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Magnetic flux, Wilson line, and orbifold

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We study torus/orbifold models with magnetic flux and Wilson line backgrounds. The number of zero modes and their profiles depend on those backgrounds. That has interesting implications from the viewpoint of particle phenomenology.

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I. INTRODUCTION

Extra-dimensional field theories and the string theory with magnetic fluxes can lead to interesting models [1–9]. A chiral theory can be realized as the four-dimensional effective field theory, because of the magnetic flux background. The number of zero modes, that is, the generation number, is determined by the magnitude of magnetic flux. Their zero-mode profiles are nontrivially quasilocalized. Such a behavior of zero-mode wave functions can lead to suppressed couplings when zero modes are quasilocalized far away each other. That would be useful to realize, e.g. suppressed Yukawa couplings for light quarks and leptons. On the other hand, when their localized points are close to each other, their couplings would be of $\mathcal{O}(1)$ and that would be useful to explain, e.g. the top Yukawa coupling. Furthermore, those localizing points on the torus background have a certain symmetry and it would become an origin of non-Abelian discrete flavor symmetries [10].[1] Furthermore, certain moduli can be stabilized by introducing magnetic fluxes [14]. Thus, magnetized brane models have phenomenologically several interesting aspects.

Indeed, magnetized D-brane models are T-duals of intersecting D-brane models [4–6,15–17]. (For a review see [18] and references therein.) Within the framework of intersecting D-brane models, many interesting models have been constructed so far.

From the magnetic backgrounds associated with orbifolds [19,20] and Wilson lines, one can also derive several interesting aspects and some of them have been studied.[2] Effects of Wilson lines on the torus with magnetic fluxes are gauge symmetry breaking and the shift of wave function profiles. Orbifolding is another way to realize a chiral theory. For the same magnetic flux, the numbers of chiral zero modes between the torus compactification and the orbifold compactification are different from each other and zero-mode profiles are different [19,20]. Adjoint matter fields remain massless on the torus with magnetic fluxes, but those are projected out on the orbifold.[3] These differences lead to phenomenologically interesting aspects [20]. However, effects due to Wilson lines have not studied on the orbifold with the magnetic flux background. Our purpose in this paper is to study more about these backgrounds such as consistency conditions, zero-mode profiles and phenomenological aspects of four-dimensional effective theory.

This paper is organized as follows: In Sec. II, we study four-dimensional effective theories derived from the torus compactification with magnetic flux and Wilson line backgrounds. Most of them are already known results. (See, e.g. [7].) However, we reconsider phenomenological implications of Wilson lines on magnetized torus models. In Sec. III, we study the orbifold background with magnetic fluxes and Wilson lines. We study zero modes under such a background and their phenomenological aspects. Section IV is devoted to a conclusion and discussions.

II. MAGNETIZED TORUS MODELS WITH WILSON LINES

A. T^2 models

Here, let us study a six-dimensional field theory with magnetic fluxes and Wilson lines. The two extra dimensions are compactified on T^2, whose area and complex structure are denoted by A and τ. We use the coordinates y_m ($m = 4, 5$) for T^2, while x_μ ($\mu = 0, \cdots, 3$) denote the four-dimensional uncompactified space-time, $R^{3,1}$. Furthermore, we often use the complex coordinate, $z = y_4 + \tau y_5$. The boundary conditions on T^2 are represented by $z \sim z + 1$ and $z \sim z + \tau$.

First, let us study $U(1)$ theory. We consider the fermion field $\lambda(x, z)$ with the $U(1)$ charge, q, and it satisfies the

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[1]Similar non-Abelian discrete flavor symmetries have been realized in heterotic orbifold models [11–13].

[2]Other backgrounds with magnetic fluxes were also studied [21–23].

[3]Within the framework of intersecting D-brane models, analogous results have been obtained by considering D6-branes wrapping rigid 3-cycles [24].

Dirac equation

$$\Gamma^M D_M \lambda(x,z) = \Gamma^M(\partial_M - iqA_M)\lambda(x,z) = 0, \quad (1)$$

with $M = 0, \cdots, 5$, where Γ^M denote the six-dimensional gamma matrices and A_M denote $U(1)$ gauge vectors. The fermion field λ and the vector fields A_M ($M = (\mu, m)$) are decomposed as

$$\begin{aligned} \lambda(x,z) &= \sum_n \chi_n(x) \otimes \psi_n(z), \\ A_\mu(x,z) &= \sum_n A_{n,\mu}(x) \otimes \phi_{n,\mu}(z), \\ A_m(x,z) &= \sum_n \varphi_{n,m}(x) \otimes \phi_{n,m}(z), \end{aligned} \quad (2)$$

with $m = 4, 5$, where $A_{n,\mu}(x)$ and $\varphi_{n,m}(x)$ correspond to four-dimensional vector fields and scalar fields, respectively. Here, the modes with $n = 0$ correspond to zero modes, while the others correspond to massive modes. Since we concentrate on zero modes, we omit the subscript corresponding to $n = 0$ hereafter.

We assume the following form of magnetic flux on T^2:

$$F = \frac{\pi i}{\mathrm{Im}\tau} m(dz \wedge d\bar{z}), \quad (3)$$

where m is an integer [25]. Such a magnetic flux can be obtained from the vector potential

$$A(z) = \frac{\pi m}{\mathrm{Im}\tau} \mathrm{Im}(\bar{z}dz). \quad (4)$$

This form of the vector potential satisfies the following relations:

$$A(z+1) = A(z) + \frac{\pi m}{\mathrm{Im}\tau} \mathrm{Im}(dz), \quad (5)$$

$$A(z+\tau) = A(z) + \frac{\pi m}{\mathrm{Im}\tau} \mathrm{Im}(\bar{\tau}dz). \quad (6)$$

Furthermore, these can be represented as the following gauge transformations:

$$A(z+1) = A(z) + d\chi_1, \qquad A(z+\tau) = A(z) + d\chi_2, \quad (7)$$

where

$$\chi_1 = \frac{\pi m}{\mathrm{Im}\tau} \mathrm{Im}(z), \qquad \chi_2 = \frac{\pi m}{\mathrm{Im}\tau} \mathrm{Im}(\bar{\tau}z). \quad (8)$$

Then, the internal part $\psi(z)$ of the fermion zero mode with the charge q must satisfy

$$\psi(z+1) = e^{iq\chi_1(z)}\psi(z), \qquad \psi(z+\tau) = e^{iq\chi_2(z)}\psi(z). \quad (9)$$

Here and hereafter, we use the $U(1)$ charge normalization such that all charges of matter fields are integers and the minimum charge is equal to $|q| = 1$. The internal part ψ is a two-component spinor

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \quad (10)$$

and we choose the gamma matrix for T^2,

$$\tilde{\Gamma}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tilde{\Gamma}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad (11)$$

in terms of the real coordinates y_m. Then, the Dirac equations for zero modes become

$$\left(\bar{\partial} + \frac{\pi q m}{2\,\mathrm{Im}(\tau)} z\right)\psi_+(z,\bar{z}) = 0, \quad (12)$$

$$\left(\partial - \frac{\pi q m}{2\,\mathrm{Im}(\tau)} \bar{z}\right)\psi_-(z,\bar{z}) = 0. \quad (13)$$

When $qm > 0$, the component ψ_+ has $M = qm$ independent zero modes and their wave functions are written as [7]

$$\Theta^{j,M}(z) = N_M e^{i\pi M z\,\mathrm{Im}(z)/\mathrm{Im}(\tau)} \vartheta\begin{bmatrix} j/M \\ 0 \end{bmatrix}(Mz, M\tau), \quad (14)$$

where N_M is a normalization factor, j denotes the flavor index, i.e. $j = 0, \cdots, M-1$ and

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\nu,\mu) = \sum_n \exp[\pi i(n+a)^2\mu + 2\pi i(n+a)(\nu+b)], \quad (15)$$

that is, the Jacobi theta function. Note that $\Theta^{0,M}(z) = \Theta^{M,M}(z)$. They satisfy the orthonormal condition

$$\int d^2z\,\Theta^{i,M}(z)(\Theta^{j,M}(z))^* = \delta_{ij}. \quad (16)$$

Furthermore, for $qm > 0$, the other component ψ_- has no zero modes. As a result, we can realize a chiral spectrum.

On the other hand, when $qm < 0$, the component ψ_- has $|qm|$ independent zero modes, but the other component ψ_+ has no zero modes.

One of the important properties of zero-mode wave functions is that we can decompose a product of two zero-mode wave functions as follows: [26,27],

$$\Theta^{i,M_1}(z)\Theta^{j,M_2}(z) = \frac{N_{M_1}N_{M_2}}{N_{M_1+M_2}} \sum_{m\in\mathbf{Z}_{M_1+M_2}} \Theta^{i+j+M_1m, M_1+M_2}(z) \times \vartheta\begin{bmatrix} \frac{M_2 i - M_1 j + M_1 M_2 m}{M_1 M_2 (M_1+M_2)} \\ 0 \end{bmatrix} \times (0, \tau_d M_1 M_2 (M_1+M_2)). \quad (17)$$

Now, let us introduce Wilson lines. The Dirac equations of the zero modes are modified by the Wilson line background, $C = C_1 + \tau C_2$ as

$$\left(\bar{\partial} + \frac{\pi q}{2\,\mathrm{Im}(\tau)}(mz + C)\right)\psi_+(z,\bar{z}) = 0, \quad (18)$$

$$\left(\partial - \frac{\pi q}{2\,\mathrm{Im}(\tau)}(m\bar{z} + \bar{C})\right)\psi_-(z,\bar{z}) = 0, \tag{19}$$

where C_1 and C_2 are real constants. That is, we can introduce the Wilson line background, $C = C_1 + \tau C_2$ by replacing χ_i in (8) as [7]

$$\chi_1 = \frac{\pi}{\mathrm{Im}\tau}\,\mathrm{Im}(mz + C), \qquad \chi_2 = \frac{\pi}{\mathrm{Im}\tau}\,\mathrm{Im}(\bar{\tau}(mz + C)). \tag{20}$$

Because of this Wilson line, the number of zero modes does not change, but their wave functions are replaced as

$$\Theta^{j,M}(z) \longrightarrow \Theta^{j,M}(z + C/m). \tag{21}$$

In general, Yukawa couplings are computed by the overlap integral of three zero-mode profiles, $\psi_i(z)$, $\psi_j(z)$, and $\psi_k(z)$,

$$y_{ijk} = g \int d^2z\, \psi_i(z)\psi_j(z)\psi_k(z), \tag{22}$$

where g denotes the corresponding coupling in the higher dimensional theory. Concretely, when Wilson lines are vanishing, the overlap integral of $\Theta^{i,M_1}(z)\Theta^{j,M_2}(z) \times (\Theta^{k,M_3}(z))^*$ for $M_3 = M_1 + M_2$ is given [7,28][4]

$$\int d^2z\,\Theta^{i,M_1}(z)\Theta^{j,M_2}(z)(\Theta^{k,M_3}(z))^* = \frac{N_{M_1}N_{M_2}}{N_{M_3}} \sum_{m\in \mathbf{Z}_{M_3}} \delta_{i+j+M_1m,k} \times \vartheta\begin{bmatrix} \frac{M_2 i - M_1 j + M_1M_2 m}{M_1M_2M_3} \\ 0 \end{bmatrix} \times (0, \tau M_1M_2M_3), \tag{23}$$

where the gauge invariance requires the third wave function must not be $\Theta^{k,M_3}(z)$ but $(\Theta^{k,M_3}(z))^*$ with the magnetic flux $M_3 = M_1 + M_2$. Here. we have used the product rule (17) and the orthogonality (16). When we introduce non-vanishing Wilson lines, the overlap integral is obtained as

$$\int d^2z\,\Theta^{i,M_1}(z + C/M_1)\Theta^{j,M_2}(z + C'/M_2) \times (\Theta^{k,M_3}(z + C''/M_3))^* = \frac{N_{M_1}N_{M_2}}{N_{M_3}} \sum_{m\in \mathbf{Z}_{M_3}} \delta_{i+j+M_1m,k} \times e^{i\pi(C\,\mathrm{Im}(C/M_1) + C'\,\mathrm{Im}(C'/M_2) + C''\,\mathrm{Im}(C''/M_3))/\mathrm{Im}\tau} \times \vartheta\begin{bmatrix} \frac{M_2 i - M_1 j + M_1M_2 m}{M_1M_2M_3} \\ 0 \end{bmatrix}((M_2C - M_1C'), \tau M_1M_2M_3), \tag{24}$$

where the gauge invariance requires $C'' = C + C'$. Furthermore, by repeating the above procedure we can compute higher order couplings [30].

[4]See also [29].

In Eqs. (23) and (24), the subscript of the Kronecker delta is defined modulo M_3, and the Kronecker delta leads to the selection rule for allowed couplings as

$$i + j - k = M_3\ell - M_1 m, \tag{25}$$

where ℓ, m are integers. When $\gcd(M_1, M_2, M_3) = g$, the above constraint becomes

$$i + j = k, \qquad (\mathrm{mod}\, g). \tag{26}$$

That implies that we can define Z_g charge for zero modes, and the allowed couplings are controlled by such a Z_g symmetry [10,30].[5] This Z_g transformation can be written as [10]

$$Z = \begin{pmatrix} 1 & & & & \\ & \rho & & & \\ & & \rho^2 & & \\ & & & \ddots & \\ & & & & \rho^{g-1} \end{pmatrix}, \tag{27}$$

where $\rho = e^{2\pi i/g}$. Furthermore, four-dimensional effective theory has a cyclic permutation symmetry

$$\Theta^{i,M_1} \longrightarrow \Theta^{i+mn_1,M_1}, \qquad \Theta^{j,M_2} \longrightarrow \Theta^{j+mn_2,M_2}, \qquad \Theta^{k,M_3} \longrightarrow \Theta^{k+mn_3,M_3}, \tag{28}$$

where $n_i = M_i/g$ and m is a universal integer, that is, another Z_g symmetry. This Z_g transformation can be written as [10]

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ & & & & \ddots & \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{29}$$

These two Z_g symmetries are noncommutable and lead to non-Abelian flavor symmetry, $(Z_g \times Z_g) \rtimes Z_g$ [10]. Its diagonal elements are written as $Z^m(Z')^n$, where

$$Z' = \begin{pmatrix} \rho & & \\ & \ddots & \\ & & \rho \end{pmatrix}. \tag{30}$$

These symmetries are also available for higher order couplings. Furthermore, when we consider vanishing Wilson lines, the Z_2 twist symmetry is enhanced by the symmetry,

$$\Theta^{i,M} \longrightarrow \Theta^{M-i,M}, \tag{31}$$

and such Z_2 can be written as

[5]See for the same selection rule in intersecting D-brane models [31,32].

$$\mathcal{P} = \begin{pmatrix} 1 & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix}. \qquad (32)$$

Then, the permutation symmetry is enhanced from Z_g to D_g, and the total symmetry becomes $(Z_g \times Z_g) \rtimes D_g$. For example, when $g = 3$, we can realize $(Z_3 \times Z_3) \rtimes Z_3 = \Delta(27)$ and $(Z_3 \times Z_3) \rtimes D_3 = \Delta(54)$.

It would be useful to consider $U(1)_a \times U(1)_b$ theory from the phenomenological viewpoint. We consider the fermion field $\lambda(x, z)$ with $U(1)_a \times U(1)_b$ charges, (q_a, q_b). We assume the following form of $U(1)_a$ magnetic flux on T^2:

$$F^a_{z\bar{z}} = \frac{\pi i}{\mathrm{Im}\tau} m_a, \qquad (33)$$

where m_a is integer, but there is no magnetic flux in $U(1)_b$. On top of that, we introduce Wilson lines C^a and C^b for $U(1)_a$ and $U(1)_b$, respectively. The zero-mode equations are written as

$$\left(\bar{\partial} + \frac{\pi}{2\,\mathrm{Im}(\tau)}(q_a(m_a z + C^a) + q_b C^b)\right)\psi_+(z, \bar{z}) = 0, \qquad (34)$$

$$\left(\partial - \frac{\pi}{2\,\mathrm{Im}(\tau)}(q_a(m_a \bar{z} + \bar{C}^a) + q_b \bar{C}^b)\right)\psi_-(z, \bar{z}) = 0. \qquad (35)$$

Then, the number of zero modes is obtained as $M = q_a m_a$ and their wave functions are written as

$$\Theta^{j,M}(z + C/m_a), \qquad (36)$$

where $C = C^a + C^b q_b/q_a$. Here, we give a few comments. All of modes with $q_a = 0$ become massive and there do not appear zero modes with $q_a = 0$. For $q_a \neq 0$, zero modes with $q_b = 0$ appear and the number of zero modes is independent of q_b. Obviously, when we introduce Wilson lines C^a and/or C^b without magnetic flux F^a, zero modes do not appear. The shift of wave functions depends on $1/m_a$ and the charge q_b. Note that although $F^b = 0$, Wilson lines C_b and charges q_b for $U(1)_b$ are also important.[6]

The above aspects of magnetic fluxes and Wilson lines are phenomenologically interesting. We consider six-dimensional super Yang-Mills theory with non-Abelian gauge group G. We introduce a magnetic flux F^a along a Cartan direction of G. Then, the gauge group breaks to $G' \times U(1)_a$ without reducing the rank. Furthermore, there appear the massless fermion fields λ', which correspond to the gaugino fields for the broken gauge group part in the fundamental representation of G' with a nonvanishing $U(1)_a$ charge. Furthermore, we introduce a Wilson line along a Cartan direction of G'. Then, the gauge group is broken to $G'' \times U(1)_a \times U(1)_b$ without reducing the rank. The gaugino fields corresponding to the broken gauge part in G' do not remain as massless modes, but they gain masses due to the Wilson line $U(1)_b$. However, the fermion fields λ' remain still massless with the same degeneracy.

Let us explain more on this aspect. Suppose that we introduce magnetic fluxes in a model with a larger group G such that they break G to a grand unified theory (GUT) group like $SU(5)$, and this model includes three families of matter fields like 10 and $\bar{5}$. Their Yukawa couplings are computed by the overlap integral of three zero-mode profiles as Eq. (22). We obtain the GUT relation among Yukawa coupling matrices when wave function profiles of matter fields in 10 ($\bar{5}$) are degenerate like Eq. (23). Then, we introduce a Wilson line along $U(1)_Y$, which breaks $SU(5)$ to $SU(3) \times SU(2) \times U(1)_Y$. Because of Wilson lines, $SU(5)$ gauge bosons except the $SU(3) \times SU(2) \times U(1)_Y$ gauge bosons become massive, and the corresponding gaugino fields become massive. However, three families of 10 and $\bar{5}$ matter fields remain massless. Importantly, this Wilson line resolves the degeneracy of wave function profiles of left-handed quarks, right-handed up-sector quarks and right-handed charged leptons in 10 and right-handed down-sector quarks and left-handed charged leptons in $\bar{5}$ as Fig. 1. That is, the GUT relation among Yukawa coupling matrices is deformed. As an illustrating model, we study the Pati-Salam model in the next subsection.

Here, we comment on some effects due to discrete values of continuous Wilson lines such as $C = k\tau$ with $k =$ integer. We find

$$\Theta^{j,M}(z + k\tau/M) = e^{\pi i k\,\mathrm{Im}(\bar{\tau}z)/\mathrm{Im}(\tau)}\Theta^{j+k,M}(z). \qquad (37)$$

Thus, the effect of such Wilson lines $C = k\tau$ is to replace the j-th zero mode by the $(j + k)$-th zero mode up to $e^{\pi i k\,\mathrm{Im}(\bar{\tau}z)/\mathrm{Im}(\tau)}$. However, when we consider 3-point and higher order couplings, the gauge invariance requires that the sum of Wilson lines of matter fields should vanish, that is, $\sum_i k_i = 0$ for allowed n-point couplings. Thus, the part $e^{\pi i k\,\mathrm{Im}(\bar{\tau}z)/\mathrm{Im}(\tau)}$ is irrelevant to four-dimensional effective theory and the resultant four-dimensional effective theory is the equivalent even when we introduce $C = k\tau$.

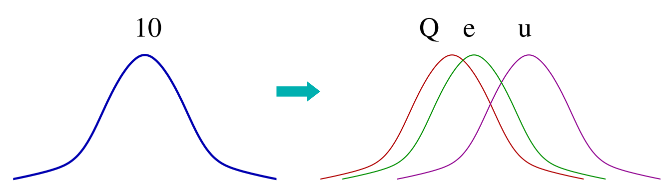

FIG. 1 (color online). Wave function splitting by Wilson lines.

[6]Wilson lines C_b and charges q_b for $U(1)_b$ are in a sense more important than Wilson lines C_a and charges q_a for $U(1)_a$, because the shift of wave functions (36) depends on q_b.

Similarly, introducing the Wilson lines $C = k$ with $k =$ integer leads to the equivalent four-dimensional effective theory.

B. Pati-Salam model

As an illustrating model, we consider the Pati-Salam model. We start with ten-dimensional $N = 1$ $U(8)$ super Yang-Mills theory with the Lagrangian

$$\mathcal{L} = -\frac{1}{4g^2}\,\mathrm{Tr}(F^{MN}F_{MN}) + \frac{i}{2g^2}\,\mathrm{Tr}(\bar{\lambda}\Gamma^M D_M \lambda), \tag{38}$$

where M, $N = 0, \cdots, 9$. We compactify the extra six dimensions on $T_1^2 \times T_2^2 \times T_3^2$, and we denote the complex coordinate for the d-th T_d^2 by z^d, where $d = 1, 2, 3$. Then, we introduce the following form of magnetic fluxes:

$$F_{z^d\bar{z}^d} = \frac{\pi i}{\mathrm{Im}\tau_d}\begin{pmatrix} m_1^{(d)}\mathbb{1}_4 & & \\ & m_2^{(d)}\mathbb{1}_2 & \\ & & m_3^{(d)}\mathbb{1}_2 \end{pmatrix}, \quad d = 1, 2, 3 \tag{39}$$

in the gauge space, where $\mathbb{1}_N$ are the unit matrices of rank N, $m_i^{(d)}$ are integers. We assume that the above background preserves four-dimensional $N = 1$ supersymmetry. Here, we denote $M_{ij}^{(d)} = m_i^{(d)} - m_j^{(d)}$ and $M_{ij} = M_{ij}^{(1)}M_{ij}^{(2)}M_{ij}^{(3)}$. This magnetic flux breaks the gauge group $U(8)$ to $U(4) \times U(2)_L \times U(2)_R$, that is the Pati-Salam gauge group up to $U(1)$ factors. The gauge sector corresponds to four-dimensional $N = 4$ supersymmetry vector multiplet, that is, there are $U(4) \times U(2)_L \times U(2)_R$ $N = 1$ vector multiplet and three adjoint chiral multiplets. In addition, there appear bifundamental matter fields like $\lambda_{(4,2,1)}$, $\lambda_{(\bar{4},1,2)}$ and $\lambda_{(1,2,2)}$, and their numbers of zero modes are equal to M_{12}, M_{31}, and M_{23}. When M_{ij} is negative, that implies their conjugate matter fields appear with the degeneracy $|M_{ij}|$. The fields $\lambda_{(4,2,1)}$ and $\lambda_{(\bar{4},1,2)}$ correspond to left-handed and right-handed matter fields, respectively, while $\lambda_{(1,2,2)}$ corresponds to up and down Higgs (Higgsino) fields. For example, we can realize three families by $M_{12}^{(d)} = (3, 1, 1)$ and $M_{31}^{(d)} = (3, 1, 1)$. That leads to $|M_{23}| = 0$ or 24. At any rate, the flavor structure is determined by the first T_1^2 in such a model. Explicitly, the zero-mode wave functions of both $\lambda_{(4,2,1)}$ and $\lambda_{(\bar{4},1,2)}$ are obtained as

$$\Theta^{j,3}(z^1)\Theta^{1,1}(z^2)\Theta^{1,1}(z^3). \tag{40}$$

Their Yukawa matrices are constrained by the Pati-Salam gauge symmetry, that is, up-sector quarks, down-sector quarks, charged leptons and neutrinos have the same Yukawa matrices with Higgs fields. Even with such a constraint, one could derive realistic quark/lepton masses and mixing angles, because this model has many Higgs fields and their vacuum expectation values generically break the up-down symmetry.

We introduce Wilson lines in $U(4)$ and $U(2)_R$ such that $U(4)$ breaks to $U(1) \times U(3)$ and $U(2)_R$ breaks $U(1) \times U(1)$. Then, the gauge group becomes the standard gauge group up to $U(1)$ factors. Furthermore, the profiles of left-handed quarks and leptons in $\lambda_{(4,2,1)}$ shift differently because of Wilson lines. Similarly, right-handed up-sector quarks, down-sector quarks, charged leptons and neutrinos in $\lambda_{(\bar{4},1,2)}$ shift differently. The flavor structure is determined by the first T_1^2. Thus, when we introduce Wilson lines in the second or third torus, the resultant Yukawa matrices are still constrained by the $SU(4) \times SU(2)_L \times SU(2)_R$. For example, we introduce Wilson lines on T_2^2. Then, zero-mode profiles of quarks, (Q, u, d) and leptons (L, e, ν) split as

$$\begin{aligned} &Q\colon\ \Theta^{j,3}(z^1)\Theta^{1,1}(z^2 + C^a)\Theta^{1,1}(z^3),\\ &L\colon\ \Theta^{j,3}(z^1)\Theta^{1,1}(z^2 - 3C^a)\Theta^{1,1}(z^3),\\ &u^c\colon\ \Theta^{j,3}(z^1)\Theta^{1,1}(z^2 - C^a + C^b)\Theta^{1,1}(z^3),\\ &d^c\colon\ \Theta^{j,3}(z^1)\Theta^{1,1}(z^2 - C^a - C^b)\Theta^{1,1}(z^3),\\ &e^c\colon\ \Theta^{j,3}(z^1)\Theta^{1,1}(z^2 + 3C^a - C^b)\Theta^{1,1}(z^3),\\ &\nu^c\colon\ \Theta^{j,3}(z^1)\Theta^{1,1}(z^2 + 3C^a + C^b)\Theta^{1,1}(z^3), \end{aligned} \tag{41}$$

where C^a and C^b are the Wilson lines to break $U(4) \to U(3) \times U(1)$ and $U(2)_R \to U(1) \times U(1)$, respectively. Those Wilson lines just change the overall factors of Yukawa matrices, but ratios among elements in each Yukawa matrix do not change. Also we can introduce Wilson lines along the same $U(1)$ directions as the magnetic fluxes (39), but they do not deform the up-down symmetry of Yukawa matrices, either.

On the other hand, when we introduce Wilson lines on the first T_1^2, the zero-mode wave functions split as

$$\begin{aligned} &Q\colon\ \Theta^{j,3}(z^1 + C^a/3)\Theta^{1,1}(z^2)\Theta^{1,1}(z^3),\\ &L\colon\ \Theta^{j,3}(z^1 - C^a)\Theta^{1,1}(z^2)\Theta^{1,1}(z^3),\\ &u^c\colon\ \Theta^{j,3}(z^1 - C^a/3 + C^b/3)\Theta^{1,1}(z^2)\Theta^{1,1}(z^3),\\ &d^c\colon\ \Theta^{j,3}(z^1 - C^a/3 - C^b/3)\Theta^{1,1}(z^2)\Theta^{1,1}(z^3),\\ &e^c\colon\ \Theta^{j,3}(z^1 + C^a - C^b/3)\Theta^{1,1}(z^2)\Theta^{1,1}(z^3),\\ &\nu^c\colon\ \Theta^{j,3}(z^1 + C^a + C^b/3)\Theta^{1,1}(z^2)\Theta^{1,1}(z^3). \end{aligned} \tag{42}$$

In this case, the flavor structure is deviated from the $SU(4) \times SU(2)_L \times SU(2)_R$ relation, that is, mass ratios and mixing angles can change. Also we can introduce Wilson lines C^a to T_2^2 and C^b to T_1^2. Then we realize

$$\begin{aligned} &Q\colon \Theta^{j,3}(z^1)\Theta^{1,1}(z^2 + C^a)\Theta^{1,1}(z^3),\\ &L\colon \Theta^{j,3}(z^1)\Theta^{1,1}(z^2 - 3C^a)\Theta^{1,1}(z^3),\\ &u^c\colon \Theta^{j,3}(z^1 + C^b/3)\Theta^{1,1}(z^2 - C^a)\Theta^{1,1}(z^3),\\ &d^c\colon \Theta^{j,3}(z^1 - C^b/3)\Theta^{1,1}(z^2 - C^a)\Theta^{1,1}(z^3),\\ &e^c\colon \Theta^{j,3}(z^1 - C^b/3)\Theta^{1,1}(z^2 + 3C^a)\Theta^{1,1}(z^3),\\ &\nu^c\colon \Theta^{j,3}(z^1 + C^b/3)\Theta^{1,1}(z^2 + 3C^a)\Theta^{1,1}(z^3). \end{aligned} \tag{43}$$

Indeed, this behavior is well known in the intersecting D-brane models, which are T-duals of magnetized D-brane models. In the intersecting D-brane side, the introduction of Wilson lines corresponds to a split of D-branes. By splitting D-branes, the gauge group breaks as $U(M+N) \to U(M) \times U(N)$, but the number of massless bifundamental modes does not change, although they are decomposed because of the gauge symmetry breaking.

III. ORBIFOLD MODELS

Here, we study orbifold models with magnetic fluxes. The T^2/Z_2 orbifold is constructed by identifying $z \sim -z$ on T^2. We also embed the Z_2 twist into the gauge space as P. Note that under the Z_2 twist, magnetic flux background is invariant. That is, we have no constraint on magnetic fluxes due to orbifolding. Furthermore, zero-mode wave functions satisfy

$$\Theta^{j,M}(-z) = \Theta^{M-j,M}(z). \tag{44}$$

Note that $\Theta^{0,M}(z) = \Theta^{M,M}(z)$. Hence, the Z_2 eigenstates are written as [19]

$$\Theta^{j,M}_{\pm}(z) = \frac{1}{\sqrt{2}}(\Theta^{j,M}(z) \pm \Theta^{M-j,M}(z)) \tag{45}$$

for $j \neq 0, M/2, M$. The wave functions $\Theta^{j,M}(z)$ for $j = 0$, $M/2$ are the Z_2 eigenstates with the Z_2 even parity. Either of $\Theta^{j,M}_{+}(z)$ and $\Theta^{j,M}_{-}(z)$ is projected out by the orbifold projection. Odd wave functions can also correspond to massless modes in the magnetic flux background, unlike the orbifold without magnetic flux, where any odd modes correspond to not massless modes, but massive modes. Before orbifolding, the number of zero modes is equal to the magnetic flux M. For example, we have to choose $M = 3$ in order to realize the three families. On the other hand, the number of zero modes on the orbifold also depends on the boundary conditions under the Z_2 twist, i.e. even or odd functions. For $M =$ even, the number of zero modes with even (odd) functions are equal to $M/2 + 1$ $(M/2 - 1)$. For $M =$ odd, the number of zero modes with even and odd functions are equal to $(M+1)/2$ and $(M-1)/2$, respectively. These results are shown in Table I. For example, when we choose even (odd) functions, the three families can be realized for $M = 4$ and 5 (7 and 8). Thus, we can obtain various three-family models in magnetized orbifold models, and those have a richer flavor structure than torus

TABLE I. The numbers of zero modes for even and odd wave functions.

	$M =$ even	$M =$ odd
Even zero modes	$M/2 + 1$	$(M+1)/2$
Odd zero modes	$M/2 - 1$	$(M-1)/2$

models with magnetic fluxes. Yukawa couplings among $\Theta^{i,M_1}_{\pm}(z)\Theta^{j,M_2}_{\pm}(z)(\Theta^{k,M_3}_{\pm}(z))^*$ are computed by use of Eq. (23).

Now, let us study the models with nontrivial orbifold twists and Wilson lines. We consider $U(1)_a \times SU(2)$ theory as the simplest example. Then we introduce magnetic flux in $U(1)_a$ like Eq. (33). In addition, we embed the Z_2 twist P into the $SU(2)$ gauge space. For example, we consider the $SU(2)$ doublet

$$\begin{pmatrix} \lambda_{1/2} \\ \lambda_{-1/2} \end{pmatrix}, \tag{46}$$

with the $U(1)_a$ charge q_a. We embed the Z_2 twist P in the gauge space as

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{47}$$

for the doublet. Obviously, we can diagonalize P as $P' = \mathrm{diag}(1, -1)$, if there is no Wilson line along another $SU(2)$ directions. However, later we will introduce a Wilson line along the Cartan direction of $SU(2)$ in the P basis. Thus, we use the above basis for P. For the $SU(2)$ gauge sector, there is no effect due to the magnetic flux. Then, in this sector the situation is the same as one on the orbifold without magnetic flux. The $SU(2)$ gauge group is broken completely by nontrivial orbifold twists and Wilson lines, that is, all of $SU(2)$ vector multiplets become massive.

Before orbifolding, the $SU(2)$ is not broken and both $\lambda_{1/2}$ and $\lambda_{-1/2}$ have $M = q_a m_a$ independent zero modes, which we denote by $\Theta^{j,M}_{1/2}(z)$ and $\Theta^{j,M}_{-1/2}(z)$, respectively. Here, we have put the indices, $1/2$ and $-1/2$ in order to make it clear that they correspond to $\lambda_{1/2}$ and $\lambda_{-1/2}$, respectively, although the forms of wave functions are the same, i.e. $\Theta^{j,M}_{1/2}(z) = \Theta^{j,M}_{-1/2}(z)$. When we impose the orbifold boundary conditions with the above P in (47), the zero modes on the orbifold without Wilson lines are written as

$$\frac{1}{\sqrt{2}}(\Theta^{j,M}_{1/2}(z) + \Theta^{M-j,M}_{-1/2}(z)) \tag{48}$$

for $j = 0, \cdots, M-1$. Note that there are M independent zero modes.[7]

[7]It may be useful to explain remaining zero modes in the basis for P'. Before orbifolding, both $\lambda'_{1/2}$ and $\lambda'_{-1/2}$ have $M = q_a m_a$ independent zero modes in the basis for P'. Then by orbifolding with P', even modes $\Theta'^{j,M}_{+}(z)$ corresponding to (45) remain for $\lambda'_{1/2}$, while $\lambda'_{-1/2}$ has only odd modes $\Theta'^{j,M}_{-}(z)$. Their total number is equal to M.

Then, we introduce the Wilson lines [33] along the Cartan direction

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{49}$$

in the P basis. The corresponding zero-mode wave functions are shifted as

$$\frac{1}{\sqrt{2}}(\Theta^{j,M}_{1/2}(z + C^b/2M) + \Theta^{M-j,M}_{-1/2}(z - C^b/2M)) \tag{50}$$

for $j = 0, \cdots, M-1$, where C^b is a continuous parameter. Note that $\lambda_{1/2}$ and $\lambda_{-1/2}$ have opposite charges under the $SU(2)$ Cartan element. Then, their wave functions are shifted to opposite directions by the same Wilson lines C^b as $\Theta^{j,M}_{1/2}(z + C^b/2M)$ and $\Theta^{j,M}_{-1/2}(z - C^b/2M)$ without changing the number of zero modes. We can also consider another Z_2 twist P in the doublet such that the following wave function

$$\frac{1}{\sqrt{2}}(\Theta^{j,M}_{1/2}(z + C^b/2M) - \Theta^{M-j,M}_{-1/2}(z - C^b/2M)) \tag{51}$$

remains.

The above aspect would be important to applications for particle phenomenology. We compute Yukawa couplings among two $SU(2)$ doublet fields and a singlet field, i.e. $(\lambda^1_{1/2}, \lambda^1_{-1/2})^T$, $(\lambda^2_{1/2}, \lambda^2_{-1/2})^T$ and λ^3_0. We assume that two $SU(2)$ doublet fields have $U(1)_a$ charges q^1_a and q^2_a, while the singlet field has the $U(1)_a$ charge q^3_a. We introduce the magnetic flux m_a in $U(1)_a$ and the same $SU(2)$ Wilson line as the above. Then, the zero-mode wave functions of two $SU(2)$ doublets and the singlet can be obtained on the orbifold as

$$\begin{aligned}
&\frac{1}{\sqrt{2}}(\Theta^{i,M_1}_{1/2}(z + C^b/2M_1) + \Theta^{M_1-i,M_1}_{-1/2}(z - C^b/2M_1)) \\
&\qquad \text{from} \begin{pmatrix} \lambda^1_{1/2} \\ \lambda^1_{-1/2} \end{pmatrix}, \\
&\frac{1}{\sqrt{2}}(\Theta^{j,M_2}_{1/2}(z + C^b/2M_2) + \Theta^{M_2-j,M_2}_{-1/2}(z - C^b/2M_2)) \\
&\qquad \text{from} \begin{pmatrix} \lambda^2_{1/2} \\ \lambda^2_{-1/2} \end{pmatrix}, \\
&\frac{1}{\sqrt{2}}(\Theta^{k,M_3}_0(z) + \Theta^{M_3-k,M_3}_0(z))^* \quad \text{from } \lambda^3_0,
\end{aligned} \tag{52}$$

where $M_i = q^i_a m_a$. Note that the Wilson line C^b has no effect on the wave functions of the $SU(2)$ singlet field λ^3_0 because λ^3_0 has no $SU(2)$ charges. Here, we have taken the same orbifold projection P as Eq. (47), but we can study other orbifold projections. Then, their Yukawa couplings are obtained by the following overlap integral:

$$\begin{aligned}
\frac{1}{2\sqrt{2}}\int d^2z\{&\Theta^{i,M_1}_{1/2}(z + C^b/2M_1)\Theta^{M_2-j,M_2}_{-1/2}(z - C^b/2M_2) \\
&\times (\Theta^{k,M_3}_0(z) + \Theta^{M_3-k,M_3}_0(z))^* \\
&+ \Theta^{M_1-i,M_1}_{-1/2}(z - C^b/2M_1)\Theta^{j,M_2}_{1/2}(z + C^b/2M_2) \\
&\times (\Theta^{k,M_3}_0(z) + \Theta^{M_3-k,M_3}_0(z))^*\}.
\end{aligned} \tag{53}$$

Note that the six-dimensional bulk Lagrangian includes only the terms corresponding to $\lambda^1_{1/2}\lambda^2_{-1/2}\lambda^3_0$ and $\lambda^1_{-1/2}\lambda^2_{1/2}\lambda^3_0$. This integral is computed as

$$\begin{aligned}
\sum_{m\in \mathbf{Z}_{M_3}} &(\delta_{i-j+M_1m,k} + \delta_{i-j+M_1m,-k})\vartheta\begin{bmatrix} \frac{M_2 i + M_1 j + M_1M_2 m}{M_1M_2M_3} \\ 0 \end{bmatrix} \\
&\times (C^b(M_1+M_2)/2, \tau M_1M_2M_3) \\
&\times (e^{i\pi C^b\,\mathrm{Im}C^b(1/M_1 - 1/M_2)/(4\,\mathrm{Im}\tau)} \\
&+ e^{i\pi C^b\,\mathrm{Im}C^b(1/M_2 - 1/M_1)/(4\,\mathrm{Im}\tau)})
\end{aligned} \tag{54}$$

up to the normalization factor $N_1N_2/(2\sqrt{2}N_3)$, where the Kronecker delta $\delta_{i-j+M_1m,k}$ in the first term means $i - j + M_1m = k$ modulo M_3 and others are defined similarly. Obviously, the result depends nontrivially on the Wilson line C^b. Thus, the Wilson lines have important effects on the Yukawa couplings.

For comparison, we study another dimensional representation, e.g. a triplet

$$\begin{pmatrix} \lambda_1 \\ \lambda_0 \\ \lambda_{-1} \end{pmatrix}, \tag{55}$$

with the $U(1)_a$ charge q_a. Suppose that we embed the Z_2 twist P in the three dimensional gauge space as

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{56}$$

for the triplet. Then, zero modes on the orbifold are written as

$$\Theta^{j,M}_1(z) + \Theta^{M-j,M}_{-1}(z), \qquad \Theta^{j,M}_0(z) + \Theta^{M-j,M}_0(z), \tag{57}$$

up to the normalization factor $1/\sqrt{2}$. The former corresponds to λ_1 and λ_{-1} and there are M zero modes. The latter corresponds to λ_0 and there are $(M/2+1)$ zero modes and $(M+1)/2$ zero modes when M is even and odd, respectively. When we introduce the continuous Wilson lines along the Cartan direction, the wave functions of these zero modes shift as

$$\begin{aligned}
&\Theta^{j,M}_1(z + C_b/M) + \Theta^{M-j,M}_{-1}(z - C_b/M), \\
&\Theta^{j,M}_0(z) + \Theta^{M-j,M}_0(z),
\end{aligned} \tag{58}$$

up to the normalization factor $1/\sqrt{2}$.

Similarly to the above, here let us compute the Yukawa couplings among the triplet $(\lambda_1^3, \lambda_0^3, \lambda_{-1}^3)^T$ and the two $SU(2)$ doublets, whose zero modes are obtained in Eq. (52). In particular, we compute the couplings including λ_1^3 and λ_{-1}^3, whose zero-mode wave functions are obtained by

$$\frac{1}{\sqrt{2}}(\Theta_1^{k,M_3}(z + C_b/M_3) + \Theta_{-1}^{M_3-k,M_3}(z - C_b/M_3))^*, \tag{59}$$

with $M_3 = q_a^3 m_a$ after orbifolding. Their Yukawa couplings are obtained by the following overlap integral:

$$\begin{aligned} \frac{1}{2\sqrt{2}}\int d^2z\{&\Theta_{1/2}^{i,M_1}(z + C^b/2M_1)\Theta_{1/2}^{j,M_2}(z + C^b/2M_2) \\ &\times (\Theta_{-1}^{M_3-k,M_3}(z - C_b/M_3))^* \\ &+ \Theta_{-1/2}^{M_1-i,M_1}(z - C^b/2M_1)\Theta_{-1/2}^{M_2-j,M_2}(z - C^b/2M_2) \\ &\times (\Theta_1^{k,M_3}(z + C_b/M_3))^*\}. \end{aligned} \tag{60}$$

This integral is computed as

$$\begin{aligned} \sum_{m\in \mathbf{Z}_{M_3}} \delta_{i+j+M_1m,k}\,&\vartheta\begin{bmatrix} \frac{M_2i-M_1j+M_1M_2m}{M_1M_2M_3} \\ 0 \end{bmatrix} \\ &\times (C^b(M_2 - M_1)/2, \tau M_1M_2M_3) \\ &\times (e^{i\pi C^b\, \mathrm{Im} C^b(1/M_1+1/M_2-4/M_3)/(4\,\mathrm{Im}\tau)} \\ &+ e^{i\pi C^b\, \mathrm{Im} C^b(4/M_3-1/M_1-1/M_2)/(4\,\mathrm{Im}\tau)}), \end{aligned} \tag{61}$$

up to the normalization factor $N_1N_2/(2\sqrt{2}N_3)$. This result is different from Eq. (54), in particular, from the viewpoint of Wilson line dependence. Thus, the Wilson lines have phenomenologically important effects, depending on the directions of Wilson lines and the representations of matter fields.

We can extend the above analysis to larger gauge groups. Here, we show a rather simple example. We consider $U(1)_a \times SU(3)$ theory with the magnetic flux in $U(1)_a$ like Eq. (33). Then, we consider the $SU(3)$ triplet

$$\begin{pmatrix} \lambda_0 \\ \lambda_{1/2} \\ \lambda_{-1/2} \end{pmatrix}, \tag{62}$$

with the $U(1)_a$ charge q_a, where the subscripts $(0, 1/2, -1/2)$ denote the $U(1)_b$ charge along one of $SU(3)$ Cartan directions. Now, we embed the Z_2 twist P in the gauge space as

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{63}$$

for the triplet. In addition, we introduce the Wilson line C^b along the $U(1)_b$ direction. The gauge group is broken as $SU(3) \to U(1)$.[8] There are M zero modes for linear combinations of $\lambda_{1/2}$ and $\lambda_{-1/2}$ with the wave functions

$$\Theta_{1/2}^{j,M}(z + C^b/2M) + \Theta_{-1/2}^{M-j,M}(z - C^b/2M), \tag{64}$$

up to the normalization factor. Also, the zero modes for λ_0 are written as

$$\Theta_0^{j,M}(z) + \Theta_0^{M-j,M}(z), \tag{65}$$

up to the normalization factor. The number of zero modes is equal to $(M/2 + 1)$ and $(M + 1)/2$ when M is even and odd, respectively. Thus, the situation is almost the same as the above $SU(2)$ case with the triplet. Although the above example is rather simple, we can consider various types of breaking for larger groups. For example, when the gauge group includes two or more $SU(2)$ subgroups, we could embed the Z_2 twist in two of $SU(2)$'s and introduce independent Wilson lines along their Cartan directions. Similarly, we can investigate such models and other types of various embedding of P and Wilson lines.

In Sec. II B, we have considered ten-dimensional theory on T^6. Also, we can consider the T^6/Z_2 orbifold, where the Z_2 twist acts as

$$Z_2\colon z_1 \to -z_1, \qquad z_2 \to -z_2, \qquad z_3 \to z_3. \tag{66}$$

For T_1^2 and T_2^2, we can introduce the type of Wilson lines, which we have considered in this section, while for T_3^2 we can introduce the type of Wilson lines, which are considered in the previous section. Then, we have a richer structure of models on the T^6/Z_2 orbifold. Furthermore, we could consider another independent Z_2' twist as

$$Z_2'\colon z_1 \to -z_1, \qquad z_2 \to z_2, \qquad z_3 \to -z_3 \tag{67}$$

on the $T^6/(Z_2 \times Z_2')$ orbifold. In this case, we can consider another independent embedding P' of Z_2' twist on the gauge space. Using these two Z_2 twist embedding and Wilson lines, we could construct various types of models. For example, when the gauge group includes two or more $SU(2)$ subgroups, we could embed P on one of $SU(2)$ and P' on the other $SU(2)$ and introduce independent Wilson lines along their Cartan directions. Other various types of model building would be possible. Thus, it would be interesting to study such model building elsewhere.

Finally, we comment on the flavor symmetry. Yukawa couplings as well as higher order couplings can be computed by use of Eq. (24). The orbifolding without Wilson lines is a procedure to choose eigenstates for $\mathcal{P}$ (32). Thus, there remains the flavor symmetry, which commutes with $\mathcal{P}$. The Z_g' symmetry (30) is commutable. The Z_g symmetry (27) is not commutable for $g =$ odd. However, when $g =$ even, the Z_2 symmetry, which is generated by $Z^{g/2}$ is

[8] This remaining $U(1)$ symmetry might be anomalous. If so, the remaining $U(1)$ would also be broken by the Green-Schwarz mechanism.

commutable with $\mathcal{P}$ and another Z_2 symmetry, which is generated by $C^{g/2}$, is also commutable with $\mathcal{P}$. For example, when $g = 4$, the generators, Z', Z^2 and C^2 are written as

$$Z' = \begin{pmatrix} i & & \\ & \ddots & \\ & & i \end{pmatrix}, \qquad Z^2 = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix},$$

$$C^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{68}$$

Here, the Z^2 and C^2 generators also commute with each other. Similarly, when $g/2 =$ even, the unbroken flavor symmetry would be obtained as $Z_g \times Z_2 \times Z_2 \times Z_2$.[9] On the other hand, when $g/2 = 3$, the generators, Z', Z^3 and C^3 are written as

$$Z' = \begin{pmatrix} \rho & & \\ & \ddots & \\ & & \rho \end{pmatrix},$$

$$Z^3 = \begin{pmatrix} 1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & 1 & \\ & & & & & -1 \end{pmatrix}, \tag{69}$$

$$C^3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

where $\rho = e^{\pi i/3}$. Here, the Z^3 and C^3 generators do not commute with each other. Thus, unbroken flavor symmetries are non-Abelian. Similarly, when $g/2 =$ odd, non-Abelian discrete flavor symmetries would remain.

[9]It is interesting to break non-Abelian flavor symmetries to Abelian symmetries by orbifolding [34].

When we introduce Wilson lines like (49), the $SU(2)$ gauge symmetry is broken at the same time as the Z_g symmetry breaking for (27). Thus, we may expect that some nontrivial linear combinations of broken Z_g and $SU(2)$ would remain. However, only the Z_2 symmetry, which is already included above, seems to remain, e.g. in the states (48). When we consider more complicated models, a new type of flavor symmetries, which are linear combinations of broken flavor symmetries and gauge symmetries, may remain. Hence, it would be interesting to investigate such models.

IV. CONCLUSION AND DISCUSSION

We have studied torus/orbifold models with magnetic fluxes and Wilson lines. These backgrounds lead to various different aspects for a particle phenomenology like the number of zero modes, their profiles, breaking patterns of flavor symmetries, etc. It would be quite interesting to construct concrete models by use of these backgrounds. We would study them elsewhere.

In addition to continuous Wilson lines studied in this paper, we can introduce discrete Wilson lines on the orbifold without magnetic fluxes, which break the gauge group without reducing its rank. It is quite important to study the possibility for introducing such discrete Wilson lines in the magnetic background and study their phenomenological implications. Furthermore, it is also important to analyze (systematically) which types of backgrounds and boundary conditions are possible in generic case.

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