mathbf{15}}_\mathbf{H}$	$\mathbf{15_H}$	$\mathbf{24_H}$
A_4	**3**	**3**	$\mathbf{3} \oplus \mathbf{1}$	$\mathbf{3} \oplus \mathbf{1}$	$\mathbf{3} \oplus \mathbf{1}$	$\mathbf{3} \oplus \mathbf{1}$	$\mathbf{3} \oplus \mathbf{1}''$	$\mathbf{3} \oplus \mathbf{1}'$	**1**

In the Higgs sector we will introduce $\mathbf{24_H}$, $\bar{\mathbf{5}}_\mathbf{H}$, $\mathbf{5_H}$ in order to break spontaneously the gauge symmetry $SU(5)$ into the standard model one and subsequently into the residual $SU(3)_C \otimes U(1)_{em}$; moreover, $\overline{\mathbf{45}}_\mathbf{H}$ and $\mathbf{45_H}$ are necessary in order to avoid the wrong prediction $M_D^T = M_E$ while $\overline{\mathbf{15}}_\mathbf{H}$ and $\mathbf{15_H}$ will generate the right path of neutrino masses through the Higgs mechanism implemented by the $SU(2)_L$ heavy scalar triplet contained in the standard model decomposition of $\mathbf{15_H}$.

The necessity to take into account the A_4 assignments as explained in the previous table is dictated by the observed phenomenology of the masses. For instance, it is easy to show that with the simpler choice of choosing $\mathbf{5_H}, \bar{\mathbf{5}}_\mathbf{H} \sim 3$ and $\mathbf{45_H}, \overline{\mathbf{45}}_\mathbf{H} \sim 3$, it is impossible to fit the measured values for the fermion masses. Although the Higgs sector of this model could seem rather cumbersome because of the introduction of four dimensional reducible representations, we stress the fact that it rests the minimal way in which we can preserve the predictivity of the A_4 flavor symmetry in the context of a renormalizable $SU(5)$ model.

III. CHARGED FERMION MASS MATRICES

The relevant operators in the Yukawa sector that generate the charged fermion mass matrices are

$$W_0 = y_1 \mathbf{10_T} \bar{\mathbf{5}}_\mathbf{T} \bar{\mathbf{5}}_H + y_2 \mathbf{10_T} \bar{\mathbf{5}}_\mathbf{T} \overline{\mathbf{45}}_\mathbf{H} + y_3 \mathbf{10_T} \mathbf{10_T} \mathbf{5_H} + y_4 \mathbf{10_T} \mathbf{10_T} \mathbf{45_H}. \quad (5)$$

As will be shown in Sec. V, in the flavor space $\bar{\mathbf{5}}_\mathbf{H}$, $\mathbf{5_H}$, $\overline{\mathbf{45}}_\mathbf{H}$, $\mathbf{45_H}$ acquire their vacuum expectation value (VEV) in the direction $\langle 1, 1, 1\rangle$. Under this condition, after spontaneous symmetry breaking the mass matrices obtained from W_0 through (5) are

$$M_f = \begin{pmatrix} h_0^f & \gamma_1^f & \gamma_2^f \\ \gamma_2^f & h_0^f & \gamma_1^f \\ \gamma_1^f & \gamma_2^f & h_0^f \end{pmatrix} = \tilde{U}_\omega M_f^{\mathrm{diag}} \tilde{U}_\omega^\dagger$$
$$\text{where } \tilde{U}_\omega = \frac{1}{\sqrt{3}} \begin{pmatrix} \omega & \omega^2 & 1 \\ \omega^2 & \omega & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad (6)$$

where we define:

$$h_0^u = 8\tilde{y}_3 \tilde{v}_5 \qquad \gamma_{1,2}^u = 4v_5(y_3^1 + y_3^2) \pm v_{45}(y_4^1 - y_4^2)$$
$$h_0^d = \tilde{y}_1 \tilde{v}_{\bar{5}} + 2\tilde{y}_2 \tilde{v}_{\overline{45}} \qquad \gamma_{1,2}^d = 4v_{\bar{5}} y_1^{2,1} + 2v_{\overline{45}} y_2^{2,1}$$
$$h_0^e = \tilde{y}_1 \tilde{v}_{\bar{5}} - 6\tilde{y}_2 \tilde{v}_{\overline{45}} \qquad \gamma_{1,2}^e = 4v_{\bar{5}} y_1^{1,2} - 6v_{\overline{45}} y_2^{1,2} \quad (7)$$

where the VEVs of the singlets from $\mathbf{4} = \mathbf{3} \oplus \mathbf{1}$ are shown with a tilde, $y_i^{j=1,2}$ refers to the two independent parameters from the singlets of $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$, as in (3), written for the y_i Yukawa coupling in (5), while $\tilde{y}_i$ refers to the singlet from $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{1}$.

Here we notice that h_0^f's, γ_1^f's, and γ_2^f's are independent parameters. The masses are given by

$$m_1^f = |h_0^f + \gamma_1^f \omega + \gamma_2^f \omega^2|$$
$$m_2^f = |h_0^f + \gamma_1^f \omega^2 + \gamma_2^f \omega| \qquad m_3^f = |h_0^f + \gamma_1^f + \gamma_2^f| \quad (8)$$

allowing a fit of experimental values as shown in [22].

As for mixing angles, since left up and down quarks have the same mass matrix (6), the V_{CKM} is unity in first approximation. In order to produce the Cabibbo angle, we now perturb the VEV directions by adding a small component in the direction $\langle 0, 0, 1\rangle$. We obtain that the mass matrices are perturbed by

$$\delta M_f = \begin{pmatrix} 0 & \epsilon_1^f & 0 \\ \epsilon_2^f & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow M_f^{\mathrm{off\ diag}} = \tilde{U}_\omega^\dagger \delta M_f \tilde{U}_\omega$$
$$= \begin{pmatrix} \omega\epsilon_1^f + \omega^2\epsilon_2^f & \epsilon_1^f + \epsilon_2^f & \omega^2\epsilon_1^f + \omega\epsilon_2^f \\ \epsilon_1^f + \epsilon_2^f & \omega^2\epsilon_1^f + \omega\epsilon_2^f & \omega\epsilon_1^f + \omega^2\epsilon_2^f \\ \omega^2\epsilon_1^f + \omega\epsilon_2^f & \omega\epsilon_1^f + \omega^2\epsilon_2^f & \epsilon_1^f + \epsilon_2^f \end{pmatrix}. \quad (9)$$

The Cabibbo angle can then be generated at least in two ways:

(1) As explained in [22], such small perturbations, if they are of order $\lambda^5 m_3^f$ (where λ is the Cabibbo angle), generate the Cabibbo angle in the quark sector and are irrelevant in the lepton sector. The crucial point is that such assumption has the consequences that our operators give negligible effects in the down and charged lepton sectors, since for the down and charged leptons $M_{d,e}^{\rm diag} + M_{d,e}^{\rm off\ diag}$ remain diagonal. On the contrary, for the up quarks we have that the off-diagonal entry $(1, 2)$ cannot be neglected: the matrix $M_u^{\rm diag} + M_u^{\rm off\ diag}$ is diagonalized by a rotation in the 12 plane with $\sin\theta_{12} \simeq \lambda$. This rotation produces the Cabibbo angle in the CKM.

(2) Another possibility is given by assuming that the Cabibbo angle comes from a rotation in the down sector. This can be the case if the perturbations of the $\mathbf{\bar{5}_H}$ and $\mathbf{\overline{45}_H}$, i.e., of order $\lambda^5 m_{\rm top} \simeq \lambda^3 m_{\rm bottom}$, are bigger than the ones of the $\mathbf{5_H}$ and $\mathbf{45_H}$, i.e., of order $\lambda^6 m_{\rm top}$. Such correction also generates a small perturbation to the tribimaximal lepton mixing matrix of order of the Cabibbo angle. In particular, if the dominant contribution comes from the $\mathbf{\bar{5}_H}$ then the tribimaximal lepton mixing matrix is multiplied on the left by $U_{\rm CKM}^\dagger$ and the net result is the presence of a nontrivial quark-lepton complementarity fully compatible with the experimental data and a prediction for the θ_{13} lepton angle [28]. On the other side, if the dominant contribution comes from the $\mathbf{\overline{45}_H}$ there is a Clebsch-Gordan coefficient between the quark and lepton mixing corrections.

IV. NEUTRINO MASS MATRIX AND LEPTON MIXING ANGLES

The relevant operators that generate the neutrino mass matrix are

$$W_1 = \gamma \mathbf{\bar{5}_T \bar{5}_T 15_H} + m_\Phi \mathbf{\overline{15}_H 15_H}. \tag{10}$$

We assume that the triplet from $\mathbf{15_H}$ acquires a small VEV in the direction $\langle 0, 0, 1\rangle$, while we use again the tilde for the VEV of the singlet. Under this condition the neutrino mass matrix obtained from W_1 is given by

$$M_\nu = \begin{pmatrix} \beta\tilde{v}_{15} & \gamma v_{15} & 0 \\ \gamma v_{15} & \omega\beta\tilde{v}_{15} & 0 \\ 0 & 0 & \omega^2\beta\tilde{v}_{15} \end{pmatrix} = V^\star M_\nu^{\rm diag} V^\dagger$$
$$\text{where } V = \begin{pmatrix} \frac{\omega}{\sqrt{2}} & 0 & -\frac{i\omega}{\sqrt{2}} \\ \frac{\omega^2}{\sqrt{2}} & 0 & \frac{i\omega^2}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \tag{11}$$

and the lepton tribimaximal mixing arises:

$$V_{\rm leptons} = \tilde{U}_\omega^\dagger \cdot V = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{12}$$

In (11) γ is the common parameter for the two singlets from $\mathbf{3 \otimes 3 \otimes 3}$, after considering that the demand for the neutrino mass matrix to be symmetric forces in (3) the relation $\gamma_1 = \gamma_2$; β is the parameter from the singlet of $\mathbf{3 \otimes 3 \otimes 1'}$ in (10). The neutrino masses are given by $\{|\omega^2\beta\tilde{v}_{15} + \gamma v_{15}|, |\omega^2\beta\tilde{v}_{15}|, |-\omega^2\beta\tilde{v}_{15} + \gamma v_{15}|\}$. Since phenomenologically we have $\delta m_{12}^2 > 0$ we obtain $|\beta\gamma v_{15}\tilde{v}_{15}| < 0$ which implies $\delta m_{13}^2 > 0$, i.e., a normal hierarchy; so as a consequence the inverted hierarchy is completely ruled out in this model because of the same underlying structure imposed by the A_4 symmetry.

Finally we predict the absolute neutrino mass value and the parameter $|m_{ee}|$ relevant for the future experiments in neutrinoless double beta decay, i.e.,

$$m_2 \geq \frac{1}{2\sqrt{2}} \frac{\delta m_{\rm atm}^2 + \delta m_{\rm sol}^2}{\sqrt{\delta m_{\rm atm}^2 - \delta m_{\rm sol}^2}} \simeq \frac{1}{2\sqrt{2}} \sqrt{\delta m_{\rm atm}^2} \simeq 0.02\ {\rm eV}, \tag{13}$$

and

$$|m_{ee}| \geq 2m_1 + m_2 \geq \frac{3}{2\sqrt{2}} \sqrt{\delta m_{\rm atm}^2} \simeq 0.05\ {\rm eV}. \tag{14}$$

V. MINIMIZATION OF THE POTENTIAL

The potential V is written in terms of the superpotential $W(\phi_i)$, which is an analytical function of the scalar fields ϕ_i, in the following way:

$$V = \sum_i \left| \frac{\partial W}{\partial \phi_i} \right|^2 + V_{D\ \rm terms} + V_{\rm soft}. \tag{15}$$

Here we are interested in the $SU(5)$ and A_4 breaking that takes place at scales of the order of the GUT scale; we can therefore neglect supersymmetry breaking terms of the order of the TeV scale, described by $V_{\rm soft}$: the latter play a crucial role in electroweak symmetry breaking, that we do not discuss. In the following we minimize the first term in (15), neglecting also D terms: minimization then amounts to imposing $\frac{\partial W}{\partial \phi_i} = 0\ \forall\ i$. After imposing this, we show that there is a finite region in parameter space where $V_{D\ \rm terms} = 0$, justifying *a posteriori* our assumption.

In order to obtain a correct $SU(5) \to SU(3) \otimes SU(2) \otimes U(1)$ symmetry breaking we impose the following structure with respect to the $SU(5)$ symmetry:

$$\langle \mathbf{45_H}\rangle_i^{i5} = v_{45}, \qquad i = 1, 2, 3; \qquad \langle \mathbf{45_H}\rangle_4^{45} = -3v_{45}; \tag{16}$$

$$\langle\overline{\mathbf{45}}_{\mathbf{H}}\rangle^{i}_{i5} = v_{\overline{45}}, \qquad i = 1, 2, 3; \qquad \langle\overline{\mathbf{45}}_{\mathbf{H}}\rangle^{4}_{45} = -3v_{\overline{45}}; \tag{17}$$

$$\langle\mathbf{24}_{H}\rangle^{\alpha}{}_{\alpha} = \mathrm{diag}v^{s}_{24}(2v^{s}_{24}, 2v^{s}_{24}, 2v^{s}_{24}, -3v^{s}_{24} + v^{t}_{24}, -3v^{s}_{24} - v^{t}_{24}); \tag{18}$$

$$\langle\mathbf{5}_{H}\rangle^{\alpha} = v_{5}(0, 0, 0, 0, 1)^{T}, \qquad \langle\bar{\mathbf{5}}_{H}\rangle_{\alpha} = v_{\bar{5}}(0, 0, 0, 0, 1)^{T}. \tag{19}$$

Moreover we assume that in flavor space the triplet from $\mathbf{15_H}$ acquires a small VEV in the direction (0, 0, 1).

Let us now come to potential minimization. The renormalizable Higgs superoperators allowed under supersymmetric $SU(5) \otimes A_4$ invariance are

$$W_2 = m_{\Sigma}\mathbf{24_H 24_H} + \lambda_{\Sigma}\mathbf{24_H 24_H 24_H} + m_5\bar{\mathbf{5}}^k_{\mathbf{H}}\mathbf{5}^k_{\mathbf{H}} + m_{\Phi}\overline{\mathbf{15}}^k_{\mathbf{H}}\mathbf{15}^k_{\mathbf{H}} + m_{45}\overline{\mathbf{45}}^k_{\mathbf{H}}\mathbf{45}^k_{\mathbf{H}} + \tilde{m}_5\tilde{\bar{\mathbf{5}}}_{\mathbf{H}}\tilde{\mathbf{5}}_{\mathbf{H}} + \tilde{m}_{\Phi}\widetilde{\overline{\mathbf{15}}}_{\mathbf{H}}\widetilde{\mathbf{15}}_{\mathbf{H}} + \tilde{m}_{45}\widetilde{\overline{\mathbf{45}}}_{\mathbf{H}}\widetilde{\mathbf{45}}_{\mathbf{H}}, \tag{20a}$$

$$\begin{aligned} W_3 = {} & \lambda_H\bar{\mathbf{5}}^k_{\mathbf{H}}\mathbf{24_H}\mathbf{5}^k_{\mathbf{H}} + c_H\bar{\mathbf{5}}^k_{\mathbf{H}}\mathbf{24_H}\mathbf{45}^k_{\mathbf{H}} + b_H\overline{\mathbf{45}}^k_{\mathbf{H}}\mathbf{24_H}\mathbf{5}^k_{\mathbf{H}} + a_H\overline{\mathbf{45}}^k_{\mathbf{H}}\mathbf{45}^k_{\mathbf{H}}\mathbf{24_H} + \tilde{\lambda}_H\tilde{\bar{\mathbf{5}}}_{\mathbf{H}}\mathbf{24_H}\tilde{\mathbf{5}}_{\mathbf{H}} + \tilde{c}_H\tilde{\bar{\mathbf{5}}}_{\mathbf{H}}\mathbf{24_H}\widetilde{\mathbf{45}}_{\mathbf{H}} \\ & + \tilde{b}_H\widetilde{\overline{\mathbf{45}}}_{\mathbf{H}}\mathbf{24_H}\tilde{\mathbf{5}}_{\mathbf{H}} + \tilde{a}_H\widetilde{\overline{\mathbf{45}}}_{\mathbf{H}}\widetilde{\mathbf{45}}_{\mathbf{H}}\mathbf{24_H}, \end{aligned} \tag{20b}$$

$$\begin{aligned} W_4 = {} & h_1\overline{\mathbf{15}}^k_{\mathbf{H}}\mathbf{24_H}\mathbf{15}^k_{\mathbf{H}} + h^{lmn}_2\overline{\mathbf{15}}^l_{\mathbf{H}}\mathbf{5}^m_{\mathbf{H}}\mathbf{5}^n_{\mathbf{H}} + h^{lmn}_3\mathbf{15}^l_{\mathbf{H}}\bar{\mathbf{5}}^m_{\mathbf{H}}\bar{\mathbf{5}}^n_{\mathbf{H}} + h^{lmn}_4\mathbf{15}^l_{\mathbf{H}}\overline{\mathbf{45}}^m_{\mathbf{H}}\overline{\mathbf{45}}^n_{\mathbf{H}} + h^{lmn}_5\overline{\mathbf{15}}^l_{\mathbf{H}}\mathbf{45}^m_{\mathbf{H}}\mathbf{45}^n_{\mathbf{H}} + \tilde{h}_1\widetilde{\overline{\mathbf{15}}}_{\mathbf{H}}\mathbf{24_H}\widetilde{\mathbf{15}}_{\mathbf{H}} \\ & + \tilde{h}_2\widetilde{\overline{\mathbf{15}}}_{\mathbf{H}}(\mathbf{5}^1_{\mathbf{H}}\mathbf{5}^1_{\mathbf{H}} + \omega^2\mathbf{5}^2_{\mathbf{H}}\mathbf{5}^2_{\mathbf{H}} + \omega\mathbf{5}^3_{\mathbf{H}}\mathbf{5}^3_{\mathbf{H}}) + \tilde{h}'_2\overline{\mathbf{15}}^k_{\mathbf{H}}\mathbf{5}^k_{\mathbf{H}}\tilde{\mathbf{5}}_{\mathbf{H}} + \tilde{h}_3\widetilde{\mathbf{15}}_{\mathbf{H}}(\bar{\mathbf{5}}^1_{\mathbf{H}}\bar{\mathbf{5}}^1_{\mathbf{H}} + \omega\bar{\mathbf{5}}^2_{\mathbf{H}}\bar{\mathbf{5}}^2_{\mathbf{H}} + \omega^2\bar{\mathbf{5}}^3_{\mathbf{H}}\bar{\mathbf{5}}^3_{\mathbf{H}}) + \tilde{h}'_3\mathbf{15}^k_{\mathbf{H}}\bar{\mathbf{5}}^k_{\mathbf{H}}\tilde{\bar{\mathbf{5}}}_{\mathbf{H}} \\ & + \tilde{h}_4\widetilde{\mathbf{15}}_{\mathbf{H}}(\overline{\mathbf{45}}^1_{\mathbf{H}}\overline{\mathbf{45}}^1_{\mathbf{H}} + \omega\overline{\mathbf{45}}^2_{\mathbf{H}}\overline{\mathbf{45}}^2_{\mathbf{H}} + \omega^2\overline{\mathbf{45}}^3_{\mathbf{H}}\overline{\mathbf{45}}^3_{\mathbf{H}}) + \tilde{h}'_4\mathbf{15}^k_{\mathbf{H}}\widetilde{\overline{\mathbf{45}}}_{\mathbf{H}}\overline{\mathbf{45}}^k_{\mathbf{H}} + \tilde{h}_5\widetilde{\overline{\mathbf{15}}}_{\mathbf{H}}(\mathbf{45}^1_{\mathbf{H}}\mathbf{45}^1_{\mathbf{H}} + \omega^2\mathbf{45}^2_{\mathbf{H}}\mathbf{45}^2_{\mathbf{H}} + \omega\mathbf{45}^3_{\mathbf{H}}\mathbf{45}^3_{\mathbf{H}}) \\ & + \tilde{h}'_5\overline{\mathbf{15}}^k_{\mathbf{H}}\widetilde{\mathbf{45}}_{\mathbf{H}}\mathbf{45}^k_{\mathbf{H}}, \end{aligned} \tag{20c}$$

where γ, β, a_H, b_H, and c_H and the y's, λ's, m's, and h's are the coupling constants of the model. The invariant combinations from $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$, e.g., as abbreviated in $h^{lmn}_2\overline{\mathbf{15}}^l_{\mathbf{H}}\mathbf{5}^m_{\mathbf{H}}\mathbf{5}^n_{\mathbf{H}}$, have to be understood following (3).

We now impose $\frac{\partial W}{\partial \phi_i} = 0 \;\forall\; i$, the superpotential W being given by the sum of the terms (20a) and (20c). The first equations we discuss are the ones obtained by imposing $\partial W/\partial \mathbf{45}^k_{\mathbf{H}} = \partial W/\partial \overline{\mathbf{45}}^k_{\mathbf{H}} = 0$:

$$v^k_{\overline{45}}A = -c_H v^s_{24} v^k_{\bar{5}}, \tag{21a}$$

$$3v^k_{\overline{45}}B = -\frac{c_H}{2}(-3v^s_{24} + v^t_{24})v^k_{\bar{5}}, \tag{21b}$$

$$v^k_{45}A = -b_H v^s_{24} v^k_{5}, \tag{21c}$$

$$3v^k_{45}B = -\frac{b_H}{2}(-3v^s_{24} + v^t_{24})v^k_{5}, \tag{21d}$$

where

$$A \equiv m_{45} + v^s_{24}\left(2a^1_H + \frac{a^2_H}{2}\right) + v^t_{24}\frac{a^2_H}{2}, \tag{22a}$$

$$B \equiv -m_{45} + 3v^s_{24}(a^1_H - a^2_H) - v^t_{24}a^1_H. \tag{22b}$$

Equations (21) imply that $v_5(v_{\bar{5}})$ is aligned with $v_{45}(v_{\overline{45}})$ in flavor space.

Then, from $\partial W/\partial\widetilde{\mathbf{15}}_{\mathbf{H}} = \partial W/\partial\widetilde{\overline{\mathbf{15}}}_{\mathbf{H}} = 0$ we obtain:

$$12\tilde{h}_4[(v^1_{\overline{45}})^2 + \omega(v^2_{\overline{45}})^2 + \omega^2(v^3_{\overline{45}})^2] + \tilde{h}_3[(v^1_{\bar{5}})^2 + \omega(v^2_{\bar{5}})^2 + \omega^2(v^3_{\bar{5}})^2] = 0 \tag{23a}$$

$$12\tilde{h}_5[(v^1_{45})^2 + \omega(v^2_{45})^2 + \omega^2(v^3_{45})^2] + \tilde{h}_2[(v^1_{5})^2 + \omega(v^2_{5})^2 + \omega^2(v^3_{5})^2] = 0. \tag{23b}$$

For generic values of the superpotential parameters $\tilde{h}_i$, these equations are identically satisfied (recall that $\omega = \exp[\frac{2\pi i}{3}]$) if $v^1_5 = v^2_5 = v^3_5$ and the same holds for v^i_{45}, $v^i_{\overline{45}}$, $v^i_{\bar{5}}$: this realizes the desired vacuum alignment since all triplets VEVs must be proportional to the direction (1, 1, 1) in flavor space.

Let us now consider the remaining equations, with the purpose of showing that a nontrivial solution indeed exists, provided certain conditions are fulfilled by the parameters of the superpotential.

(i) From $\partial W/\partial \mathbf{5}^k_{\mathbf{H}} = 0$ and $\partial W/\partial \bar{\mathbf{5}}^k_{\mathbf{H}} = 0$ we obtain:

$$v^k_5\alpha = 3c_H v^k_{45}(-5v^s_{24} + v^t_{24}), \tag{24a}$$

$$v^k_{\bar{5}}\alpha = 3b_H v^k_{\overline{45}}(-5v^s_{24} + v^t_{24}); \tag{24b}$$

(ii) from $\partial W/\partial \widetilde{\mathbf{45}}_H = 0$ and $\partial W/\partial \widetilde{\overline{\mathbf{45}}}_{\mathbf{H}} = 0$:

$$\tilde{v}_{\overline{45}}\tilde{A} = -\tilde{c}_H v^s_{24}\tilde{v}_{\bar{5}}, \tag{25a}$$

$$3\tilde{v}_{\overline{45}}\tilde{B} = -\frac{\tilde{c}_H}{2}(-3v^s_{24} + v^t_{24})\tilde{v}_{\bar{5}}, \tag{25b}$$

$$\tilde{v}_{45}\tilde{A} = -\tilde{b}_H v^s_{24} v_5, \tag{25c}$$

$$3\tilde{v}_{45}\tilde{B} = -\frac{\tilde{b}_H}{2}(-3v^s_{24} + v^t_{24})\tilde{v}_5; \tag{25d}$$

(iii) from $\partial W/\partial \tilde{\mathbf{5}}_{\mathbf{H}} = 0$ and $\partial W/\partial \tilde{\bar{\mathbf{5}}}_{\mathbf{H}} = \mathbf{0}$:

$$\tilde{v}_5\tilde{\alpha} = 3\tilde{c}_H\tilde{v}_{45}(-5v^s_{24} + v^t_{24}), \tag{26a}$$

$$\tilde{v}_{\bar{5}}\tilde{\alpha} = 3\tilde{b}_H\tilde{v}_{\overline{45}}(-5v^s_{24} + v^t_{24}); \tag{26b}$$

(iv) from $\partial W/\partial \mathbf{15}^k_{\mathbf{H}} = 0$ and $\partial W/\partial \overline{\mathbf{15}}^k_{\mathbf{H}} = 0$ for every $l \neq m \neq k$ we obtain:

$$12(h^1_4 + h^2_4)v^l_{\overline{45}}v^m_{\overline{45}} + 12\tilde{h}^l_4\tilde{v}_{\overline{45}}v^k_{\overline{45}} + (h^1_3 + h^2_3)v^l_{\bar{5}}v^m_{\bar{5}} + \tilde{h}^l_3\tilde{v}_{\bar{5}}v^k_{\bar{5}} = 0, \tag{27a}$$

$$12(h^1_5 + h^2_5)v^l_{45}v^m_{45} + 12\tilde{h}^l_5\tilde{v}_{45}v^k_{45} + (h^1_2 + h^2_2)v^l_5 v^m_5 + \tilde{h}^l_2\tilde{v}_5 v^k_5 = 0, \tag{27b}$$

(v) from $\partial W/\partial \mathbf{24_H} = 0$:

$$2v^s_{24}\beta_1 + \sum_{k=1}^{3}[(2a^1_H - a^2_H)v^k_{\overline{45}}v^k_{45} + b_H v^k_{\overline{45}}v^k_5 + c_H v^k_{45}v^k_{\bar{5}}] + (2\tilde{a}^1_H - \tilde{a}^2_H)\tilde{v}_{\overline{45}}\tilde{v}_{45} + \tilde{b}_H\tilde{v}_{\overline{45}}\tilde{v}_5 + \tilde{c}_H\tilde{v}_{45}\tilde{v}_{\bar{5}} = 0 \tag{28}$$

$$(-3v^s_{24} + v^t_{24})\beta_2 + 3\Big\{\sum_{k=1}^{3}[3(2a^1_H - a^2_H)v^k_{\overline{45}}v^k_{45} - b_H v^k_{\overline{45}}v^k_5 - c_H v^k_{45}v^k_{\bar{5}}] + 3(2\tilde{a}^1_H - \tilde{a}^2_H)\tilde{v}_{\overline{45}}\tilde{v}_{45} - \tilde{b}_H\tilde{v}_{\overline{45}}\tilde{v}_5 - \tilde{c}_H\tilde{v}_{45}\tilde{v}_{\bar{5}}\Big\} = 0$$

$$(-3v^s_{24} - v^t_{24})\beta_3 + \sum_{k=1}^{3}[\lambda_H v^k_{\bar{5}}v^k_5 - 12a^2_H v^k_{\overline{45}}v^k_{45}] + \tilde{\lambda}_H\tilde{v}_{\bar{5}}\tilde{v}_5 - 12\tilde{a}^2_H\tilde{v}_{\overline{45}}\tilde{v}_{45} = 0 \tag{29}$$

$$(-3v^s_{24} - v^t_{24})\beta_3 + \sum_{k=1}^{3}[\lambda_H v^k_{\bar{5}}v^k_5 - 12a^2_H v^k_{\overline{45}}v^k_{45}] + \tilde{\lambda}_H\tilde{v}_{\bar{5}}\tilde{v}_5 - 12\tilde{a}^2_H\tilde{v}_{\overline{45}}\tilde{v}_{45} = 0, \tag{30}$$

where we have defined the following combinations:

$$\alpha \equiv m_5 - \lambda_H(3v^s_{24} + v^t_{24}), \tag{31a}$$

$$\beta_1 \equiv (2m_\Sigma + 6\lambda_\Sigma v^s_{24}), \tag{31b}$$

$$\beta_2 \equiv [2m_\Sigma + 3\lambda_\Sigma(-3v^s_{24} + v^t_{24})], \tag{31c}$$

$$\beta_3 \equiv [2m_\Sigma + 3\lambda_\Sigma(-3v^s_{24} - v^t_{24})], \tag{31d}$$

with similar relations for $\tilde{A}$, $\tilde{B}$ [see Eqs. (22)] and $\tilde{\alpha}$, obtained considering the substitutions of the "nontilded" parameters with the "tilded" ones.

Comparing the first equation in (21) with the second one, as well as the third with the fourth, and performing the same analysis with (25), we obtain the relations:

$$\begin{cases} 6Bv^s_{24} = A(-3v^s_{24} + v^t_{24}) \\ 6\tilde{B}v^s_{24} = \tilde{A}(-3v^s_{24} + v^t_{24}) \end{cases} \rightarrow \frac{B}{A} = \frac{\tilde{B}}{\tilde{A}}; \tag{32}$$

from (21) and (24) we have, instead:

$$\begin{cases} 3b_H c_H v^s_{24}(-5v^s_{24} + v^t_{24}) = -\alpha A \\ 3\tilde{b}_H\tilde{c}_H v^s_{24}(-5v^s_{24} + v^t_{24}) = -\tilde{\alpha}\tilde{A} \end{cases} \rightarrow \frac{\alpha A}{b_H c_H} = \frac{\tilde{\alpha}\tilde{A}}{\tilde{b}_H\tilde{c}_H}; \tag{33}$$

it is possible, at this point, to use the system of (32) and (33) in order to obtain $v^{s,t}_{24}$ as functions of the parameters in the superpotential. The allowed solutions are

$$v^t_{24} = \frac{3\eta \pm \sqrt{2\varphi}}{4\sigma}, \qquad v^s_{24} = \frac{\eta \mp \sqrt{2\varphi}}{12\sigma}, \tag{34}$$

where

$$\eta \equiv (2a^1_H - a^2_H)m_5 b_H c_H + (2a^1_H - a^2_H)^2 m_5\lambda_H, \tag{35a}$$

$$\sigma \equiv [b_H c_H + \lambda_H(2a^1_H - a^2_H)]^2, \tag{35b}$$

$$\varphi \equiv m_5\sigma[3m_{45}b_H c_H + (2a^1_H - a^2_H) \times (3m_{45}\lambda_H + m_5(a^1_H + a^2_H))]. \tag{35c}$$

From (27) we obtain:

$$\tilde{v}_5 = -\frac{\tilde{A}}{A}\left[\frac{12(h^1_5 + h^2_5)(b_H v^s_{24})^2 + A^2(h^1_2 + h^2_2)}{12\tilde{h}^l_5 b_H\tilde{b}_H(v^s_{24})^2 + \tilde{h}^l_2 A\tilde{A}}\right]v_5 \equiv -\frac{\tilde{A}}{A}\Theta_1 v_5, \tag{36a}$$

$$\tilde{v}_{\bar{5}} = -\frac{\tilde{A}}{A}\left[\frac{12(h^1_4 + h^2_4)(c_H v^s_{24})^2 + A^2(h^1_3 + h^2_3)}{12\tilde{h}^l_4 c_H\tilde{c}_H(v^s_{24})^2 + \tilde{h}^l_3 A\tilde{A}}\right]v_{\bar{5}} \equiv -\frac{\tilde{A}}{A}\Theta_2 v_{\bar{5}} \tag{36b}$$

and

$$\tilde{v}_{45} = -\frac{\tilde{b}_H v^s_{24}}{A}\Theta_1 v_5, \tag{37a}$$

$$\tilde{v}_{\overline{45}} = -\frac{\tilde{c}_H v^s_{24}}{A}\Theta_2 v_{\bar{5}}. \tag{37b}$$

These relations allow us to express v_{45}, $\tilde{v}_{45}$, and $\tilde{v}_5$ as functions of v_5, as well as $v_{\overline{45}}$, $\tilde{v}_{\overline{45}}$, and $\tilde{v}_{\bar{5}}$ as functions of $v_{\bar{5}}$. Considering the relations obtained in (36) and (37), we can rewrite the three equations from (28)–(30) as three compatible relations that allow us to write the product $v_5 v_{\bar{5}}$ as a function of the parameters of the superpotential.

We now show that it is possible to choose the (super) potential parameters in such a way that the D term's contributions appearing in (15) are zero. For a supersymmetric gauge theory the D terms can be written as

$$\frac{1}{2}\sum_G \sum_\alpha \sum_{i,j} g_G^2 (\phi_i^\dagger T_G^\alpha \phi_i)(\phi_j^\dagger T_G^\alpha \phi_j), \tag{38}$$

where we take into account that, for the minimal supersymmetric standard model, $G = SU(3)_C, SU(2)_L, U(1)_Y$, with different couplings g_G and generators T_G.

Let us first consider contributions for $\mathbf{5_H}, \mathbf{\bar{5}_H}$ representations. The following decomposition holds:

$$\begin{aligned} \mathbf{5_H} &= (\mathbf{3}, \mathbf{1}, -\mathbf{1/3}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{1/2}), \\ \mathbf{\bar{5}_H} &= (\mathbf{\bar{3}}, \mathbf{1}, \mathbf{1/3}) \oplus (\mathbf{1}, \mathbf{\bar{2}}, -\mathbf{1/2}). \end{aligned} \tag{39}$$

Since we only consider contributions to D terms coming from the VEVs $\langle \mathbf{5_H} \rangle, \langle \mathbf{\bar{5}_H} \rangle$, only the $SU(2) \otimes U(1)$ doublet in (39) contributes. Moreover, the off-diagonal $SU(2)$ generators T_1, T_2 also give zero contribution, so we need only to consider the effect of T_3 and the hypercharge Y. Also taking into account that in flavor space the VEVs have the structure $\langle \mathbf{5_H}, \mathbf{\bar{5}_H} \rangle = v_{5,\bar{5}}(1, 1, 1)$, a straightforward calculation gives:

$$\begin{aligned} &\langle |\mathbf{5_H^\dagger} T_5^\alpha \mathbf{5_H}|_{SU(2)_L} + |\mathbf{\bar{5}_H^\dagger} T_{\bar{5}}^\alpha \mathbf{\bar{5}_H}|_{SU(2)_L} \rangle \\ &\quad = \frac{3}{2}(-|v_5|^2 + |v_{\bar{5}}|^2) + \frac{1}{2}(-|\tilde{v}_5|^2 + |\tilde{v}_{\bar{5}}|^2) \end{aligned} \tag{40}$$

while the $U(1)$ contribution reads:

$$\begin{aligned} &\langle |\mathbf{5_H^\dagger} T_5^\alpha \mathbf{5_H}|_{U(1)_Y} + |\mathbf{\bar{5}_H^\dagger} T_{\bar{5}}^\alpha \mathbf{\bar{5}_H}|_{U(1)_Y} \rangle \\ &\quad = \frac{3}{2}(|v_5|^2 - |v_{\bar{5}}|^2) + \frac{1}{2}(|\tilde{v}_5|^2 - |\tilde{v}_{\bar{5}}|^2). \end{aligned} \tag{41}$$

Similar considerations hold for the $\mathbf{45_H}, \mathbf{\overline{45}_H}$ representations, decomposed as

$$\begin{aligned} \mathbf{45_H} = {} & (\mathbf{8}, \mathbf{2}, \mathbf{1/2}) \otimes (\mathbf{\bar{6}}, \mathbf{1}, -\mathbf{1/3}) \otimes (\mathbf{3}, \mathbf{3}, -\mathbf{1/3}) \\ & \otimes (\mathbf{\bar{3}}, \mathbf{2}, -\mathbf{7/6}) \otimes (\mathbf{3}, \mathbf{1}, -\mathbf{1/3}) \oplus (\mathbf{\bar{3}}, \mathbf{1}, \mathbf{4/3}) \\ & \otimes (\mathbf{1}, \mathbf{2}, \mathbf{1/2}) \end{aligned} \tag{42}$$

and for which only the doublet component contributes. The $\mathbf{24_H}$ instead

$$\begin{aligned} \mathbf{24_H} = {} & (\mathbf{8}, \mathbf{1}, \mathbf{0}) \oplus (\mathbf{1}, \mathbf{3}, \mathbf{0}) \oplus (\mathbf{3}, \mathbf{2}, -\mathbf{5/6}) \oplus (\mathbf{\bar{3}}, \mathbf{\bar{2}}, \mathbf{5/6}) \\ & \oplus (\mathbf{1}, \mathbf{1}, \mathbf{0}) \end{aligned} \tag{43}$$

acquires a nonzero VEV along the $(\mathbf{1}, \mathbf{1}, \mathbf{0})$ component, which is an isospin singlet with zero hypercharge and therefore does not contribute to D terms. Overall, D terms can be written as

$$\begin{aligned} \frac{g^2 + g'^2}{2} \bigg\{ \bigg[& \frac{3}{2}(-|v_5|^2 + |v_{\bar{5}}|^2) + \frac{1}{2}(-|\tilde{v}_5|^2 + |\tilde{v}_{\bar{5}}|^2) \\ & + \frac{3}{2}(-|v_{45}|^2 + |v_{\overline{45}}|^2) + \frac{1}{2}(-|\tilde{v}_{45}|^2 + |\tilde{v}_{\overline{45}}|^2) \bigg] \bigg\}^2. \end{aligned} \tag{44}$$

Since all VEVs appearing in (44) are expressed as functions of v_5 and $v_{\bar{5}}$ through Eqs. (21), (36), and (37), imposing vanishing D terms implies:

$$\begin{aligned} & |v_5|^2 \bigg(\frac{3}{2} + \frac{1}{2} \bigg| \frac{\tilde{A}}{A} \Theta_2 \bigg|^2 + \frac{3}{2} \bigg| \frac{c_H}{A} v_{24}^2 \bigg|^2 + \frac{1}{2} \bigg| \frac{\tilde{c_H}}{A} \Theta_2 \bigg|^2 \bigg) \\ & = |v_{\bar{5}}|^2 \bigg(\frac{3}{2} + \frac{1}{2} \bigg| \frac{\tilde{A}}{A} \Theta_1 \bigg|^2 + \frac{3}{2} \bigg| \frac{b_H}{A} v_{24}^2 \bigg|^2 + \frac{1}{2} \bigg| \frac{\tilde{b_H}}{A} \Theta_1 \bigg|^2 \bigg). \end{aligned} \tag{45}$$

So, while the minimization conditions discussed above fix the value of the product $v_5 v_{\bar{5}}$, requiring vanishing D terms adds Eq. (45) and fixes the value of v_5 and $v_{\bar{5}}$ (and therefore of all the remaining VEVs) as functions of the potential parameters.

As a conclusion we have that vacuum alignment in flavor space $v_i \propto (1, 1, 1)$, that allows us to obtain the correct phenomenology in the context of the considered model, can be obtained in a finite region of the parameter space of the superpotential, under the condition that the triplet from the $\mathbf{15_H}$ acquires a VEV in the direction $(0, 0, 1)$, and that this VEV can be neglected in comparison with the other scales of the model.

VI. CONCLUSIONS

In this paper we have achieved the possibility to reproduce the nice features of the A_4 group, with regard to the mixing of leptons, inside a renormalizable $SU(5)$ theory. Even if the GUT scale is very close to the Planck scale, in fact, we think that renormalizability has to be a fundamental characteristic of the considered unification theory, in order to avoid the presence of higher dimensional operators as fundamental blocks in the construction of the mass matrices and to improve the predictivity of the model.

In our model the neutrino mass matrix comes from the presence of a heavy $SU(2)_L$ scalar triplet embedded into the $\mathbf{15_H}$ representation of $SU(5)$, while in order to obtain the correct phenomenology at GUT scale we need to introduce an extended Higgs sector, as described in Sec. II where the presence of the four dimensional reducible representations of A_4 is claimed. As expected [14,29] we are not able to reproduce with only the aid of A_4 symmetry the hierarchy between the masses; on the contrary, the mixing angles in the CKM matrix and in the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix, the latter being described by the tribimaximal mixing, are reproduced in a very clear way. With respect to our previous work [24], where a combination of type I and type III seesaw mechanisms was considered for generating neutrino masses, the model considered here with a type II mechanism constitutes an improvement, since under the assumption that $\mathbf{15_H}$ acquires a VEV in the direction (0, 0, 1), the alignment of the remaining triplets $v_i \propto$ (1, 1, 1) is obtained in a natural way in a finite region of the parameter space of the superpotential.

ACKNOWLEDGMENTS

Two of us (M. P. and A. U.) would like to thank the organizers and the participants of the ''XLIVth Rencontres de Moriond 2009—Electroweak Interactions and Unified Theories'' where part of this work has been performed.

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