Computation of the p^6 order chiral Lagrangian coefficients

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We present results of computing the p^6 order low energy constants in the normal part of chiral Lagrangian both for two and three flavor pseudoscalar mesons. This is a generalization of our previous work on calculating the p^4 order coefficients of the chiral Lagrangian in terms of the quark self-energy $\Sigma(p^2)$. We show that most of our results are consistent with those we can find in the literature.

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I. INTRODUCTION

The chiral Lagrangian for low lying pseudoscalar mesons [1,2] as the most successful effective field theory is now widely used in various strong, weak, and electromagnetic processes. To match the increasing demand for higher precision in a low energy description of QCD, the applications of the low energy expansion of the chiral Lagrangian are extended from early time discussions on the leading p^2 and next to leading p^4 orders to present p^6 order. For the latest review, see Ref. [3]. In the chiral Lagrangian, there are many unknown phenomenological low energy constants (LECs) which appear in front of each Goldstone field dependent operator and the number of the LECs increases rapidly when we go to the higher orders of the low energy expansion. For example for the three flavor case, the p^2 and p^4 order chiral Lagrangians have two and ten LECs respectively, while the normal part of the p^6 order chiral Lagrangian has 90 LECs. Such a large number of LECs is very difficult to fix from the experiment data. This badly reduces the predictive power of the chiral Lagrangian and blurs the check of its convergence. The area of estimating p^6 order LECs is where most improvement is needed in the future of higher order chiral Lagrangian calculations.

A way to increase the precision of the low energy expansion and improve the present embarrassed situation is studying the relation between the chiral Lagrangian and the fundamental principles of QCD. We expect that this relation will be helpful for understanding the origin of these LECs and further offer us their values. In a previous paper [4], based on an earlier study of deriving the chiral Lagrangian from the first principles of QCD [5] in which LECs are defined in terms of certain Green's functions in QCD, we have developed techniques and calculated the p^2 and p^4 order LECs. Our simple approach involves the approximations of taking the large-N_c limit, the leading order in dynamical perturbation theory, and the improved ladder approximation, thereby the relevant Green's functions related to LECs are expressed in terms of the quark self-energy $\Sigma(p^2)$. The resulting chiral Lagrangian in terms of the quark self-energy is proved equivalent to a gauge invariant, nonlocal, dynamical (GND) quark model [6]. By solving the Schwinger-Dyson equation (SDE) for $\Sigma(p^2)$, we obtain the approximate LECs which are consistent with the experimental values. With these results, generalization of the calculations to p^6 order LECs becomes the next natural step. Considering that the algebraic derivations for those formulas to express LECs in terms of the quark self-energy at p^4 order are lengthy (they need at least several months of handwork), it is almost impossible to achieve the similar works for the p^6 order calculations just by hand. Therefore, to realize the calculations for the p^6 order LECs, we need to computerize the original calculations and this is a very hard task. The key difficulty comes from the fact that the formulation developed in Ref. [7] and exploited in Ref. [4] does not automatically keep the local chiral covariance of the theory, and one has to adjust the calculation procedure by hand to realize the covariance of the results. To match with the computer program, we need to change the original formulation to a chiral covariant one. In Refs. [8–10], we have built and developed such a formulation, followed by next several year's efforts, we now successfully encode the formulation into computer programs. With the help of these computer codes we can reproduce analytical results on the computer originally derived by hand in Ref. [4] within 15 minutes now. This not only confirms the reliability of the program itself, but also checks the correctness of our original formulas. Based on this progress, in this paper we generalize our previous works on calculating the p^4 order LECs to computing the p^6 order LECs of chiral Lagrangian both for two and three flavor pseudoscalar mesons. This generalization not only produces new numerical predictions for the p^6 order LECs, but also forces us to reexamine our original formulation from a new angle in dealing with p^2 and p^4 order LECs.

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This paper is organized as follows: In Sec. II, we review our previous calculations on the p^2 and p^4 order LECs. Then, in Sec. III, based on the technique developed in Ref. [8], we reformulate the original low energy expansion used in Ref. [4] into a chiral covariant one suitable for computer derivation. In Sec. IV, from the present p^6 order viewpoint, we reexamine the formulation we had taken before and show that if we sum all higher order anomaly part contribution terms together, their total contributions to the normal part of the chiral Lagrangian vanish. This leads to a change in the role of finite p^4 order anomaly part contributions which originally are subtracted in the chiral Lagrangian in Ref. [4] and now must be used to cancel divergent higher order anomaly part contributions. We reexhibit the numerical result of the p^4 order LECs without subtraction of p^4 order anomaly part contributions. In Sec. V, we present the general p^6 order chiral Lagrangian in terms of rotated sources and express the p^6 order LECs in terms of the quark self-energy. Section VI is where we give numerical results for p^6 order LECs in the normal part of the chiral Lagrangian both for two and three flavor pseudoscalar mesons. In Sec. VII, we apply and compare with our results to those of some individuals and combinations of LECs proposed and estimated in the literature, checking the correctness of our numerical predictions. Section VIII is a summary. In the Appendices, we list some necessary relations among our symbols and those used in the literature.

II. REVIEW OF THE CALCULATIONS ON THE p^2 AND p^4 ORDER LECS

Theoretically, the action of the chiral Lagrangian at the large N_c limit derived from the first principle of QCD takes the form [5]

$$\begin{aligned} S_{\rm eff} = & -iN_c \,{\rm Tr}\ln[i\not\partial + J_\Omega - \Pi_{\Omega c}] + iN_c\,{\rm Tr}\ln[i\not\partial + J_\Omega] \\ & - iN_c\,{\rm Tr}\ln[i\not\partial + J] + N_c\,{\rm Tr}[\Phi_{\Omega c}\Pi^T_{\Omega c}] \\ & + N_c\sum_{n=2}^{\infty}\int d^4x_1\cdots d^4x'_n\frac{(-i)^n(N_c g_s^2)^{n-1}}{n!} \\ & \times \bar{G}^{\sigma_1\cdots\sigma_n}_{\rho_1\cdots\rho_n}(x_1,x'_1,\cdots,x_n,x'_n) \\ & \times \Phi^{\sigma_1\rho_1}_{\Omega c}(x_1,x'_1)\cdots\Phi^{\sigma_n\rho_n}_{\Omega c}(x_n,x'_n) + O\Big(\frac{1}{N_c}\Big) \end{aligned} \tag{1}$$

in which J_Ω is the external source J including currents and densities after Goldstone field dependent chiral rotation Ω:

$$\begin{aligned} J_\Omega &= [\Omega P_R + \Omega^\dagger P_L][J + i\not\partial][\Omega P_R + \Omega^\dagger P_L] \\ &= \not v_\Omega + \not a_\Omega\gamma_5 - s_\Omega + ip_\Omega\gamma_5 \\ J &= \not v + \not a\gamma_5 - s + ip\gamma_5 \qquad U = \Omega^2. \end{aligned} \tag{2}$$

$\Phi_{\Omega c}$ and $\Pi_{\Omega c}$ are two point rotated quark Green's functions and the interaction part of two point rotated quark vertices in the presence of external sources, respectively; $\Phi_{\Omega c}$ is defined by

$$\begin{aligned} \Phi^{\sigma\rho}_{\Omega c}(x,y) &\equiv \frac{1}{N_c}\langle\bar{\psi}^\sigma_\Omega(x)\psi^\rho_\Omega(y)\rangle \\ &= -i[(i\not\partial + J_\Omega - \Pi_{\Omega c})^{-1}]^{\rho\sigma}(y,x) \\ \psi_\Omega(x) &\equiv [\Omega(x)P_L + \Omega^\dagger(x)P_R]\psi(x) \end{aligned} \tag{3}$$

with subscript $_c$ denoting the classical field and $\psi(x)$ being light quark fields. $\bar{G}^{\sigma_1\cdots\sigma_n}_{\rho_1\cdots\rho_n}(x_1,x'_1,\cdots,x_n,x'_n)$ is the effective gluon n-point Green's function and g_s is the coupling constant of QCD. It can be shown that the term in the third to fifth lines and the last term in the second line of the right-hand side (rhs) of Eq. (1) are independent of pseudoscalar meson field U or Ω and therefore are just irrelevant constants in the effective action, while the second term in the first line and the first term in the second line of the rhs of Eq. (1) are anomaly part contributions, since they represent the variations of the path integral measure for light quark fields ψ. The remaining first term is called normal part contribution which relies on $\Pi_{\Omega c}$. The $\Phi_{\Omega c}$ and $\Pi_{\Omega c}$ are related by the first equation of (3) and determined by

$$\begin{aligned} [\Phi_{\Omega c} + \tilde{\Xi}]^{\sigma\rho} &+ \sum_{n=1}^{\infty}\int d^4x_1 d^4x'_1\cdots d^4x_n d^4x'_n \\ & \times\frac{(-i)^{n+1}(N_c g_s^2)^n}{n!}\bar{G}^{\sigma\sigma_1\cdots\sigma_n}_{\rho\rho_1\cdots\rho_n}(x,y,x_1,x'_1,\cdots,x_n,x'_n) \\ & \times\Phi^{\sigma_1\rho_1}_{\Omega c}(x_1,x'_1)\cdots\Phi^{\sigma_n\rho_n}_{\Omega c}(x_n,x'_n) = O\Big(\frac{1}{N_c}\Big), \end{aligned} \tag{4}$$

where $\tilde{\Xi}$ is a Lagrangian multiplier which ensures the constraint ${\rm tr}_l[\gamma_5\Phi^T_{\Omega c}(x,x)] = 0$. Equation (4) is the SDE in the presence of the rotated external source. In Ref. [4], we have assumed the solution of (4) approximately by

$$\begin{aligned} \Pi^{\sigma\rho}_{\Omega c}(x,y) &= [\Sigma(\bar{\nabla}^2_x)]^{\sigma\rho}\delta^4(x-y) \\ \bar{\nabla}^\mu_x &= \partial^\mu_x - iv^\mu_\Omega(x), \end{aligned} \tag{5}$$

where Σ is the quark self-energy which satisfies SDE (4) with vanishing rotated external source. Under the ladder approximation, this SDE in Euclidean space-time is reduced to the standard form of

$$\Sigma(p^2) - 3C_2(R)\int\frac{d^4q}{4\pi^3}\frac{\alpha_s[(p-q)^2]}{(p-q)^2}\frac{\Sigma(q^2)}{q^2+\Sigma^2(q^2)} = 0, \tag{6}$$

where $C_2(R)$ is the second order Casimir operator of the quark representation R, in our case, quark is belonging to the $SU(N_c)$ fundamental representation, therefore $C_2(R) = (N_c^2-1)/2N_c$ and in the large N_c limit, we will neglect the second term of it. $\alpha_s(p^2)$ is the running coupling constant of QCD which depends on N_c and quark flavor. With these strong and not everywhere justified approximations, the resulting action (1) of the chiral Lagrangian becomes the GND model introduced in Ref. [6],

$$S_{\rm eff} \approx S_{\rm GND} + O\Big(\frac{1}{N_c}\Big)$$
$$S_{\rm GND} \equiv -iN_c \,{\rm Tr}\ln[i\not\partial + J_\Omega - \Sigma(\bar{\nabla}^2)] + iN_c\,{\rm Tr}\ln[i\not\partial + J_\Omega] - iN_c\,{\rm Tr}\ln[i\not\partial + J]. \quad (7)$$

in which the third term on the rhs of (7) is independent of pseudoscalar field U; therefore it only affects the contact term of the chiral Lagrangian. In fact, for the contact term part, we can take $\Omega = 1$ in (7), then

$$S_{\rm eff}|_{\rm contact} \approx -iN_c\,{\rm Tr}\ln\{i\not\partial + J - \Sigma[(\partial - iv)^2)]\} + O\Big(\frac{1}{N_c}\Big). \quad (8)$$

For the noncontact terms concerned in this paper, we can ignore the third term on the rhs of (7) and the next key element is to compute term ${\rm Tr}\ln[i\not\partial + J_\Omega - \Sigma(\bar{\nabla}^2)]$. The remaining term ${\rm Tr}\ln[i\not\partial + J_\Omega]$ in our previous work is obtained by further taking the limit $\Sigma \to 0$ in ${\rm Tr}\ln[i\not\partial + J_\Omega - \Sigma(\bar{\nabla}^2)]$.[1] Since anomaly terms are at least the p^4 order and at this order, anomaly is the well-known Wess-Zumino terms which have no unknown LECs (In Ref. [11], we have derived such terms from $S_{\rm GND}$). All unknown LECs at p^2 and p^4 orders are in the normal part of the chiral Lagrangian, so to calculate the p^2 and p^4 order LECs, we only need to discuss the normal part of the chiral Lagrangian which is in fact the real part of ${\rm Tr}\ln(\cdots)$. With the help of the Schwinger proper time method [7], this real part in Euclidean space-time[2] with metric tensor $g^{\mu\nu} = {\rm diag}(1, 1, 1, 1)$ can be written as

$$\begin{aligned}
{\rm Re\,Tr}\ln[\not\partial - i\not v_\Omega - i\not a_\Omega\gamma_5 - s_\Omega + ip_\Omega\gamma_5 + \Sigma(-\bar{\nabla}^2)] &= \frac{1}{2}{\rm Tr}\ln[[\not\partial - i\not v_\Omega - i\not a_\Omega\gamma_5 - s_\Omega + ip_\Omega\gamma_5 + \Sigma(-\bar{\nabla}^2)]^\dagger \\
&\quad \times[\not\partial - i\not v_\Omega - i\not a_\Omega\gamma_5 - s_\Omega + ip_\Omega\gamma_5 + \Sigma(-\bar{\nabla}^2)]] \\
&= -\frac{1}{2}\lim_{\Lambda\to\infty}\int_{1/\Lambda^2}^{\infty}\frac{d\tau}{\tau}\int d^4x\,{\rm tr}\langle x|\exp[-\tau[\bar{E} - (\bar{\nabla} - ia_\Omega)^2 \\
&\quad + \Sigma^2(-\bar{\nabla}^2) + \hat{I}_\Omega\Sigma(-\bar{\nabla}^2) + \Sigma(-\bar{\nabla}^2)\tilde{I}_\Omega - \not d\Sigma(-\bar{\nabla}^2)]]|x\rangle \\
&= -\frac{1}{2}\lim_{\Lambda\to\infty}\int_{1/\Lambda^2}^{\infty}\frac{d\tau}{\tau}\int d^4x\int\frac{d^4k}{(2\pi)^4} \\
&\quad \times{\rm tr}\exp[-\tau[\bar{E} + (k + i\bar{\nabla}_x + a_\Omega)^2 + \Sigma^2((k + i\bar{\nabla}_x)^2) \\
&\quad + \hat{I}_\Omega\Sigma((k + i\bar{\nabla}_x)^2) + \Sigma((k + i\bar{\nabla}_x)^2)\tilde{I}_\Omega - \not d\Sigma((k + i\bar{\nabla}_x)^2)]],
\end{aligned} \quad (9)$$

in which

$$\bar{E} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]R^{\mu\nu} + \gamma_\mu d^\mu(s_\Omega - ip_\Omega\gamma_5) + i\gamma_\mu[a_\Omega^\mu\gamma_5(s_\Omega - ip_\Omega\gamma_5) + (s_\Omega - ip_\Omega\gamma_5)a_\Omega^\mu\gamma_5] + s_\Omega^2 + p_\Omega^2 - [s_\Omega, p_\Omega]i\gamma_5$$
$$d^\mu\mathcal{O} \equiv \partial^\mu\mathcal{O} - i[v_\Omega^\mu, \mathcal{O}] \quad (\mathcal{O} = \text{any operator}) \qquad R^{\mu\nu} = V_\Omega^{\mu\nu} - i[a_\Omega^\mu, a_\Omega^\nu] + (d^\mu a_\Omega^\nu - d^\nu a_\Omega^\mu)\gamma_5$$
$$V_{\Omega,\mu\nu} = i[\bar{\nabla}_\mu, \bar{\nabla}_\nu] \qquad \bar{\nabla}_x^\mu = \partial^\mu - iv_\Omega^\mu(x) \qquad \hat{I}_\Omega = -i\not a_\Omega\gamma_5 - s_\Omega - ip_\Omega\gamma_5$$
$$\tilde{I}_\Omega = -i\not a_\Omega\gamma_5 - s_\Omega + ip_\Omega\gamma_5 \qquad \not d\Sigma(-\bar{\nabla}^2) = \gamma_\mu d^\mu\Sigma(-\bar{\nabla}^2).$$

In (9), a cutoff Λ is introduced into the theory to regularize the possible ultraviolet divergences. In practical calculations, we treat it as the physical cutoff of the theory. Taking the low energy expansion for (9), we can finally express ${\rm Tr}\ln[i\not\partial + J_\Omega - \Sigma(\bar{\nabla}^2)]$ in terms of power expansion of external sources with coefficients being Σ dependent functions. Further vanishing Σ, we obtain ${\rm Tr}\ln[i\not\partial + J_\Omega]$. Then the rhs of (7) is expressed in terms of power expansion of rotated external sources; compare the result with the parametrization of the effective action without applying the equations of motion for pseudoscalar mesons,

[1]This will cause some confusion and we are going to discuss them in Sec. IV.

[2]Our extension from Minkovski space to Euclidean space takes $x^0|_M \to -ix^4|_E$, $x^i|_M \to x^i|_M$, $\gamma^0|_M \to \gamma^4|_E$, $\gamma^i|_M \to i\gamma^i|_E$, with $i = 1, 2, 3$ being space indices and there γ_E^μ are Hermitian. v_Ω^μ, a_Ω^μ transform as x^μ. $\gamma_5|_M \to \gamma_5|_E$, $s|_M \to -s|_E$, $p|_M \to -p|_E$.

$$S_{\rm eff} = \int d^4x\,{\rm tr}_f[F_0^2 a_\Omega^2 + F_0^2 B_0 s_\Omega - \mathcal{K}_1^{({\rm norm},\Pi_{\Omega c}\neq 0)}[d_\mu a_\Omega^\mu]^2 - \mathcal{K}_2^{({\rm norm},\Pi_{\Omega c}\neq 0)}(d^\mu a_\Omega^\nu - d^\nu a_\Omega^\mu)(d_\mu a_{\Omega,\nu} - d_\nu a_{\Omega,\mu}) + \mathcal{K}_3^{({\rm norm},\Pi_{\Omega c}\neq 0)}[a_\Omega^2]^2 + \mathcal{K}_4^{({\rm norm},\Pi_{\Omega c}\neq 0)} a_\Omega^\mu a_\Omega^\nu a_{\Omega,\mu} a_{\Omega,\nu} + \mathcal{K}_5^{({\rm norm},\Pi_{\Omega c}\neq 0)} a_\Omega^2\,{\rm tr}_f[a_\Omega^2] + \mathcal{K}_6^{({\rm norm},\Pi_{\Omega c}\neq 0)} a_\Omega^\mu a_\Omega^\nu\,{\rm tr}_f[a_{\Omega,\mu} a_{\Omega,\nu}] + \mathcal{K}_7^{({\rm norm},\Pi_{\Omega c}\neq 0)} s_\Omega^2 + \mathcal{K}_8^{({\rm norm},\Pi_{\Omega c}\neq 0)} s_\Omega\,{\rm tr}_f[s_\Omega] + \mathcal{K}_9^{({\rm norm},\Pi_{\Omega c}\neq 0)} p_\Omega^2 + \mathcal{K}_{10}^{({\rm norm},\Pi_{\Omega c}\neq 0)} p_\Omega\,{\rm tr}_f[p_\Omega] + \mathcal{K}_{11}^{({\rm norm},\Pi_{\Omega c}\neq 0)} s_\Omega a_\Omega^2 + \mathcal{K}_{12}^{({\rm norm},\Pi_{\Omega c}\neq 0)} s_\Omega\,{\rm tr}_f[a_\Omega^2] - \mathcal{K}_{13}^{({\rm norm},\Pi_{\Omega c}\neq 0)} V_\Omega^{\mu\nu} V_{\Omega,\mu\nu} + i\mathcal{K}_{14}^{({\rm norm},\Pi_{\Omega c}\neq 0)} V_\Omega^{\mu\nu} a_{\Omega,\mu} a_{\Omega,\nu} + \mathcal{K}_{15}^{({\rm norm},\Pi_{\Omega c}\neq 0)} p_\Omega d_\mu a_\Omega^\mu] + O(p^6) + U\text{-independent source terms.} \quad (10)$$

We can read out F_0^2, B_0, and $\mathcal{K}_i^{({\rm norm},\Pi_{\Omega c}\neq 0)}$ for $i = 1,\ldots,15$ as functions of Σ. $\mathcal{K}_i^{({\rm norm},\Pi_{\Omega c}\neq 0)}$ relate to the conventional p^4 order LECs through (25) of Ref. [4]. A superscript $^{({\rm norm},\Pi_{\Omega c}\neq 0)}$ on each of $\mathcal{K}_i$ denotes the property that when $\Pi_{\Omega c} = \Sigma = 0$, all $\mathcal{K}_i$ vanish, i.e.

$$\mathcal{K}_i^{({\rm norm},\Pi_{\Omega c}\neq 0)} = \mathcal{K}_i^{({\rm norm})} - \mathcal{K}_i^{({\rm norm},\Pi_{\Omega c}=0)} \qquad \mathcal{K}_i^{({\rm norm})} \overset{\Sigma=0}{=} \mathcal{K}_i^{({\rm norm},\Pi_{\Omega c}=0)} \qquad i = 1,\ldots,15, \quad (11)$$

where $\mathcal{K}_i^{({\rm norm})}$ and $-\mathcal{K}_i^{({\rm norm},\Pi_{\Omega c}=0)}$ are the contributions to the effective action from the first and second terms and the third terms in the rhs of (7), respectively. Replacing superscript $^{({\rm norm},\Pi_{\Omega c}\neq 0)}$ with $^{({\rm norm})}$ in the rhs of (10), we obtain term $-iN_c\,{\rm Tr}\ln[i\not\partial + J_\Omega - \Sigma(\bar\nabla^2)]$. Replacing superscript $^{({\rm norm},\Pi_\Omega\neq 0)}$ with $^{({\rm norm},\Pi_\Omega=0)}$ and vanishing F_0^2 in the rhs of (10), we obtain term $-iN_c\,{\rm Tr}\ln[i\not\partial + J_\Omega] + iN_c\,{\rm Tr}\ln[i\not\partial + J]$. The resulting formulas for $F_0^2B_0$, F_0^2, and $\mathcal{K}_i^{({\rm norm})}$ expressed in terms of Σ are explicitly given in (34), (35), and (36) in Ref. [4].

With the analytical formulas for LECs of F_0^2, B_0, and $\mathcal{K}_i^{({\rm norm},\Pi_{\Omega c}\neq 0)}$ for $i = 1,\ldots,15$ as functions of Σ, we can suitably choose running coupling constant $\alpha_s(p^2)$, solve SDE (6) numerically obtaining quark self-energy Σ, then calculate the numerical values of all p^2 and p^4 order LECs. To obtain the final numerical result in Ref. [4], we have assumed $F_0 = f_\pi = 93$ MeV as input[3] to fix the dimensional parameter $\Lambda_{\rm QCD}$ appearing in running coupling constant $\alpha_s(p^2)$ and taken cutoff parameter Λ appearing in (9) equal to infinity and 1 GeV, respectively. The final obtained values are consistent with those fixed phenomenologically.

III. CHIRAL COVARIANT LOW ENERGY EXPANSION

Equation (9) is the starting point of our reformulation in this section. In Ref. [4], we expand (9) up to the p^4 order and obtain the analytical result. This expansion is not explicitly chiral covariant, since the operator $\bar\nabla_x^\mu$ which appears in the formula is not always covariant under the local chiral symmetry transformations. For example, when $\bar\nabla_x^\mu$ acts on a constant number 1, it gives $\bar\nabla_x^\mu 1 = -iv_\Omega^\mu(x)$ which is not covariant since $v_\Omega^\mu(x)$ itself behaves as the gauge field in the local chiral symmetry transformations. Only when they combined into commutators, such as $[\bar\nabla_x^\mu, \bar\nabla_x^\nu]$ or $[\bar\nabla_x^\mu, a_\Omega^\nu(x)]$, the covariance recovers back. Therefore in the detailed calculation, we need to confirm that all $\bar\nabla_x^\mu$ appearing in the result can be arranged into some commutators. This is a conjecture. In the original work of Ref. [4], we have found that this conjecture is valid up to some terms with coefficients being expressed as integration over some total derivatives, i.e. form of $\int d^4k \frac{\partial}{\partial k^\mu} g(k)$. If we ignore these total derivative terms, up to order of p^4, we can explicitly prove the conjecture. At the stage of our earlier works, we did not question the reason why we can drop out those total derivative terms [In fact, in Eq. (74) of Ref. [5], we have shown that in order to obtain the well-known Pagels-Stokar formula, a total derivative term must be dropped out.] This leads to further discussions on the role of total derivative terms in the quantum field theory [10]. Later in this section, we will give the correct reason for dropping out those total derivative terms. Arranging various $\bar\nabla_x^\mu$ into commutators is a very tricky and complex task which is very hard to be achieved by computer. In order to computerize the calculation, we need to find a way which can automatically arrange all $\bar\nabla_x^\mu$ into some commutators. This leads the developments given in Refs. [8–10], where we have introduced

$$k^\mu + i\bar\nabla_x^\mu = e^{i\bar\nabla_x\cdot\partial/\partial k}\bigg(k^\mu + \tilde F^\mu\bigg(\bar\nabla, \frac{\partial}{\partial k}\bigg)\bigg)e^{-i\bar\nabla_x\cdot\partial/\partial k}, \quad (12)$$

in which

[3] Later we will use a changed value $F_0 = 87$ for the two-flavor case. For detail, see the discussion of Eq. (58).

$$\tilde{F}^{\mu}\left(\bar{\nabla},\frac{\partial}{\partial k}\right) \equiv -e^{Ad(-i\bar{\nabla}_x\cdot\partial/\partial k)}\left(F\left[Ad\left(i\bar{\nabla}_x\cdot\frac{\partial}{\partial k}\right)\right](i\bar{\nabla}_x^{\mu})\right)$$
$$= \frac{1}{2}(\nu\mu)\frac{\partial}{\partial k^{\nu}} - \frac{i}{3}(\lambda\nu\mu)\frac{\partial^2}{\partial k^{\lambda}\partial k^{\nu}} - \frac{1}{8}(\rho\lambda\nu\mu)\frac{\partial^3}{\partial k^{\rho}\partial k^{\lambda}\partial k^{\nu}} + \frac{i}{30}(\sigma\rho\lambda\nu\mu)\frac{\partial^4}{\partial k^{\sigma}\partial k^{\rho}\partial k^{\lambda}\partial k^{\nu}}$$
$$+ \frac{1}{144}(\delta\sigma\rho\lambda\nu\mu)\frac{\partial^5}{\partial k^{\delta}\partial k^{\sigma}\partial k^{\rho}\partial k^{\lambda}\partial k^{\nu}} + O(p^7),$$
$$F(z) = \sum_{n=2}^{\infty}\frac{z^{n-1}}{n!} \qquad [Ad(B)]^n(C) \equiv \underbrace{[B,[B,\cdots,[B}_{n\text{ times}},C]\cdots]],$$
$$(\mu_n\mu_{n-1}\cdots\mu_2\mu_1) \equiv [\bar{\nabla}_x^{\mu_n},[\bar{\nabla}_x^{\mu_{n-1}},\cdots,[\bar{\nabla}_x^{\mu_2},\bar{\nabla}_x^{\mu_1}]\cdots]], \tag{13}$$

where the default set of Lorentz indices for $(\mu_n\mu_{n-1}\cdots\mu_2\mu_1)$ is the superscripts. In some cases, we need subscripts. We will use $\underline{\mu}$ to denote the corresponding subscript for μ. Note that in the present notation for $(\mu_n\mu_{n-1}\cdots\mu_2\mu_1)$, we do not explicitly write $\bar{\nabla}_x$, but only their Greek superscripts for short. If we use other symbols, such as s_Ω which appeared in (μs_Ω) and a_Ω^ν in (μa_Ω^ν), then we take the definition that $(\mu s_\Omega) \equiv [\bar{\nabla}_x^\mu, s_\Omega]$ and $(\mu a_\Omega^\nu) \equiv [\bar{\nabla}_x^\mu, a_\Omega^\nu]$.

Substitute (12) into (9); we change (9) to

$$\mathrm{Re\,Tr\,ln}[\not\partial - i\not v_\Omega - i\not a_\Omega\gamma_5 - s_\Omega + ip_\Omega\gamma_5 + \Sigma(-\bar{\nabla}^2)]$$
$$= -\frac{1}{2}\lim_{\Lambda\to\infty}\int_{1/\Lambda^2}^{\infty}\frac{d\tau}{\tau}\int d^4x\int\frac{d^4k}{(2\pi)^4}\mathrm{tr}e^{i\bar{\nabla}_x^\mu\cdot\partial/\partial k}$$
$$\times\exp\{-\tau[\tilde{E} + (k+\tilde{F})^2 + \tilde{a}_{\underline{\mu}}\gamma_5(k^\mu + \tilde{F}^\mu)$$
$$+ (k^\mu + \tilde{F}^\mu)\tilde{a}_{\underline{\mu}}\gamma_5 + \tilde{a}^2 + \Sigma^2((k+\tilde{F})^2) + \tilde{J}\Sigma((k+\tilde{F})^2)$$
$$+ \Sigma((k+\tilde{F})^2)\tilde{K} - \gamma_{\underline{\mu}}[\tilde{\bar{\nabla}}_x^{\mu},\Sigma((k+\tilde{F})^2)]]\}\cdot 1 \tag{14}$$

with tilde operation defined as

$$\tilde{\mathcal{O}} \equiv \mathcal{O} - i(\nu\mathcal{O})\frac{\partial}{\partial k^\nu} - \frac{1}{2}(\lambda\nu\mathcal{O})\frac{\partial^2}{\partial k^\lambda\partial k^\nu} + \frac{i}{6}(\rho\lambda\nu\mathcal{O})$$
$$\times\frac{\partial^3}{\partial k^\rho\partial k^\lambda\partial k^\nu} + \frac{1}{24}(\sigma\rho\lambda\nu\mathcal{O})\frac{\partial^4}{\partial k^\sigma\partial k^\rho\partial k^\lambda\partial k^\nu}$$
$$- \frac{i}{120}(\delta\sigma\rho\lambda\nu\mathcal{O})\frac{\partial^5}{\partial k^\delta\partial k^\sigma\partial k^\rho\partial k^\lambda\partial k^\nu} + O(p^7), \tag{15}$$

where $\tilde{\mathcal{O}} \equiv (\tilde{E},\tilde{J},\tilde{K},\tilde{a}^\mu,\tilde{a}^2,\tilde{\bar{\nabla}}_x^\mu)^T$ and $\mathcal{O} \equiv (E,\hat{I},\tilde{I},a_\Omega^\mu,a_\Omega^2,\bar{\nabla}_x^\mu)^T$. Note that for finite cutoff Λ, the value of parameter τ must be real and larger than zero; the term $e^{-\tau k^2}$ in (14) then provides a natural suppression factor for the momentum integration and this leads the convergence of the integration. For a converged integration, we can replace the term $e^{i\bar{\nabla}_x^\mu\cdot\partial/\partial k}$ in front of the integration kernel in (14) by 1, since the difference $(e^{i\bar{\nabla}_x^\mu\cdot\partial/\partial k} - 1)\cdots$ is some momentum total derivative terms which vanish as long as we have nontrivial suppression factor $e^{-\tau k^2}$. With these considerations, (14) becomes

$$\mathrm{Re\,Tr\,ln}[\not\partial - i\not v_\Omega - i\not a_\Omega\gamma_5 - s_\Omega + ip_\Omega\gamma_5 + \Sigma(-\bar{\nabla}^2)]$$
$$= -\frac{1}{2}\lim_{\Lambda\to\infty}\int_{1/\Lambda^2}^{\infty}\frac{d\tau}{\tau}\int d^4x\int\frac{d^4k}{(2\pi)^4}\mathrm{tr}e^B\cdot 1, \tag{16}$$

$$B \equiv -\tau[\tilde{E} + (k+\tilde{F})^2 + \tilde{a}_{\underline{\mu}}\gamma_5(k^\mu + \tilde{F}^\mu) + (k^\mu + \tilde{F}^\mu)\tilde{a}_{\underline{\mu}}\gamma_5$$
$$+ \tilde{a}^2 + \Sigma^2((k+\tilde{F})^2) + \tilde{J}\Sigma((k+\tilde{F})^2)$$
$$+ \Sigma((k+\tilde{F})^2)\tilde{K} - \gamma_{\underline{\mu}}[\tilde{\bar{\nabla}}_x^\mu,\Sigma((k+\tilde{F})^2)]]. \tag{17}$$

From (17), we see that all $\bar{\nabla}^\mu$ in (16) appear as commutators, therefore (16) and (17) offer a covariant formulation which matches the general result that the real part of $\mathrm{Tr\,ln}\cdots$ should be invariant under local chiral transformations. The price is that we need to handle many momentum derivatives on the exponential and the resulting computations become extremely lengthy. But as long as our reformulation is suitable to computerize, it is worth paying such a price. To deal with the next problem of derivatives on the exponential, we first take the low energy expansion on B,

$$B = B_0 + B_1 + \frac{1}{2}B_2 + \frac{1}{3!}B_3 + \frac{1}{4!}B_4 + \frac{1}{5!}B_5 + \frac{1}{6!}B_6 + \cdots, \tag{18}$$

where $\frac{1}{n!}B_n$ is the p^n order part of B. We further introduce a parameter t dependent $B(t)$ as

$$B(t) = B_0 + tB_1 + \frac{t^2}{2}B_2 + \frac{t^3}{3!}B_3 + \frac{t^4}{4!}B_4 + \frac{t^5}{5!}B_5 + \frac{t^6}{6!}B_6 + \cdots$$
$$B = B(t)|_{t=1}. \tag{19}$$

Then take Taylor expansion of $e^{B(t)}$ at point $t = 0$,

$$e^B = e^{B(t)}|_{t=1}$$
$$= e^{B_0} + \left[\frac{d}{dt}e^{B(t)}\right]_{t=0} + \frac{1}{2!}\left[\frac{d^2}{dt^2}e^{B(t)}\right]_{t=0}$$
$$+ \frac{1}{3!}\left[\frac{d^3}{dt^3}e^{B(t)}\right]_{t=0} + \frac{1}{4!}\left[\frac{d^4}{dt^4}e^{B(t)}\right]_{t=0} + \cdots. \tag{20}$$

With the help of identities

$$\left[\frac{d}{d\tau}e^{B}\right]e^{-B} = f(Ad(B))\left(\frac{dB}{d\tau}\right)$$
$$f(z) = \frac{e^z - 1}{z} = 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots. \quad (21)$$

One can explicitly work out $\frac{1}{n!}[\frac{d^n}{dt^n}e^{B(t)}]_{t=0}$, for several lowest orders

$$\frac{d}{dt}e^{B(t)}|_{t=0} = e^{B_0}f[Ad(-B_0)](B_1), \quad (22)$$

$$\frac{d^2}{dt^2}e^{B(t)}|_{t=0} = \frac{d}{dt}e^{B(t)}|_{t=0}f[Ad(-B_0)](B_1) + e^{B_0}\frac{df}{dt}[Ad(-B(t))]|_{t=0}(B_1) + e^{B_0}f[Ad(-B_0)](B_2), \quad (23)$$

where

$$\frac{d^m f}{dt^m}[Ad(-B(t))] = \sum_{n=0}^{\infty}\frac{1}{(n+1)!}\frac{d^m}{dt^m}[Ad(-B(t))]^n. \quad (24)$$

For the more higher orders needed in our computations, to save space we do not write them down in the present paper; the results of $\frac{d^3}{dt^3}e^{B(t)}|_{t=0}$, $\frac{d^4}{dt^4}e^{B(t)}|_{t=0}$, $\frac{d^5}{dt^5}e^{B(t)}|_{t=0}$, and $\frac{d^6}{dt^6}e^{B(t)}|_{t=0}$ can be found in Appendix A of Ref. [12].

With the help of (22), (23), and (A1)–(A4) of Ref. [12], as long as the B_0, B_1, B_2, B_3, B_4, B_5, B_6 are known, (20) is known and we can substitute it back into (16) to calculate the real part of $-iN_c\,\mathrm{Tr}\ln[i\not\partial + J_\Omega - \Sigma(\bar{\nabla}^2)]$ order by orders up to the p^6 order in the low energy expansion. To obtain B_i, (17) tells us that the difficulty is the low energy expansion for $\Sigma((k+\tilde{F})^2)$. To achieve it, we expand the argument of $\Sigma((k+\tilde{F})^2)$ as

$$(k+\tilde{F})^2 = k^2 + \frac{1}{2}A_2 + \frac{1}{6}A_3 + \frac{1}{24}A_4 + \frac{1}{120}A_5 + \frac{1}{720}A_6 + O(p^{5,6})|_{\mathrm{traceless}} + O(p^7), \quad (25)$$

in which

$$A_2 = -2(\mu\nu)k_\mu\frac{\partial}{\partial k^\nu}, \quad (26)$$

$$A_3 = 4i(\mu\nu\lambda)k_\nu\frac{\partial^2}{\partial k^\mu\partial k^\lambda} + 2i(\mu\underline{\mu}\nu)\frac{\partial}{\partial k^\nu}, \quad (27)$$

$$A_4 = 6(\mu\nu\lambda\rho)k_\lambda\frac{\partial^3}{\partial k^\mu\partial k^\nu\partial k^\rho} + 6(\mu\nu)(\underline{\mu}\lambda)\frac{\partial^2}{\partial k^\nu\partial k^\lambda} + 3(\mu\nu\underline{\nu}\lambda)\frac{\partial^2}{\partial k^\mu\partial k^\lambda} + 3(\mu\nu\underline{\mu}\lambda)\frac{\partial^2}{\partial k^\nu\partial k^\lambda}, \quad (28)$$

$$A_5 = 0, \quad (29)$$

$$A_6 = -90(\mu\nu\lambda\rho)(\underline{\lambda}\sigma)\frac{\partial^4}{\partial k^\mu\partial k^\nu\partial k^\rho\partial k^\sigma} - 80(\mu\nu\lambda)(\rho\underline{\nu}\sigma)\frac{\partial^4}{\partial k^\mu\partial k^\lambda\partial k^\rho\partial k^\sigma}. \quad (30)$$

Since we are only interested in the terms not higher than p^6, we find that those traceless terms of p^5 and p^6 orders will not make contributions to the final result. So to save space and simplify the computations, we do not explicitly write down the detailed structure of them, just represent these terms with symbol $O(p^{5,6})|_{\mathrm{traceless}}$ and remove the traceless term in A_5 and A_6. We further introduce $A(t)$ as

$$A(t) \equiv k^2 + \frac{t^2}{2}A_2 + \frac{t^3}{6}A_3 + \frac{t^4}{24}A_4 + \frac{t^5}{120}A_5 + \frac{t^6}{720}A_6 + O(p^{5,6})|_{\mathrm{traceless}} + O(p^7)$$
$$A(t) = \begin{cases} k^2 & t = 0 \\ (k+\tilde{F})^2 & t = 1. \end{cases} \quad (31)$$

Then

$$\Sigma((k+\tilde{F})^2) = \Sigma(k^2) + \left[\frac{d}{dt}\Sigma[A(t)]\right]_{t=0} + \frac{1}{2!}\left[\frac{d^2}{dt^2}\Sigma[A(t)]\right]_{t=0} + \frac{1}{3!}\left[\frac{d^3}{dt^3}\Sigma[A(t)]\right]_{t=0} + \frac{1}{4!}\left[\frac{d^4}{dt^4}\Sigma[A(t)]\right]_{t=0} + \cdots. \quad (32)$$

Now, we need to know $[\frac{d^m}{dt^m}\Sigma[A(t)]]_{t=0}$, using the following formula:

$$\Sigma[A(t)] = \Sigma[s + A(t)]|_{s=0} = e^{A(t)[\partial/\partial s]}\Sigma(s)e^{-A(t)[\partial/\partial s]}|_{s=0} = e^{A(t)[\partial/\partial s]}\Sigma(s)|_{s=0}, \quad (33)$$

then

$$\left[\frac{d^m}{dt^m}\Sigma[A(t)]\right]_{t=0} = \left[\frac{d^m}{dt^m}e^{A(t)[\partial/\partial s]}\right]_{t=0}\Sigma(s)|_{s=0}. \quad (34)$$

Therefore to compute $[\frac{d^m}{dt^m}\Sigma[A(t)]]_{t=0}$, we only need to calculate $[\frac{d^m}{dt^m}e^{A(t)\partial/\partial s}]_{t=0}\Sigma(s)|_{s=0}$ which is just equivalent to replacing $B_l \to A_l\frac{\partial}{\partial s}$ in (22), (23), and (A1)–(A4) of Ref. [12], followed by multiplying an extra factor $\Sigma(s)$ at the rhs and vanishing parameter s after finishing all differential operations. Following this calculation road map, the detailed calculation gives

$$\left[\frac{d}{dt}\Sigma[A(t)]\right]_{t=0} = e^{Ad\{A_0[\partial/\partial s]\}}\left(f\left[Ad\left(-A_0\frac{\partial}{\partial s}\right)\right]A_1\right) \times \Sigma'(s + A_0)|_{s=0} = 0, \quad (35)$$

$$\frac{1}{2}\left[\frac{d^2}{dt^2}\Sigma[A(t)]\right]_{t=0} = \frac{1}{2}e^{Ad\{A_0[\partial/\partial s]\}}\left[\left(e^{-A_0[\partial/\partial s]}\frac{d}{dt}e^{A(t)[\partial/\partial s]}|_{t=0}f\left[Ad\left(-A_0\frac{\partial}{\partial s}\right)\right]\left(A_1\frac{\partial}{\partial s}\right)+\frac{df}{dt}\left[Ad\left(-A(t)\frac{\partial}{\partial s}\right)\right]\Big|_{t=0}\right.\right.$$
$$\left.\left.\times\left(A_1\frac{\partial}{\partial s}\right)+f\left[Ad\left(-A_0\frac{\partial}{\partial s}\right)\right]\left(A_2\frac{\partial}{\partial s}\right)\right)\right]\Sigma(s+A(t))|_{s=0}$$
$$= -(\mu,\nu)k_\mu\Sigma'_k\frac{\partial}{\partial k^\nu}, \tag{36}$$

where $\Sigma_k \equiv \Sigma(k^2)$. For more higher orders, we list the results of $[\frac{d^3}{dt^3}\Sigma[A(t)]]_{t=0}$, $[\frac{d^4}{dt^4}\Sigma[A(t)]]_{t=0}$, $[\frac{d^5}{dt^5}\Sigma[A(t)]]_{t=0}$, and $[\frac{d^6}{dt^6}\Sigma[A(t)]]_{t=0}$ in Appendix A of Ref. [12]. With these results, we finally obtain the low energy expansion of B:

$$B_0 = -\tau(k^2+\Sigma_k^2), \tag{37}$$

$$B_1 = 2\tau(-a_\Omega^\mu k_\mu + ia_\Omega^\mu\gamma_\mu\Sigma_k)\gamma_5, \tag{38}$$

$$B_2 = -2a_\Omega^2\tau + (\mu\nu)\gamma_\mu\gamma_\nu\tau - a_\Omega^\mu a_\Omega^\nu[\gamma_\mu,\gamma_\nu]\tau - i(d^\mu a_\Omega^\nu - d^\nu a_\Omega^\mu)\gamma_\mu\gamma_\nu\gamma_5\tau + 4s_\Omega\tau\Sigma_k + 2(\mu\nu)\tau k_\mu\frac{\partial}{\partial k^\nu}$$
$$+ 4(\mu a_\Omega^\nu)\gamma_\nu\gamma_5\tau k_\mu\Sigma'_k + 4(\mu a_\Omega^\nu)\gamma_\nu\gamma_5\tau\Sigma_k\frac{\partial}{\partial k^\mu} + 2i(\mu a_{\Omega\mu})\gamma_5\tau + 4i(\mu a_\Omega^\nu)\gamma_5\tau k_\nu\frac{\partial}{\partial k^\mu}$$
$$+ 4i(\mu\nu)\gamma_\mu\tau k_\nu\Sigma'_k + 4(\mu\nu)\tau k_\mu\Sigma_k\Sigma'_k\frac{\partial}{\partial k^\nu}. \tag{39}$$

We list B_3, B_4, B_5, and B_6 in Appendix A of Ref. [12]. With these explicit expressions for B_0, B_1, B_2, B_3, B_4, B_5, B_6, using (22), (23), and (A1)–(A4) of Ref. [12], we get (20) and further substitute (20) back into (16); we can obtain the real part of $-iN_c\,\mathrm{Tr}\ln[i\not{\partial} + J_\Omega - \Sigma(\bar{\nabla}^2)]$ order by orders up to the p^6 order in the low energy expansion. The analytical results of p^2 and p^4 orders are the same as those given by (34), (35), and (36) in Ref. [4], except some total derivative terms which, as we mentioned before, can be ignored as long as we take finite cutoff Λ.

IV. AMBIGUITIES IN THE ANOMALY PART CONTRIBUTIONS TO THE CHIRAL LAGRANGIAN

In the last section, we introduced a chiral covariant method to calculate $\mathrm{Tr}\ln[i\not{\partial} + J_\Omega - \Sigma(\bar{\nabla}^2)]$ which is already computerized now. With the help of the computer, for the p^2 and p^4 order analytical formulas in the low energy expansion, we can get results within 15 minutes, while for the p^6 order terms, we need roughly 13 hours to output all expansion results. From our general result (7), the term $-iN_c\,\mathrm{Tr}\ln[i\not{\partial} + J_\Omega - \Sigma(\bar{\nabla}^2)]$ is the normal part. To get the full result of the chiral Lagrangian, we need to calculate the remaining anomaly part contributions $iN_c\,\mathrm{Tr}\ln[i\not{\partial} + J_\Omega] - iN_c\,\mathrm{Tr}\ln[i\not{\partial} + J]$. As the discussion of Ref. [13], in the 1980s there was a class of works (see Ref. [14]) identifying this part as the full chiral Lagrangian, and in Ref. [13] we refer to them as the anomaly approach of calculating LECs. In our previous work [4], we pointed out that these anomaly part contributions are completely canceled by the normal part contribution, left nontrivial pure Σ dependent terms contribute to the chiral Lagrangian.

For the anomaly part contributions, the key is to calculate the U field dependent term $\mathrm{Tr}\ln[i\not{\partial} + J_\Omega]$ which, as we mentioned before, can be obtained by vanishing Σ in $\mathrm{Tr}\ln[i\not{\partial} + J_\Omega - \Sigma(\bar{\nabla}^2)]$. In practice, the limit was taken by first assuming Σ as a constant mass m and then letting $m \to 0$. For p^2 order, this operation gives a null result, while for p^4 order, it gives the result originally presented in the anomaly approach. Now in this work, naively what we need to do is to generalize the calculation to p^6 order. But to our surprise, we get many terms with divergent coefficients. Checking the calculation carefully, we find that the reason for appearance of infinities is due to the fact that most of the coefficients in front of the p^6 order operators have dimension of $1/m^2$ which goes to infinity when we take the limit $\Sigma = m \to 0$. Note that the p^6 terms may also have coefficients of $1/\Lambda^2$ which are finite in the limit of $m \to 0$, although they vanish when we take $\Lambda \to \infty$. These terms are irrelevant to our discussion on the divergence of p^6 order terms and therefore we do not need to care about them. Applying the argument on $1/m^2$ dependence of the p^6 order coefficients back to the p^2 and p^4 order results we discussed before, coefficients in front of p^2 order operators have dimension of m^2 which goes to zero; this explains the phenomena that anomaly approach cannot produce p^2 order terms. For p^4 order, the coefficients in front of operators are dimensionless and therefore the m dependence is at most logarithmic of the form $\ln m/\Lambda$ which implies existence of a logarithmic ultraviolet divergence. Since we know that in the large N_c limit, the p^4 order LECs (noncontact coefficients) are not divergent, the $\ln m/\Lambda$ term then cannot appear in the final expression of these LECs, therefore in p^4 order, anomaly approach leads finite result LECs. In general for a p^{2n} order operator, the corresponding coefficient should have dimension

$1/m^{2(n-2)}$. This implies that the infinity in the anomaly part contributions will be a general phenomena, when we go to the higher orders of the low energy expansion, since the higher the order is, the more negative powers of m dependence the coefficient will have and these negative powers of m will result in infinities as we take limit $\Sigma = m \to 0$.

The appearance of these high order infinities provides more evidence that the anomaly approach is not a correct formulation in calculating LECs, at least not for the p^6 and more higher order LECs. Since the high order divergence term is an additional part of our general result (7), we cannot avoid them in our computations. How should we deal with these high order infinities from negative powers of m? There exists an alternative way, not relying on the low energy expansion, to examine these anomaly part contributions in which we must exploit the first equation of (2) and we find

$$\begin{aligned}\mathrm{Tr}\ln[i\not\partial + J_\Omega] - \mathrm{Tr}\ln[i\not\partial + J] &= \ln\mathrm{Det}[i\not\partial + J_\Omega] - \ln\mathrm{Det}[i\not\partial + J] \\ &= \ln\mathrm{Det}[[\Omega P_R + \Omega^\dagger P_L][J + i\not\partial][\Omega P_R + \Omega^\dagger P_L]] \\ &\quad - \ln\mathrm{Det}[i\not\partial + J] \\ &= \ln\mathrm{Det}[[\Omega P_R + \Omega^\dagger P_L][\Omega P_R + \Omega^\dagger P_L]] \\ &= \mathrm{Tr}\ln[[\Omega P_R + \Omega^\dagger P_L][\Omega P_R + \Omega^\dagger P_L]]. \end{aligned} \tag{40}$$

For our interests, we are only interested in the real part of it, then

$$\begin{aligned}\mathrm{Re}\ \mathrm{Tr}\ln[i\not\partial + J_\Omega] - \mathrm{Re}\,\mathrm{Tr}\ln[i\not\partial + J] &= \tfrac{1}{2}\mathrm{Tr}\ln[[\Omega P_R + \Omega^\dagger P_L][\Omega P_R + \Omega^\dagger P_L]] \\ &\quad + \tfrac{1}{2}\mathrm{Tr}\ln[[\Omega P_R + \Omega^\dagger P_L]^\dagger[\Omega P_R + \Omega^\dagger P_L]^\dagger] \\ &= \tfrac{1}{2}\mathrm{Tr}\ln[[\Omega P_R + \Omega^\dagger P_L][\Omega P_R + \Omega^\dagger P_L]] \\ &\quad + \tfrac{1}{2}\mathrm{Tr}\ln[[\Omega^\dagger P_R + \Omega P_L][\Omega^\dagger P_R + \Omega P_L]] \\ &= \tfrac{1}{2}\mathrm{Tr}\ln[[P_R + P_L][P_R + P_L]] = \tfrac{1}{2}\mathrm{Tr}\ln 1 = 0, \end{aligned} \tag{41}$$

which shows that the compact form of anomaly part contributions to the normal part of the chiral Lagrangian is zero.

How can this null result be consistent with another divergent result obtained from the low energy expansion? The only possible explanation is that the p^4 order finite term plus all those higher order infinities results a zero ! Is it possible? A well-known positive example is the expansion $1/(1+x) = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - \cdots$ goes to zero when x is very large, which implies that the summation of series $x - x^2 + x^3 - x^4 + x^5 - x^6 + \cdots$ converges to 1 when x is very large and each individual term in the series diverges. In the following we take a more realistic but simplified example to show that this really happens in our formulation. Our example starts from (9) for the case of Σ equal to a constant mass m:

$$\begin{aligned}\mathrm{Re}\ \mathrm{Tr}\ln[\not\partial - i\not v_\Omega - i\not a_\Omega\gamma_5 - s_\Omega + ip_\Omega\gamma_5 + m] &= -\frac{1}{2}\lim_{\Lambda\to\infty}\int_{1/\Lambda^2}^{\infty}\frac{d\tau}{\tau} \\ &\quad \times \int d^4x\int\frac{d^4k}{(2\pi)^4}\,\mathrm{tr}e^{-\tau(k^2 + k\cdot b' + m^2 + bm + C)}, \end{aligned} \tag{42}$$

$$b'^\mu \equiv 2i\bar{\nabla}_x^\mu + 2a_\Omega^\mu \qquad b \equiv \hat{I}_\Omega + \tilde{I}_\Omega \qquad C \equiv \bar{E} + (i\bar{\nabla}_x + a_\Omega)^2. \tag{43}$$

For simplicity, we ignore the contributions from b' which does not change the key result of our discussion. Then our example becomes investigating the following integration:

$$I \equiv \int_{1/\Lambda^2}^{\infty}\frac{d\tau}{\tau}\int_0^\infty k^2dk^2e^{-\tau k^2 - \tau m^2 - \tau bm - \tau C} \tag{44}$$

with b and C not commuting each other. We will show that high order terms in the low energy expansion of the above integration I will go to infinity when we take $m \to 0$, but if summing all the expansion terms together, we get a finite result which corresponds to the previous null result of summing all higher order terms of anomaly part contributions into a compact form. We use three different methods to finish the above integration and explain our point. The first method is to vanish m first and then to finish the integration, i.e.,

$$\begin{aligned} I|_{m=0} &= \int_{1/\Lambda^2}^{\infty}\frac{d\tau}{\tau}\int_0^\infty k^2dk^2e^{-\tau k^2 - \tau C} = \int_{1/\Lambda^2}^{\infty}\frac{d\tau}{\tau^3}e^{-\tau C} \\ &= \frac{\Lambda^4}{2}e^{-C/\Lambda^2} - \frac{\Lambda^2}{2}Ce^{-C/\Lambda^2} - \frac{1}{2}C^2\mathbf{Ei}\left(-\frac{C}{\Lambda^2}\right), \end{aligned} \tag{45}$$

where

$$\begin{aligned}\mathbf{Ei}(-x) &\equiv -\int_x^\infty\frac{e^{-u}}{u}du \\ &= \gamma + \ln x + \sum_{n=1}^{\infty}\frac{(-1)^n}{n!n}x^n \qquad |x| < \infty. \end{aligned} \tag{46}$$

The second method is first finishing the integration and then vanishing m,

$$I = \int_{1/\Lambda^2}^{\infty}\frac{d\tau}{\tau}\int_0^{\infty}k^2dk^2e^{-\tau(k^2+m^2+bm+C)} = \int_{1/\Lambda^2}^{\infty}\frac{d\tau}{\tau^3}e^{-\tau(m^2+bm+C)}$$
$$= \frac{\Lambda^4}{2}e^{-(m^2+bm+C)^2/\Lambda^2} - \frac{\Lambda^2(m^2+bm+C)}{2}e^{-(m^2+bm+C)^2/\Lambda^2} - \frac{1}{2}(m^2+bm+C)^2\mathbf{Ei}\left(-\frac{m^2+bm+C}{\Lambda^2}\right)$$
$$\overset{m\to 0}{=}\frac{\Lambda^4}{2}e^{-C/\Lambda^2} - \frac{\Lambda^2}{2}Ce^{-C/\Lambda^2} - \frac{1}{2}C^2\mathbf{Ei}\left(-\frac{C}{\Lambda^2}\right). \tag{47}$$

We obtain the same result as that obtained in the first method, therefore interchanging the order of integration and the $m \to 0$ limit does not change the result.

The third method is first taking Taylor expansion in terms of the power of b and C which corresponds to performing the low energy expansion and then finishing integration, finally vanishing m,

$$I = \int_{1/\Lambda^2}^{\infty}\frac{d\tau}{\tau}\int_0^{\infty}k^2dk^2e^{-\tau k^2-\tau m^2}\sum_{n=0}^{\infty}\frac{1}{n!}(-\tau bm-\tau C)^n = \int_{1/\Lambda^2}^{\infty}d\tau e^{-\tau m^2}\sum_{n=0}^{\infty}\frac{1}{n!}\tau^{n-3}(-bm-C)^n$$
$$= m^4\left(\frac{\Lambda^4}{2m^4}e^{-m^2/\Lambda^2} - \frac{\Lambda^2}{2m^2}e^{-m^2/\Lambda^2} - \frac{1}{2}\mathbf{Ei}\left(-\frac{m^2}{\Lambda^2}\right)\right) - (bm^3+Cm^2)\left(\frac{\Lambda^2}{m^2}e^{-m^2/\Lambda^2} + \mathbf{Ei}\left(-\frac{m^2}{\Lambda^2}\right)\right)$$
$$-\frac{1}{2}(bm+C)^2\mathbf{Ei}\left(-\frac{m^2}{\Lambda^2}\right) + \sum_{n=0}^{\infty}\frac{(-\frac{b}{m}-\frac{C}{m^2})^{n+3}}{(n+3)!}m^4\Gamma\left(n+1,\frac{m^2}{\Lambda^2}\right)$$
$$\overset{m\to 0}{=}\frac{\Lambda^4}{2} - C\Lambda^2 - \frac{1}{2}C^2\mathbf{Ei}\left(-\frac{m^2}{\Lambda^2}\right) + \sum_{n=0}^{\infty}\frac{n!}{(n+3)!}\left(-\frac{b}{m}-\frac{C}{m^2}\right)^{n+3}m^4e^{-m^2/\Lambda^2}\sum_{k=0}^{n}\frac{1}{k!}\left(\frac{m^2}{\Lambda^2}\right)^k\Bigg|_{m\to 0}. \tag{48}$$

We see that there are negative powers of m terms which will cause divergence when we take limit $m \to 0$. This is just what has happened for the high order terms in the anomaly part contributions. So if we calculate term by terms in above expansion, we will meet infinities which seems to contradict with results obtained in the first two methods. The only way left to escape this contradiction is to sum all these divergences together; to see what will happen after the summation, we introduce a series,

$$g(x,c) \equiv \sum_{n=0}^{\infty}\frac{n!}{(n+3)!}x^{n+3}\sum_{k=0}^{n}\frac{c^k}{k!}, \tag{49}$$

in which $c = m^2/\Lambda^2$ and $x = -\frac{b}{m} - \frac{C}{m^2}$, which will go to negative infinity when $m \to 0$. With the help of relation $\frac{d}{dx}\mathbf{Ei}(-x) = \frac{e^{-x}}{x}$ and boundary condition $g''(0,c) = g'(0,c) = g(0,c) = 0$, we find

$$g'''(x,c) = \sum_{n=0}^{\infty}x^n\sum_{k=0}^{n}\frac{c^k}{k!} = \frac{e^{cx}}{1-x},$$
$$g''(x,c) = e^c[-\mathbf{Ei}(cx-c) + \mathbf{Ei}(-c)],$$
$$g'(x,c) = (x-1)e^c[-\mathbf{Ei}(cx-c) + \mathbf{Ei}(-c)] + \frac{1}{c}(e^{cx}-1),$$
$$g(x,c) = \frac{1}{2}(x-1)^2e^c[-\mathbf{Ei}(cx-c) + \mathbf{Ei}(-c)] + \frac{x-1}{2c}e^{cx} + \frac{1}{2c} + \frac{1}{2c^2}(e^{cx}-1) - \frac{x}{c}. \tag{50}$$

Then (48) becomes

$$I|_{m=0} = \lim_{m\to 0}\left[\frac{\Lambda^4}{2} - C\Lambda^2 - \frac{1}{2}C^2\mathbf{Ei}\left(-\frac{m^2}{\Lambda^2}\right) + m^4e^{-m^2/\Lambda^2}g\left(-\frac{b}{m}-\frac{C}{m^2},\frac{m^2}{\Lambda^2}\right)\right]$$
$$= \lim_{m\to 0}\left[\frac{\Lambda^4}{2} - C\Lambda^2 - \frac{1}{2}C^2\mathbf{Ei}\left(-\frac{m^2}{\Lambda^2}\right) + \frac{1}{2}(bm+C+m^2)^2\left[-\mathbf{Ei}\left(\frac{-bm-C-m^2}{\Lambda^2}\right) + \mathbf{Ei}\left(-\frac{m^2}{\Lambda^2}\right)\right]\right.$$
$$\left.+\frac{1}{2}\Lambda^2(-bm-C-m^2)e^{(-bm-C-m^2)/\Lambda^2} + \frac{1}{2}\Lambda^2m^2e^{-m^2/\Lambda^2} + \frac{1}{2}\Lambda^4(e^{(-bm-C-m^2)/\Lambda^2} - e^{-m^2/\Lambda^2}) + \Lambda^2(bm+C)\right]$$
$$= -\frac{1}{2}C^2\mathbf{Ei}\left(-\frac{C}{\Lambda^2}\right) - \frac{1}{2}\Lambda^2Ce^{-C/\Lambda^2} + \frac{1}{2}\Lambda^4e^{-C/\Lambda^2}. \tag{51}$$

It is the same as the results obtained from the first two methods, i.e. summing all those infinities together, we obtain a correct finite result.

With the above discussion, our result now is that the *total anomaly part contributions to the normal part of the chiral Lagrangian vanish*. Just taking several individual terms cannot reflect the true result of the full action. In fact, finite result of the p^4 order plays a role to cancel the summations of all higher order terms. Finally, it gets the total summation zero. In this sense, in order to make sense for the p^6 and more higher order divergent terms, we must sum them together and then we get the p^4 order result with an extra minus sign. To avoid the appearance of divergences in p^6 and higher orders terms, what we need to do is to drop out all anomaly part contributions, since divergences from high order terms are finally canceled by p^4 order terms. In this view, our general result (7) must be changed to

$$S_{\mathrm{eff}}\big|_{\text{normal part}} \approx -iN_c \,\mathrm{Re}\,\mathrm{Tr}\ln[i\not{\bar{\nabla}} + J_\Omega - \Sigma(\bar{\nabla}^2)]. \quad (52)$$

In fact in Ref. [11], we already show that including the total effective action in the anomalous part takes the form [see Eq. (21) in Ref. [11]]

$$S_{\mathrm{GND}} = -iN_c \,\mathrm{Tr}\ln[i\not{\bar{\nabla}} + J_\Omega - \Sigma(\bar{\nabla}^2)] + \text{Wess-Zumino terms.} \quad (53)$$

With this new viewpoint on all anomaly part contributions, we need to modify our original numerical results on p^4 order LECs, since it takes into account the finite values of anomaly part contributions, and now we know that these nontrivial values must be used to cancel the infinities coming from all higher order terms. In Table I, we list our modified p^4 order LECs for cutoff $\Lambda = 1000^{+100}_{-100}$ MeV. The 10% variation of the cutoff is considered in our calculation to examine the effects of cutoff dependence and the result change can be treated as the error of our calculations. The result LECs are taken the values at $\Lambda = 1$ GeV. The superscript is the LEC's difference caused at $\Lambda = 1.1$ GeV and the subscript is the difference caused at $\Lambda = 0.9$ GeV, i.e.,

$$L_{\Lambda=1\ \mathrm{GeV}}\Big|^{L_{\Lambda=1.1\ \mathrm{GeV}}-L_{\Lambda\neq1\ \mathrm{GeV}}}_{L_{\Lambda=0.9\ \mathrm{GeV}}-L_{\Lambda=1\ \mathrm{GeV}}}$$
$$\bar{l}_{\Lambda=1\ \mathrm{GeV}}\Big|^{\bar{l}_{\Lambda=1.1\ \mathrm{GeV}}-\bar{l}_{\Lambda=1\ \mathrm{GeV}}}_{\bar{l}_{\Lambda=0.9\ \mathrm{GeV}}-\bar{l}_{\Lambda=1\ \mathrm{GeV}}} \quad \text{or} \quad l_{\Lambda=1\ \mathrm{GeV}}\Big|^{l_{\Lambda=1.1\ \mathrm{GeV}}-l_{\Lambda=1\ \mathrm{GeV}}}_{l_{\Lambda=0.9\ \mathrm{GeV}}-l_{\Lambda=1\ \mathrm{GeV}}}. \quad (54)$$

In obtaining the result, we have taken the running coupling constant as model A given by (40) of Ref. [4] and the low energy value of this α_s is already chosen well above the critical value to trigger the $S\chi SB$ of the theory. It should be noted that α_s depends on the number of quark flavors, so does for Σ from SDE. In fixing the Λ_{QCD}, we have taken input $F_0 = 87$ MeV. The reason for taking this value is that if the final F_π is around the value of 93 MeV, then our formula shows F_0 must be located around 87 MeV. In Sec. VI, we will exhibit this phenomena explicitly.

V. p^6 ORDER OF CHIRAL LAGRANGIAN: NORMAL PART

The general form of the p^6 order chiral Lagrangian was first introduced in Ref. [15] and then discussed in Ref. [16]. Now we can express the normal part of it in terms of our rotated sources as we have done in (10) for the p^4 order terms. Considering that our computation is done under the large N_c limit, within this approximation, terms in the chiral Lagrangian with two and more traces vanish when we do not apply the equation of motion. To avoid unnecessary complicities, in this paper we only write down those terms with one trace,

$$S_{\mathrm{eff}}\big|_{p^6,\mathrm{normal}} = \int d^4x \left[\sum_{n=1}^{94} \mathcal{Z}_n \,\mathrm{tr}_f[\bar{O}_n] + O\left(\frac{1}{N_c}\right)\right], \quad (55)$$

with $\bar{O}_n$ being the p^6 order operator we could get in our calculation and $\mathcal{Z}_n$ being the corresponding coefficient; $O(\frac{1}{N_c})$ consist of most multitrace terms. Our computations then give the explicit expressions of $\mathcal{Z}_n$ in terms of quark self-energy. The detailed expressions are given in (B1) of Ref. [12]. The definitions of operators $\bar{O}_n$ for $n = 1, 2, \ldots, 94$ are given in Table II, where some operators have i in front of them to ensure their coefficients being real. In Ref. [16], the p^6 order operator was denoted by Y_i for the case of n flavor with coefficient K_i [17], O_i for the case of three flavor with coefficient C_i, and P_i for the case of two flavors with coefficient c_i,

TABLE I. The obtained values of the p^4 coefficients $L_1 \ldots, L_{10}$ for three flavor quarks and $\bar{l}_1 \ldots, \bar{l}_6, l_7$ for two flavor quarks where $l_i = \frac{1}{32\pi^2}\gamma_i(\bar{l}_i + \ln\frac{M_\pi^2}{\mu^2})$ for $i = 1, \ldots, 7$, $\mu = 770$ MeV, and γ_i are given in Ref. [2]. Since $\gamma_7 = 0$, we calculate l_7 instead of $\bar{l}_7$. We collect the experimental values given in Ref. [2] and our old result given in Ref. [4] for comparisons. Λ_{QCD}, and Λ and $-\langle\bar{\psi}\psi\rangle^{1/3}$ are in units of MeV, and $L_1 \ldots, L_{10}, l_7$ are in units of 10^{-3}.

	Λ_{QCD}	$-\langle\bar{\psi}\psi\rangle^{1/3}$	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
$\Lambda = 1000^{+100}_{-100}$	453^{-6}_{+12}	260^{-8}_{+9}	$1.23^{+0.03}_{-0.04}$	$2.46^{+0.05}_{-0.08}$	$-6.85^{-0.14}_{+0.21}$	$0.0^{+0.0}_{-0.0}$	$1.48^{-0.01}_{-0.03}$	$0.0^{+0.0}_{-0.0}$	$-0.51^{+0.05}_{-0.06}$	$1.02^{-0.06}_{+0.06}$	$8.86^{+0.24}_{-0.37}$	$-7.40^{-0.29}_{+0.44}$
Ref. [4]:	484	296	0.403	0.805	−3.47	0	1.47	0	−0.792	1.83	2.28	−4.08
Experiment		250	0.9 ± 0.3	1.7 ± 0.7	-4.4 ± 2.5	0 ± 0.5	2.2 ± 0.5	0 ± 0.3	-0.4 ± 0.15	1.1 ± 0.3	7.4 ± 0.7	-6.0 ± 0.7

	Λ_{QCD}	$-\langle\bar{\psi}\psi\rangle^{1/3}$	$\bar{l}_1$	$\bar{l}_2$	$\bar{l}_3$	$\bar{l}_4$	$\bar{l}_5$	$\bar{l}_6$	l_7
$\Lambda = 1000^{+100}_{-100}$	465^{-6}_{+12}	227^{-6}_{+8}	$-4.77^{-0.17}_{+0.24}$	$8.01^{+0.09}_{-0.14}$	$1.97^{+0.29}_{-0.35}$	$4.34^{-0.01}_{-0.02}$	$17.35^{+0.53}_{-0.80}$	$19.98^{+0.44}_{-0.67}$	$-8.18^{+0.50}_{-0.43}$
Experiment		250	-2.3 ± 3.7	6.0 ± 1.3	2.9 ± 2.4	4.3 ± 0.9	13.9 ± 1.3	16.5 ± 1.1	$O(5)$

$$
S_{\mathrm{eff}}\big|_{p^6,\mathrm{normal}} = \int d^4x \begin{cases} \sum_{i=1}^{112} K_i Y_i + 3 \text{ contact terms} & n \text{ flavors} \\ \sum_{i=1}^{90} C_i O_i + 4 \text{ contact terms} & \text{three flavors} \\ \sum_{i=1}^{53} c_i P_i + 4 \text{ contact terms} & \text{two flavors.} \end{cases} \tag{56}
$$

We list down the relations among symbols used in the present paper and those in Ref. [16] in Table XV of Appendix A. Consider that our parametrization of the p^6 order chiral Lagrangian (55) is general to the case of n flavor quarks; there exist some relations among our coefficients and n flavor coefficients given in (56). With the help of computer derivations, we have worked out these relations and list them in Appendix B of Ref. [12]. As a check, we vanish the quark self-energy Σ in the codes which correspond to taking $m = 0$ before other further calculations and find the null p^6 result. This verifies the analytical result discussed in Sec. IV that the anomaly part contributions do not contribute to the normal part of the chiral Lagrangian. Another consistency check is done for those operators which have two terms combined together by C and P symmetry requirements. For the n flavor case, such operators are $\bar{O}_9$, $\bar{O}_{11}$, $\bar{O}_{12}$, $\bar{O}_{14}$, $\bar{O}_{17}$, $\bar{O}_{18}$, $\bar{O}_{19}$, $\bar{O}_{20}$, $\bar{O}_{28}$, $\bar{O}_{29}$, $\bar{O}_{32}$, $\bar{O}_{34}$, $\bar{O}_{35}$, $\bar{O}_{37}$, $\bar{O}_{38}$, $\bar{O}_{39}$, $\bar{O}_{40}$, $\bar{O}_{44}$, $\bar{O}_{45}$, $\bar{O}_{46}$, $\bar{O}_{47}$, $\bar{O}_{48}$, $\bar{O}_{49}$, $\bar{O}_{73}$, $\bar{O}_{76}$, $\bar{O}_{77}$, $\bar{O}_{78}$, $\bar{O}_{80}$, $\bar{O}_{82}$, $\bar{O}_{84}$, $\bar{O}_{85}$. Since in each of these operators there are two terms, we can compute the coefficients in front of each term and check if they are the same. We have done the checks for all of these operators and all obtain the same analytical expressions for the two terms in the same operator. This partly verifies the correctness of our result given in (B1) of Ref. [12]. From n flavors to three flavors, there are some extra constraints [see (B1) in Ref. [16]] which make some operators depending on others. Further from three flavors to two flavors, there are also some extra constraints [see (B3) in Ref. [16]] which make some more operators depending on

TABLE II. p^6 order operators.

n	$\bar{O}_n$	n	$\bar{O}_n$	n	$\bar{O}_n$
1	$(a_\Omega^2)^3$	33	$a_\Omega^\mu a_\Omega^\nu a_{\Omega\mu} d_\nu p_\Omega$	65	$d^2 a_\Omega^\nu d_\nu p_\Omega$
2	$a_\Omega^2 a_\Omega^\nu a_\Omega^\lambda a_{\Omega\nu} a_{\Omega\lambda}$	34	$a_\Omega^\mu a_\Omega^\nu (d_\mu a_{\Omega\nu} p_\Omega + p_\Omega d_\nu a_{\Omega\mu})$	65	$d^\mu d^\nu a_{\Omega\nu} d_\mu p_\Omega$
3	$a_\Omega^2 a_\Omega^\nu a_\Omega^2 a_{\Omega\nu}$	35	$a_\Omega^\mu a_\Omega^\nu (d_\nu a_{\Omega\mu} p_\Omega + p_\Omega d_\mu a_{\Omega\nu})$	67	$d^\mu s_\Omega d_\mu s_\Omega$
4	$a_\Omega^\mu a_\Omega^\nu a_{\Omega\mu} a_\Omega^\lambda a_{\Omega\nu} a_{\Omega\lambda}$	36	$a_\Omega^\mu p_\Omega a_{\Omega\mu} d^\nu a_{\Omega\nu}$	68	$d^\mu p_\Omega d_\mu p_\Omega$
5	$a_\Omega^\mu a_\Omega^\nu a_\Omega^\lambda a_{\Omega\mu} a_{\Omega\nu} a_{\Omega\lambda}$	37	$a_\Omega^\mu p_\Omega a_\Omega^\nu (d_\mu a_{\Omega\nu} + d_\nu a_{\Omega\mu})$	69	$iV_\Omega^{\mu\nu} V_{\Omega\mu}{}^\lambda V_{\Omega\nu\lambda}$
6	$a_\Omega^2 (d^\nu a_{\Omega\nu})^2$	38	$a_\Omega^\mu (d_\mu a_\Omega^\nu d_\nu s_\Omega + d^\nu s_\Omega d_\mu a_{\Omega\nu})$	70	$V_\Omega^{\mu\nu} V_{\Omega\mu\nu} a_\Omega^2$
7	$a_\Omega^2 d^\nu a_\Omega^\lambda d_\nu a_{\Omega\lambda}$	39	$a_\Omega^\mu (d^\nu a_{\Omega\mu} d_\nu s_\Omega + d^\nu s_\Omega d_\nu a_{\Omega\mu})$	71	$V_\Omega^{\mu\nu} V_{\Omega\mu}{}^\lambda a_{\Omega\nu} a_{\Omega\lambda}$
8	$a_\Omega^2 d_\nu a_\Omega^\lambda d_\lambda a_{\Omega\nu}$	40	$a_\Omega^\mu (d^\nu a_{\Omega\nu} d_\mu s_\Omega + d_\mu s_\Omega d^\nu a_{\Omega\nu})$	72	$V_\Omega^{\mu\nu} V_{\Omega\mu}{}^\lambda a_{\Omega\lambda} a_{\Omega\nu}$
9	$a_\Omega^\mu a_\Omega^\nu (d_\mu a_{\Omega\nu} d^\lambda a_{\Omega\lambda} + d^\lambda a_{\Omega\lambda} d_\nu a_{\Omega\mu})$	41	$(a_\Omega^2)^2 s_\Omega$	73	$V_\Omega^{\mu\nu} (a_{\Omega\mu} V_{\Omega\nu}{}^\lambda a_{\Omega\lambda} - a_\Omega^\lambda V_{\Omega\mu\lambda} a_{\Omega\nu})$
10	$a_\Omega^\mu a_\Omega^\nu d_\mu a_\Omega^\lambda d_\nu a_{\Omega\lambda}$	42	$a_\Omega^\mu a_\Omega^\nu a_{\Omega\mu} a_{\Omega\nu} s_\Omega$	74	$V_\Omega^{\mu\nu} a_\Omega^\lambda V_{\Omega\mu\nu} a_{\Omega\lambda}$
11	$a_\Omega^\mu a_\Omega^\nu (d_\mu a_\Omega^\lambda d_\lambda a_{\Omega\nu} + d^\lambda a_{\Omega\mu} d_\nu a_{\Omega\lambda})$	43	$a_\Omega^\mu a_\Omega^2 a_{\Omega\mu} s_\Omega$	75	$V_\Omega^{\mu\nu} V_{\Omega\mu\nu} s_\Omega$
12	$a_\Omega^\mu a_\Omega^\nu (d_\nu a_{\Omega\mu} d^\lambda a_{\Omega\lambda} + d^\lambda a_{\Omega\lambda} d_\mu a_{\Omega\nu})$	44	$ia_\Omega^\mu (d_\mu a_\Omega^\nu d^\lambda V_{\Omega\nu\lambda} + d^\nu V_{\Omega\nu}{}^\lambda d_\mu a_{\Omega\lambda})$	76	$iV_\Omega^{\mu\nu} (a_{\Omega\mu} d_\nu p_\Omega + d_\mu p_\Omega a_{\Omega\nu})$
13	$a_\Omega^\mu a_\Omega^\nu d_\nu a_\Omega^\lambda d_\mu a_{\Omega\lambda}$	45	$ia_\Omega^\mu (d^\nu a_{\Omega\mu} d^\lambda V_{\Omega\nu\lambda} + d^\nu V_{\Omega\nu}{}^\lambda d_\lambda a_{\Omega\mu})$	77	$iV^{\mu\nu} (p_\Omega d_\mu a_{\Omega\nu} - d_\mu a_{\Omega\nu} p_\Omega)$
14	$a_\Omega^\mu a_\Omega^\nu (d_\nu a_\Omega^\lambda d_\lambda a_{\Omega\mu} + d^\lambda a_{\Omega\nu} d_\mu a_{\Omega\lambda})$	46	$ia_\Omega^\mu (d^\nu a_{\Omega\nu} d^\lambda V_{\Omega\mu\lambda} - d^\nu V_{\Omega\mu\nu} d^\lambda a_{\Omega\lambda})$	78	$iV_\Omega^{\mu\nu} (d_\mu a_{\Omega\nu} d^\lambda a_{\Omega\lambda} - d^\lambda a_{\Omega\lambda} d_\mu a_{\Omega\nu})$
15	$a_\Omega^\mu a_\Omega^\nu d^\lambda a_{\Omega\mu} d_\lambda a_{\Omega\nu}$	47	$ia_\Omega^\mu (d^\nu a_\Omega^\lambda d_\mu V_{\Omega\nu\lambda} - d_\mu V_\Omega^{\nu\lambda} d_\nu a_{\Omega\lambda})$	79	$iV_\Omega^{\mu\nu} d_\mu a_\Omega^\lambda d_\nu a_{\Omega\lambda}$
16	$a_\Omega^\mu a_\Omega^\nu d^\lambda a_{\Omega\nu} d_\lambda a_{\Omega\mu}$	48	$ia_\Omega^\mu (d^\nu a_\Omega^\lambda d_\nu V_{\Omega\mu\lambda} - d^\nu V_{\Omega\mu}{}^\lambda d_\nu a_{\Omega\lambda})$	80	$iV_\Omega^{\mu\nu} (d_\mu a_\Omega^\lambda d_\lambda a_{\Omega\nu} + d^\lambda a_{\Omega\mu} d_\nu a_{\Omega\lambda})$
17	$a_\Omega^\mu (d_\mu a_\Omega^\nu a_{\Omega\nu} + d^\nu a_{\Omega\mu} a_{\Omega\nu}) d^\lambda a_{\Omega\lambda}$	49	$ia_\Omega^\mu (d^\nu a_\Omega^\lambda d_\lambda V_{\Omega\mu\nu} - d^\nu V_{\Omega\mu}{}^\lambda d_\lambda a_{\Omega\nu})$	81	$iV_\Omega^{\mu\nu} d^\lambda a_{\Omega\mu} d_\lambda a_{\Omega\nu}$
18	$a_\Omega^\mu (d_\mu a_\Omega^\nu a_\Omega^\lambda d_\nu a_{\Omega\lambda} + d^\nu a_\Omega^\lambda a_{\Omega\nu} d_\lambda a_{\Omega\mu})$	50	$d^\mu V_{\Omega\mu}{}^\nu d^\lambda V_{\Omega\nu\lambda}$	82	$iV_\Omega^{\mu\nu} (a_{\Omega\mu} a_{\Omega\nu} s_\Omega + s_\Omega a_{\Omega\mu} a_{\Omega\nu})$
19	$a_\Omega^\mu (d_\mu a_\Omega^\nu a_\Omega^\lambda d_\lambda a_{\Omega\nu} + d^\nu a_\Omega^\lambda a_{\Omega\nu} d_\mu a_{\Omega\lambda})$	51	$d^\mu V_\Omega^{\nu\lambda} d_\mu V_{\Omega\nu\lambda}$	83	$iV_\Omega^{\mu\nu} a_{\Omega\mu} s_\Omega a_{\Omega\nu}$
20	$a_\Omega^\mu (d^\nu a_{\Omega\mu} a_\Omega^\lambda d_\nu a_{\Omega\lambda} + d^\nu a_\Omega^\lambda a_{\Omega\lambda} d_\nu a_{\Omega\mu}$	52	$d^\mu V_\Omega^{\nu\lambda} d_\nu V_{\Omega\mu\lambda}$	84	$iV_\Omega^{\mu\nu} (a_{\Omega\mu} a_{\Omega\nu} a_\Omega^2 + a_\Omega^2 a_{\Omega\mu} a_{\Omega\nu})$
21	$a_\Omega^\mu d^\nu a_{\Omega\nu} a_{\Omega\mu} d^\lambda a_{\Omega\lambda}$	53	$d^2 a_\Omega^\nu d_\nu d^\lambda a_{\Omega\lambda}$	85	$iV_\Omega^{\mu\nu} (a_{\Omega\mu} a_\Omega^\lambda a_{\Omega\nu} a_{\Omega\lambda} + a_\Omega^\lambda a_{\Omega\mu} a_{\Omega\lambda} a_{\Omega\nu})$
22	$a_\Omega^\mu d^\nu a_\Omega^\lambda a_{\Omega\mu} d_\nu a_{\Omega\lambda}$	54	$d^2 a_\Omega^\nu d^\lambda d_\nu a_{\Omega\lambda}$	86	$iV_\Omega^{\mu\nu} a_{\Omega\mu} a_\Omega^2 a_{\Omega\nu}$
23	$a_\Omega^\mu d^\nu a_\Omega^\lambda a_{\Omega\mu} d_\lambda a_{\Omega\nu}$	55	$d^2 a_\Omega^\nu d^2 a_{\Omega\nu}$	87	$iV_\Omega^{\mu\nu} a_\Omega^\lambda a_{\Omega\mu} a_{\Omega\nu} a_{\Omega\lambda}$
24	$a_\Omega^2 s_\Omega^2$	56	$d^\mu d^\nu a_{\Omega\mu} d_\nu d^\lambda a_{\Omega\lambda}$	88	s_Ω^3
25	$a_\Omega^2 p_\Omega^2$	57	$d^\mu d^\nu a_{\Omega\mu} d^\lambda d_\nu a_{\Omega\lambda}$	89	$s_\Omega p_\Omega^2$
26	$a_\Omega^\mu s_\Omega a_{\Omega\mu} s_\Omega$	58	$a^\mu d^\nu a_{\Omega\nu} d_\mu d^\lambda a_{\Omega\lambda}$	90	$s_\Omega p_\Omega d^\mu a_{\Omega\mu}$
27	$a_\Omega^\mu p_\Omega a_{\Omega\mu} p_\Omega$	59	$d^\mu d^\nu a_\Omega^\lambda d_\mu d_\nu a_{\Omega\lambda}$	91	$s_\Omega d^\mu a_{\Omega\mu} p_\Omega$
28	$a_\Omega^\mu (s_\Omega d_\mu p_\Omega + d_\mu p_\Omega s_\Omega)$	60	$d^\mu d^\nu a_\Omega^\lambda d_\mu d_\lambda a_{\Omega\nu}$	92	$s_\Omega d^\mu a_\Omega^\nu d_\mu a_{\Omega\nu}$
29	$a_\Omega^\mu (p_\Omega d_\mu s_\Omega + d_\mu s_\Omega p_\Omega)$	61	$d^\mu d^\nu a_\Omega^\lambda d_\nu d_\mu a_{\Omega\lambda}$	93	$s_\Omega d^\mu a_\Omega^\nu d_\nu a_{\Omega\mu}$
30	$a_\Omega^2 d^\nu a_{\Omega\nu} p_\Omega$	62	$d^\mu d^\nu a_\Omega^\lambda d_\nu d_\lambda a_{\Omega\mu}$	94	$s_\Omega (d^\mu a_{\Omega\mu})^2$
31	$a_\Omega^2 p_\Omega d^\nu a_{\Omega\nu}$	63	$d^\mu d^\nu a_\Omega^\lambda d_\lambda d_\nu a_{\Omega\mu}$		
32	$a_\Omega^2 a_\Omega^\nu d_\nu p_\Omega + a_\Omega^\mu a_\Omega^2 d_\mu p_\Omega$	64	$d^\mu d^\nu a_{\Omega\mu} d_\nu p\Omega$		

others. The sequence number for n flavors, three flavors, and two flavors are different; their comparisons are listed in Table 2 in Ref. [16].

VI. NUMERICAL VALUES OF p^6 ORDER LECS: NORMAL PART

With all above preparations in the previous sections, we now come to the stage of giving numerical values to the p^6 order LECs in the normal part of the chiral Lagrangian. Note that the necessary input and process of the present computations for the p^6 order LECs are the same as those for the p^2 and p^4 order LECs given in the end of Sec. IV, we list the numerical results in Table III. As done in Table I, the result LECs are taken for the values at $\Lambda = 1$ GeV with superscript the difference caused at $\Lambda = 1.1$ GeV and subscript the difference caused at $\Lambda = 0.9$ GeV,

TABLE III. The obtained values of the p^6 order LECs $C_1 \ldots, C_{90}$ for three flavors and $c_1 \ldots, c_{53}$ for two flavors. The LECs are in units of 10^{-3} GeV^{-2}. The resulting LECs are taken as the values at $\Lambda = 1$ GeV with the superscript the difference caused at $\Lambda = 1.1$ GeV and the subscript the difference caused at $\Lambda = 0.9$ GeV. The value $\equiv 0$ means that the constants vanish at the large N_c limit.

i	C_i	j	c_j	i	C_i	j	c_j	i	C_i	j	c_j
1	$3.79^{+0.10}_{-0.17}$	1	$3.58^{+0.09}_{-0.15}$	31	$-0.63^{+0.05}_{-0.09}$	17	$-1.10^{+0.12}_{-0.19}$	61	$2.88^{-0.22}_{+0.28}$	34	$2.84^{-0.22}_{+0.26}$
2	$\equiv 0$			32	$0.18^{-0.03}_{+0.04}$	18	$0.43^{-0.07}_{+0.08}$	62	$\equiv 0$		
3	$-0.05^{+0.01}_{-0.01}$	2	$-0.03^{+0.01}_{-0.01}$	33	$0.09^{-0.00}_{+0.03}$	19	$0.41^{-0.06}_{+0.10}$	63	$2.99^{-0.24}_{+0.30}$		
4	$3.10^{+0.09}_{-0.15}$	3	$2.89^{+0.08}_{-0.13}$	34	$1.59^{-0.10}_{+0.16}$	20	$1.56^{-0.10}_{+0.17}$	64	$\equiv 0$		
5	$-1.01^{+0.08}_{-0.11}$	4	$1.21^{-0.07}_{+0.06}$	35	$0.17^{-0.12}_{+0.17}$	21	$0.29^{-0.18}_{+0.24}$	65	$-2.43^{+0.15}_{-0.16}$	35	$3.39^{-0.32}_{+0.41}$
6	$\equiv 0$			36	$\equiv 0$			66	$1.71^{+0.07}_{-0.12}$	36	$1.57^{+0.06}_{-0.10}$
7	$\equiv 0$			37	$-0.56^{+0.09}_{-0.11}$			67	$\equiv 0$		
8	$2.31^{-0.16}_{+0.18}$			38	$0.41^{-0.08}_{+0.07}$	22	$-1.32^{+0.18}_{-0.25}$	68	$\equiv 0$		
9	$\equiv 0$			39	$\equiv 0$	23	$0.86^{-0.12}_{+0.15}$	69	$-0.86^{-0.04}_{+0.06}$	38	$-0.68^{-0.03}_{+0.05}$
10	$-1.05^{+0.08}_{-0.09}$	5	$-0.98^{+0.07}_{-0.09}$	40	$-6.35^{-0.18}_{+0.32}$	24	$-4.84^{-0.14}_{+0.25}$	70	$1.73^{-0.08}_{+0.07}$	39	$1.81^{-0.08}_{+0.07}$
11	$\equiv 0$			41	$\equiv 0$			71	$\equiv 0$		
12	$-0.34^{+0.02}_{-0.01}$	6	$-0.33^{+0.01}_{-0.01}$	42	$0.60^{+0.00}_{-0.00}$			72	$-3.30^{+0.05}_{-0.00}$	40	$-3.17^{+0.05}_{-0.02}$
13	$\equiv 0$			43	$\equiv 0$			73	$0.50^{+0.43}_{-0.56}$	41	$0.30^{+0.42}_{-0.54}$
14	$-0.83^{+0.12}_{-0.19}$	7	$-1.72^{+0.25}_{-0.35}$	44	$6.32^{+0.20}_{-0.36}$	25	$6.03^{+0.19}_{-0.33}$	74	$-5.07^{-0.16}_{+0.27}$	42	$-4.74^{-0.14}_{+0.24}$
15	$\equiv 0$	8	$0.86^{-0.12}_{+0.15}$	45	$\equiv 0$			75	$\equiv 0$		
16	$\equiv 0$			46	$-0.60^{-0.02}_{+0.04}$	26	$-1.14^{-0.05}_{+0.07}$	76	$-1.44^{-0.23}_{+0.31}$	43	$-1.29^{-0.23}_{+0.30}$
17	$0.01^{-0.01}_{-0.01}$	9	$-0.84^{+0.12}_{-0.17}$	47	$0.08^{+0.01}_{-0.00}$			77	$\equiv 0$		
18	$-0.56^{+0.09}_{-0.11}$			48	$3.41^{+0.06}_{-0.10}$			78	$17.51^{+1.02}_{-1.59}$	44	$16.16^{+0.94}_{-1.45}$
19	$-0.48^{+0.09}_{-0.13}$	10	$-0.37^{+0.07}_{-0.10}$	49	$\equiv 0$			79	$-0.56^{-0.30}_{+0.40}$	45	$0.26^{-0.26}_{+0.34}$
20	$0.18^{-0.03}_{+0.04}$	11	$\equiv 0$	50	$8.71^{+0.78}_{-1.12}$	27	$13.57^{+1.41}_{-2.00}$	80	$0.87^{-0.04}_{+0.03}$	46	$0.85^{-0.04}_{+0.02}$
21	$-0.06^{+0.01}_{-0.01}$			51	$-11.49^{+0.18}_{-0.09}$	28	$0.93^{+0.98}_{-1.25}$	81	$\equiv 0$		
22	$0.27^{+0.19}_{-0.25}$	12	$0.15^{+0.18}_{-0.24}$	52	$-5.04^{-0.67}_{+0.93}$			82	$-7.13^{-0.32}_{+0.51}$	47	$-6.73^{-0.29}_{+0.47}$
23	$\equiv 0$			53	$-11.99^{-0.87}_{+1.33}$	29	$-11.01^{-0.81}_{+1.23}$	83	$0.07^{+0.20}_{-0.27}$	48	$-0.22^{+0.18}_{-0.25}$
24	$1.62^{+0.04}_{-0.07}$			54	$\equiv 0$			84	$\equiv 0$		
25	$-5.98^{-0.49}_{+0.72}$	13	$-5.39^{-0.45}_{+0.66}$	55	$16.79^{+0.96}_{-1.49}$	30	$15.72^{+0.89}_{-1.38}$	85	$-0.82^{+0.03}_{-0.02}$	49	$-0.78^{+0.03}_{-0.01}$
26	$3.35^{+0.29}_{-0.47}$	14	$4.17^{+0.30}_{-0.49}$	56	$19.34^{+0.52}_{-0.98}$	31	$17.57^{+0.42}_{-0.82}$	86	$\equiv 0$		
27	$-1.54^{+0.15}_{-0.18}$	15	$-2.71^{+0.21}_{-0.25}$	57	$7.92^{+1.34}_{-1.85}$	32	$7.18^{+1.28}_{-1.76}$	87	$7.57^{+0.37}_{-0.60}$	50	$7.18^{+0.34}_{-0.55}$
28	$0.30^{+0.01}_{-0.01}$			58	$\equiv 0$			88	$-5.47^{-0.73}_{+1.03}$	51	$-4.85^{-0.69}_{+0.97}$
29	$-3.08^{-0.26}_{+0.32}$	16	$-2.22^{-0.22}_{+0.27}$	59	$-22.49^{-1.21}_{+1.89}$	33	$-21.19^{-1.12}_{+1.76}$	89	$34.74^{+1.61}_{-2.62}$	52	$32.19^{+1.46}_{-2.37}$
30	$0.60^{+0.02}_{-0.03}$			60	$\equiv 0$			90	$2.44^{-0.38}_{+0.46}$	53	$2.51^{-0.37}_{+0.46}$

$$C_{\Lambda=1\ \text{GeV}}\Big|^{C_{\Lambda=1.1\ \text{GeV}}-C_{\Lambda=1\ \text{GeV}}}_{C_{\Lambda=0.9\ \text{GeV}}-C_{\Lambda=1\ \text{GeV}}}$$

$$c_{\Lambda=1\ \text{GeV}}\Big|^{c_{\Lambda=1.1\ \text{GeV}}-c_{\Lambda=1\ \text{GeV}}}_{c_{\Lambda=0.9\ \text{GeV}}-c_{\Lambda=1\ \text{GeV}}}.$$

We further list results of the p^6 order LECs at $\Lambda = \infty$ in Table IV. Consider that in the limit of $\Lambda = \infty$, dropping out momentum total derivative terms in Eq. (14) is problematic, we only take resulting LECs at $\Lambda = \infty$ as a reference. Since the terms of three flavors and two flavors may have different sequence numbers, as done in Ref. [16], we put them in the same line in our table. Since the number of independent operators in the two flavors is smaller than that in the three flavors, there are some operators in three flavors being independent operators, but being dependent in two flavors. Then these operators will not have their two flavor counterparts in our table, these leave the rhs some empty blanks in the corresponding two flavor columns. For the two flavor case, Ref. [18] further proposes a new relation among operators,

$$\begin{aligned} 0 = {} & 8P_1 - 2P_2 + 6P_3 - 12P_{13} + 8P_{14} - 3P_{15} - 2P_{16} \\ & - 20P_{24} + 8P_{25} + 12P_{26} - 12P_{27} - 28P_{28} + 8P_{36} \\ & - 8P_{37} - 8P_{39} + 2P_{40} + 8P_{41} - 8P_{42} - 6P_{43} \\ & + 4P_{48}, \end{aligned} \tag{57}$$

which implies that one of the operators appearing in the above formula should be a further dependent operator. Because of ignorance of the values of the coefficients in front of these operators, Ref. [18] arbitrarily chooses this operator being P_{27}. Now our computations show that $c_{27} \neq 0$, so the original choice is not suitable. Considering that $c_{37} = 0$ in our computation, we instead now take P_{37} as that dependent operator. P_{37} now is a dependent operator; its name then is deleted in our Table III.

To verify our choice of $F_0 = 87$ MeV really results in the experimental value of F_π, we exploit the relation between F_0 and F_π given in Ref. [19]:

TABLE IV. p^6 order LECs the same as the result given in Table III, but at $\Lambda = \infty$ $C_1 \ldots, C_{90}$ are for three flavor and $c_1 \ldots, c_{53}$ are for two flavors in units of 10^{-3} GeV^{-2}. The value $\equiv 0$ means that the constants vanish at the large N_c limit.

i	C_i	j	c_j	i	C_i	j	c_j	i	C_i	j	c_j
1	3.61	1	3.39	31	−0.22	17	−0.22	61	1.36	34	1.45
2	≡ 0			32	0.02	18	0.09	62	≡ 0		
3	−0.01	2	0.00	33	0.08	19	0.09	63	1.41		
4	2.98	3	2.77	34	1.03	20	0.97	64	≡ 0		
5	−0.51	4	0.66	35	−0.40	21	−0.46	65	−1.28	35	1.56
6	≡ 0			36	≡ 0			66	1.73	36	1.58
7	≡ 0			37	−0.06			67	≡ 0		
8	1.16			38	−0.01	22	−0.25	68	≡ 0		
9	≡ 0			39	≡ 0	23	0.20	69	−0.90	38	−0.71
10	−0.49	5	−0.49	40	−6.10	24	−4.70	70	0.91	39	1.08
11	≡ 0			41	≡ 0			71	≡ 0		
12	−0.19	6	−0.20	42	0.49			72	−2.43	40	−2.37
13	≡ 0			43	≡ 0			73	2.47	41	2.08
14	−0.26	7	−0.42	44	6.17	25	5.86	74	−4.96	42	−4.61
15	≡ 0	8	0.20	45	≡ 0			75	≡ 0		
16	≡ 0			46	−0.58	26	−1.11	76	−2.33	43	−2.08
17	−0.15	9	−0.29	47	0.08			77	≡ 0		
18	−0.06			48	3.13			78	18.97	44	17.41
19	−0.08	10	−0.09	49	≡ 0			79	−1.81	45	−0.89
20	0.02	11	≡ 0	50	10.73	27	17.28	80	0.49	46	0.52
21	−0.01			51	−8.65	28	4.93	81	≡ 0		
22	1.11	12	0.91	52	−7.24			82	−7.27	47	−6.83
23	≡ 0			53	−13.65	29	−12.49	83	0.96	48	0.59
24	1.55			54	≡ 0			84	≡ 0		
25	−7.21	13	−6.46	55	18.10	30	16.83	85	−0.47	49	−0.49
26	3.93	14	4.68	56	17.99	31	16.33	86	≡ 0		
27	−0.60	15	−1.42	57	12.69	32	11.45	87	7.83	50	7.39
28	0.29			58	≡ 0			88	−7.83	51	−6.96
29	−3.81	16	−2.78	59	−23.88	33	−22.35	89	35.69	52	32.93
30	0.58			60	≡ 0			90	0.25	53	0.51

$$\frac{F_\pi}{F_0} = 1 + x_2(l_4^r - L) + x_2^2\Big[\frac{1}{16\pi^2}\Big(-\frac{1}{2}l_1^r - l_2^r + \frac{29}{12}L\Big) - \frac{13}{192}\frac{1}{(16\pi^2)^2} + \frac{7}{4}k_1 + k_2 - 2l_3^r l_4^r + 2(l_4^r)^2 - \frac{5}{4}k_4 + r_F^r\Big] + O(x_2^3), \tag{58}$$

$$x_2 = \frac{M_\pi^2}{F_\pi^2} \qquad L = \frac{1}{16\pi^2}\ln\frac{M_\pi^2}{\mu^2} \qquad k_i = (4l_i^r - \gamma_i L)L$$
$$r_F^r = (8c_7 + 16c_8 + 8c_9)F_0^2, \tag{59}$$

in which r_F^r is from Ref. [17] and l_i^r and γ_i for $i = 1, 2, \ldots, 7$ are defined in Ref. [2]. Scale μ is taken to be ρ mass $\mu = M_\rho = 770$ MeV. Numerical calculations show that for $\Lambda = 1000^{+100}_{-100}$ MeV, the contributions up to order of p^4 [ignoring x_2^2 terms in (58)] give result $F_\pi = 92.99^{+.00}_{-.03}$ MeV and the contributions up to the order of p^6 [ignoring x_2^3 terms in (58)] give result $F_\pi = 92.97^{+.00}_{-.04}$ MeV with $r_F^r = -5.036^{+0.730}_{-1.290} \times 10^{-5}$. We see that the p^6 order contributions to F_π are very small and $F_0 = 87$ MeV is directly related to $F_\pi = 93$ MeV.

VII. COMPARISONS WITH EXPERIMENT AND MODEL RESULTS

As we have mentioned in the Introduction of this paper, present experiment data is far enough to fix the p^6 order LECs. But there do exist some combinations of the LECs which already have their experiment or model calculation values. Usually, these LECs are labeled by dimensionless parameters with convention[4] of $C_i^r \equiv C_i F_0^2$ or $c_i^r \equiv c_i F_0^2$. In this section, we collect those combinations of LECs in the literature which have their experiment or model calculation values and compare them with our numerical results obtained in the last section with finite cutoff[5] as the check of our computations.

A. $\pi\pi$ and πK scattering

From the investigation of $\pi\pi$ scattering amplitudes, one can work out the values of some combinations of p^6 order LECs. Reference [17] introduces the following combinations:

$$\begin{aligned}
r_1^r &= 64c_1^r - 64c_2^r + 32c_4^r - 32c_5^r + 32c_6^r - 64c_7^r - 128c_8^r - 64c_9^r + 96c_{10}^r + 192c_{11}^r - 64c_{14}^r + 64c_{16}^r + 96c_{17}^r + 192c_{18}^r \\
r_2^r &= -96c_1^r + 96c_2^r + 32c_3^r - 32c_4^r + 32c_5^r - 64c_6^r + 32c_7^r + 64c_8^r + 32c_9^r - 32c_{13}^r + 32c_{14}^r - 64c_{16}^r \\
r_3^r &= 48c_1^r - 48c_2^r - 40c_3^r + 8c_4^r - 4c_5^r + 8c_6^r - 8c_{12}^r + 20c_{13}^r \qquad r_4^r = -8c_3^r + 4c_5^r - 8c_6^r + 8c_{12}^r - 4c_{13}^r \\
r_5^r &= -8c_1^r + 10c_2^r + 14c_3^r \qquad r_6^r = 6c_2^r + 2c_3^r
\end{aligned} \tag{60}$$

and gives the values of them by two theoretical methods of the resonance-saturation (RS) [20] and pure dimensional analysis (ND) which only accounts for the order of magnitude and in Table V. We see that all coefficients obtained from our calculations are consistent with those more precise RS results given in Ref. [17]. With our predictions for p^4 and p^6 order LECs, we can directly calculate the scattering lengths a_l^I and slope parameters b_l^I which relate to p^4 and p^6 order LECs through formulas given in Appendices C and D of Ref. [20]. We list experimental and our results in Table VI. In our results, as done in Table III, we take $\mu = 770$ MeV, but to match the result given in Ref. [20] where μ is taken at $\mu = 1$ GeV, we also take $\mu = 1000$ MeV for comparison. We take two options, one only includes p^4 order contributions and the other combines in p^6 order contributions. For p^6 order contributions, for comparison, we consider the cases of without and with r_i^r coefficients. We see that the contributions from p^6 order LECs are rather small and only change the third digit of the result.

Further, Ref. [22] introduces coefficients in πK scattering:

$$\begin{aligned}
c_{01}^- &= 32m_{K^+}^3(-C_1^r + 2C_3^r + 2C_4^r), \qquad c_{20}^- = 6m_{K^+}(-C_1^r + 2C_3^r + 2C_4^r), \\
c_{11}^+ &= 8m_{K^+}^2(3C_1^r + 6C_3^r - 2C_4^r), \qquad c_{30}^+ = \tfrac{1}{2}(-7C_1^r - 32C_2^r + 2C_3^r + 10C_4^r), \\
c_{01}^+ &= 16m_{K^+}^2 m_{\pi^+}^2(C_6^r + C_8^r + C_{10}^r + 2C_{11}^r - 2C_{12}^r - 2C_{13}^r + 2C_{22}^r + 4C_{23}^r) \\
&\quad + 16m_{K^+}^4(C_5^r + 2C_6^r + C_{10}^r + 4C_{11}^r - 2C_{12}^r - 4C_{13}^r + 2C_{22}^r + 4C_{23}^r), \\
c_{10}^- &= 8m_{K^+} m_{\pi^+}^2(-4C_4^r - C_6^r - C_8^r + C_{10}^r + 2C_{11}^r - 2C_{12}^r - 6C_{13}^r + 2C_{22}^r - 2C_{25}^r) \\
&\quad + 8m_{K^+}^3(-4C_4^r - C_5^r - 2C_6^r + C_{10}^r + 4C_{11}^r - 2C_{12}^r - 12C_{13}^r + 2C_{22}^r - 2C_{25}^r), \\
c_{20}^+ &= m_{K^+}^2(12C_1^r + 48C_2^r - 8C_4^r + C_5^r + 10C_6^r + 8C_7^r + 4C_8^r + C_{10}^r + 4C_{11}^r - 2C_{12}^r - 4C_{13}^r + 2C_{22}^r - 4C_{23}^r + 4C_{25}^r) \\
&\quad + m_{\pi^+}^2(12C_1^r + 48C_2^r - 8C_4^r + 4C_5^r + 5C_6^r + 8C_7^r + C_8^r + C_{10}^r + 2C_{11}^r - 2C_{12}^r - 10C_{13}^r + 2C_{22}^r - 4C_{23}^r + 4C_{25}^r).
\end{aligned} \tag{61}$$

[4]An alternative convention is that C_i^r and c_i^r are used to denote the renormalized LECs in some literature.

[5]If the LECs at finite cutoff are replaced with those at $\Lambda = \infty$, we have checked that qualitative feature of the comparison results of this section will not change.

TABLE V. The obtained values for the combinations of the p^6 order LECs from $\pi\pi$ scattering and our work. The coefficients in the table are in units of 10^{-4}.

	r_1^r	r_2^r	r_3^r	r_4^r	r_5^r	r_6^r
RS in Ref. [17]	-0.6	1.3	-1.7	-1.0	1.1	0.3
ND in Ref. [17]	80	40	20	3	6	2
Ours	$-9.32^{-2.62}_{+3.51}$	$8.93^{+3.12}_{-4.27}$	$-3.06^{-0.81}_{+1.11}$	$-0.12^{+0.22}_{-0.29}$	$0.87^{+0.04}_{-0.06}$	$0.42^{+0.02}_{-0.03}$

TABLE VI. The obtained values for a_i^I and b_i^I in $\pi\pi$ scattering from experimental values given by Ref. [21] and our work.

	a_0^0	b_0^0	$-10a_0^2$	$-10b_0^2$	$10a_1^1$	$10^2b_1^1$	$10^2a_2^0$	$10^3a_2^2$
Ref. [21]	0.26 ± 0.05	0.25 ± 0.03	0.28 ± 0.12	0.82 ± 0.08	0.38 ± 0.02		0.17 ± 0.03	0.13 ± 0.30
p^4 $\mu = 10^3$ MeV	$0.210^{-0.000}_{+0.000}$	$0.260^{-0.000}_{+0.000}$	$0.406^{+0.001}_{-0.001}$	$0.662^{-0.002}_{+0.003}$	$0.405^{+0.001}_{-0.002}$	$0.772^{+0.015}_{-0.020}$	$0.264^{+0.002}_{-0.003}$	$0.195^{-0.009}_{+0.012}$
p^4 $\mu = 770$ MeV	$0.204^{-0.000}_{+0.000}$	$0.248^{-0.000}_{+0.000}$	$0.411^{+0.001}_{-0.001}$	$0.685^{-0.002}_{+0.003}$	$0.401^{+0.001}_{-0.002}$	$0.772^{+0.015}_{-0.020}$	$0.235^{+0.002}_{-0.003}$	$0.076^{-0.009}_{+0.012}$
p^6 $\mu = 10^3$ MeV $r_i^r \neq 0$	$0.237^{-0.000}_{+0.000}$	$0.307^{-0.000}_{+0.000}$	$0.394^{+0.001}_{-0.001}$	$0.637^{-0.004}_{+0.005}$	$0.447^{+0.002}_{-0.003}$	$1.255^{+0.029}_{-0.037}$	$0.421^{+0.005}_{-0.008}$	$0.339^{-0.011}_{+0.011}$
p^6 $\mu = 10^3$ MeV $r_i^r = 0$	$0.237^{-0.000}_{+0.000}$	$0.305^{-0.000}_{-0.000}$	$0.392^{+0.001}_{-0.001}$	$0.629^{-0.003}_{+0.004}$	$0.445^{+0.002}_{-0.002}$	$1.217^{+0.019}_{-0.024}$	$0.409^{+0.004}_{-0.006}$	$0.337^{-0.005}_{+0.003}$
p^6 $\mu = 770$ MeV $r_i^r \neq 0$	$0.228^{-0.000}_{+0.000}$	$0.287^{-0.000}_{+0.000}$	$0.402^{+0.001}_{-0.001}$	$0.665^{-0.003}_{+0.005}$	$0.435^{+0.002}_{-0.003}$	$1.164^{+0.028}_{-0.037}$	$0.336^{+0.005}_{+0.008}$	$0.212^{-0.012}_{+0.012}$
p^6 $\mu = 770$ MeV $r_i^r = 0$	$0.227^{-0.000}_{+0.000}$	$0.285^{-0.000}_{-0.000}$	$0.400^{+0.001}_{-0.001}$	$0.657^{-0.003}_{+0.004}$	$0.433^{+0.002}_{-0.002}$	$1.125^{+0.018}_{-0.023}$	$0.352^{+0.004}_{-0.006}$	$0.210^{-0.006}_{+0.004}$

In the Table I of Ref. [23], in terms of c_{30}^+, c_{11}^+, c_{20}^-, c_{01}^-, three constraints of p^6 order LECs are fixed from πK subthreshold parameters, $\pi\pi$ amplitude and a resonance model. In the Table II of Ref. [23], in terms of c_{20}^+, c_{01}^+, c_{10}^-, another three constraints of p^6 order LECs are fixed from the dispersive calculations and a resonance model, In which for the left-hand side (lhs) of Table VII, except C_2, all other LECs or combinations of LECs obtained by us have the same signs and orders of magnitudes as those from Ref. [23]. While for the rhs of Table VII, our results are not consistent with those obtained from the dispersive calculations.

B. Form factors

In Ref. [17], in dealing with the vector form factor of the pion, r_{V1}^r and r_{V2}^r are introduced into theory which relate to p^6 order LECs through

$$r_{V1}^r = -16c_6^r - 4c_{35}^r - 8c_{53}^r, \qquad r_{V2}^r = -4c_{51}^r - 4c_{53}^r. \tag{62}$$

For the scalar form factor, people introduce r_{S2}^r and r_{S3}^r relating to p^6 order LECs by

$$r_{S2}^r = 32c_6^r + 16c_7^r + 32c_8^r + 16c_9^r + 16c_{20}^r, \qquad r_{S3}^r = -8c_6^r. \tag{63}$$

In Ref. [17], discussion of the decay of $\pi(p) \to e\nu\gamma(q)$ further introduces r_{A1}^r and r_{A2}^r relating to p^6 order LECs by

$$r_{A1}^r = 48c_6^r - 16c_{34}^r + 8c_{35}^r - 8c_{44}^r + 16c_{46}^r - 16c_{47}^r + 8c_{50}^r, \qquad r_{A2}^r = 8c_{44}^r - 16c_{50}^r + 4c_{51}^r. \tag{64}$$

In Ref. [24], a naive estimation of C_{12}^r is made from scalar meson dominance of the pion scalar form factor and $2C_{12}^r + C_{34}^r$ is estimated through λ_0 in K_{l3} measurements [see Eq. (8.11) in Ref. [24]], while in Ref. [25], C_{12}^r and $C_{12}^r + C_{34}^r$ are also estimated from the πK form factors. In Table VIII, we list the numerical results for the above combinations of LECs given by our calculations based on Table III in the last section and by Refs. [17,24,25], from which we see that among ten parameters between our

TABLE VII. The obtained values for the combinations of the p^6 order LECs from πK, $\pi\pi$ scattering, and our work. The coefficients in the lhs of the table are in units of 10^{-4} GeV^{-2}.

	$C_1 + 4C_3$	C_2	$C_4 + 3C_3$	$C_1 + 4C_3 + 2C_2$		$c_{20}^+ \frac{m_\pi^4}{F_\pi^4}$	$c_{01}^+ \frac{m_\pi^2}{F_\pi^4}$	$c_{10}^- \frac{m_\pi^3}{F_\pi^4}$
Input c_{30}^+, c_{11}^+, c_{20}^-	20.7 ± 4.9	-9.2 ± 4.9	9.9 ± 2.5	2.3 ± 10.8				
Input c_{30}^+, c_{11}^+, c_{01}^-	28.1 ± 4.9	-7.4 ± 4.9	21.0 ± 2.5	13.4 ± 10.8	Dispersive	0.024 ± 0.006	2.07 ± 0.10	0.31 ± 0.01
$\pi\pi$ amplitude			23.5 ± 2.3	18.8 ± 7.2				
Resonance model	7.2	-0.5	10.0	6.2	Resonance model	0.003	3.8	0.09
Ours	$35.9^{+1.3}_{-2.1}$	$0.0^{+1.3}_{-0.0}$	$29.5^{+1.1}_{-1.9}$	$35.9^{+1.3}_{-2.1}$	Ours	$0.006^{-0.002}_{+0.003}$	$-0.159^{+0.133}_{-0.178}$	$0.020^{+0.037}_{-0.050}$

TABLE VIII. The obtained values for the combinations of the p^6 order LECs appear in vector and scalar form factor of pion. The coefficients in the table are in units of 10^{-4}.

	Ours	Ref. [17]		Ours	Ref. [17]		Ours	Ref. [17]
r^r_{V1}	$-2.13^{+0.30}_{-0.39}$	-2.5	r^r_{S2}	$0.07^{+0.05}_{-0.08}$	-0.3	r^r_{A1}	$1.14^{+0.07}_{-0.09}$	-0.5
r^r_{V2}	$2.23^{+0.10}_{-0.16}$	2.6	r^r_{S3}	$0.20^{-0.01}_{+0.01}$	0.6	r^r_{A2}	$-0.38^{-0.06}_{+0.08}$	1.1

	Ours	Ref. [24]		Ours	Ref. [25]
C^r_{12}	$-0.026^{+0.001}_{-0.001}$	-0.1	C^r_{12}	$-0.026^{+0.001}_{-0.001}$	$(0.3 \pm 5.4) \times 10^{-3}$
$2C^r_{12} + C^r_{34}$	$0.068^{-0.006}_{+0.010}$	-0.10 ± 0.17	$C^r_{12} + C^r_{34}$	$0.094^{-0.007}_{+0.011}$	$(3.2 \pm 1.5) \times 10^{-2}$

predictions and values given in the literature, four of them have the same orders of magnitudes and signs (r^r_{V1}, r^r_{V2}, r^r_{S3}, and $C^r_{12} + C^r_{34}$), another one of them has different orders of magnitudes but the same signs (C^r_{12} in Ref. [24]), the left five of them have opposite signs (r^r_{S2}, r^r_{A1}, r^r_{A2}, $2C^r_{12} + C^r_{34}$, and C^r_{12} in Ref. [25]).

Further in Fig. 1, we compare the experimental data for vector form factors collected in Figs. 4 and 5 of Ref. [19] with our results. In obtaining our numerical predictions, we have exploited the formula given by Eq. (3.16) in Ref. [19] which especially depends on p^6 order LECs through r^r_{V1}, r^r_{V2} defined in (62) and we input the formula p^4 and p^6 LECs obtained in Tables I and III of previous sections.

From Fig. 1, we see that p^6 order LECs explicitly improve the p^4 and p^2 order chiral perturbation predictions, making them more consistent with experimental data.

C. Photon-photon collisions

In Ref. [26], discussion of the photon-photon collision $\gamma\gamma \to \pi^0\pi^0$ introduces a^r_1, a^r_2, and b^r relating to p^6 order LECs by

$$\begin{aligned} a^r_1 &= 4096\pi^4(-c^r_{29} - c^r_{30} + c^r_{34}) \\ a^r_2 &= 256\pi^4(8c^r_{29} + 8c^r_{30} + c^r_{31} + c^r_{32} + 2c^r_{33}) \\ b^r &= -128\pi^4(c^r_{31} + c^r_{32} + 2c^r_{33}). \end{aligned} \tag{65}$$

In Ref. [27], calculation of the photon-photon collision $\gamma\gamma \to \pi^+\pi^-$ introduces another type of a^r_1, a^r_2, and b^r relating to p^6 order LECs by

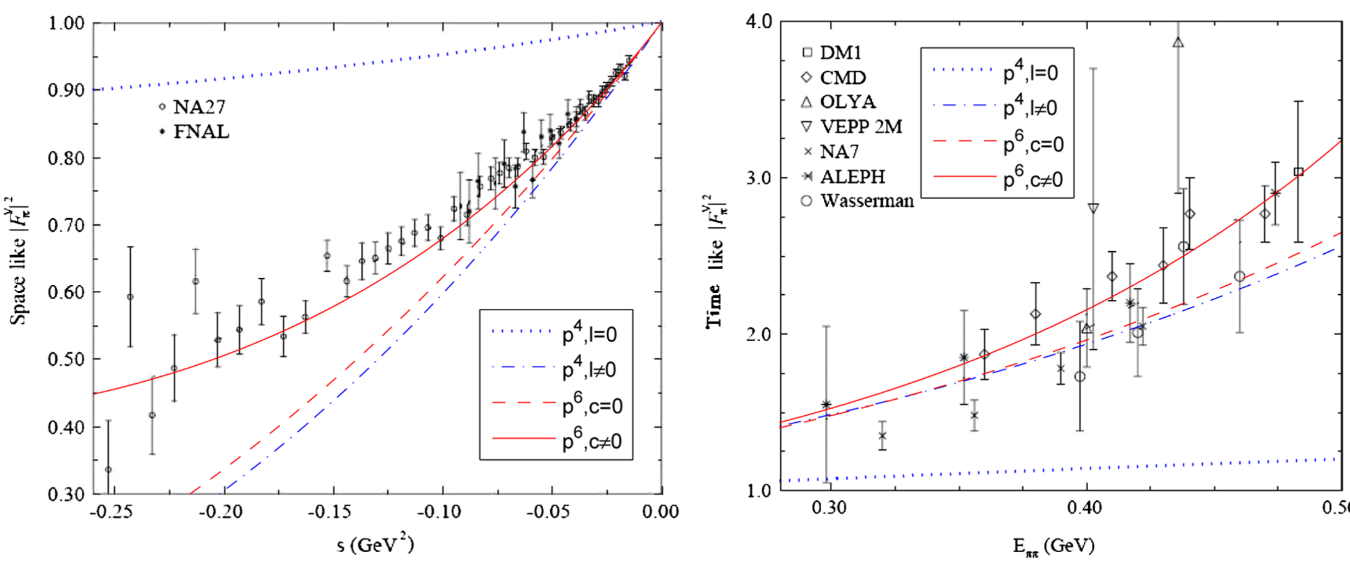

FIG. 1 (color online). The spacelike and timelike data for the vector form factor. The red solid curve corresponds to predictions from chiral perturbation up to p^6 order with LECs obtained in Table III of this paper. The red dashed line is the result by vanishing p^6 order LECs in corresponding red solid curve. The blue dot-dashed curve corresponds to predictions from chiral perturbation up to p^4 order with LECs obtained in Table I of this paper. The blue dotted line is the result by vanishing p^4 order LECs in corresponding blue dot-dashed curve. The black x axis with $|F^V_\pi|^2 = 1.0$ corresponds to predictions from p^2 order chiral perturbation.

TABLE IX. The obtained values for the combinations of the p^6 order LECs appear in photon-photon collisions.

	Ours	Ref. [26]		Ours	Ref. [27]
a_1^r	$-5.65^{-0.91}_{+1.23}$	-14 ± 5	a_1^r	$-5.86^{-0.49}_{+0.58}$	-3.2
a_2^r	$3.79^{+0.02}_{-0.05}$	7 ± 3	a_2^r	$-0.98^{-0.07}_{+0.12}$	0.7
b^r	$1.66^{+0.05}_{-0.09}$	3 ± 1	b^r	$-0.23^{-0.01}_{+0.02}$	0.4

$$
\begin{aligned}
a_1^r &= -4096\pi^4(6c_6^r + c_{29}^r - c_{30}^r - 3c_{34}^r + c_{35}^r + 2c_{46}^r \\
&\quad - 4c_{47}^r + c_{50}^r), \\
a_2^r &= 256\pi^4(8c_{29}^r - 8c_{30}^r + c_{31}^r + c_{32}^r - 2c_{33}^r \\
&\quad + 4c_{44}^r + 8c_{50}^r - 4c_{51}^r), \\
b^r &= -128\pi^4(c_{31}^r + c_{32}^r - 2c_{33}^r - 4c_{44}^r). \qquad (66)
\end{aligned}
$$

In Table IX, we list the numerical results for the above combinations of LECs given by our calculations based on Table III in the last section and by Refs. [26,27], for which we see that among six parameters between our predictions and values given in the literature, except two have opposite signs, the other four all have the same orders of magnitudes and signs.

D. Radiative pion decay

In Ref. [28], through reanalysis of the radiative pion decay, a group of p^6 order LECs is fixed. From Table X, we find that all LECs and combinations of LECs from our predictions have the same signs and orders of magnitudes as those from experiment values.

E. Model calculations

Except the above phenomenological estimations on the values of some LECs, there are model calculations for some others of them and most of these analyses use a (single) resonance approximation. In contrast, our calculations do not rely on the assumption of existence of resonances. In this subsection, we list these calculation values we can collect from the literature and compare with our results.

Reference [29] estimates values of some LECs. The comparison between their results and our results is given in Table XI.

For C_{63}^r and C_{65}^r, Ref. [30] gives the value for their combination $2C_{63}^r - C_{65}^r = (1.8 \pm 0.7) \times 10^{-5}$ which, compared to our result of $6.36^{-0.48}_{+0.56} \times 10^{-5}$, is at the same order of magnitude and has the same sign.

For C_{87}^r, there are several works to estimate its values; we list them in Table XII. Where C_{87}^r given in Refs. [32,33] are in the form of C_{87} in units of GeV^{-2}, we have transformed them into our expression of C_{87}^r with $C_{87}^r = C_{87}F_0^2$.

Further, Ref. [34] exploits resonance Lagrangian estimates values of LECs C_{78}, C_{82}, C_{87}, C_{88}, C_{89}, C_{90}. From

TABLE X. The obtained values for the combinations of the p^6 order LECs from pion radiative decay and our work. The coefficients in the table are in units of 10^{-5}.

	C_{12}^r	C_{13}^r	C_{61}^r	C_{62}^r	$2C_{63}^r - C_{65}^r$	C_{64}^r
Ref. [28]	-0.6 ± 0.3	0 ± 0.2	1.0 ± 0.3	0 ± 0.2	1.8 ± 0.7	0 ± 0.2
ours	$-0.26^{+0.01}_{-0.01}$	$0.0^{+0.0}_{-0.0}$	$2.18^{-0.17}_{+0.20}$	$0.0^{+0.0}_{-0.0}$	$6.36^{-0.42}_{+0.58}$	$0.0^{+0.0}_{-0.0}$

	C_{78}^r	C_{80}^r	C_{81}^r	C_{82}^r	C_{87}^r	C_{88}^r
Ref. [28]	10.0 ± 3.0	1.8 ± 0.4	0 ± 0.2	-3.5 ± 1.0	3.6 ± 1.0	-3.5 ± 1.0
Ours	$13.26^{+0.77}_{-1.20}$	$0.66^{-0.03}_{+0.02}$	$0.0^{+0.0}_{-0.0}$	$-5.39^{-0.24}_{+0.39}$	$5.73^{+0.28}_{-0.45}$	$-4.14^{-0.55}_{+0.78}$

TABLE XI. The obtained values for the p^6 order LEC in Ref. [29] and our works. The coefficients in the table are in units of 10^{-3} GeV^{-2}.

	C_{14}	C_{19}	C_{38}	C_{61}	C_{80}	C_{87}
Ref. [29]	-4.3	-2.8	1.2	1.9	1.9	7.6
Ours	$-0.83^{+0.12}_{-0.19}$	$-0.48^{+0.09}_{-0.13}$	$0.41^{-0.08}_{+0.07}$	$2.88^{-0.22}_{+0.26}$	$0.87^{-0.04}_{+0.03}$	$7.57^{+0.37}_{-0.60}$

TABLE XII. The obtained values for the p^6 order LEC C^r_{87}. The coefficients in the table are in units of 10^{-5}.

	Ours	Ref. [31]	Ref. [32]	Ref. [33]
C^r_{87}	$5.73^{+0.28}_{-0.45}$	3.1 ± 1.1	4.3 ± 0.4	3.70 ± 0.14

TABLE XIII. The obtained values for the p^6 order LEC from resonance Lagrangian given by Ref. [34] and our work. The coefficients in the table are in units of $10^{-4}/F_0^2$.

	C_{78}	C_{82}	C_{87}	C_{88}	C_{89}	C_{90}
Lowest meson dominance	1.09	−0.36	0.40	−0.52	1.97	0.0
Resonance Lagrangian I	1.09	−0.29	0.47	−0.16	2.29	0.33
Resonance Lagrangian II	1.49	−0.39	0.65	−0.14	3.22	0.51
Ours	$1.326^{+0.077}_{-0.120}$	$-0.539^{-0.024}_{+0.039}$	$0.573^{+0.028}_{-0.045}$	$-0.414^{-0.055}_{+0.078}$	$2.630^{+0.122}_{-0.198}$	$0.185^{-0.029}_{+0.035}$

TABLE XIV. The obtained values for the p^6 order LEC from our work. The coefficients in the table are in units of 10^{-3} GeV^{-2}.

C_{20}	$-3C_{21}$	C_{32}	$\frac{1}{6}C_{35}$	C_{24}	$6C_{28}$	$3C_{30}$
$0.18^{-0.03}_{+0.04}$	$0.18^{-0.03}_{+0.03}$	$0.18^{-0.03}_{+0.04}$	$0.028^{-0.020}_{+0.028}$	$1.62^{+0.04}_{-0.07}$	$1.80^{+0.06}_{-0.06}$	$1.80^{+0.06}_{-0.09}$

Table XIII, we find that our results are consistent with those obtained from the resonance Lagrangian.

Reference [35] estimates the value of C_{38} and gives $C^r_{38} = (2 \pm 6) \times 10^{-6}$ which is also consistent with our result of $C^r_{38} = 3.1^{-0.6}_{+0.6} \times 10^{-6}$.

In terms of resonance exchange, Ref. [36] proposes some relations among different p^6 order LECs,

$$C_{20} = -3C_{21} = C_{32} = \tfrac{1}{6}C_{35} \qquad C_{24} = 6C_{28} = 3C_{30}. \tag{67}$$

To check the validity of these relations for our results, in Table XIV, we write corresponding values obtained in our calculations We see that except C_{35}, all the other LECs satisfy the relations.

VIII. SUMMARY

In this paper, we revise our original formulation of calculating LECs to a chiral covariant one suitable to computerize. With the help of the computer, we successfully obtain the analytical expressions for all the p^6 order LECs in the normal part of the chiral Lagrangian for pseudoscalar mesons on the quark self-energy $\Sigma(k^2)$. The ambiguities for the anomaly part contributions to the normal part of the chiral Lagrangian are clarified and we prove that this part totally should vanish and therefore need not be considered in our computations. Since our calculation is done under the large N_c limit, only operators of p^6 order with one trace and some multitraces from the equation of motion survive in our formulation. We set up relations among the coefficients in front of these operators and LECs defined in Ref. [16]. Then with input of $F_0 =$ 87 MeV to fix the $\Lambda_{\rm QCD}$ in the running coupling constant of $\alpha_s(k^2)$ appear in the kernel of SDE and choose cutoff of the theory being $\Lambda = 1000^{+100}_{-100}$ MeV and $\Lambda = \infty$, we calculate all p^6 order LECs numerically both for two flavor and three flavor cases. Comparing our resulting LECs with those combinations which we can find experimental or model calculation values in the literature, we find that except for a few of them havng wrong signs, most of our predicted combinations of p^6 order LECs have the same signs and orders of magnitudes with experiment or model calculation values. This sets the solid basis for our p^6 order computations. For those combinations with wrong signs or wrong order of magnitudes with experiment values, we need further investigation. Based on these obtained p^6 order LECs, we expect a very large number of predictions for various pseudoscalar meson physics in the near future.

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APPENDIX A: RELATIONS AMONG OUR SYMBOLS AND THOSE USED IN REF. [16]

To help in understanding the mutual relation between the definition of symbols in our formulation and those in Ref. [16], in Table XV we give a comparison.

TABLE XV. Comparisons between the symbols introduce in Ref. [16] (the first and third columns) and corresponding ones defined in present paper (the second and fourth columns).

Ref. [16]	Present paper	Ref. [16]	Present paper
g_R	V_R	χ_-^μ	$4iB_0 d^\mu p_\Omega - 4iB_0 s_\Omega a_\Omega^\mu - 4iB_0 a_\Omega^\mu s_\Omega$
g_L	V_L	$F_{L,R}^{\mu\nu}$	$f_{L,R}^{\mu\nu}$
$u(\varphi)$	Ω	$f_+^{\mu\nu}$	$2V^{\mu\nu} - 2i(a_\Omega^\mu a_\Omega^\nu - a_\Omega^\nu a_\Omega^\mu)$
u^μ	$2a_\Omega^\mu$	$\nabla^\lambda f_+^{\mu\nu}$	$2d^\lambda V^{\mu\nu} - 2id^\lambda(a_\Omega^\mu - a_\Omega^\nu a_\Omega^\mu)$
$u\cdot u = u^\mu u^\mu$	$4a_\Omega^2$	$f_-^{\mu\nu}$	$-2(d^\mu a_\Omega^\nu - d^\nu a_\Omega^\mu)$
χ	χ	$\nabla^\lambda f_-^{\mu\nu}$	$-2(d^\lambda d^\mu a_\Omega^\nu - d^\lambda d^\nu a_\Omega^\mu)$
χ_+	$4B_0 s_\Omega$	$h^{\mu\nu}$	$2(d^\mu a_\Omega^\nu + d^\nu a_\Omega^\mu)$
χ_+^μ	$4B_0 d^\mu s_\Omega + 4B_0 p_\Omega a_\Omega^\mu + 4B_0 a_\Omega^\mu p_\Omega$	∇^μ	d^μ
χ_-	$4iB_0 p_\Omega$	Γ^μ	$-iv_\Omega^\mu$

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