

Slowly varying dilaton cosmologies and their field theory dualsAdel Awad,^{1,2,*} Sumit R. Das,^{3,†} Archisman Ghosh,^{3,‡} Jae-Hyuk Oh,^{3,§} and Sandip P. Trivedi^{4,5,6,||}¹*Center for Theoretical Physics, British University of Egypt, Sherouk City 11837, P.O. Box 43, EGYPT*²*Department of Physics, Faculty of Science, Ain Shams University, Cairo, 11566, EGYPT*³*Department of Physics and Astronomy, University of Kentucky, Lexington, Kentucky 40506, USA*⁴*Tata Institute of Fundamental Research, Mumbai 400005, India*⁵*Stanford Institute of Theoretical Physics, Stanford, California 94305, USA*⁶*SLAC, Stanford University, Stanford, California 94309, USA*

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We consider a deformation of the $\text{AdS}_5 \times S^5$ solution of IIB supergravity obtained by taking the boundary value of the dilaton to be time dependent. The time dependence is taken to be slowly varying on the anti-de Sitter (AdS) scale thereby introducing a small parameter ϵ . The boundary dilaton has a profile which asymptotes to a constant in the far past and future and attains a minimum value at intermediate times. We construct the supergravity (sugra) solution to first nontrivial order in ϵ , and find that it is smooth, horizon-free, and asymptotically $\text{AdS}_5 \times S^5$ in the far future. When the intermediate values of the dilaton becomes small enough the curvature becomes of order the string scale and the sugra approximation breaks down. The resulting dynamics is analyzed in the dual $SU(N)$ gauge theory on S^3 with a time dependent coupling constant which varies slowly. When $N\epsilon \ll 1$, we find that a quantum adiabatic approximation is applicable, and use it to argue that at late times the geometry becomes smooth $\text{AdS}_5 \times S^5$ again. When $N\epsilon \gg 1$, we formulate a classical adiabatic perturbation theory based on coherent states which arises in the large N limit. For large values of the 't Hooft coupling this reproduces the supergravity results. For small 't Hooft coupling the coherent state calculations become involved and we cannot reach a definite conclusion. We argue that the final state should have a dual description which is mostly smooth AdS_5 space with the possible presence of a small black hole.

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I. INTRODUCTION

The AdS/CFT correspondence [1–3] provides us with a nonperturbative formulation of quantum gravity. One hopes that it will shed some light on the deep mysteries of quantum gravity, in particular, on the question of singularity resolution.

Motivated by this hope we consider a class of time dependent solutions in this paper which can be viewed as deformations of the $\text{AdS}_5 \times S^5$ background in IIB string theory. These solutions are obtained by taking the boundary value of the dilaton in anti-de Sitter (AdS) space to become time dependent.¹ We are free to take the boundary value of the dilaton to be any time dependent function. To keep the solutions under analytical control though we take the rate of time variation of the dilaton to be small compared to the radius of AdS space, R_{AdS} . This introduces a small parameter ϵ and we construct the bulk solution in perturbation theory in ϵ . The resulting solutions are found to be well behaved. In particular one finds that no black

hole horizon forms in the course of time evolution. The metric and dilaton respond on a time scale of order R_{AdS} which is nearly instantaneous compared to the much slower time scale at which the boundary value of the dilaton varies. For dilaton profiles which asymptote to a constant in the far future one finds that all the energy that is sent in comes back out and the geometry settles down eventually to that of AdS space. What makes these solutions nontrivial is that by waiting for a long enough time, of order $\frac{R_{\text{AdS}}}{\epsilon}$, a big change in the boundary dilaton can occur. The solutions probe the response of the bulk to such big changes.

Consider an example of this type where the boundary dilaton undergoes a big change making the 't Hooft coupling² of order unity or smaller at intermediate times,

$$\lambda \equiv g_s N \leq O(1), \quad (1)$$

when³ $t \simeq 0$, before becoming large again in the far future. As was mentioned above, the bulk responds rapidly to the changing boundary conditions and within a time of order R_{AdS} the dilaton everywhere in the bulk then becomes

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¹It is important in the subsequent discussion that we work in global AdS_5 with the boundary $S^3 \times R$.²When we refer to the 't Hooft coupling we have the gauge theory in mind and accordingly by the dilaton in this context we will always mean its boundary value.³Here N is the number of units of flux in the bulk and the rank of the gauge group in the boundary theory.

small and meets the condition, Eq. (1). Now the supergravity solution receives α' corrections in string theory, these are important when R_{AdS} becomes of order the string scale. Using the well-known relation,

$$R_{\text{AdS}}/l_s \sim (g_s N)^{1/4} \quad (2)$$

we then find that once Eq. (1) is met the curvature becomes of order the string scale everywhere along a spacelike slice which intersects the boundary. As a result the supergravity approximation breaks down along this slice and the higher derivative corrections become important for the subsequent time development. This break down of the supergravity approximation is the sense in which a singularity arises in these solutions.

In contrast the curvature in units of the 10-dimensional Planck scale l_{Pl} (or the 5-dimensional Planck scale) remains small for all time. The radius R_{AdS} in l_{Pl} units is given by

$$R_{\text{AdS}}/l_{\text{Pl}} \sim N^{1/4}. \quad (3)$$

We keep N to be fixed and large throughout the evolution; this then keeps the curvature small in Planck units.⁴ The solutions we consider can therefore be viewed in the following manner: the curvature in Planck units in these solutions stays small for all time, but for a dilaton profile which meets the condition Eq. (1) the string scale in length grows and becomes of order the curvature scale at intermediate times. At this stage the geometry gets highly curved on the string scale. We are interested in whether a smooth space-time geometry can emerge again in the future in such situations.

It is worth relating this difference in the behavior of the curvature as measured in string and Planck scales to another fact. We saw that when the curvature becomes of order the string scale α' corrections become important. The second source of corrections to the supergravity approximation are quantum loop corrections. Their importance is determined by the parameter $1/N$. Since N is kept fixed and large, these corrections are always small. From Eq. (3) we see that this ties into the fact that the AdS radius stays large in Planck units.

To understand the evolution of the system once the curvature gets to be of order the string scale we turn to the dual gauge theory. The gauge theory lives on an S^3 of radius R and the slowly varying dilaton maps to a Yang-Mills coupling which varies slowly compared to R . Since these are the only two length scales in the system, the slow time variation suggests that one can understand the resulting dynamics in terms of an adiabatic approximation.

In fact we find it useful to consider two different adiabatic perturbation theories. The first, which we call quantum adiabatic perturbation theory, is a good approximation

⁴The backreaction corrects the curvature but these corrections are suppressed in ϵ .

when the parameter ϵ satisfies the condition,

$$N\epsilon \ll 1. \quad (4)$$

Once this condition is met, the rate of change of the Hamiltonian is much smaller than the energy gap between the ground state and the first excited state in the gauge theory. As a result, the standard text book adiabatic approximation in quantum mechanics applies and the system at any time is, to good approximation, in the ground state of the *instantaneous* Hamiltonian. In the far future, when the time dependence turns off, the state settles into the ground state of the resulting $\mathcal{N} = 4$ super Yang-Mills (SYM) theory, and admits a dual description as a smooth AdS space.

Note that this argument holds even when the 't Hooft coupling at intermediate times becomes of order unity or smaller. The fact that the states of the time independent $\mathcal{N} = 4$ SYM theory furnish a unitary representation of the conformal group guarantees that the spectrum has a gap of order $1/R$ for all values of the Yang-Mills coupling [4]; see also, [5,6]. Thus as long as Eq. (4) is met, the conditions for this perturbation theory apply. As a result, we learn that for very slowly varying dilaton profiles which meet the condition, Eq. (4), the geometry after becoming of order the string scale at intermediate times, again opens out into a smooth AdS space in the far future.

The supergravity solutions we construct are controlled in the approximation,

$$\epsilon \ll 1. \quad (5)$$

This is different, and much less restrictive, than the condition stated above in Eq. (4) for the validity of the quantum adiabatic perturbation theory. In fact one finds that a different perturbation theory can also be formulated in the gauge theory. This applies when the conditions,

$$N\epsilon \gg 1, \quad \epsilon \ll 1, \quad (6)$$

are met. This approximation is classical in nature and arises because the system is in the large N limit [otherwise Eq. (6) cannot be met]. We will call this approximation the “large N classical adiabatic perturbation theory” (LNCAPT) below. The behavior of the system in this approximation reproduces the behavior of the supergravity solutions for cases where the 't Hooft coupling is large for all times.

Let us now discuss this approximation in more detail. Each gauge invariant operator in the boundary theory gives rise to an infinite tower of coupled oscillators whose frequency grows with the growing mode number. The gauge invariant operators are dual to bulk modes. The infinite tower of oscillators which arises for each operator is dual to the infinite number of modes, with different radial wave functions and different frequency, which arise for each bulk field. Of particular importance is the operator dual to the dilaton \hat{O} and the modes which arise from it. The

time varying boundary dilaton results in a driving force for these oscillators. When $N\epsilon \gg 1$, these oscillators are excited by the driving force into a coherent state with a large mean occupation number of quanta, of order $N\epsilon$, and therefore behave classically. This is a reflection of the fact that at large N , the system behaves classically: coherent states of these oscillators correspond to classical configurations (see, e.g., Ref [7]).

Usually a reformulation of the boundary theory in terms of such oscillators is not very useful, since these oscillators would have a nontrivial operator algebra which would signify that the bulk modes are interacting. Simplifications happen in low dimensional situations like matrix quantum mechanics [8] where one is led to a collective field theory in $1 + 1$ dimensions as an explicit construction of the holographic map [9]. Even in this situation, the collective field theory is a nontrivial interacting theory, i.e., the oscillators are coupled. In our case there are an infinite number of collective fields which would seem to make the situation hopeless.

In our setup, however, the slowness of the driving force simplifies the situation drastically. The source couples directly to the dilaton in the bulk, and when $\epsilon \ll 1$, to lowest order the response of the dilaton as well as the other fields is *linear* and independent of each other. This will be clear in the supergravity solutions we present below. This implies that to lowest order in ϵ , the oscillators which are dual to these modes are really *harmonic oscillators* which are decoupled from each other.

The resulting dynamics is then well approximated by the classical adiabatic perturbation theory, which we refer to as the LNCAPT as mentioned above. The criterion for its applicability is that the driving force varies on a time scale much slower than the frequency of each oscillator. In particular if the frequency of the driving force is of order that of the oscillators, one would be close to resonance and the perturbation theory would break down. In our case this condition for the driving force to vary slowly compared to the frequency of the oscillators becomes Eq. (5). When this condition is met, the adiabatic approximation is valid for all modes—even those with the lowest frequency. The expectation value of the energy and the operator dual to the dilaton, \hat{O} , can then be calculated in the resulting perturbation theory and we find that the leading order answers in ϵ agree with the supergravity calculations.⁵

Having understood the supergravity solutions in the gauge theory language we turn to asking what happens if the 't Hooft coupling becomes of order unity or smaller at intermediate times [while still staying in the parametric

regime Eq. (6)]. The new complication is that additional oscillators now enter the analysis. These oscillators correspond to string modes in the bulk. When the 't Hooft coupling becomes of order unity their frequencies can become small and comparable to the oscillators which are dual to supergravity modes.

At first sight one is tempted to conclude that these additional oscillators do not change the dynamics in any significant manner and the system continues to be well approximated by the large N classical adiabatic approximation. The following arguments support this conclusion. First, the anharmonic terms continue to be of order ϵ and thus are small, so that the oscillators are approximately decoupled. Second, the existence of a gap of order $1/R$ for all values of the 't Hooft coupling, which we referred to above, ensures that the driving force varies much more slowly than the frequency of the additional oscillators, thus keeping the system far from resonance. Finally, one still expects that in the parametric regime, Eq. (6), an $O(N\epsilon)$ number of quanta are produced keeping the system classical. These arguments suggest that the system should continue to be well approximated by the LNCAPT. In fact, since the additional oscillators do not directly couple to the driving force produced by the time dependent dilaton, but rather couple to it only through anharmonic terms which are subdominant in ϵ , their effects should be well controlled in an ϵ expansion. If these arguments are correct, the energy which is pumped into the system initially should then get completely pumped back out and the system should settle into the ground state of the final $\mathcal{N} = 4$ theory in the far future. The dual description in the far future would then be a smooth AdS₅ space-time.

However, further thought suggests another possibility for the resulting dynamics which is of a qualitatively different kind. This possibility arises because, as was mentioned above, when the 't Hooft coupling becomes of order unity, string modes can get as light as supergravity modes. This means that the frequency of some of the oscillators dual to string modes can become comparable to oscillators dual to supergravity modes, and thus the string mode oscillators can get activated. Now there are many more string mode oscillators than there are supergravity mode oscillators, since the supergravity modes correspond to chiral operators in the gauge theory which are only $O(1)$ in number, while the string modes correspond to nonchiral operators which are $O(N^2)$ in number. Thus once string mode oscillators can get activated there is the possibility that many new degrees of freedom enter the dynamics.

With so many degrees of freedom available the system could thermalize at least in the large N limit. In this case the energy which is initially present in the oscillators that directly couple to the dilaton would get equipartitioned among all the degrees of freedom. The subsequent evolution would be dissipative and this energy would not be recovered in the far future. At late times, when the 't Hooft

⁵More precisely, both the supergravity and the forced oscillator calculations need to be renormalized to get a finite answer. One finds that after the counterterms are chosen to get agreement for the standard two-point function (which measures the response for a small amplitude dilaton perturbation) the expectation value of the energy and \hat{O} agree.

coupling becomes big again, the gravity description of the dissipative behavior depends on how small is ϵ . From the calculations done in the supergravity regime, one knows that the total energy that is produced is of order $N^2\epsilon^2$. When $N\epsilon \gg 1$, but $\epsilon \ll (g_{\text{YM}}^2 N)^{-7/8}$ the result is likely to be a gas of string modes. However if $\epsilon > (g_{\text{YM}}^2 N)^{-7/8}$, the energy is sufficient to form a small black hole (with horizon radius smaller than R_{AdS}). A big black hole cannot form since this would require an energy of the order of N^2 , and $\epsilon \ll 1$ always. Thus, in the far future, once the 't Hooft coupling becomes large again, the strongest departure from normal space-time would be the presence of a small black hole in AdS space. The small black hole would eventually disappear by emitting Hawking radiation but that would happen on a much longer time scale of order $N^2 R_{\text{AdS}}$.

It is difficult for us to settle here which of the two possibilities discussed above, either adiabatic nondissipative behavior well described by the LNCAPT, or dissipative behavior with organized energy being lost in heat, is the correct one. One complication is that the rate of time variation which is set by ϵ is also the strength of the anharmonic couplings between the oscillators. In thermodynamics, working in the microcanonical ensemble, it is well known that with energy of order $N^2\epsilon^2$ the configuration which entropically dominates is a small black hole.⁶ This suggests that if the time variation in the problem were much smaller than the anharmonic terms, a small black hole would form. However, in our case their being comparable makes it a more difficult question to decide. One should emphasize that regardless of which possibility is borne out our conclusion is that most of the space-time in the far future is smooth AdS, with the possible presence of a small black hole.

Let us end with some comments on related work. The spirit of our investigation is close to the work on AdS cosmologies in [10] and related work in [11–14]. See also [15–17] for additional work. Discussion of cosmological singularities in the context of the matrix theory appears in [18].

The supergravity analysis we describe is closely related to the strategy which was used in the paper [19], for finding forced fluid dynamics solutions; in that case one worked with an infinite brane at temperature T and the small parameter was the rate of variation of the dilaton (or metric) compared to T . Our regime of interest is complementary to that in [20] where the dilaton was chosen to be small in amplitude, but with arbitrary time dependence and which leads to formation of black holes in supergravity for a suitable regime of parameters.

This paper is organized as follows. In Sec. II we find the supergravity solutions and use them to find the expectation value of operators in the boundary theory like the stress

⁶At least when the 't Hooft coupling is big enough so that supergravity can be trusted.

energy and \hat{O} in Sec. III. The quantum adiabatic perturbation theory is discussed in Sec. IV. A forced harmonic oscillator is discussed in Sec. V. This simple system helps illustrate the difference between the two kinds of perturbation theory and sets the stage for the discussion of the large N classical adiabatic approximation in Sec. VI. Conclusions and future directions are discussed in Sec. VII. There are three appendices which contains details of derivation of some of the formulas in the main text.

II. THE BULK RESPONSE

In this section we will calculate the deformation of the supergravity solution in the presence of a slowly varying time dependent but spatially homogeneous dilaton specified on the boundary. This will be a reliable description of the time evolution of the system so long as $e^{\Phi(t)}$ never becomes small.

A. Some general considerations

IIB supergravity in the presence of the Ramond-Ramond five form flux is well known to have an $\text{AdS}_5 \times S^5$ solution. In global coordinates this takes the form

$$ds^2 = -\left(1 + \frac{r^2}{R_{\text{AdS}}^2}\right)dt^2 + \frac{dr^2}{1 + \frac{r^2}{R_{\text{AdS}}^2}} + r^2 d\Omega_3^2 + R_{\text{AdS}}^2 d\Omega_5^2. \quad (7)$$

Here R_{AdS} is given by

$$R_{\text{AdS}} = (4\pi g_s N)^{1/4} l_s \sim N^{1/4} l_{\text{pl}} \quad (8)$$

where l_s is the string scale and $l_{\text{pl}} \sim g_s^{1/4} l_s$ is the 10-dimensional Planck scale. g_s is the value of the dilaton, which is constant and does not vary with time or spatial position,

$$e^{\Phi} = g_s. \quad (9)$$

In the time dependent situations we consider below N will be held fixed. Let us discuss some of our conventions before proceeding. We will find it convenient to work in the 10-dimensional Einstein frame. Usually one fixes l_{pl} to be of order unity in this frame. Instead for our purposes it will be convenient to set

$$R_{\text{AdS}} = 1. \quad (10)$$

From Eq. (8) this means setting $l_{\text{pl}} \sim 1/N^{1/4}$. The $\text{AdS}_5 \times S^5$ solution then becomes

$$ds^2 = -(1 + r^2)dt^2 + \frac{1}{(1 + r^2)}dr^2 + r^2 d\Omega_3^2 + d\Omega_5^2, \quad (11)$$

for any constant value of the dilaton, Eq. (9). Let us also mention that when we turn to the boundary gauge theory

we will set the radius R of the S^3 on which it lives to also be unity.

The essential idea in finding the solutions we describe is the following. Consider a situation where Φ varies with time slowly compared to R_{AdS} . Since the solution above exists for any value of g_s and the dilaton varies slowly one expects that the resulting metric at any time t is well approximated by the $\text{AdS}_5 \times S^5$ metric given in Eq. (11). This zeroth order metric will be corrected due to the varying dilaton which provides an additional source of stress energy in the Einstein equations. However these changes should be small for a slowly varying dilaton and should therefore be calculable order by order in perturbation theory.

Let us make this more precise. Consider as the starting point of this perturbation theory the AdS_5 metric given in Eq. (11) and a dilaton profile,

$$\Phi = \Phi_0(t) \quad (12)$$

which is a function of time alone. We take $\Phi_0(t)$ to be of the form

$$\Phi_0 = f\left(\frac{\epsilon t}{R_{\text{AdS}}}\right) \quad (13)$$

where $f\left(\frac{\epsilon t}{R_{\text{AdS}}}\right)$ is a dimensionless function of time and ϵ is a small parameter,

$$\epsilon \ll 1. \quad (14)$$

The function f satisfies the property that

$$f'\left(\frac{\epsilon t}{R_{\text{AdS}}}\right) \sim O(1) \quad (15)$$

where prime indicates the derivative with respect to the argument of f .

When $\epsilon = 0$, the dilaton is a constant and the solution reduces to $\text{AdS}_5 \times S^5$. When ϵ is small,

$$\frac{d\Phi_0}{dt} = \frac{\epsilon}{R_{\text{AdS}}} f'\left(\frac{\epsilon t}{R_{\text{AdS}}}\right) \sim \frac{\epsilon}{R_{\text{AdS}}} \quad (16)$$

so that the dilaton is varying slowly on the scale R_{AdS} , and the contribution that the dilaton makes to the stress tensor is parametrically suppressed.⁷ In such a situation the backreaction can be calculated order by order in ϵ . The time dependent solutions we consider will be of this type and ϵ will play the role of the small parameter in which we carry out the perturbation theory. A simple rule to count powers of ϵ is that every time derivative of Φ_0 comes with a factor of ϵ .

The profile for the dilaton we have considered in Eq. (12) is S^5 symmetric. It is consistent to assume that

⁷The more precise statement for the slowly varying nature of the dilaton, as will be discussed in a footnote before Eq. (85), is that its Fourier transform has support at frequencies much smaller than $1/R_{\text{AdS}}$.

the backreacted metric will also be S^5 symmetric with the radius of the S^5 being equal to R_{AdS} . The interesting time dependence will then unfold in the remaining five directions of AdS space and we will focus on them in the following analysis.

The zeroth order metric in these directions is given by

$$ds^2 = -(1+r^2)dt^2 + \frac{1}{(1+r^2)}dr^2 + r^2 d\Omega_3^2. \quad (17)$$

And the zeroth order dilaton is given by Eq. (12),

$$\Phi_0 = f(\epsilon t). \quad (18)$$

We can now calculate the corrections to this solution order by order in ϵ .

Let us make two more points at this stage. First, we will consider a dilaton profile Φ_0 which approaches a constant as $t \rightarrow -\infty$. This means that in the far past the corrections to the metric and the dilaton which arise as a response to the time variation of the dilaton must also vanish. Second, the perturbation theory we have described above is a derivative expansion. The solutions we find can only describe slowly varying situations. This stills allows for a big change in the amplitude of the dilaton and the metric though, as long as such changes accrue gradually. It is this fact that makes the solutions nontrivial.

B. Corrections to the dilaton

Let us first calculate the corrections to the dilaton. We can expand the dilaton as

$$\Phi(t) = \Phi_0(t) + \Phi_1(r, t) + \Phi_2(r, t) \cdots, \quad (19)$$

where Φ_0 is the zeroth order profile we start with, given in Eq. (13). Φ_1 is of order ϵ , Φ_2 is of order ϵ^2 , and so on. The metric can be expanded as

$$g_{ab} = g_{ab}^{(0)} + g_{ab}^{(1)} + g_{ab}^{(2)} + \cdots \quad (20)$$

where $g_{ab}^{(0)}$ is the zeroth order metric given in Eq. (17) and $g_{ab}^{(1)}, g_{ab}^{(2)} \cdots$ are the first order, second order, etc. corrections.

The dilaton satisfies the equation

$$\nabla^2 \Phi = 0. \quad (21)$$

Expanding this we find that to order ϵ^2 ,

$$\nabla_0^2 \Phi_0 + \nabla_0^2 \Phi_1 + \nabla_1^2 \Phi_0 + \nabla_1^2 \Phi_1 + \nabla_0^2 \Phi_2 = 0. \quad (22)$$

Here ∇_0^2 is the Laplacian which arises from the zeroth order metric, and ∇_1^2, ∇_2^2 are the corrections to the Laplacian to order ϵ, ϵ^2 , respectively, which arise due to the corrections in the metric. The first term on the left-hand side is of order ϵ^2 , since it involves two time derivatives acting on Φ_0 . The second term is of order⁸ ϵ , and so is the third term.

⁸It is easy to see that Φ_1 , if nonvanishing, must depend on the radial coordinate; this makes $\nabla_0^2 \Phi_1$ of order ϵ . Φ_1 would be r dependent for the same reason that Φ_2 in in Eq. (25).

However, we see in Sec. II C that the $O(\epsilon)$ correction to the metric and thus ∇_1^2 vanishes. So the second term is the only one of $O(\epsilon)$ and we learn that

$$\Phi_1 = 0. \quad (23)$$

The first correction to the dilaton therefore arises at $O(\epsilon^2)$. Equation (22) now becomes

$$\nabla_0^2 \Phi_0 + \nabla_0^2 \Phi_2 = 0. \quad (24)$$

Since Φ_0 preserves the S^3 symmetry of AdS_5 , Φ_2 will also be S^3 symmetric and must therefore only be a function of t, r . Further since Φ_2 is $O(\epsilon^2)$ any time derivative on it would be of higher order and can be dropped. Solving Eq. (24) then gives

$$\begin{aligned} \Phi_2(r, t) = & \int^r \frac{dr'}{(r')^3(1+(r')^2)} \left[\int^{r'} \frac{y^3}{1+y^2} dy \ddot{\Phi}_0(t) \right. \\ & \left. + a_1(t) \right] + a_2(t). \end{aligned} \quad (25)$$

Here $a_1(t), a_2(t)$ are two functions of time which arise as integration ‘‘constants.’’

The integrations in (25) can be performed, leading to

$$\begin{aligned} \Phi_2(r, t) = & \frac{1}{4} \ddot{\Phi}_0(t) \left[\frac{1}{r^2} \log(1+r^2) - \frac{1}{2} (\log(1+r^2))^2 \right. \\ & \left. - \text{dilog}(1+r^2) \right] \\ & + a_1(t) \frac{1}{2} \left[\log(1+r^2) - \frac{1}{r^2} - 2 \log r \right] + a_2(t). \end{aligned} \quad (26)$$

The first term in Φ_2 is regular at $r = 0$, while the term multiplying $a_1(t)$ diverges here. To find a self-consistent solution in perturbation theory Φ_2 must be small compared to Φ_0 for all values of r , we therefore set $a_1 = 0$. The first term in $\Phi_2(r, t)$ has the following expansion for large values of r ,

$$\ddot{\Phi}_0(t) \left[\frac{\pi^2}{24} - \frac{1}{4r^2} + \left(\frac{3}{16} + \frac{1}{4} \log r \right) \frac{1}{r^4} + \dots \right]. \quad (27)$$

Since we are solving for the dilaton with a specified boundary value $\Phi_0(t)$, $\Phi_2(r, t)$ should vanish at the boundary. This determines $a_2(t)$ to be

$$a_2(t) = -\frac{\pi^2}{24} \ddot{\Phi}_0(t), \quad (28)$$

leading to the final solution

$$\begin{aligned} \Phi_2(r, t) = & \frac{1}{4} \ddot{\Phi}_0(t) \left[\frac{1}{r^2} \log(1+r^2) - \frac{1}{2} (\log(1+r^2))^2 \right. \\ & \left. - \text{dilog}(1+r^2) - \frac{\pi^2}{6} \right]. \end{aligned} \quad (29)$$

The solution is regular everywhere. Since $\text{Lim}_{t \rightarrow -\infty} \ddot{\Phi}_0(t)$,

$\ddot{\Phi}_0(t) = 0$, the correction vanishes in the far past, as required.

C. Corrections to the metric

The time varying dilaton provides an additional source of stress energy. The lowest order contribution due to this stress energy is $O(\epsilon)^2$ as we will see below. It then follows, after a suitable coordinate transformation if necessary, that the $O(\epsilon)$ corrections to the metric vanish and the first nonvanishing corrections to it arise at order ϵ^2 . The essential point here is that any $O(\epsilon)$ correction to the metric must be r dependent and thus would lead to a contribution to the Einstein tensor of order ϵ , which is not allowed. This is illustrated by the dilaton calculation above, where a similar argument leads to the $O(\epsilon)$ contribution, Φ_1 , vanishing. In this subsection we calculate the leading $O(\epsilon^2)$ corrections to the metric.

Before we proceed it is worth discussing the boundary conditions which must be imposed on the metric. As was discussed in the previous subsection, we consider a dilaton source, Φ_0 , which approaches a constant value in the far past, $t \rightarrow -\infty$. The corrections to the metric that arise from such a source should also vanish in the far past. Thus we see that as $t \rightarrow -\infty$ the metric should approach that of AdS_5 space-time. Also the solutions we are interested in correspond to the gauge theory living on a time independent $S^3 \times R$ space-time in the presence of a time dependent Yang-Mills coupling (dilaton). This means the leading behavior of the metric for large r should be that of AdS_5 space. Changing this behavior corresponds to turning on a non-normalizable component of the metric and is dual to changing the metric of the space-time on which the gauge theory lives.

We expect that these boundary conditions, which specify both the behavior as $t \rightarrow -\infty$ and as $r \rightarrow \infty$ should lead to a unique solution to the super gravity equations. The former determine the normalizable modes and the latter the non-normalizable modes. This is dual to the fact that in the gauge theory the response should be uniquely determined once the time dependent Lagrangian is known (this corresponds to the fixing the non-normalizable modes) and the state of the system is known in the far past (this corresponds to fixing the normalizable modes).

Since Φ_0 is S^3 symmetric, we can consistently assume that the corrections to the metric will also preserve the S^3 symmetry. The resulting metric can then be written as

$$\begin{aligned} ds^2 = & -g_{tt}(t, r) dt^2 + g_{rr}(t, r) dr^2 + 2g_{tr}(t, r) dt dr \\ & + R^2 d\Omega^2. \end{aligned} \quad (30)$$

Now as is discussed in Appendix A up to $O(\epsilon^2)$ we can consistently set $g_{tr} = 0$. In addition to this order we can set $R^2 = r^2$. Below we also use the notation

$$g_{tt} \equiv e^{2A(t,r)}, \quad (31)$$

$$g_{rr} \equiv e^{2B(t,r)}. \quad (32)$$

The metric then takes the form

$$ds^2 = -e^{2A(t,r)}dt^2 + e^{2B(t,r)}dr^2 + r^2d\Omega^2. \quad (33)$$

The trace reversed Einstein equation is

$$R_{AB} = \Lambda g_{AB} + \frac{1}{2}\partial_A\Phi\partial_B\Phi. \quad (34)$$

In our conventions,

$$\Lambda = -4. \quad (35)$$

To order ϵ^2 we can set $\Phi = \Phi_0$ in the second term on the right-hand side (rhs).

A few simple observations make the task of computing the curvature components to $O(\epsilon^2)$ much simpler. As we mentioned above, the first corrections to the metric should arise at $O(\epsilon^2)$. To order ϵ^2 the metric is then

$$g_{ab}(t,r) = g_{ab}^{(0)}(r) + g_{ab}^{(2)}(t,r). \quad (36)$$

Now the zeroth order metric, $g_{ab}^{(0)}$, is time independent. The time derivatives of $g_{ab}^{(2)}$ are nonvanishing but of order ϵ^3 and thus can be neglected for calculating the curvature tensor to this order. As a result for calculating the curvature components to order ϵ^2 we can neglect all time derivatives of the metric, Eq. (36).

Before proceeding we note that the comments above imply that the equations determining the second order metric components schematically take the form

$$\hat{O}(r)g_{ab}^{(2)} = f_{ab}(r)\dot{\Phi}_0^2 \quad (37)$$

where $\hat{O}(r)$ is a second order differential operator in the radial variable, r . As a result the solution will be of the form

$$g_{ab}^{(2)} = \mathcal{F}(r)_{ab}\dot{\Phi}_0^2, \quad (38)$$

where $\mathcal{F}(r)$ are functions of r which arise by inverting $\hat{O}(r)$. We see that the corrections to the metric at time t are determined by the dilaton source Φ_0 at the same instant of time t . Note also that since we are only considering a dilaton source Φ_0 which vanishes in the far past, the solution Eq. (38) correctly imposes the boundary condition that $g_{ab}^{(2)}$ vanishes in far past and the metric becomes that of AdS₅.

Bearing in mind the discussion above, the curvature components are now easy to calculate. The $t-t$ component of Eq. (34) gives

$$\frac{(A'e^{(A-B)})'}{e^{(A+B)}} + 3\frac{A'e^{-2B}}{r} = \frac{\dot{\Phi}_0^2}{2}e^{-2A} + 4. \quad (39)$$

The $r-r$ component gives

$$-\frac{(A'e^{(A-B)})'}{e^{(A+B)}} + 3\frac{B'e^{-2B}}{r} = -4. \quad (40)$$

The component with legs along the S^3 gives

$$\frac{B' - A'}{e^{2B}r} + \frac{2}{r^2}(1 - e^{-2B}) = -4. \quad (41)$$

In these equations primes indicate derivatives with respect to r and dots indicate derivatives with respect to time.

Adding the $t-t$ and $r-r$ equations gives

$$3(A' + B')\frac{e^{-2B}}{r} = \frac{\dot{\Phi}_0^2}{2}e^{-2A}. \quad (42)$$

Equations (41) and (42) then lead to

$$\frac{2B'e^{-2B}}{r} - \frac{1}{6}\dot{\Phi}_0^2e^{-2A} + \frac{2}{r^2}(1 - e^{-2B}) = -4. \quad (43)$$

This is a first order equation in B . Integrating we get to order ϵ^2 ,

$$e^{-2B} = 1 + r^2 + \frac{c_1}{r^2} - \frac{1}{6}\frac{\dot{\Phi}_0^2}{r^2}\left[\int_0^r e^{-2A_0}r^3 dr\right]. \quad (44)$$

Here c_1 is an integration constant and $e^{2A_0} = 1 + r^2$ is the zeroth order value of e^{2A} . We require that the metric become that of AdS₅ space as $t \rightarrow -\infty$ this sets $c_1 = 0$.⁹ A negative value of c_1 would mean starting with a black hole in AdS₅ in the far past.

The integral within the square brackets on the rhs in Eq. (44) is given by

$$\int_0^r e^{-2A_0}r^3 dr = \frac{1}{2}[r^2 - \ln(1 + r^2) + d_1]. \quad (45)$$

This gives

$$e^{-2B} = 1 + r^2 - \frac{1}{12}\frac{\dot{\Phi}_0^2}{r^2}[r^2 - \ln(1 + r^2) + d_1]. \quad (46)$$

A solution which is regular for all values of r is obtained by setting d_1 to vanish. This gives

$$e^{-2B} = 1 + r^2 - \frac{1}{12}\dot{\Phi}_0^2\left[1 - \frac{1}{r^2}\ln(1 + r^2)\right]. \quad (47)$$

We can obtain e^{2A} from Eq. (42). To second order in ϵ^2 this equation becomes

$$A' = \frac{1}{6}r\dot{\Phi}_0^2e^{-2(A_0-B_0)} - B', \quad (48)$$

which gives

$$A = -B + \frac{1}{12}\dot{\Phi}_0^2\left[-\frac{1}{1+r^2} + d_3\right], \quad (49)$$

with d_3 being a general function of time. Equations (47) and (49) lead to

⁹Note that c_1 could be a function of time and still solve Eq. (43); recall though that the equations above were derived by neglecting all time derivatives of the metric, Eq. (36). Only a time independent constant c_1 is consistent with this assumption. A similar argument will also apply to the other integration constants we obtain as we proceed.

$$e^{2A} = 1 + r^2 + \dot{\Phi}_0^2 \left[-\frac{1}{4} + \frac{1}{12} \frac{\ln(1+r^2)}{r^2} + \frac{d_3}{6} (1+r^2) \right]. \quad (50)$$

The last term on the right-hand side changes the leading behavior of e^{2A} as $r \rightarrow \infty$, if d_3 does not vanish, and therefore corresponds to turning on a non-normalizable mode of the metric. As was discussed above, we want solutions where this mode is not turned on, and we therefore set d_3 to vanish.

This finally gives

$$e^{2A} = 1 + r^2 - \frac{1}{4} \dot{\Phi}_0^2 + \frac{1}{12} \dot{\Phi}_0^2 \frac{\ln(1+r^2)}{r^2}. \quad (51)$$

Equations (47) and (51) are the solutions to the metric, Eq. (33), to second order. Note that the Einstein equations give rise to three equations, Eqs. (39)–(41). We have used only two linear combinations out of these to find A , B . One can show that the remaining equation is also solved by the solution given above.

In summary we note that the Einstein equations can be solved consistently to second order in ϵ^2 . The resulting solution is horizon-free and regular for all values of the radial coordinate and satisfies the required boundary conditions discussed above. The second order correction to the metric is parametrically suppressed by ϵ^2 compared to the leading term for all values of r , thereby making the perturbation theory self-consistent.

Let us end by commenting on the choice of integration constants made in obtaining the solution above. The boundary conditions, as $t \rightarrow -\infty$ and $r \rightarrow \infty$, determine most of the integration constants. One integration constant d_1 which appears in the solution for e^{2B} , Eq. (46) is fixed by regularity at $r \rightarrow 0$.¹⁰ For $d_1 = 0$ the second order correction is small compared to the leading term, and the use of perturbation theory is self-consistent. Moreover we expect that the boundary conditions imposed here lead to a unique solution to the supergravity equations, as was discussed at the beginning of this subsection. Thus the solution obtained by setting $d_1 = 0$ should be the correct one.

The solution above is regular and has no horizon. It has these properties due to the slowly varying nature of the boundary dilaton. The dual field theory in this case is in a nondissipative phase. Once the dilaton begins to change sufficiently rapidly with time we expect that a black hole is formed, corresponding to the formation of a strongly dissipative phase in the dual field theory. In [20] the effect of a *small amplitude* time dependent dilaton with arbitrary time dependence was studied. Indeed it was found that when the time variation is fast enough, there are no regular horizon-free solutions and a black hole is formed.

¹⁰Similar in solving for the dilaton perturbation the integration constant a_1 is fixed by requiring regularity at $r = 0$, Eq. (25).

Finally, the analysis of this section holds when e^Φ is large enough to ensure applicability of supergravity. The fact that a black hole is not formed in this regime does not preclude formation of black holes from stringy effects when e^Φ becomes small enough. In fact we will argue in later sections that the latter is a distinct possibility.

D. Effective decoupling of modes

An important feature of the lowest order calculation of this section is that the perturbations of the dilaton and the metric are essentially linear and do not couple to each other. To this order, the dilaton perturbation is simply a solution of the linear d'Alembertian equation in AdS₅. Similarly the metric perturbations also satisfy the linearized equations of motion in AdS, albeit in the presence of a source provided by the energy-momentum tensor of the dilaton. This is a feature present only in the leading order calculation. As explained above, this arises because of the smallness of the parameter ϵ . We will use this feature to compare leading order supergravity results with gauge theory calculations in a later section.

III. CALCULATION OF STRESS TENSOR AND OTHER OPERATORS

In this section we calculate the boundary stress tensor and the expectation value of the operator dual to the dilaton, staying in the supergravity approximation. This will be done using standard techniques of the holographic renormalization group [21–28].

A. The energy-momentum tensor

The metric is of the form of Eqs. (33), (47), and (51). For calculating the stress tensor, a boundary is introduced at large and finite radial location, $r = r_0$. The induced metric on the boundary is

$$\begin{aligned} ds_B^2 &\equiv h_{\mu\nu} dx^\mu dx^\nu \\ &= -e^{2A} dt^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2 \\ &\quad + \sin^2 \theta \sin^2 \phi d\psi^2). \end{aligned} \quad (52)$$

The 5-dimensional action is given by

$$\begin{aligned} S_5 &= \frac{1}{16\pi G_5} \int_M d^5x \sqrt{-g} \left(R + 12 - \frac{1}{2} (\nabla\Phi)^2 \right) \\ &\quad - \frac{1}{8\pi G_5} \int_{r=r_0} d^4x \sqrt{-h} \Theta. \end{aligned} \quad (53)$$

Here $h_{\mu\nu}$ is the induced metric on the boundary, and Θ is the trace of the extrinsic curvature of the boundary. In our conventions, with $R_{\text{AdS}} = 1$,

$$G_5 = \frac{\pi}{2N^2}. \quad (54)$$

A counterterm needs to be added, it is

$$S_{\text{ct}} = -\frac{1}{8\pi G_5} \int_{\partial M} d^4x \sqrt{-h} \left[3 + \frac{\mathcal{R}}{4} - \frac{1}{8} (\nabla\Phi)^2 - \log(r_0) a_{(4)} \right]. \quad (55)$$

The last term is needed to cancel logarithmic divergences which arise in the action; it is well known and is discussed in, e.g., [21,27]. From Eq. (24) of [27] we have¹¹ that

$$a_{(4)} = \frac{1}{8} R_{\mu\nu} R^{\mu\nu} - \frac{1}{24} R^2 - \frac{1}{8} R^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{24} R h^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{16} (\nabla^2 \Phi)^2 + \frac{1}{48} \{(\nabla\Phi)^2\}^2. \quad (56)$$

Here ∇ is a covariant derivative with respect to the metric $h_{\mu\nu}$.

Varying the total action $S_T = S_5 + S_{\text{ct}}$ gives the stress energy,

$$\begin{aligned} T^{\mu\nu} &= \frac{2}{\sqrt{-h}} \frac{\delta S_T}{\delta h_{\mu\nu}} \\ &= \frac{1}{8\pi G_5} \left[\Theta^{\mu\nu} - \Theta h^{\mu\nu} - 3h^{\mu\nu} + \frac{1}{2} G^{\mu\nu} - \frac{1}{4} \nabla^\mu \Phi \nabla^\nu \Phi + \frac{1}{8} h^{\mu\nu} (\nabla\Phi)^2 + \dots \right]. \end{aligned} \quad (57)$$

Here $G^{\mu\nu}$ is the Einstein tensor with respect to the metric $h_{\mu\nu}$. The ellipses stand for extra terms obtained by varying the last term in Eq. (55) proportional to $a_{(4)}$. While these terms are not explicitly written down in Eq. (57), we do include them in the calculations below.

The expectation value of the stress tensor in the boundary theory is then given by

$$\langle T_\nu^\mu \rangle = r_0^4 T_\nu^\mu. \quad (58)$$

Carrying out the calculation gives a finite answer,

$$\begin{aligned} \langle T_t^t \rangle &= \frac{N^2}{4\pi^2} \left[-\frac{3}{8} - \frac{\dot{\Phi}_0^2}{16} \right] \\ \langle T_\theta^\theta \rangle = \langle T_\psi^\psi \rangle = \langle T_\phi^\phi \rangle &= \frac{N^2}{4\pi^2} \left[\frac{1}{8} - \frac{\dot{\Phi}_0^2}{16} \right] \end{aligned} \quad (59)$$

where we have used Eq. (54). We remind the reader that in our conventions the radius of the S^3 on which the boundary gauge theory lives has been set equal to unity. The first term on the right-hand side of (59) arises due to the Casimir effect. The second term is the additional contribution due to the varying Yang-Mills coupling.

From Eq. (59) the total energy in the boundary theory can be calculated. We get

$$E = -\langle T_t^t \rangle V_{S^3} = \frac{3N^2}{16} + \frac{N^2 \dot{\Phi}_0^2}{32}, \quad (60)$$

where $V_{S^3} = 2\pi^2$ is the volume of a unit three-sphere. Note that the varying dilaton gives rise to a positive contribution to the mass, as one would expect. Moreover this additional contribution vanishes when the $\dot{\Phi}$ vanishes. In particular, for a dilaton profile which in the far future, as $t \rightarrow \infty$, again approaches a constant value (which could be different from the starting value it had at $t \rightarrow -\infty$) the net energy produced due to the varying dilaton vanishes.

B. Expectation value of the operator dual to the dilaton

The operator dual to the dilaton has been discussed explicitly in [3,10,29].

Its expectation value is given by

$$\langle \hat{\mathcal{O}}_{l=0} \rangle = \frac{\delta S_T}{\delta \Phi_B} \Big|_{\Phi_B \rightarrow 0}. \quad (61)$$

Here S_T is the total action including the boundary terms, Eq. (55). Since Φ_B is a function of t alone the left-hand side (lhs) is the $l=0$ component of the operator dual to the dilaton which we denote by $\hat{\mathcal{O}}_{l=0}$.

The steps involved are analogous to those above for the stress tensor and yield,

$$\langle \hat{\mathcal{O}}_{l=0} \rangle = -\frac{N^2}{16} \dot{\Phi}_0. \quad (62)$$

Note that the lhs refers to the expectation value for the dual operator integrated over the boundary S^3 . In obtaining Eq. (63) we have removed all the divergent terms and only kept the finite piece. A quadratically divergent piece is removed by the third term in Eq. (55) proportional to $(\nabla\Phi)^2$, and a log divergence is removed by a contribution from the last term in Eq. (55) proportional to $a_{(4)}$.

C. Additional comments

Let us end this section with a few comments.

The only source for time dependence in the boundary theory is the varying Yang-Mills coupling. A simple extension of the usual Noether procedure for the energy, now in the presence of this time dependence, tells us that

$$\frac{dE}{dt} = -\dot{\Phi}_0 \langle \hat{\mathcal{O}}_{l=0} \rangle. \quad (63)$$

It is easy to see that the answers obtained above in Eqs. (60) and (62) satisfy this relation. The relation Eq. (63) is a special case of a more general relation which applies for a dilaton varying both in space and time; this was discussed in Appendix A of [19].

In general, for a slowly varying dilaton one can expand $\langle \hat{\mathcal{O}}_{l=0} \rangle$ in a power series in $\dot{\Phi}_0$. For constant dilaton, the solution is AdS₅ where one knows that the $\langle \hat{\mathcal{O}}_{l=0} \rangle$ vanishes. Thus one can write

¹¹Note that our definition of the dilaton Φ is related to $\phi_{(0)}$ in [27] by $\phi_{(0)} = \Phi/2$.

$$\langle \hat{\mathcal{O}}_{l=0} \rangle = c_1 \dot{\Phi}_0 + c_2 \ddot{\Phi}_0 + c_3 (\dot{\Phi}_0)^2 \dots \quad (64)$$

where the ellipses stand for higher powers of derivatives of the dilaton. Comparing with the answer in Eq. (62) one sees that in the supergravity limit c_1 and c_3 vanish. As a result $\frac{dE}{dt}$ is a total derivative, and as was discussed above if the dilaton asymptotes to a constant in the far future there is no net gain in energy.

It is useful to contrast this with what happens in the case of an infinite black brane at temperature T subjected to a time dependent dilaton which is slowly varying compared to the temperature T . This situation was analyzed extensively in [19]. In that case [see Eq. (2.13), Eq. (3.20), and section 7.2 of Ref. [19]] the leading term in Eq. (64) proportional to $\dot{\Phi}_0$ does not vanish. The temperature then satisfies an equation,

$$\frac{dT}{dt} = \frac{1}{12\pi} \dot{\Phi}_0^2. \quad (65)$$

As a result any variation in the dilaton leads to a net increase in the temperature, and the energy density. Note the first term in Eq. (64) contains only one derivative with respect to time and breaks time reversal invariance. It can only arise in a dissipative system. In the case of a black hole, the formation of a horizon breaks time reversal invariance and turns the system dissipative allowing this term to arise. In the solution we construct no horizons form and are consistent that the first term is absent.

We see in the solution discussed above that the second order corrections to the dilaton and metric arise in an instantaneous manner—at some time t , and for all values of r , they are determined by the boundary value of the dilaton at the same instant of time t . This might seem a little puzzling at first since one would have expected the effects of the changing boundary conditions to be felt in a retarded manner. Note though that in AdS space a light ray can reach any point in the bulk from the boundary within a time of order R_{AdS} . When $\epsilon \ll 1$ this is much smaller than the time taken for the boundary conditions to change appreciably. This explains why the leading corrections arise in an instantaneous manner. Some of the corrections which arise at higher order would turn this instantaneous response into a retarded one.

From the solution and the expectation values of the energy and $\hat{\mathcal{O}}_{l=0}$ it follows that in the far future the system settles down into an AdS₅ solution again. The near instantaneous nature of the solution means that this happens quickly on the times scale of order R_{AdS} . This agrees with general expectations. The supergravity modes carry an energy of order $1/R_{\text{AdS}}$ and should give rise to a response time of order R_{AdS} .

Also note that in our units, where $R_{\text{AdS}} = 1$, each supergravity mode carries an energy of order unity. The total energy at intermediate times is of order $N^2 \epsilon^2$, so we see that an $O(N^2 \epsilon^2)$ number of quanta are excited by the time

varying boundary dilaton. This can be a big number when $N\epsilon \gg 1$. In fact the energy is really carried by the various dilaton modes. The metric perturbations are S^3 symmetric and thus contain no gravitons (in the sense of genuine propagating modes). One can think of this energy as being stored in a spatial region of order R_{AdS} in size located at the center of AdS space. This is what one would expect, since the supergravity modes which are produced by the time varying boundary dilaton have a size of order R_{AdS} and their gravitational redshift is biggest at the center of AdS space.¹²

In summary, the response in the bulk to the time varying boundary dilaton is characteristic of a nondissipative adiabatic system which is being driven much more slowly than its own fast internal time scale of response.

IV. GAUGE THEORY : QUANTUM ADIABATIC APPROXIMATION

We now turn to analyzing the behavior of the system in the dual field theory. The motivation behind this is to be able to extend our understanding to situations in which the 't Hooft coupling at intermediate time becomes of order one or smaller, so that the geometry in the bulk becomes of order the string scale. In such situations the supergravity calculation presented in the previous section breaks down and higher derivative corrections become important. The gauge theory description continues to be valid, however. Using this description one can then hope to answer how the system evolves in the region of string scale curvature, and, in particular, whether by waiting for enough time a smooth geometry with small curvature emerges again on the gravity side.

We saw in the previous subsection that the bulk response was characteristic of an adiabatic system which was being driven slowly compared to the time scale of its own internal response. This suggests that in the gauge theory, also an adiabatic perturbation theory should be valid and should prove useful in understanding the response. A related observation is the following. The bulk solutions we have considered correspond to keeping the radius R of the S^3 on which the gauge theory lives to be constant and independent of time. We will choose conventions in which $R = R_{\text{AdS}} = 1$. The Yang-Mills coupling is related to the boundary dilaton by

$$g_{\text{YM}}^2 = e^{\Phi_0(t)}. \quad (66)$$

The dilaton profile Eq. (18) also means that Yang-Mills coupling in the gauge theory varies slowly compared to the radius R . Since this is the only other scale in the system,

¹²AdS is of course a homogeneous space-time, but our boundary conditions pick out a particular notion of time. The center of AdS, where the energy is concentrated, is the region as mentioned above where the redshift in the corresponding energy is the biggest.

this also suggests that an adiabatic approximation should be valid in the boundary theory.

We will discuss two different kinds of adiabatic perturbation theory below. The first, which we call quantum adiabatic perturbation theory, is studied in this section. This is the adiabatic perturbation theory one finds discussed in a standard text book of quantum mechanics; see [30,31]. Its validity, we will see below, requires the condition, $N\epsilon \ll 1$, to be met. We will argue that once this condition is met the gauge theory analysis allows us to conclude that, even in situations where the curvature becomes of order the string scale at intermediate times, a dual smooth AdS₅ geometry emerges as a good approximation in the far future.

The supergravity calculations, however, required only the condition $\epsilon \ll 1$, which is much less restrictive than the condition $N\epsilon \ll 1$. Understanding the supergravity regime on the gauge theory side leads us to formulate another perturbation theory, which we call “large N classical adiabatic perturbation theory.” To explain this we find it useful to first discuss the example of a driven harmonic oscillator, as considered in Sec. V. Following this, we discuss LNCAPT in the gauge theory in Sec. VI. We find that its validity requires that the conditions of Eq. (6) are met. Using it we will get agreement with the supergravity calculations of Secs. II and III, when the ’t Hooft coupling remains large for all times.

Towards the end of Sec. VI, we discuss what happens in the gauge theory when the conditions of Eq. (6) are met but with the ’t Hooft coupling becoming small at intermediate times. Two qualitatively different behaviors are possible, and we will not be able to decide between them here. Either way, at late times a mostly smooth AdS description becomes good on the gravity side, with the possible presence of a small black hole.

In the discussion below we will consider the following type of profile for the boundary dilaton: it asymptotes in the far past and future to constant values such that the initial and final values of the ’t Hooft coupling, λ , are big, and attains its minimum value near $t = 0$. If this minimum value of $\lambda \leq 1$, the supergravity approximation will break down. We will also take the initial state of the system to be the ground state of the $\mathcal{N} = 4$ theory; on S^3 the spectrum of the gauge theory is gapped and this state is well defined.

A. The quantum adiabatic approximation

1. General features

It is well known that the spectrum of the $\mathcal{N} = 4$ theory on S^3 has a gap between the energy of the lowest state and the first excited state. This gap is of order $1/R$ and thus is of order unity in our conventions. The existence of this gap follows very generally just from the fact that the spectrum must provide a unitary representation of the conformal group [4], and the gap is therefore present for *all* values of the Yang-Mills coupling constant. In the supergravity

approximation, the spectrum can be calculated using the gravity description and is consistent with the gap, the lowest lying states have an energy $E = 2$. This is also true at very weak ’t Hooft coupling.

Now for a slowly varying dilaton, Eq. (18), we see that the Yang-Mills coupling and therefore the externally imposed time dependence varies slowly compared to this gap. There is a well-known adiabatic approximation which is known to work in such situations; see, e.g., [30,31] whose treatment we closely follow. We will refer to this as the quantum adiabatic approximation below and study the Yang-Mills theory in this approximation.

The essential idea behind this approximation is that when a system is subjected to a time dependence which is slow compared to its internal response time, the system can adjust itself very quickly and as a result to good approximation stays in the ground state of the instantaneous Hamiltonian.

More precisely, consider a time dependent Hamiltonian $H(\zeta(t))$, where $\zeta(t)$ is the time varying parameter. Now consider the one parameter family of time *independent* Hamiltonians given by $H(\zeta)$. To make our notation clear, a different value of ζ corresponds to a different Hamiltonian in this family, but each Hamiltonian is time independent. Let $|\phi_m(\zeta)\rangle$ be a complete set of eigenstates of the Hamiltonian $H(\zeta)$ satisfying

$$H(\zeta)|\phi_m(\zeta)\rangle = E_m(\zeta)|\phi_m(\zeta)\rangle, \quad (67)$$

in particular let the ground state of $H(\zeta)$ be given by $|\phi_0(\zeta)\rangle$. We take $|\phi_m(\zeta)\rangle$ to have the unit norm. Then the adiabatic theorem states that if $\zeta \rightarrow \zeta_0$ in the far past, and we start with the state $|\phi_0\rangle$ which is the ground state of $H(\zeta_0)$ in the far past, the state at any time t is well approximated by

$$|\psi^0(t)\rangle \simeq |\phi_0(\zeta)\rangle e^{-i \int_{-\infty}^t E_0(\zeta) dt'}. \quad (68)$$

Here $|\phi_0(\zeta)\rangle$ is the ground state of the time independent Hamiltonian corresponding to the value $\zeta = \zeta(t)$. Similarly, in the phase factor $E_0(\zeta)$ is the value of the ground state energy for $\zeta = \zeta(t)$.

Corrections can be calculated by expanding the state at time t in a basis of energy eigenstates at the instantaneous value of the parameter ζ . The first corrections take the form

$$|\psi^1(t)\rangle = \sum_{n \neq 0} a_n(t) |\phi_n(\zeta)\rangle e^{-i \int_{-\infty}^t E_n dt'} \quad (69)$$

where the coefficient $a_n(t)$ is

$$a_n(t) = - \int_{-\infty}^t dt' \frac{\langle \phi_n(\zeta) | \frac{\partial H}{\partial \zeta} | \phi_0(\zeta) \rangle}{E_0 - E_n} \dot{\zeta} e^{-i \int_{-\infty}^t (E_0 - E_n) dt'}. \quad (70)$$

In the formula above on the rhs $|\phi_n(\zeta)\rangle$, $\frac{\partial H}{\partial \zeta}$, $E_n(\zeta)$ are all functions of time, through the time dependence of ζ .

2. Conditions for validity

For the adiabatic approximation to be good, the first corrections must be small. To ensure this we impose the condition

$$\left| \langle \phi_n | \frac{\partial H}{\partial \zeta} | \phi_0 \rangle \dot{\zeta} \right| \ll (E_1 - E_0)^2 \quad (71)$$

where $(E_1 - E_0)$ is the energy gap between the ground state and the first excited state and $|\phi_n\rangle$ is any excited state. [This would then imply that the lhs in Eq. (71) is smaller than $(E_n - E_0)^2$ for all n .] This condition is imposed for all time for the adiabatic approximation to be valid.¹³

In our case the role of the parameter ζ is played by the dilaton Φ_0 (with the gauge coupling $g_{\text{YM}}^2 = e^{\Phi_0}$). Thus Eq. (71) takes the form

$$\left| \langle \phi_n | \frac{\partial H}{\partial \Phi_0} | \phi_0 \rangle \dot{\Phi}_0 \right| \ll (E_1 - E_0)^2. \quad (72)$$

Now, as we will see below in Subsec. IV C, $\frac{\partial H}{\partial \Phi_0}$ is, up to a sign, exactly the operator $\hat{\mathcal{O}}_{l=0}$ which is dual to the modes of the dilaton which are spherically symmetric on the S^3 . Therefore Eq. (72) becomes

$$|\langle \phi_n | \hat{\mathcal{O}}_{l=0} | \phi_0 \rangle \dot{\Phi}_0| \ll (E_1 - E_0)^2. \quad (73)$$

We have argued above that the rhs is of order unity in our conventions due to the existence of a robust gap. On the lhs, $\dot{\Phi}_0 \sim O(\epsilon)$, and as we will argue below the matrix element, $|\langle \phi_n | \hat{\mathcal{O}}_{l=0} | \phi_0 \rangle| \sim O(N)$. Thus Eq. (73) becomes

$$N\epsilon \ll 1. \quad (74)$$

B. Highly curved geometries

Equation (74) is the required condition then for the applicability of quantum perturbation theory. When this condition is met, we can continue to trust the quantum adiabatic approximation in the gauge theory even when the 't Hooft coupling becomes of order unity or smaller at intermediate times. All the conditions which are required for the validity of this approximation continue to be hold in this case. First, as was discussed above the gap of order unity continues to exist. Second, the matrix elements which enter are in fact independent of λ since they correspond to the two-point function of dilaton which is a chiral operator. Thus the system continues to be well described in the quantum adiabatic approximation so long as Eq. (74) is met. It follows then that in the far future the state of the system to good approximation is the ground state of the

¹³The actual condition is that the corrections to $|\psi^0\rangle$ must be small. This means that at first order $\langle \psi^1 | \psi^1 \rangle$ should be small. When Eq. (71) is met $|a_n|$ is small, but in some cases that might not be enough and the requirement that the sum $\sum |a_n|^2$ is small imposes extra restrictions. There could also be additional conditions which arise at second order, etc.

$\mathcal{N} = 4$ theory. This implies that the dual description in the far future is a smooth AdS₅ geometry.

There is one important caveat to the above conclusion. It is possible that at $\lambda \sim O(1)$ there are several states in the spectrum, scaling as a positive power of N , which accumulate near the first excited state. This does not happen for $\lambda \gg 1$ and for $\lambda \ll 1$ (where the spectrum of the free theory is of course known) but it remains a logical possibility. If this is true, the conditions for the adiabatic approximation will have to be revised so that the dilaton varies even more slowly as a power of N . This is a question which can be settled in principle once the spectrum of the $\mathcal{N} = 4$ theory is known for all λ . Similarly, the possibility for unexpected surprises at higher orders can also be examined once enough is known about the $\mathcal{N} = 4$ theory. The point is simply that in this approximation all matrix elements and conditions can be phrased as statements in the time independent $\mathcal{N} = 4$ theory. As our knowledge of the $\mathcal{N} = 4$ theory grows, we will be able to check for any such unexpected surprises.

Let us also mention before proceeding that when the condition Eq. (74) is met and for a dilaton profile where the 't Hooft coupling stays large for all time, the metric is to good approximation smooth AdS₅ for all time. However the small corrections to this metric and dilaton cannot be calculated reliably in the classical approximation used in Sec. II. This is because in this regime it is very difficult to even produce one supergravity quantum as an excitation above the adiabatic vacuum. Therefore quantum effects are important in calculating these corrections.

C. More comments

We close this section by discussion two points relevant to the analysis leading up to condition Eq. (74).

First, let us argue why $\frac{\partial H}{\partial \Phi_0} = -\hat{\mathcal{O}}_{l=0}$. The argument is sketched out below, more details can be found in [10]. The action of the $\mathcal{N} = 4$ theory is given by

$$S = \int dt d\Omega_3 \sqrt{-g} \left(-\frac{1}{4e^{\Phi_0}} \right) \text{Tr} F_{\mu\nu} F^{\mu\nu} + \dots \quad (75)$$

where the ellipses indicate extra terms coming from scalars and fermions. Varying with respect to Φ_0 gives us the operator dual to the dilaton

$$\hat{\mathcal{O}} = \sqrt{-g} \left(\frac{1}{4e^{\Phi_0}} \right) \text{Tr} F_{\mu\nu} F^{\mu\nu} + \dots \quad (76)$$

where the ellipses denote extra terms which arise from the terms left out in Eq. (75). Henceforth, to emphasize the key argument, we neglect the additional terms coming from the ellipses.

Working in $A_0 = 0$ gauge, the Hamiltonian density \mathcal{H} is given by

$$\mathcal{H} = e^{\Phi_0} \frac{\pi_i \pi^i}{2} + \frac{e^{-\Phi_0}}{4} F_{ij} F^{ij} \quad (77)$$

where

$$\pi_i = e^{-\Phi_0} \partial_0 A_i \quad (78)$$

is the momentum conjugate to A_i . Varying with respect to Φ_0 gives

$$\frac{\partial \mathcal{H}}{\partial \Phi_0} = \frac{\pi_i \pi^i}{2} e^{\Phi_0} - \frac{e^{-\Phi_0}}{4} F_{ij} F^{ij}. \quad (79)$$

Substituting from Eq. (78) one sees that this agrees (up to a sign) with the operator $\hat{\mathcal{O}}$ given in Eq. (76). When the dilaton depends on time alone, we can integrate the above equations over S^3 , which leads to the relation $\frac{\partial H}{\partial \Phi_0} = -\hat{\mathcal{O}}_{l=0}$, where H now stands for the Hamiltonian (rather than the Hamiltonian density).

Second, we estimate how the matrix element, $\langle \phi_n | \hat{\mathcal{O}}_{l=0} | \phi_0 \rangle$, which appears in Eq. (73), scales with N . It is useful to first recall that the $\mathcal{N} = 4$ theory, which is conformally invariant, has an operator state correspondence. The states $|\phi_n\rangle$ can be thought of as being created from the vacuum by the insertion of a local operator. This makes it clear that the only states having a nonzero matrix element, $\langle \phi_n | \hat{\mathcal{O}}_{l=0} | \phi_0 \rangle$, are those which can be created from the vacuum by inserting $\hat{\mathcal{O}}_{l=0}$, since the only operator with which $\hat{\mathcal{O}}_{l=0}$ has a nonzero two-point function is $\hat{\mathcal{O}}_{l=0}$ itself.

Now in terms of powers of N the two-point function scales like

$$\langle \hat{\mathcal{O}}_{l=0} \hat{\mathcal{O}}_{l=0} \rangle \sim N^2. \quad (80)$$

The state $|\phi_n\rangle$, which appears in the matrix element in Eq. (73), has the unit norm and is therefore created from the vacuum by the operator

$$|\phi_n\rangle \sim \frac{1}{N} \hat{\mathcal{O}}_{l=0} |0\rangle. \quad (81)$$

From Eqs. (80) and (81), we then see that the matrix element scales like

$$\langle \phi_n | \hat{\mathcal{O}}_{l=0} | \phi_0 \rangle \sim N \quad (82)$$

as was mentioned above.

Our discussion leading up to the estimate of the matrix element has been imprecise in some respects. First, strictly speaking the operator state correspondence we used is a property of the Euclidean theory on R^4 , where as we are interested in the Minkowski theory on $S^3 \times R$. However, this is a technicality which can be taken care of by first relating the matrix element in the Minkowski theory to that in Euclidean $S^3 \times R$ space and then relating the latter to that on R^4 by a conformal transformation.

More importantly, the state created by $\hat{\mathcal{O}}_{l=0}$ is not an eigenstate of energy, but is in fact a sum over an infinite number of states labeled by an integer n with energies $\omega_n = 4 + 2n$. This can be understood as follows. The

operator $\hat{\mathcal{O}}$ can be expanded into positive and negative frequency modes, A_n, A_n^\dagger , respectively, for an infinite set n , and acting with any of the A_n^\dagger 's gives a state

$$|\varphi_n\rangle \simeq A_n^\dagger |0\rangle. \quad (83)$$

One must therefore worry about the dependence on the mode number n in the matrix element and the effects of summing up the contributions for all of these modes. We will return to address this issue in more detail in Subsecs. **VIB** and **VIC**, when we describe the operators A_n, A_n^\dagger more explicitly and discuss renormalization. For now, let us state that after more careful treatment we will find that the condition for the quantum adiabatic approximation Eq. (74) goes through unchanged. The physical reason is simply this: we are interested here in the very low-frequency response of the system and its very high frequency modes are not relevant for this.

V. THE SLOWLY DRIVEN HARMONIC OSCILLATOR

The supergravity calculations required the condition $\epsilon \ll 1$. To understand this regime in the dual gauge theory it is first useful to consider a quantum mechanical harmonic oscillator with frequency ω_0 driven by a time dependent source $J(t)$. We will see that in this case a classical adiabatic perturbation theory becomes valid when¹⁴

$$\frac{j}{j\omega_0} \ll 1, \quad (84)$$

$$j \gg \omega_0^{5/2}. \quad (85)$$

Having understood this system we then return to the gauge theory in the following subsection.

The Hamiltonian is given by

$$H = \frac{1}{2} \dot{X}^2 + \frac{1}{2} \omega_0^2 \left(X + \frac{J(t)}{\omega_0^2} \right)^2. \quad (86)$$

In the quantum adiabatic approximation one considers the instantaneous Hamiltonian. At time t_0 this is given by

$$H_0 = \frac{1}{2} \dot{X}^2 + \frac{1}{2} \omega_0^2 \left(X + \frac{J(t_0)}{\omega_0^2} \right)^2 \quad (87)$$

where $J(t_0)$ is also regarded as a time independent constant in H_0 .

¹⁴Equations (84) and (85) clearly cannot hold when j vanishes. The more precise versions of these conditions are as follows. Equation (84) is really the requirement that J is slowly varying. By this one means that the Fourier transform of J has support, up to say exponentially small corrections, only for small frequencies compared to ω_0 . Equation (85) is the requirement that the coherent state parameter, $\lambda(t)$ given in Eq. (99), is large.

The ground state of H_0 is a coherent state. We define

$$X = \frac{a + a^\dagger}{\sqrt{2\omega_0}}, \quad P = -i\sqrt{\omega_0} \left(\frac{a - a^\dagger}{\sqrt{2}} \right) \quad (88)$$

to be the conventional creation and destruction operators. Here,

$$P = \dot{X} \quad (89)$$

is the conjugate momentum. The ground state is

$$|\phi_0\rangle = N_\alpha e^{\alpha a^\dagger} |0\rangle. \quad (90)$$

Here N_α is a normalization constant, determined by requiring that $\langle \phi_0 | \phi_0 \rangle = 1$. The state $|0\rangle$ is the vacuum annihilated by a , i.e.,

$$a|0\rangle = 0, \quad (91)$$

and

$$\alpha = -\frac{J}{\sqrt{2\omega_0^3}}. \quad (92)$$

The ground state energy is

$$E_0 = \frac{1}{2}\omega_0; \quad (93)$$

it is independent of time.

A quick way to derive these results is to work with the shifted creation and destruction operators,

$$\tilde{a} = a - \alpha, \quad \tilde{a}^\dagger = a^\dagger - \alpha \quad (94)$$

where α is given in Eq. (92). The Hamiltonian takes the form

$$H = \omega_0(\tilde{a}^\dagger \tilde{a}) + \frac{1}{2}\omega_0. \quad (95)$$

It is clear then that the ground state is annihilated by \tilde{a} , leading to Eq. (90) and the ground state energy is Eq. (93).

For the quantum adiabatic theorem to be valid, the condition in Eq. (71) must hold. For the harmonic oscillator it is easy to see that this gives

$$j \ll \omega_0^{5/2}. \quad (96)$$

In fact the time evolution in this case can be exactly solved. We consider the case where $J(t) \rightarrow 0$, $t \rightarrow -\infty$. Starting with the state $|0\rangle$ in the far past, which is the vacuum of the Hamiltonian in the far past, we then find that the state at any time t is given by

$$|\psi(t)\rangle = N(t) e^{\lambda(t)a^\dagger} |\phi_0\rangle \quad (97)$$

where $|\phi_0\rangle$ is the adiabatic vacuum given in Eq. (90), $N(t)$ is a normalization constant, and the coherent state parameter is $\lambda(t)$. Imposing the Schrödinger equation one gets

$$i\dot{\lambda} = i\frac{j}{\sqrt{2\omega_0^3}} + \omega_0\lambda. \quad (98)$$

The solution for $\lambda(t)$ with initial condition $\lambda(-\infty) = 0$ is given by

$$\lambda(t) = \frac{e^{-i\omega_0 t}}{\sqrt{2\omega_0^3}} \int_{-\infty}^t J(t') e^{i\omega_0 t'} dt'. \quad (99)$$

Some details leading to Eq. (99) are given in Appendix B. This state will behave like a classical state when the coherent state parameter is big in magnitude, i.e., when

$$|\lambda| \gg 1. \quad (100)$$

The integral on the rhs of Eq. (99) can be done by parts [we set $J(-\infty) = 0$],

$$\int_{-\infty}^t dt' J e^{i\omega_0 t'} = J(t) \frac{e^{i\omega_0 t}}{i\omega_0} - \int_{-\infty}^t dt' \dot{J} \frac{e^{i\omega_0 t'}}{i\omega_0}. \quad (101)$$

Subsequent iterations obtained by further integrations by parts give rise to a series expansion¹⁵ for λ in terms of higher derivatives of J . The higher order terms are small if J is slowly varying compared to the frequency of the oscillator ω_0 . Evaluating the second term which arises in his expansion, for example, and requiring it to be smaller than the first term in Eq. (101) gives

$$\frac{j}{J\omega_0} \ll 1. \quad (102)$$

We assume now that J is slowly varying and the first term on the rhs of Eq. (101) is a good approximation to the integral. This tells us that for Eq. (100) to be true the condition which must be met is

$$j \gg \omega_0^{5/2}. \quad (103)$$

Note that this condition is opposite to the one needed for the quantum adiabatic theorem to apply Eq. (97).

The answer for the $\langle X \rangle$ can be easily obtained by inserting the expression for λ obtained in Eq. (99) in the wave function, Eq. (97). Let us obtain it here in a slightly different manner. When Eq. (100) is true the system behaves classically and its response to the driving force can be obtained by solving the classical equation of motion for the forced oscillator. In terms of the Fourier transform of J this gives

$$X(t) = \int \frac{J(\omega)}{\omega^2 - \omega_0^2} e^{-i\omega t} d\omega. \quad (104)$$

The correct pole prescription on the rhs is that for a retarded propagator.

When the source is slowly varying compared to ω_0 , the denominator $\omega^2 - \omega_0^2$ in Eq. (104) can be expanded in a power series in $\frac{\omega^2}{\omega_0^2}$ and the resulting Fourier transforms can be expressed as time derivatives of J . The first two terms

¹⁵In general one expects this to be an asymptotic rather than convergent series.

give

$$X = -\frac{J(t)}{\omega_0^2} + \frac{\dot{J}}{\omega_0^4} + \dots \quad (105)$$

The first term on the rhs is the location of the instantaneous minimum. The second term is the first correction due to the time dependent source. Subsequent corrections are small if the source is slowly varying and the condition Eq. (102) is met. It is useful to express this result as

$$X + \frac{J(t)}{\omega_0^2} = \frac{\dot{J}}{\omega_0^4} + \dots \quad (106)$$

The left-hand side is the expectation value of X after adding a shift to account for the instantaneous minimum of the potential. The right-hand side we see now only contains time derivatives of J . Before proceeding let us note that expanding the denominator in Eq. (104) in a power series in $\frac{\omega^2}{\omega_0^2}$ gives a good approximation only if $J(\omega)$ has most of its support for $\omega \ll \omega_0$. This is how the more precise condition mentioned in the footnote before Eq. (84) arises.

It is also useful to discuss the energy. From Eq. (105) and the Hamiltonian we see that the leading contribution comes from the Kinetic energy term and is given to leading order by

$$E = \frac{1}{2} \frac{\dot{J}^2}{\omega_0^4} \quad (107)$$

(strictly speaking this is the energy above the ground state energy).

The external source driving the oscillator changes its energy. Noether's argument in the presence of the time dependent source leads to the conclusion that

$$\frac{\partial H}{\partial t} = j \left(X + \frac{J}{\omega_0^2} \right) \quad (108)$$

[this also directly follow from the Hamiltonian, Eq. (86)]. From Eqs. (106) and (107) we see that this condition is indeed true. Let us also note that the rate of change in energy can be expressed in terms of the shifted operators, Eq. (94), as

$$\frac{\partial H}{\partial t} = j \left(\frac{\tilde{a} + \tilde{a}^\dagger}{\sqrt{2\omega_0}} \right); \quad (109)$$

this form will be useful in our discussion below.

To summarize, we find that when the conditions of Eqs. (102) and (103) are met, the driven harmonic oscillator behaves like a classical system. Its response, for example, $\langle X \rangle$, and the energy, E , can be calculated in an expansion in time derivatives of J , which is controlled when Eq. (102) is valid and the source is slowly varying. We will refer to this perturbation expansion as the classical adiabatic perturbation approximation below. Note that the condition of Eq. (103) is opposite to the one required for the quantum adiabatic perturbation theory to hold. In the

next subsection we will discuss how a similar classical adiabatic approximation arises in the gauge theory.

VI. GAUGE THEORY: LARGE N CLASSICAL ADIABATIC PERTURBATION THEORY

We now return to the gauge theory and formulate a large N classical adiabatic approximation based on coherent states in this theory. This will allow us to obtain results in the gauge theory which agree with those obtained using supergravity in Secs. II and III.

A. Adiabatic approximation in terms of coherent states

The supergravity solution in Sec. II describes *classical solutions* rather than states which contain a small number of bulk particles. The AdS/CFT correspondence implies that bulk classical solutions correspond to coherent states in the boundary gauge theory with a large number of particles in which operators like $\hat{\mathcal{O}}$ have nontrivial expectation values. On the other hand, states obtained by the action of a few factors of $\hat{\mathcal{O}}$ on the vacuum are few-particle states in the bulk. The quantum adiabatic approximation described in Sec. IV attempts to determine the wave function in a basis formed out of such single particle states and does not apply to the supergravity solution in Sec. II.

We, therefore, need to formulate an adiabatic approximation in terms of coherent states of gauge invariant operators in the boundary theory to try and understand the supergravity solutions of Sec. II in a dual description. As is well known, these coherent states become classical in a smooth fashion in the $N \rightarrow \infty$ limit. (See, e.g., [7]). Consider a complete (usually overcomplete) set of gauge invariant operators in the Schrödinger picture, $\hat{\mathcal{O}}^I$. A general coherent state is of the form

$$|\Psi(t)\rangle = \exp \left[i\chi(t) + \sum_I \lambda^I(t) \hat{\mathcal{O}}_{(+)}^I \right] |0\rangle_A. \quad (110)$$

Here $\hat{\mathcal{O}}_{(+)}^I$ denotes the creation part of the operator and $|0\rangle_A$ denotes the adiabatic vacuum corresponding to some instantaneous value of the dilation Φ_0 ,

$$H[\Phi_0] |0\rangle_A = E_{\Phi_0} |0\rangle_A \quad (111)$$

with the ground state energy E_{Φ_0} .

The algebra of operators $\hat{\mathcal{O}}^I$, together with the Schrödinger equation then leads to a differential equation which determines the time evolution of the coherent state parameters $\lambda^I(t)$ in terms of the time dependent source $\Phi_0(t)$. The idea is then to solve this equation in an expansion in time derivatives of $\Phi_0(t)$. This is the coherent state adiabatic approximation we are seeking.

In general it is almost impossible to implement this program practically, since the operators $\hat{\mathcal{O}}^I$ have a non-trivial operator algebra which mixes all of them. The coherent state (110) is in the coadjoint orbit of this algebra

[7]. The resulting theory of fields conjugate to these operators would be in fact the full interacting string field theory in the bulk. In our case, however, the situation drastically simplifies for large 't Hooft coupling at the lowest order of an expansion in Φ_0 . This is because these various operators decouple and their algebra essentially reduces to free oscillator algebras.

We have already found this decoupling in our supergravity calculation. The departure of the solution from $\text{AdS}_5 \times S^5$ is due to the time dependence of the boundary value of the dilaton, and are small when the time variations are small, controlled by the parameter ϵ . To lowest order in ϵ [which is $O(\epsilon^2)$] the deformation of the bulk dilaton in fact satisfied a linear equation in the AdS_5 background in the presence of a source provided by the boundary value $\Phi_0(t)$. This equation does not involve the deformation of the metric. Similarly, the equation for metric deformation does not involve the dilaton deformation to lowest order.

This allows us to treat each supergravity field and its dual operator separately. With this understanding we will now consider the coherent state (110) with only the operator dual to the dilaton, $\hat{\mathcal{O}}$. Since our source is spherically symmetric and higher point functions of the operators are not important in this lowest order calculation, we can restrict this operator to its spherically symmetric part.

B. Large N classical adiabatic perturbation theory

Let us now elaborate in more detail on the LNCAPT.

The linearized approximation in the gravity theory means that only the two-point function is nontrivial and all connected higher point functions vanish. The nonlinear terms correspond to nontrivial higher order correlations. In this approximation the gauge theory simplifies a great deal. Each gauge invariant operator—which is dual to a bulk mode—gives rise to a tower of harmonic oscillators. The response of the gauge theory can be understood from the response of these oscillators.

In fact, in the quadratic approximation, the only oscillators which are excited are those which couple directly to the dilaton and so we only have to discuss their dynamics. We have already discussed the operator dual to the dilaton in Sec. IV C. The dilaton excitations we consider are S^3 symmetric and correspondingly the only modes of $\hat{\mathcal{O}}$ which when excited are S^3 symmetric. Here we denote these by $\hat{\mathcal{O}}_{l=0}$.

In the Heisenberg picture $\hat{\mathcal{O}}_{l=0}$ can be expanded in terms of time dependent modes; this is dual to the fact that the S^3 symmetric dilaton can be expanded in terms of modes with different radial and related time dependence in the bulk. One finds, as is discussed in Appendix C, that only even integer frequencies appear in the time dependence giving

$$\hat{\mathcal{O}}_{l=0} = N \sum_{n=1}^{\infty} F(2n) [A_{2n} e^{-i2nt} + A_{2n}^\dagger e^{i2nt}]. \quad (112)$$

Here A_{2n} , A_{2n}^\dagger are canonically normalized creation and destruction operators satisfying the relations

$$[A_m, A_n] = [A_m^\dagger, A_n^\dagger] = 0 \quad [A_m, A_n^\dagger] = \delta_{m,n}. \quad (113)$$

Their commutators with the gauge theory Hamiltonian are

$$[H, A_{2n}^\dagger] = (2n)A_{2n}^\dagger \quad [H, A_{2n}] = -(2n)A_{2n}. \quad (114)$$

The normalization factor $F(2n)$ may be computed by comparing with the standard two-point function as is detailed in Appendix C. The result is

$$|F(2n)|^2 = \frac{A\pi^4}{3} n^2(n^2 - 1) \quad (115)$$

for $n \geq 2$. $F(0)$ and $F(2)$ vanish, so this means that the sum in Eq. (112) receives its first contribution at $n = 2$. It also means that the lowest energy state which can be created by acting with $\hat{\mathcal{O}}_{l=0}$ on the vacuum has energy equal to 4. This is what we expect on general grounds, since the energies of states created by an operator with conformal dimension Δ are given by

$$\omega(n, l) = \Delta + 2n + l(l + 2) \quad n = 0, 1, 2 \dots \quad (116)$$

The constant A in Eq. (115) is the normalization of the two-point function which may be determined, e.g., from a bulk calculation. Before proceeding let us also note that $F(2n)$ grows like $F(2n) \sim n^2$, Eq. (115), for large mode number n . This enhances the coupling of the higher frequency modes to the dilaton and will be important in our discussion of renormalization below.

From now onwards we will find it convenient to work in the Schrödinger representation, in which operators are time independent. The operator $\hat{\mathcal{O}}_{l=0}$ in this representation is given by

$$\hat{\mathcal{O}}_{l=0} = N \sum_n F(2n) [A_{2n} + A_{2n}^\dagger]. \quad (117)$$

From Eq. (114) it follows that the Hamiltonian for A_{2n} , A_{2n}^\dagger modes can be written as

$$H = \sum_n 2n A_{2n}^\dagger A_{2n}. \quad (118)$$

Note this Hamiltonian measures the energy above that of the ground state.

The operators, A_{2n}^\dagger , A_{2n} create and destroy a single quantum of excitation when acting on the vacuum of the $\mathcal{N} = 4$ theory with the instantaneous value of $g_{\text{YM}}^2 = e^{\Phi_0}$. Thus they are the analogue of the shifted creation and destruction operators we had defined in the harmonic oscillator case, \tilde{a} , \tilde{a}^\dagger . The Hamiltonian, Eq. (118), is the analogue of the Hamiltonian, Eq. (95) in the harmonic oscillator case.

The time dependence of the Hamiltonian due to the varying dilaton can be expressed as follows:

$$\frac{\partial H}{\partial t} = \frac{\partial H}{\partial \Phi} \dot{\Phi}_0 = -\hat{\mathcal{O}}_{l=0} \dot{\Phi}_0 \quad (119)$$

leading to

$$\frac{\partial H}{\partial t} = -\hat{\mathcal{O}}_{l=0} \dot{\Phi}_0 = -N \sum_n F(2n) [A_{2n} + A_{2n}^\dagger] \dot{\Phi}_0, \quad (120)$$

where we have used Eq. (117). It is useful to write this as

$$\frac{\partial H}{\partial t} = -N \sum_n F(2n) \sqrt{4n} \dot{\Phi}_0 \left[\frac{A_{2n} + A_{2n}^\dagger}{\sqrt{4n}} \right], \quad (121)$$

which is analogous to the time dependence in the forced oscillator system, Eq. (109).

So we see that the gauge theory, in the quadratic approximation maps to a tower of oscillators, with frequencies, $\omega_n = 2n$. Comparing with Eq. (109) we see that the oscillator with energy $2n$ couples to a source,

$$J_n = -NF(2n)\sqrt{4n}\dot{\Phi}_0. \quad (122)$$

The analysis of the harmonic oscillator now directly applies. The resulting state is a coherent state,

$$|\psi\rangle = \hat{N}(t) e^{\sum_n \lambda_n A_{2n}^\dagger} |\phi_0\rangle. \quad (123)$$

Here $|\phi_0\rangle$ is the adiabatic vacuum, which is in the ground state of the $\mathcal{N} = 4$ theory with coupling $g_{\text{YM}}^2 = e^{\Phi_0}$. $\hat{N}(t)$ is a normalization constant and the coherent state parameter λ_n is given from Eq. (99) by

$$\lambda_n = \frac{e^{-i\omega_n t}}{\sqrt{2\omega_n^3}} \int_{-\infty}^t J_n(t') e^{i\omega_n t'} dt'. \quad (124)$$

The condition that the source is varying slowly, Eq. (102), becomes,

$$\left| \frac{\ddot{\Phi}_0}{n\dot{\Phi}_0} \right| \ll 1 \quad \forall n. \quad (125)$$

It is clearly sufficient to satisfy this condition for $n = 1$,

$$\left| \frac{\ddot{\Phi}_0}{\dot{\Phi}_0} \right| \sim \epsilon \ll 1. \quad (126)$$

This condition is met for the dilaton profile we have under consideration.¹⁶ When this condition is true λ_n can be evaluated by keeping the first term in Eq. (101). The condition that the state is classical, is that $\lambda_n \gg 1$, this gives¹⁷

¹⁶This condition is analogous to Eq. (84) for the driven harmonic oscillator. As discussed in that context in the footnote before Eq. (84) there is a more precise version of this condition. It is the statement that for all modes, n , the Fourier transform of J_n must have essentially all its support at frequencies much smaller than the oscillator frequency, $2n$.

¹⁷The more precise condition is simply that $\lambda_n \gg 1$, $\forall n$. This gives Eq. (127) provided that the integral in Eq. (124) can be approximated by the first term of the derivative approximation.

$$|NF(2n)\sqrt{4n}\dot{\Phi}_0| \gg (2n)^{5/2}. \quad (127)$$

Noting from Eq. (115) that $F(2n) \sim n^2$ for large n we see that the factors of n cancel out on both sides, leading to the conclusion that when

$$|N\dot{\Phi}_0| \sim N\epsilon \gg 1 \quad (128)$$

all the oscillators are in a classical state. In this way we recover the first condition discussed in Eq. (6).

The summary is that when the two conditions,

$$\epsilon \ll 1, \quad N\epsilon \gg 1 \quad (129)$$

are both valid, the gauge theory is described to leading order in ϵ as a system of harmonic oscillators. The oscillators which couple to the dilaton are excited by it and are in a classical state.

This description can be used to calculate the resulting expectation value of operators. The calculation for $\langle \frac{A_{2n} + A_{2n}^\dagger}{\sqrt{4n}} \rangle$ is analogous to that for $\langle X + \frac{J}{\omega} \rangle$ in the harmonic oscillator case [since the A_{2n}, A_{2n}^\dagger are analogous to the shifted operators, $\tilde{a}, \tilde{a}^\dagger$ Eq. (94)]. From Eqs. (106) and (122) we get that to leading order in ϵ ,

$$\left\langle \frac{A_{2n} + A_{2n}^\dagger}{\sqrt{4n}} \right\rangle = -N \frac{F(2n)\sqrt{4n}}{(2n)^4} \ddot{\Phi}_0. \quad (130)$$

Substituting in Eq. (117) next gives

$$\langle \hat{\mathcal{O}}_{l=0} \rangle = -CN^2 \ddot{\Phi}_0 \quad (131)$$

where C is

$$C = \sum \frac{F(2n)^2}{4n^3}. \quad (132)$$

The functional dependence on Φ_0 and N in Eq. (130) agrees with what we found in the supergravity calculation, Eq. (62). The constant of proportionality C is in fact quadratically divergent. This follows from noting that for large n , $F(2n) \sim n^2$.

A little thought tells us that the divergence should in fact have been expected. The supergravity calculation also had a divergence and the finite answer in Eq. (62) was obtained only after regulating this divergence and renormalizing. Therefore it is only to be expected that a similar divergence will also appear in the description in terms of the oscillators. In the subsection which follows we will discuss the issue of renormalization in more detail. The bottom line is that counterterms can be chosen so that the coefficient in Eq. (62) agrees with that in the supergravity calculation.

It is also important to discuss how the energy behaves. From Eqs. (107) and (122) we see that the energy above the ground state is

$$\langle E \rangle - E_{\text{gnd}} = \frac{1}{2} CN^2 \dot{\Phi}_0^2. \quad (133)$$

We note that the functional dependence on $\dot{\Phi}_0, N$ match with those obtained in the supergravity calculations,

Eq. (60). The constant of proportionality which is obtained by summing over the oscillator modes in the case of the energy is the same as C defined above, Eq. (132). It is also therefore quadratically divergent.

The fact that the two constants of proportionality in Eqs. (133) and (131) are the same, follows on general grounds. Noether's argument in the presence of the time dependence means that each oscillator satisfies the relation, Eq. (108). On summing over all of them we then get the relation

$$\left\langle \frac{dE}{dt} \right\rangle = -\dot{\Phi}_0 \langle \hat{\mathcal{O}}_{l=0} \rangle \quad (134)$$

leading to the equality of the two constants. Earlier we had also seen that the supergravity calculation satisfies this relation, Eq. (63). It follows from these observations that if after renormalization the answer for $\langle \hat{\mathcal{O}}_{l=0} \rangle$ agrees between the supergravity theory and the oscillator description developed here, then the expectation value for E will also agree in the two cases.

Here we have analyzed the gauge theory to leading order in ϵ . Going to higher orders introduces anharmonic couplings between the different oscillators. These couplings arise because of connected three-point and higher point correlations in the gauge theory. The three-point function, for example, is suppressed by $1/N$, the four-point function by $1/N^2$, and so on. For computations in the ground state, these would therefore be suppressed in the large N limit. However as we have seen here the time dependence results in a coherent state which contains $O(N\epsilon)^2$ quanta being produced. The three-point function in such a state is suppressed by $O(\epsilon)$ and not by $O(1/N)$. Since $\epsilon \ll 1$, this is still enough though to justify our neglect of the cubic terms to leading order in ϵ . Similarly the effect of four-point correlators in the coherent state are suppressed by $O(\epsilon)^2$, etc. This is in agreement with the supergravity calculation, where the cubic terms in the equations of motion are suppressed by $O(\epsilon)$, etc.

To go to higher orders in ϵ using the oscillator description the effect of the anharmonic couplings induced by the higher order correlations would have to be introduced. In addition one would have to keep the contributions from the quadratic approximation to the required order in ϵ . As long as the 't Hooft coupling stays big for all times and the supergravity approximation is valid, there is no reason to believe that these effects will be significant and the behavior of the system should be well described by the leading harmonic oscillator description, in agreement with what we saw in supergravity. When the 't Hooft coupling begins to get small though the anharmonic couplings, one could potentially significantly change the behavior of the system, as we will discuss in Sec. VID.

C. Renormalization

Let us now return to the constant C Eq. (132). One would like to know if it can be made to agree with the supergrav-

ity answer Eq. (62). Since the mode sum in C diverges, at first sight it would seem that by suitably removing the infinities this can always be done. To be explicit, imposing a cutoff on the mode sum in C one gets from Eq. (132),

$$C = \sum \frac{F(2n)^2}{4n^3} = c_1 n_{\max}^2 + c_2 \ln(n_{\max}) + \text{finite term} \quad (135)$$

(a term linear in n_{\max} can always be removed by shifting n_{\max}). Removing the infinities would mean removing the first two terms, but by changing n_{\max} by a finite amount the finite term left over will clearly change and can be made equal to any answer we want.

However this seems too superficial of an answer. One would like to ensure that the freedom to adjust C corresponds to the freedom to add local counterterms in the theory, and also that once the counterterms are chosen so that C agrees no other discrepancy appears with supergravity.

This is in fact true and can be easily seen by relating the calculation for $\langle \hat{\mathcal{O}} \rangle$ in Eq. (131) to the two-point function for the dilaton. In fact we will only need the two-point function of the S -wave dilaton which is equal to the two-point function of $\langle \hat{\mathcal{O}}_{l=0} \hat{\mathcal{O}}_{l=0} \rangle$ in the gauge theory. Since the S -wave dilaton couples directly to $\hat{\mathcal{O}}_{l=0}$, we have

$$\langle \hat{\mathcal{O}}_{l=0}(t) \rangle = \int dt' \langle \hat{\mathcal{O}}_{l=0}(t) \hat{\mathcal{O}}_{l=0}(t') \rangle \Phi(t'). \quad (136)$$

Using Eq. (112) we find that

$$\begin{aligned} \langle \hat{\mathcal{O}}_{l=0}(t) \hat{\mathcal{O}}_{l=0}(t') \rangle &= N^2 \sum_n F(2n)^2 (4n) \int \frac{d\omega}{2\pi i} \\ &\times \frac{e^{-i(t-t')\omega}}{(\omega^2 - (2n)^2)} \end{aligned} \quad (137)$$

where we have expressed the answer in terms of a Fourier transform in frequency space. We are not being explicit about the pole prescription here, this will determine which propagator (Feynman, retarded, etc.) one requires. From Eq. (137) the propagator in frequency space can be read off to be

$$G(\omega) = N^2 \sum_n \frac{F(2n)^2 (4n)}{(\omega^2 - (2n)^2)}. \quad (138)$$

Since $F(2n) \sim n^2$ the sum over modes on the rhs is *quartically* divergent.

For purposes of comparing with the adiabatic approximation we expand this propagator in powers in ω^2 . This gives

$$\begin{aligned} \frac{G(\omega)}{N^2} = & - \sum \frac{F(2n)^2(4n)}{(2n)^2} - \omega^2 \sum \frac{F(2n)^2(4n)}{((2n)^2)^2} \\ & - \omega^4 \sum \frac{F(2n)^2(4n)}{((2n)^2)^3} + \dots \end{aligned} \quad (139)$$

The terms within the ellipses contain powers higher than ω^4 and are not divergent. The first term on the rhs must be set to zero after renormalization to preserve conformal invariance, otherwise the vacuum expectation value for $\langle \hat{O} \rangle$ in the $\mathcal{N} = 4$ theory with constant coupling would not vanish. The leading contribution to $\langle \hat{O} \rangle$ in the adiabatic approximation then arises from the second term which is quadratically divergent. After Fourier transforming the ω^2 dependence of this term gives rise to the second derivative with respect to the time of the dilaton. And the sum over modes is the same as that in C , Eq. (132).

Now the point is that all divergences in the two-point function can be removed by local counterterms since they correspond to contact terms. In fact the gravity calculation also needed counterterms and from our discussion in Sec. III A we know that these counterterms are of the form given in Eq. (55). In particular, the third term in Eq. (55) proportional to $(\nabla\Phi)^2$ cancels the quadratic divergence while the last term in Eq. (55), $a_{(4)}$, contains terms which cancel the subleading logarithmic divergence. Also once the counterterms are chosen so that C agrees no other discrepancy can appear. The point here is that the leading order in ϵ calculations are only sensitive to the two-point function. And the finite terms in the two-point function are well known to agree between the gravity and gauge theory sides. In fact the finite two-point function is just determined by conformal invariance and since the anomalous dimension of \hat{O} does not get renormalized, it can be calculated in the free field limit itself.

The bottom line then is that using the freedom to adjust the counterterms, C can be made to agree with the supergravity calculations in Sec. III.

Let us end by pointing out that the supergravity value for C , Eq. (62) is

$$C_{\text{sugra}} = \frac{1}{16} \quad (140)$$

which means that the effect of renormalization is to only include the contributions of modes with mode number $n \sim O(1)$. This makes good physical sense; we are dealing with the low-frequency response of the system here, and the high-frequency modes should not be relevant for this purpose.

This last comment also has a bearing on our discussion in Sec. IV of the quantum adiabatic perturbation theory. The criterion for the validity of this approximation was stated in Eq. (74). Now what this condition really ensures is that the amplitude to excite the system to a state $|\phi_n\rangle = A_n^\dagger|0\rangle$ containing any one single oscillator excitation is small. However there are an infinite number of such single

excitation states, corresponding to the infinite number of values that n takes, and one might be worried that this condition is not sufficient. Even though the amplitude to excite the system into any given state $|\phi_n\rangle$ is small, the sum of these amplitudes, more correctly the norm of the first order correction of the wave function $\langle \psi^1 | \psi^1 \rangle$, Eq. (69), is still large and in fact would diverge when summed over all the modes. This would invalidate the approximation. The reason this concern does not arise is tied to our discussion above. After renormalization only a few low-frequency modes contribute to the response of the system and one is only interested in how the wave function changes for these modes. For this purpose, the condition in Eq. (74) is enough and we see that when it is met the quantum adiabatic approximation is indeed valid.

D. Highly curved geometry

So far we have considered what happens in the parametric regime, Eq. (129), when the 't Hooft coupling stays big at all times. In this case, the supergravity description is always valid. We saw above that the gauge theory can be described in this regime in terms of approximately decoupled classical harmonic oscillators and this reproduces the supergravity results.

Now let us consider what happens when the dilaton takes a larger excursion so that the 't Hooft coupling at intermediate times becomes of order unity or even smaller. Some of the resulting discussion is already contained in the Introduction above.

A natural expectation is that the description in terms of the classical adiabatic system of weakly coupled oscillators should continue to apply even when the 't Hooft coupling becomes small. There are several reasons to believe this. First, anharmonic terms continue to be of order ϵ and thus are small. The leading anharmonic terms arise from three-point correlations, $\langle \hat{O}_1 \hat{O}_2 \hat{O}_3 \rangle$. In the vacuum these go like $1/N$. In the coherent state produced by the time dependence these go like ϵ . The enhancement by $N\epsilon$ arises because the coherent state contains $O((N\epsilon)^2)$ quanta, so that the probability goes as $(N\epsilon)^2/N^2 \sim \epsilon^2$.¹⁸ Four-point functions give rise to terms going like $O(\epsilon^2)$ and so on, these are even smaller. In the absence of anharmonic terms the theory should reduce to a system of oscillators. Second, the existence of a gap of order $1/R$ means that for each oscillator the time dependence is slow compared to its frequency. Therefore the system continues to be very far from resonance and should evolve adiabatically. Finally, in the parametric regime, Eq. (129), the analysis of the previous subsections should then apply leading to the con-

¹⁸The probability $|\langle \phi | \hat{O} \hat{O} \hat{O} | \phi \rangle|^2$ is proportional to $\frac{1}{N^2} \times (N^2 \epsilon^2)^3$, with each factor of $N^2 \epsilon^2$ as an estimate of the contribution for each of the operators \hat{O} . The contribution of the 2-point function $|\langle \phi | \hat{O} \hat{O} | \phi \rangle|^2$ is just proportional to $(N^2 \epsilon^2)^2$, resulting in a relative suppression of $O(\epsilon^2)$.

clusion that an $O(N\epsilon) \gg 1$ quanta is produced making the coherent state a good classical state.

If this expectation is borne out, the system should settle back into the ground state of the final $\mathcal{N} = 4$ theory in the far future and should have a good description in terms of smooth AdS space then.

However, as discussed in the Introduction, there are reasons to worry that this expectation is not borne out. New features could enter the dynamics when the 't Hooft coupling becomes small at intermediate times, and these could change the qualitative behavior of the system. These new features have to do with the fact that string modes can start getting excited in the bulk when the curvature becomes of order the string scale. These modes correspond to nonchiral operators in the gauge theory and the corresponding oscillators have a time dependent frequency. When the 't Hooft coupling is big these frequencies are much bigger than those of the supergravity modes and as a result the string mode oscillators are not excited. But when the 't Hooft coupling becomes of order unity some of the frequencies of these string modes become of order the supergravity modes and hence these oscillators can begin to get excited.¹⁹ In fact the string modes are many more in number than the supergravity modes, since there is an order unity worth of chiral operators in the gauge theory and an $O(N^2)$ worth of nonchiral ones.

The worry then is that if a significant fraction of these string oscillators gets excited the correct picture which could describe the ensuing dynamics is one of thermalization rather than classical adiabatic evolution. In this case the energy pumped into the system initially would get equipartitioned among all the different degrees of freedom. Subsequent evolution would then be dissipative, and the energy would increase in a monotonic manner, as it does for a large black hole, Eq. (65).

Because of the dissipative behavior the energy which is initially pumped in would not be recovered in the future. Rather one would expect that when the 't Hooft coupling becomes large again, the energy, which is of order $N^2\epsilon^2$ remains in the system. The gravity description of the resulting thermalized state depends on the value of ϵ relative to $\lambda \equiv g_{\text{YM}}^2 N$ and N . In this late time regime of the large 't Hooft coupling, the various possibilities can be figured out from entropic considerations in supergravity (see, e.g., Sec. 3.4 of [6]). The result in our case is the following. For $\epsilon \ll (g_{\text{YM}}^2 N)^{5/4}/N$ a gas of supergravity modes is favored. For $(g_{\text{YM}}^2 N)^{5/4}/N < \epsilon \ll (g_{\text{YM}}^2 N)^{-7/8}$ one would have a gas of massive string modes. For $(g_{\text{YM}}^2 N)^{-7/8} < \epsilon \ll 1$ one gets a small black hole, i.e., a black hole whose size is much smaller than R_{AdS} . A big black hole requires $O(N^2)$ energy which is parametrically much larger. Thus, the strongest departure from AdS

space-time in the far future would be the presence of small black holes. Such black holes would eventually evaporate by emitting Hawking radiation. However this takes an $O(N^2 R_{\text{AdS}})$ amount of time which is much longer than the time scale $O(R_{\text{AdS}}/\epsilon)$ on which the 't Hooft coupling evolves. As a result for a long time after the 't Hooft coupling has become big again the gravity description would be that of a small black hole in AdS space.

An important complication in deciding between these two possibilities is that the rate of time variation is ϵ which is also the strength of the anharmonic couplings between the supergravity oscillators and string oscillators. If the rate of time variation could have been made much smaller, thermodynamics would become a good guide for how the system evolves. In the microcanonical ensemble, which is the correct one to use for our purpose, with energy $N^2\epsilon^2$ the entropically dominant configurations are as discussed in the previous paragraph, and this would suggest that dissipation would indeed set in. However, as emphasized above, this conclusion is far from obvious here since the time variation is parametrically identical to the strength of the anharmonic couplings.

In fact we know that the guidance from thermodynamics is misleading in the supergravity regime, where the 't Hooft coupling stays large for all times. In this case we have explicitly found the solution in Sec. II. It does not contain a black hole. Moreover, it does not suffer from any tachyonic instability—since it is a small correction from AdS space which does not have any tachyonic instability.²⁰ The only way a black hole could form is due to a tunneling process, but this would be highly suppressed in the supergravity regime.

One reason for this suppression is that the energy in the supergravity solution discussed in Sec. II is carried by supergravity quanta which have a size of order R_{AdS} . This energy would have to be concentrated in a much smaller region of order the small black hole's horizon to form the black hole and this is difficult to do. In contrast, away from the supergravity regime this could happen more easily. When the 't Hooft coupling becomes small at intermediate times, strings become large and floppy, of order R_{AdS} , at intermediate times. If a significant fraction of the energy gets transferred to these strings at intermediate times, it could find itself concentrated within a small black hole horizon once the 't Hooft coupling becomes large again.

In summary, we do not have a clean conclusion for the future fate of the system in the parametric regime, Eq. (129). Note however that in both possibilities discussed above most of space-time in the far future is smooth AdS space, with the possible presence of a small black hole. Hopefully, the framework developed here will be useful to think about this issue further.

¹⁹The primary reason for them getting excited are the anharmonic terms which couple them to the modes dual to the dilaton.

²⁰Note that we are working on S^3 here.

VII. CONCLUSIONS

In this paper, we examined the behavior of the $\text{AdS}_5 \times S^5$ solution of IIB supergravity when it is subjected to a time dependent boundary dilaton. This is dual to the behavior of the $\mathcal{N} = 4$ super Yang-Mills theory subjected to a time dependent gauge coupling. The AdS_5 solution was studied in global coordinates and the dual field theory lives on an S^3 of fixed radius R . We worked in units where $R_{\text{AdS}} = R = 1$. Three parameters are relevant for describing the resulting dynamics:

- (1) N —which is the number of units of flux and is dual to the rank of the gauge group. This was held fixed during the evolution.
- (2) $\lambda = e^{\Phi(t)}N$ —which determines the value of R_{AdS} in string units is the 't Hooft coupling in the gauge theory. Especially relevant is its minimum value λ_{min} during the time evolution. When $\lambda_{\text{min}} \gg 1$ supergravity is a good approximation for all times. When $\lambda_{\text{min}} \leq O(1)$ supergravity breaks down at intermediate times.

$\epsilon \sim \dot{\Phi}$ —which determines the rate of change of the boundary dilaton in units of R_{AdS} . Throughout the analysis we worked in the slowly varying regime where $\epsilon \ll 1$.

 - (i) Our results are as follows:

When $N\epsilon \ll 1$ the dynamics can be described by a quantum adiabatic approximation. The gauge theory stays in the ground state of the instantaneous Hamiltonian to good approximation. At late times the system is well described by smooth AdS_5 space-time. This is true even when $\lambda_{\text{min}} \leq 1$ as discussed in Sec. IV.
 - (ii) When $N\epsilon \gg 1$ and $\lambda_{\text{min}} \gg 1$, the system is well described by a supergravity solution, which consists of AdS_5 space-time with corrections which are suppressed in ϵ . The gauge theory provides an alternate description in terms of weakly coupled harmonic oscillators which are modes of gauge invariant operators dual to supergravity modes. These oscillators are subjected to a driving force that is slowly varying compared to their frequency. A classical adiabatic perturbation theory, the LNCAPT, describes the dynamics of the system. This dual description reproduces the supergravity answers for the energy and $\langle \hat{O} \rangle$, as discussed in Secs. VIA and VIB.
 - (iii) When $N\epsilon \gg 1$, and $\lambda_{\text{min}} \leq O(1)$, supergravity breaks down. In this case we do not have a clean conclusion for the final state of the system. Additional oscillators which correspond to string modes can now get activated. There are two possibilities: either the description in terms of classical adiabatic dynamics for the oscillators continues to apply, or a qualitative new feature of thermalization sets in. In the former case, space-time in the far future is well approximated by smooth AdS space. In the latter case, the gravity description depends on

the value of ϵ and may consist of a string gas or small black holes. This is discussed in Sec. VIC.

- (iv) We have not addressed here what happens when the dilaton begins to vary more rapidly and ϵ becomes $\sim O(1)$. It is natural to speculate that a black hole forms eventually in this case. The oscillators in the gauge theory now become strongly coupled with $O(1)$ anharmonic couplings.

If $\lambda_{\text{min}} \gg 1$ this parametric regime can be studied in supergravity itself. When $\epsilon \ll 1$ the calculations in Sec. II showed that no black hole forms. As ϵ increases the natural expectation is that eventually a black hole should begin to form at some critical value. The size of this black hole should then grow with ϵ , leading to a big black hole with a radius bigger than the AdS scale. Very preliminary indications for this come from the calculations in Sec. II where we see that as ϵ increases the value of $|g_{tt}|$ becomes smaller at the center of AdS Eq. (51), suggesting that a horizon would eventually form at $\epsilon \sim O(1)$. Better evidence comes from studying a region of parameter space where $\epsilon \gg 1$ but where the total amplitude of the dilaton variation is small. In this case²¹ one finds that a boundary variation of the dilaton, which is sufficiently fast compared to its amplitude, always produces a black hole.

When $\lambda_{\text{min}} \leq O(1)$, and ϵ becomes $\sim O(1)$, supergravity breaks down at intermediate times. If thermalization has already set in the parametric regime, $N\epsilon \gg 1$, $\epsilon \ll 1$, as discussed above, then one expects that the small black hole which has formed for $\epsilon \ll 1$ would grow and become of order the AdS scale or bigger when $\epsilon \geq O(1)$. If thermalization does not set in when $\epsilon \ll 1$, then at some critical value $\epsilon \sim O(1)$ one would expect that this does happen leading to the formation of a black hole whose mass then grows as ϵ further increases.

It will be interesting to try and analyze this regime further in subsequent work.

- (v) Finally, one can consider a regime where $\epsilon \rightarrow \infty$ at time $t \rightarrow 0$. This regime was considered in [10] where the dilaton was taken to vanish like $e^{\Phi} \sim (t)^p$ as $t \rightarrow 0$, leading to a diverging value for $\dot{\Phi}$. In a toy quantum mechanics model, it was argued that the response of the system in this case is singular, suggesting that this singularity is a genuine pathology which is not smoothed out. However the conclusions for the toy model do not directly apply to the field theory. Important questions regarding the renormalization of this time dependent field theory remain and could invalidate this conclusion.

One is hesitant to try and draw general conclusions about the possibility of emergence of a smooth space-

²¹The results reported in [20] are for the case of AdS_{d+1} space-times with d odd.

time from string scale curved regions on the basis of the very limited analysis presented here. One lesson which has emerged is that, at least for the kind of time dependence studied in this paper, AdS space has a tendency to form a black hole.²² This fate can be avoided (as in the case when $N\epsilon \ll 1$) but it requires slow time variation or perhaps more generally rather finely tuned conditions. To understand in greater detail when this fate of black hole formation can be avoided requires a deeper understanding of the process of thermalization in the dual field theory.

In this paper we analyzed the effects of a time dependent dilaton. It will be interesting to extend this to other supergravity modes as well by making their boundary values time dependent—e.g., making the radius of the S^3 on which the gauge theory lives time dependent or introducing time dependence along the other exactly flat directions in the $\mathcal{N} = 4$ theory besides the dilaton. Also, we have kept the parameter N fixed in this work. As was discussed in the Introduction, N measures the strength of quantum corrections and is also the value of R_{AdS} in Planck units, Eq. (3). It would be interesting to consider cases where N changes and becomes smaller thereby increasing the strength of quantum effects and making the curvature of order l_{Pl} . One way to do this might be by introducing time dependence that moves the system onto the Coulomb branch. This could reduce the effective value of N in the interior. For recent interesting work, see [32], also the related earlier work, [33,34]. Finally, a length scale was introduced in the gauge theory by working on S^3 here. Instead one could consider a confining gauge theory like the Klebanov-Strassler kind²³ [35], which has a mass gap on R^3 . In this case one could consider the response of the system to time dependence slow compared to the confining scale and hope to use an adiabatic approximation to understand this response.

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²²AdS space is of course homogeneous so the reader might be puzzled about where the black hole forms. The point is that the time dependence imposed on the boundary picks out a particular notion of time and the black hole forms where the redshift factor for this time is smallest; this is the “center of AdS space” in global coordinates.

²³We thank M. Mulligan for a related discussion.

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APPENDIX A: COMMENTS ON METRIC TO $O(\epsilon^2)$

We are interested in calculating the backreaction on the metric to $O(\epsilon^2)$ that arises due to the dilaton Φ_0 . Without loss of generality we can assume that the metric is S^3 symmetric and therefore of form

$$ds^2 = -g_{tt}dt^2 + g_{rr}dr^2 + 2g_{tr}dtdr + R^2d\Omega^2 \quad (\text{A1})$$

where the metric coefficients are functions of r, t . The zeroth order metric is that of AdS₅, Eq. (7). We argued above that the backreaction to the dilaton source arises at order ϵ^2 . Thus g_{tr} in Eq. (A1) is of order ϵ^2 .

We now show that by doing a suitable coordinate transformation, the mixed component g_{tr} can be set to vanish up to order ϵ^2 . The coordinate transformation is, from (t, r) to (t, \tilde{r}) , where

$$r = \tilde{r} - \frac{g_{tr}}{g_{rr}}t, \quad (\text{A2})$$

which leads to

$$dr = d\tilde{r} - \left(\frac{g_{tr}}{g_{rr}}\right)'t d\tilde{r} - \frac{g_{tr}}{g_{rr}}dt + O(\epsilon^3). \quad (\text{A3})$$

The prime above indicates derivatives with respect to r . We can drop the ϵ^3 terms for our purpose; these originate from additional time derivatives on the metric components. Substituting in Eq. (A1), we see that in the new coordinates the $g_{t\tilde{r}}$ components of the metric vanish up to $O(\epsilon^3)$ corrections which we are neglecting anyways. To avoid clutter we will henceforth drop the tilde on the r coordinate and write the metric as

$$ds^2 = -g_{tt}dt^2 + g_{rr}dr^2 + R^2d\Omega^2. \quad (\text{A4})$$

Next we show that up to $O(\epsilon^2)$ we can set R equal to the coordinate r without reintroducing the mixed components. First we define,

$$\tilde{r} = R \quad (\text{A5})$$

leading to

$$d\tilde{r} = R'dr + \dot{R}dt \quad (\text{A6})$$

where the dot indicates a time derivative. Now any time dependence in R arises only due to the dilaton and therefore is of order ϵ^2 . This means that \dot{R} is $O(\epsilon^3)$ and can be

neglected. So up to $O(\epsilon^2)$ no mixed components arise in the metric due to this coordinate transformation. We now drop the bar on the radial coordinate and write the final metric as

$$ds^2 = -g_{tt}dt^2 + g_{rr}dr^2 + r^2d\Omega^2. \quad (\text{A7})$$

APPENDIX B: MORE ON THE DRIVEN HARMONIC OSCILLATOR

In this appendix we provide the steps leading to (98) and (99). The time derivative of the state vector $|\psi(t)\rangle$ in (97) is

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = i(\dot{\lambda} + \dot{\alpha})a^\dagger|\psi(t)\rangle + i\left(\frac{\dot{N}}{N} + \frac{\dot{N}_\alpha}{N_\alpha}\right)|\psi(t)\rangle \quad (\text{B1})$$

where we have used the expression for $|\phi_0\rangle$ in (90). The action of the Hamiltonian H on the state is easily obtained by noting that

$$[H, e^{\lambda a^\dagger}] = \left(\omega_0\lambda a^\dagger + \frac{J\lambda}{\sqrt{2\omega_0}}\right)e^{\lambda a^\dagger}. \quad (\text{B2})$$

This leads to

$$H|\psi(t)\rangle = \left(\omega_0\lambda a^\dagger + \frac{J\lambda}{\sqrt{2\omega_0}}\right)|\psi(t)\rangle + \frac{\omega_0}{2}|\psi(t)\rangle. \quad (\text{B3})$$

It may easily be checked that the states $|\psi(t)\rangle$ and $a^\dagger|\psi(t)\rangle$ are linearly independent. Equating the coefficients of $a^\dagger|\psi(t)\rangle$ in Eqs. (B1) and (B3) and using Eq. (92) then leads to Eq. (98). Equating the coefficients of $|\psi(t)\rangle$ in Eqs. (B1) and (B3) gives an equation that determines $N(t)$. Note that $|N(t)|$ is determined directly from the requirement that $\langle\psi|\psi\rangle = 1$.

APPENDIX C: THE NORMALIZATION FACTOR $F(2n)$

In computing the normalization $F(2n)$ in (115) it is best to first continue to Euclidean signature and then perform a conformal transformation from $R \times S^3$ to R^4 . The radial coordinate on the R^4 is given by $r = e^\tau$, where τ is the Euclidean time in $R \times S^3$. Then the Heisenberg picture operator on R^4 is given by

$$\hat{O}_{l=0} = \sum_{m=-\infty}^{\infty} \frac{\mathcal{O}_m}{r^{m+4}}. \quad (\text{C1})$$

The factor of r^{m+4} in the denominator reflects the fact that the operator $\hat{O}_{l=0}$ has dimension 4. The conformally invariant vacuum satisfies

$$\mathcal{O}_m|0\rangle = 0 \quad m \geq -3 \quad \langle 0|\mathcal{O}_m = 0 \quad m \leq 3. \quad (\text{C2})$$

Then the radial time ordered two-point function is given by

$$\langle \hat{O}_{l=0}(r)\hat{O}_{l=0}(r') \rangle = \sum_{m=4}^{\infty} \sum_{n=-\infty}^{-4} \frac{\langle 0|\mathcal{O}_m\mathcal{O}_n|0\rangle}{r^{m+4}(r')^{n+4}}. \quad (\text{C3})$$

The two-point function only involves the central term in the operator algebra. This means we can write

$$\begin{aligned} \mathcal{O}_m &= NF(m)A_m \quad (m > 0) \\ \mathcal{O}_{-m} &= NF^*(m)A_m^\dagger \quad (m > 0) \end{aligned} \quad (\text{C4})$$

where the operators A_m, A_m^\dagger satisfy an operator algebra and $F(m)$ is a normalization

$$[A_m, A_n] = [A_m^\dagger, A_n^\dagger] = 0 \quad [A_m, A_n^\dagger] = \delta_{mn}. \quad (\text{C5})$$

Note that because of (C3) only terms for $n \geq 4$ contribute to the sum. This leads to the result

$$\langle \hat{O}_{l=0}(r)\hat{O}_{l=0}(r') \rangle = \frac{N^2}{r^8} \sum_{m=4}^{\infty} |F(m)|^2 \left(\frac{r'}{r}\right)^{m-4}. \quad (\text{C6})$$

On the other hand since the dimension of the operator $\hat{O}^\Phi(r, \Omega_3)$ is 4 we know the two-point function on R^4 . This is given by

$$\langle \hat{O}(r, \Omega_3)\hat{O}(r', \Omega'_3) \rangle = \frac{AN^2}{|\vec{r} - \vec{r}'|^8} \quad (\text{C7})$$

where A is a order one numerical constant. Here $\vec{r} = (r, \Omega)$, etc. is the location of the operator on R^4 . Integrating over Ω_3, Ω'_3 we get

$$\begin{aligned} &\int d\Omega_3 \int d\Omega'_3 \langle \hat{O}(r, \Omega_3)\hat{O}(r', \Omega'_3) \rangle \\ &= AN^2(8\pi^3) \int_0^\pi \frac{\sin^2\theta d\theta}{(r^2 + (r')^2 - 2rr'\cos\theta)^4}. \end{aligned} \quad (\text{C8})$$

The integral can be performed. The result is for $r > r'$

$$\begin{aligned} &\int d\Omega_3 \int d\Omega'_3 \langle \hat{O}(r, \Omega_3)\hat{O}(r', \Omega'_3) \rangle \\ &= N^2 \frac{4A\pi^4}{r^8} \frac{(r/r')^2 + 1}{(1 - (r/r')^2)^5}. \end{aligned} \quad (\text{C9})$$

Using the power series expansion

$$\frac{1+x}{(1-x)^5} = \sum_{m=0}^{\infty} \frac{1}{12} (m+1)(m+2)^2(m+3)x^m \quad (\text{C10})$$

we finally get

$$\begin{aligned} &\int d\Omega_3 \int d\Omega'_3 \langle \hat{O}(r, \Omega_3)\hat{O}(r', \Omega'_3) \rangle \\ &= N^2 \frac{A\pi^4}{3} \frac{1}{r^8} \sum_{m=0}^{\infty} (m+1)(m+2)^2(m+3) \left(\frac{r'}{r}\right)^{2m}. \end{aligned} \quad (\text{C11})$$

The result clearly shows that only operators with even mode numbers are present in the expansion (C1). Comparing (C11) and (C6) we get

$$F(2m+1) = 0 \quad |F(2m)|^2 = \frac{A\pi^4}{3} m^2(m^2-1) \quad (\text{C12})$$

which is the result in Eq. (115).

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