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Effective actions at finite temperature

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This is a more detailed version of our recent paper where we proposed, from first principles, a direct method for evaluating the exact fermion propagator in the presence of a general background field at finite temperature. This can, in turn, be used to determine the finite temperature effective action for the system. As applications, we discuss the complete one loop finite temperature effective actions for $0 + 1$ dimensional QED as well as for the Schwinger model in detail. These effective actions, which are derived in the real time (closed time path) formalism, generate systematically all the Feynman amplitudes calculated in thermal perturbation theory and also show that the retarded (advanced) amplitudes vanish in these theories. Various other aspects of the problem are also discussed in detail.

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I. INTRODUCTION

The effective action for a system of fermions interacting with a background field, which incorporates all the one loop corrections in the theory, is an important fundamental concept in quantum field theory. At zero temperature we know that the n-point amplitudes (involving the background fields) at one loop are, in general, divergent and, consequently, the evaluation of the effective action at $T = 0$ needs a regularization. Effective actions can, of course, be evaluated perturbatively. However, a beautiful method due to Schwinger [1], also known as the proper time formalism, is quite useful in evaluating one loop effective actions at zero temperature with a gauge invariant regularization (in the case of gauge backgrounds). We note that the effective action for a fermion with mass m interacting with a general background is given by

$$\Gamma_{\rm eff}[A] = -i\,{\rm Tr}\,\ln(i\partial - m - gA) = -i\,{\rm Tr}\,\ln H, \qquad (1)$$

where A denotes the background field and g represents the coupling to the background, and we have identified

$$H = i\partial - m - gA. \qquad (2)$$

Here we have suppressed the Lorentz structure of the kinetic term as well as the background to allow for generality. For example, the background field can be a scalar or a gauge background. Similarly, the system under consideration may be a ($0 + 1$ dimensional) quantum mechanical system, in which case the fermion kinetic term will have no structure (or contraction with Dirac gamma matrices).

Schwinger expressed the effective action (1) in a regularized integral form,

$$\Gamma_{\rm eff}[A] = \lim_{\nu\to 0} i\int_0^\infty \frac{d\tau}{\tau^{1-\nu}}\,{\rm Tr}e^{-\tau H}, \qquad (3)$$

where τ is known as the "proper time" parameter. The idea here is that the operator $e^{-\tau H}$ in the integrand can be thought of as the evolution operator in the Euclidean time τ, with H denoting the (Hamiltonian) generator for the evolution. (The proper time can also be made Minkowskian with an appropriate $i\epsilon$ prescription.) As a result, we can write the proper time evolution equations

$$\frac{dx^\mu}{d\tau} = -i[x^\mu, H], \qquad \frac{dp_\mu}{d\tau} = -i[p_\mu, H]. \qquad (4)$$

If these equations can be solved and $x^\mu(\tau)$ [or $p_\mu(\tau)$] can be determined in a closed form, then one can evaluate the trace in (3) in the eigenbasis $|x^\mu(\tau)\rangle$ [or $|p_\mu(\tau)\rangle$] and evaluate the (gauge invariant) regularized effective action in a closed form as well (or at least give an integral representation for it). This has been profitably used to calculate the imaginary part of the effective action for fermions interacting with a constant background electromagnetic field which describes the decay rate of the vacuum [1]. However, solving the dynamical equations in (4) is, in general, not easy when interactions are present. When the dynamical equations cannot be solved in a closed form, the method due to Schwinger leads to a perturbative determination of the effective action.

In the past couple of decades, there have been several attempts [2,3] to generalize the method due to Schwinger to finite temperature [4,5] and to determine the imaginary part of the effective action leading to conflicting results [2]. In [6] we have presented an alternative method for determining finite temperature effective actions for fermions interacting with an arbitrary background field. We believe that since the amplitudes at finite temperature are ultraviolet finite unlike those at zero temperature, it is not necessary to generalize the method due to Schwinger to finite temperature. After all, the proper time method was designed to provide a (gauge invariant) regularization which is not necessary at finite temperature. Therefore, we have

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proposed [6] a direct method for evaluating finite temperature effective actions based mainly on the general properties of systems at finite temperature. In this paper we give a detailed description of this method along with various other aspects not discussed in [6].

As we have emphasized in [6], we believe that the real time formalism [5] (we use the closed time path formalism due to Schwinger [7]) is more suited for this purpose. We note that, in general, the imaginary time formalism (the Matsubara formalism [8]) leads naturally to retarded and advanced amplitudes, but the Feynman (time ordered) amplitudes (beyond the two point function) cannot be consistently generated in this formalism [9]. (We emphasize that this is true for space-time dimensions $d \geq 2$. In $0+1$ dimension the Feynman and retarded amplitudes coincide and, therefore, the imaginary time formalism naturally leads to Feynman amplitudes.) On the other hand, the effective action that we are interested in is precisely the one that generates Feynman amplitudes. In contrast to the imaginary time formalism, the effective action, when evaluated properly in the real time formalism, leads naturally not only to the Feynman amplitudes, but also to the retarded and advanced amplitudes as we will show in examples. Furthermore, as we have emphasized earlier in [5,10], the real time calculations can be carried out quite easily in the mixed space where the spatial coordinates have been Fourier transformed, as we will describe in the following examples.

The present paper is organized as follows. In Sec. II we recapitulate our proposal [6] for evaluating effective actions at finite temperature. In Sec. III we apply the method to evaluating the complete effective action at finite temperature for the $0+1$ dimensional QED. From the structure of this effective action, we show that all the temperature dependent retarded (advanced) amplitudes vanish. The temperature dependent effective action for the Schwinger model, the $1+1$ dimensional massless QED, is discussed in detail in Sec. IV where we show that the temperature dependent retarded (advanced) amplitudes vanish in this theory as well. We present our conclusions and summarize future directions in Sec. V.

II. PROPOSAL

From the definition of the effective action (1) for a system of massive fermions interacting with an arbitrary background, it is straightforward to obtain

$$\frac{\partial \Gamma_{\text{eff}}}{\partial m} = \int dt d\mathbf{x} S(t, \mathbf{x}; t, \mathbf{x}), \tag{5}$$

where $S(t, \mathbf{x}; t', \mathbf{x}')$ denotes the complete Feynman propagator for the fermion (including the factor i) in the presence of the background field. However, keeping in mind that the fermion may not always have a mass (say, for example, in the Schwinger model [11]), we use, alternatively, the fact that the variation of the effective action with respect to the background field leads to the generalized fermion "propagator" at coincident points [even though we use the same symbol as in (5), the exact meaning of S below depends on the nature of the background field as we explain],

$$\frac{\delta \Gamma_{\text{eff}}}{\delta A(t, \mathbf{x})} = g S(t, \mathbf{x}; t, \mathbf{x}), \tag{6}$$

where we are suppressing the Lorentz structure of the background field as well as that of the generalized propagator. We note that for a scalar background, S in (6) indeed denotes the complete fermion propagator of the interacting theory at coincident coordinates. On the other hand, for a gauge field background, the right-hand side in (6) determines the current density of the theory which is related to the complete fermion propagator of the theory through a Dirac trace involving the Dirac matrix. In either case, we note that it is the fermion propagator that is relevant in (6) for the evaluation of the effective action. We note that in the mixed space (where the spatial coordinates $\mathbf{x}$ have been Fourier transformed), we can write (6) as

$$\frac{\delta \Gamma_{\text{eff}}}{\delta A(t, -\mathbf{p})} = g S(t, t; \mathbf{p}). \tag{7}$$

Since the effective action is so intimately connected with the fermion propagator, our proposal is to determine the complete fermion propagator at finite temperature directly such that

(i) it satisfies the appropriate equations for the complete propagator of the theory,

(ii) it satisfies the necessary symmetry properties of the theory such as the Ward identity,

(iii) and most importantly, it satisfies the antiperiodicity property associated with a finite temperature fermion propagator [5].

In fact, it is the third requirement that is quite important in a direct determination of the propagator. We note that this last condition is missing at zero temperature, which makes it difficult to determine the complete propagator (independent of the problem of divergence). When the theory is free of ultraviolet divergence (so that it does not need a regularization at zero temperature), this propagator will be the exact fermion propagator of the theory and would lead to the complete effective action including the correct zero temperature part. On the other hand, if the theory needs to be regularized at zero temperature, this propagator will not yield the correct zero temperature effective action. However, we note that our interest is in the finite temperature part of the effective action which does not need to be regularized (it is not ultraviolet divergent) and will be determined correctly in this approach. We illustrate the method with two examples.

III. 0 + 1 DIMENSIONAL QED

Let us consider the 0 + 1 dimensional QED described by the Lagrangian

$$L = \bar{\psi}(t)(i\partial_t - m - eA(t))\psi(t), \tag{8}$$

where the fermion mass can be thought of as a chemical potential, and in 0 + 1 dimension, the gauge potential has only a single component (which we suppress). There are no Dirac matrices, and as a result, the gauge background behaves like a scalar background. This is a simple model which has been studied exhaustively [12,13] in connection with large gauge invariance [14] at finite temperature, but it is also quite useful in clarifying various concepts involved in our proposal before we generalize it to higher dimensions.

A. Finite temperature propagator

As we noted earlier, we use the closed time path formalism where the path in the complex time plane has the form shown in Fig. 1. In the closed time path formalism (in any real time formalism) [5], the degrees of freedom need to be doubled, and we denote the background fields on the $C_\pm$ branches of the contour as $A_\pm(t)$, respectively. The two branches labeled by $C_\pm$ lead to the doubling of the degrees of freedom, while the branch $C_\perp$ along the imaginary axis decouples from any physical amplitude.

Since t is the only coordinate on which field variables depend in this theory, there is no need for a mixed space propagator. We note that the complete fermion propagator of the theory (ordered along the contour in Fig. 1) satisfies the equations

$$\begin{aligned}(i\partial_t - m - eA_c(t))S_c(t,t') &= i\delta_c(t-t'),\\ S_c(t,t')(i\overleftarrow{\partial}_{t'} + m + eA_c(t')) &= -i\delta_c(t-t'),\end{aligned} \tag{9}$$

where the subscript "c" characterizes a function on the contour. On the contour, the step function is defined naturally as [5]

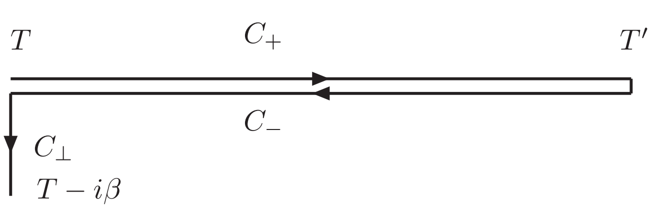

FIG. 1. The closed time path contour in the complex t plane. Here $T \to -\infty$, while $T' \to \infty$ and β denotes the inverse temperature (in units of the Boltzmann constant k) [5].

$$\theta_c(t-t') = \begin{cases} \theta(t-t') & \text{if both } t, t' \in C_+ \\ \theta(t'-t) & \text{if both } t, t' \in C_- \text{ or } \in C_\perp \\ 1 & \text{if } t \in C_- (\text{or } \in C_\perp) \text{ and } t' \in C_+ \\ 0 & \text{if } t \in C_+ \text{ and } t' \in C_- (\text{or } \in C_\perp), \end{cases} \tag{10}$$

and the delta function on the contour is defined in the standard manner as

$$\delta_c(t-t') = \partial_t\theta_c(t-t'). \tag{11}$$

Equations (9) can be solved exactly, subject to our three requirements in the following manner. First we note that a general solution of (9) can be written (in the contour ordered form) as

$$\begin{aligned}S_c(t,t') = {}& (\theta_c(t-t')C - \theta_c(t'-t)D)\\ & \times e^{-im(t-t') - ie\int_{t'}^{t} dt''_c A_c(t''_c)},\end{aligned} \tag{12}$$

where C and D are independent of t, t', satisfying

$$C + D = 1, \tag{13}$$

in order for (12) to satisfy (9). If we now impose the antiperiodicity condition

$$S_c(-\infty, t') = -S_c(-\infty - i\beta, t'), \tag{14}$$

where β is the inverse of temperature in units of the Boltzmann constant (k), we determine

$$D = Ce^{-\beta m - ie\int_{-\infty}^{-\infty - i\beta} dt'_c A_c(t'_c)} = Ce^{-\beta m - ie(a_+ - a_-)}. \tag{15}$$

Here we have defined

$$a_\pm = \int_{-\infty}^{\infty} dt A_\pm(t), \tag{16}$$

and have used the fact that the vertical branch decouples from amplitudes.

Together with (13) the relation in (15) determines

$$\begin{aligned}C &= 1 - n_{\rm F}\Big(m + \frac{ie}{\beta}(a_+ - a_-)\Big),\\ D &= n_{\rm F}\Big(m + \frac{ie}{\beta}(a_+ - a_-)\Big),\end{aligned} \tag{17}$$

where $n_{\rm F}$ denotes the Fermi distribution function and leads to the contour ordered propagator of the form

$$\begin{aligned}S_c(t,t') = {}& e^{-imt - ie\int d\bar{t}\theta_c(t-\bar{t})A_c(\bar{t})}\Big(\theta_c(t-t') - n_{\rm F}\Big(m + \frac{ie}{\beta}\\ & \times (a_+ - a_-)\Big)\Big)e^{imt' + ie\int d\bar{t}\theta_c(t'-\bar{t})A_c(\bar{t})}.\end{aligned} \tag{18}$$

In fact, this propagator satisfies antiperiodicity in both its arguments, namely, (14) as well as the condition

$$S_c(t, -\infty) = -S_c(t, -\infty - i\beta). \tag{19}$$

Although the phases in (18) can be combined to write them in a simpler form as in (12), in this case, we have chosen to write them in this suggestive form which generalizes naturally to higher dimensions where the propagator will carry spinor indices. When t, t' are restricted to the appropriate branches of the contour, (18) determines all the components of the full 2×2 matrix propagator of the theory,

$$\begin{aligned} S_{++}(t,t') &= (\theta(t-t') - D)e^{-im(t-t')-ie\int_{t'}^{t} dt'' A_+(t'')}, \\ S_{+-}(t,t') &= -De^{-im(t-t')+ie(\int_t^\infty dt'' A_+(t'')+\int_{t'}^\infty dt'' A_-(t''))}, \\ S_{-+}(t,t') &= Ce^{-im(t-t')-ie(\int_t^\infty dt'' A_-(t'')+\int_{t'}^\infty dt'' A_+(t''))}, \\ S_{--}(t,t') &= -(\theta(t-t') - C)e^{-im(t-t')+ie\int_{t'}^{t} dt'' A_-(t'')}, \end{aligned} \tag{20}$$

where the constants C, D are defined in (17).

B. Lippmann-Schwinger equation

In addition to satisfying the antiperiodicity conditions, it can be checked that the propagator in (18) satisfies the required Ward identity. Furthermore, it also satisfies the Lippmann-Schwinger equation (the perturbation expansion) [15] for the propagator, which can be seen as follows. For simplicity of discussion, let us consider the propagator only in the C_+ branch of the contour in the complex t plane and factor out the exponential factor $e^{-im(t-t')}$ which trivially factors out in a product. In this case, we can write (suppressing the thermal index $+$)

$$S(t,t') = e^{-ie\Phi(t)}(\theta(t-t') - D)e^{ie\Phi(t')}, \tag{21}$$

where we have defined [D is defined in (17)]

$$\Phi(t) = \int dt'' \theta(t-t'')A(t''), \tag{22}$$

with

$$\Phi(\infty) = a = \int_{-\infty}^{\infty} dt A(t), \qquad \Phi(-\infty) = 0. \tag{23}$$

The propagator, in the absence of interactions, is given by

$$S_0(t,t') = \theta(t-t') - n_F(m). \tag{24}$$

The Lippmann-Schwinger equation can be written as

$$S(t,t') = (1 + ieS_0A)^{-1}S_0(t,t'), \tag{25}$$

which can also be expressed as

$$\begin{aligned} &(1 + ieS_0A)S(t,t') = S_0(t,t') \quad \text{or} \\ &S(t,t') - S_0(t,t') + ieS_0AS(t,t') = 0. \end{aligned} \tag{26}$$

Using the definition (22) we note that we can write

$$A(t) = \frac{d\Phi(t)}{dt}, \tag{27}$$

which leads to

$$ieS_0AS(t,t') = -\int dt'' S_0(t,t'') \frac{de^{-ie\Phi(t'')}}{dt''} \times (\theta(t''-t') - D)e^{ie\Phi(t')}. \tag{28}$$

Integrating the right-hand side of (28) (by parts) and using (23) as well as identities associated with Fermi distribution functions, we obtain

$$ieS_0AS(t,t') = -S(t,t') + S_0(t,t'). \tag{29}$$

Using this in (26) we conclude that the propagator (21) satisfies the Lippmann-Schwinger equation to all orders. This argument can be carried over to show that the complete 2×2 matrix propagator satisfies the Lippmann-Schwinger equation. In other words, the complete propagator (18) satisfies exactly the perturbative expansion (for the propagator) to all orders in the interaction at finite temperature.

C. Effective action

The $0 + 1$ dimensional theory is free from ultraviolet divergences and, therefore, (18) represents the complete fermion propagator of the theory in the presence of a background gauge field. We can now take the coincident limit ($t = t'$) in (18) to obtain

$$S_c(t,t) = \frac{1}{2}\left(1 - 2n_F\left(m + \frac{ie}{\beta}(a_+ - a_-)\right)\right). \tag{30}$$

It is interesting to note that in the coincident limit the propagator is independent of the time coordinate. This is, in fact, a consequence of the Ward identity of the theory (see [13]).

Using (30) we can integrate (6) or (7) (note that there is no momentum in this $0 + 1$ dimensional example) to obtain the normalized effective action of the theory, which has the form

$$\Gamma_{\text{eff}}[a_+, a_-] = -i \ln\left[\cos\frac{e(a_+ - a_-)}{2} + i \tanh\frac{\beta m}{2} \times \sin\frac{e(a_+ - a_-)}{2}\right]. \tag{31}$$

This is the complete effective action of the theory which reduces to the well studied action [12,13] on C_+ when we set $a_- = 0$. However, being the complete effective action, (31) contains all the information about retarded, advanced, and other amplitudes as well.

For example, from the structure of the complete effective action in (31) we can now show that all the retarded (advanced) amplitudes in this theory vanish at finite temperature. Let us recall that the retarded n-point amplitude in a theory can be expressed as [16]

$$\Gamma_R^{(n)} = \Gamma_{++\cdots+} + \Gamma_{+-+\cdots+} + \Gamma_{++-+\cdots+} \cdots + \cdots + \Gamma_{+--\cdots-+} + \Gamma_{+--\cdots--}. \tag{32}$$

Namely, it is the sum of all amplitudes, where the first index is held fixed to be + and all the $(n-1)$ other thermal indices are permuted over the two values $\pm$. Let us note from (31) that since $\Gamma_{\rm eff} = \Gamma_{\rm eff}[a_+ - a_-]$, it now follows from (32) that the retarded n-point amplitude of the theory can be written as [since the amplitudes are time independent, taking the derivative with respect to the background field $A_\pm(t)$ is equivalent to taking the derivative with respect to $a_\pm$]

$$\begin{aligned}\Gamma_R^{(n)} &= \sum_{m=0}^{n-1} {}^{n-1}C_m \frac{d^{n-1-m}}{da_+^{n-1-m}} \frac{d^m}{da_-^m} \frac{d\Gamma_{\rm eff}[a_+ - a_-]}{da_+}\bigg| \\ &= \sum_{m=0}^{n-1} {}^{n-1}C_m \frac{d^{n-1-m}}{da_+^{n-1-m}} \left(-\frac{d}{da_+}\right)^m \frac{d\Gamma_{\rm eff}[a_+ - a_-]}{da_+}\bigg| \\ &= (1-1)^{n-1} \frac{d^n\Gamma_{\rm eff}[a_+ - a_-]}{da_+^n}\bigg| = 0, \qquad n \geq 2, \end{aligned} \tag{33}$$

where the restriction stands for setting all the background fields to zero. Therefore, all the retarded (advanced) amplitudes vanish in this theory. (The one point amplitude is, by definition, a Feynman amplitude.)

IV. SCHWINGER MODEL

After the derivation of the complete effective action in the $0+1$ dimensional theory, let us next consider the fermion sector of the Schwinger model [11] or massless QED in $1+1$ dimensions described by the Lagrangian density

$$\mathcal{L} = \bar{\psi}(t,x)\gamma^\mu(i\partial_\mu - eA_\mu(t,x))\psi(t,x). \tag{34}$$

At zero temperature, this model is soluble and describes free massive photons. The effective action for this model (for an arbitrary gauge background) has also been studied perturbatively at finite temperature [17], even in the presence of a chemical potential [18]. Here we will derive the closed form expression of the finite temperature effective action following our method. We note here that the two point function in the Schwinger model needs to be regularized at zero temperature (this is the only nonvanishing amplitude in the Schwinger model at zero temperature) and, consequently, the zero temperature part of the effective action following from our propagator will not coincide with the regularized zero temperature effective action. However, our interest is in the finite temperature part of the effective action which is free from ultraviolet divergences. For completeness we note that the simple point-splitting regularization of the fermion propagator is sufficient to regularize the theory and can be carried out even in our method. However, we will not do this here since our main interest is in the finite temperature part of the effective action (which does not need a regularization).

The theory (34) is best studied in the natural basis of right-handed and left-handed fermion fields (although everything that we say can be carried out covariantly as well as in the presence of a chemical potential). Defining [19,20]

$$\begin{aligned} &\psi_{\rm R} = \frac{1}{2}(\mathbb{1} + \gamma_5)\psi, \qquad \psi_{\rm L} = \frac{1}{2}(\mathbb{1} - \gamma_5)\psi, \\ &x^\pm = \frac{x^0 \pm x^1}{2}, \qquad p_\pm = p_0 \pm p_1, \\ &\partial_\pm = \partial_0 \pm \partial_1, \qquad A_\pm = A_0 \pm A_1, \end{aligned} \tag{35}$$

the Lagrangian density (34) naturally decomposes into two decoupled sectors described by

$$\mathcal{L} = \psi_{\rm R}^\dagger(i\partial_+ - eA_+)\psi_{\rm R} + \psi_{\rm L}^\dagger(i\partial_- - eA_-)\psi_{\rm L}, \tag{36}$$

where $\psi_{\rm R}$, $\psi_{\rm L}$ denote only the component spinor fields (there is no spinor index left anymore). While the zero temperature regularization mixes the two sectors through the two point function (anomaly), at finite temperature we do not have divergences and, therefore, we do not expect the two sectors to mix. Therefore, we can study the finite temperature effective action in each of the two sectors separately.

A. Propagator

Let us consider the theory only in the sector of the right-handed fermions in (36). This is very much like the $0+1$ dimensional theory. However, there is one essential difference which makes the derivation much more difficult; namely, the field variables depend on two coordinates (t, x) or, equivalently, on (x^+, x^-). We would like to emphasize here that although we use the light-cone coordinates for simplicity, the theory is still quantized on the equal-time surface and the propagator is defined through the time ordered Green's function (namely, we do not use the statistical mechanics of the light-front [21]). As we mentioned earlier, the finite temperature derivations become a lot simpler in the mixed space. Thus, Fourier transforming the x^- coordinate, the action for the right-handed fermions takes the form (the conjugate variables to x^- should be written as p_-, k_-, which we write as p, k for simplicity)

$$\begin{aligned} S_{\rm R} = 2\int dx^+ \frac{dp}{2\pi}\psi_{\rm R}^\dagger(x^+, -p)\bigg[i\partial_+\psi_{\rm R}(x^+, p) \\ - e\int \frac{dk}{2\pi} A_+(x^+, p-k)\psi_{\rm R}(x^+, k)\bigg]. \end{aligned} \tag{37}$$

As a result, we recognize that the equations for the propagator will involve a convolution. They are best described by introducing the following operator notations for the propagator as well as the gauge potential:

$$\begin{aligned} S(x^+, x'^+; p, k) &= \langle p|\hat{S}(x^+, x'^+)|k\rangle, \\ A_+(x^+, p-k) &= \langle p|\hat{A}_+(x^+)|k\rangle. \end{aligned} \tag{38}$$

For example, the propagator equation following from (37),

$$i\partial_+ S(x^+, x'^+; p, k) - e\int \frac{dq}{2\pi} A_+(x^+, p-q) S(x^+, x'^+; q, k) = \frac{i}{2}\delta(x^+ - x'^+) 2\pi\delta(p-k), \tag{39}$$

can be written in the compact form

$$(i\partial_+ - e\hat{A}_+(x^+))\hat{S}(x^+, x^-) = \frac{i}{2}\delta(x^+ - x'^+), \tag{40}$$

with the operator notation. Here we have assumed the normalization of the momentum states to be

$$\langle p|k\rangle = 2\pi\delta(p-k). \tag{41}$$

With the operator notation in (38) the equations for the propagator ordered along the contour take the (operator) forms [see also (9)]

$$\begin{aligned}(i\partial_+ - e\hat{A}_c(x^+))\hat{S}_c(x^+, x'^+) &= \frac{i}{2}\delta_c(x^+ - x'^+),\\ \hat{S}_c(x^+, x'^+)(i\overleftarrow{\partial}'_+ + e\hat{A}_c(x'^+)) &= -\frac{i}{2}\delta_c(x^+ - x'^+).\end{aligned} \tag{42}$$

We note from (7) and (37) that, in the present case, we can identify

$$S_c(x^+, x'^+; p) = \int \frac{dk}{2\pi}\langle k+p|\hat{S}_c(x^+, x'^+)|k\rangle. \tag{43}$$

The general solution of (42) can be written in the form

$$\hat{S}_c(x^+, x'^+) = \tfrac{1}{4} e^{-ie\int d\bar{x}^+ \theta_c(x^+ - \bar{x}^+)\hat{A}_{+c}(\bar{x}^+)}(\mathrm{sgn}_c(x^+ - x'^+) + \hat{\mathcal{O}}) \times e^{ie\int d\bar{x}^+ \theta_c(x'^+ - \bar{x}^+)\hat{A}_{+c}(\bar{x}^+)}, \tag{44}$$

where

$$\mathrm{sgn}_c(x^+ - x'^+) = \theta_c(x^+ - x'^+) - \theta_c(x'^+ - x^+), \tag{45}$$

and $\hat{\mathcal{O}}$, which contains all the nontrivial information about interactions and temperature, is independent of the coordinates x^+, x'^+. Since x^-, x'^- have been Fourier transformed, the antiperiodicity can be imposed only at the level of matrix elements in the mixed space by requiring

$$\langle p|\hat{S}\Big(\frac{-\infty + x}{2}, x'^+\Big)|k\rangle = -e^{-(\beta p)/2}\langle p|\hat{S}\Big(\frac{-\infty - i\beta + x}{2}, x'^+\Big)|k\rangle. \tag{46}$$

The factor $e^{-(\beta p)/2}$ in (46) arises basically from the exponential factor in the Fourier transform. The requirement of antiperiodicity (46) determines

$$\begin{aligned}\hat{\mathcal{O}} &= 1 - 2(\hat{\mathcal{O}}_+ + 1)^{-1},\\ \hat{\mathcal{O}}_+ &= e^{(ie(\hat{a}_{+(+)} - \hat{a}_{+(-)}))/2} e^{(\beta\hat{K})/2} e^{(ie(\hat{a}_{+(+)} - \hat{a}_{+(-)}))/2},\end{aligned} \tag{47}$$

where $\hat{K}$ denotes the momentum operator satisfying

$$\hat{K}|p\rangle = p|p\rangle, \tag{48}$$

and [$(\pm)$ with the parentheses denote the thermal indices, while $+$ without the parentheses represents the light-cone component of the background field]

$$\hat{a}_{+(\pm)} = \int_{-\infty}^{\infty} dx^+ \hat{A}_{+(\pm)}(x^+). \tag{49}$$

Therefore, the contour ordered propagator satisfying the Ward identity as well as the appropriate antiperiodicity condition has the form

$$\hat{S}_c(x^+, x'^+) = \tfrac{1}{4} e^{-ie\int d\bar{x}^+ \theta_c(x^+ - \bar{x}^+)\hat{A}_{+c}(\bar{x}^+)}(\mathrm{sgn}_c(x^+ - x'^+) + 1 - 2(\hat{\mathcal{O}}_+ + 1)^{-1}) e^{ie\int d\bar{x}^+ \theta_c(x'^+ - \bar{x}^+)\hat{A}_{+c}(\bar{x}^+)}. \tag{50}$$

It can be checked that this propagator also satisfies the antiperiodicity condition on the second variable, namely,

$$\langle p|\hat{S}\Big(x^+, \frac{-\infty + x'}{2}\Big)|k\rangle = -e^{(\beta k)/2}\langle p|\hat{S}\Big(x^+, \frac{-\infty - i\beta + x'}{2}\Big)|k\rangle. \tag{51}$$

Therefore, it satisfies all the requirements of our proposal.

B. Lippmann-Schwinger equation

As in $0+1$ dimension, we can show that the complete propagator (50) satisfies the Lippmann-Schwinger equation. Once again, for simplicity, we will restrict ourselves to the propagator on the C_+ branch (and we suppress the thermal index $+$). Let us note from (50) that in the absence of interactions, the propagator at finite temperature can be written as

$$\begin{aligned}S_0(x^+, x'^+) &= \frac{1}{4}(\mathrm{sgn}(x^+ - x'^+) + \hat{O}_0)\\ &= \frac{1}{4}\Big(\mathrm{sgn}(x^+ - x'^+) + 1 - 2n_{\mathrm{F}}\Big(\frac{\hat{K}}{2}\Big)\Big).\end{aligned} \tag{52}$$

Furthermore, we define

$$\hat{\Phi}(x^+) = \int d\bar{x}^+ \theta(x^+ - \bar{x}^+)\hat{A}_+(\bar{x}^+), \tag{53}$$

satisfying

$$\hat{\Phi}(\infty) = \hat{a}_+ = \int dx^+ A_+(x^+), \qquad \hat{\Phi}(-\infty) = 0, \tag{54}$$

so that the complete propagator at finite temperature (50) can be written as

$$S(x^+, x'^+) = e^{-ie\hat{\Phi}(x^+)}\tfrac{1}{4}(\mathrm{sgn}(x^+ - x'^+) + \hat{O}) e^{ie\hat{\Phi}(x'^+)}, \tag{55}$$

where $\hat{O}$ is defined in (47).

In terms of these operators the Lippmann-Schwinger equation can be written as [we point out here that the unconventional factor of 2 in the interaction term is a consequence of the use of light-cone coordinates, which can also be understood from the factor of $\frac{1}{2}$ in Eq. (42)]

$$S(x^+, x'^+) = (1 + 2ieS_0A)^{-1}S_0(x^+, x'^+), \tag{56}$$

which also has the equivalent description

$$(1+2ieS_0A)S(x^+,x'^+)=S_0(x^+,x'^+) \text{ or}$$
$$S(x^+,x'^+)-S_0(x^+,x'^+)+2ieS_0AS(x^+,x'^+)=0. \tag{57}$$

Using the identity

$$\hat{A}_+(x^+) = \frac{d\hat{\Phi}(x^+)}{dx^+}, \tag{58}$$

we note that we can write

$$2ieS_0\hat{A}_+S(x^+, x'^+) = -2\int dx''^+ S_0(x^+, x''^+)\frac{de^{-ie\hat{\Phi}(x''^+)}}{dx''^+} \times \frac{1}{4}(\text{sgn}(x''^+ - x'^+) + \hat{O})e^{ie\hat{\Phi}(x'^+)}. \tag{59}$$

Integrating this by parts, as in the 0 + 1 dimensional case, it is straightforward to show that (57) holds; namely, the Lippmann-Schwinger equation holds so that the complete propagator agrees with its perturbative expansion to all orders.

C. Effective action

We note that the propagator (50) involves operators which do not commute, in general. Nonetheless, it can be checked that at the coincident point, $x^+ = x'^+$, the two exponential factors cancel each other and the propagator becomes independent of x^+, much like in 0 + 1 dimension. Let us recall that the effective action can be obtained from the propagator at the coincident limit [see, for example, (7)]. This coordinate independence is expected from the equations of motion (42) as well as from the Ward identity of the theory. However, since the cancellation is a bit more involved than in 0 + 1 dimension, we discuss this in some detail.

To show the cancellation of phases in (50) to all orders when $x^+ = x'^+$, let us recall the definitions in (38) and (53) which lead to

$$\langle k + p|\hat{\Phi}(x^+)|k\rangle = \Phi(x^+, p) = \int dx''^+ \theta_c(x^+ - x''^+)A_{+c}(x''^+, p). \tag{60}$$

With this, let us look at the contributions coming only from the phases in the propagator (50) or (55) at any fixed order N when $x^+ = x'^+$, namely,

$$S_c(x^+; p)| = S_c(x^+, x^+; p)| = \int \frac{dk}{8\pi}\langle k + p|e^{-ie\hat{\Phi}(x^+)}\hat{O}e^{ie\hat{\Phi}(x^+)}|k\rangle|. \tag{61}$$

Here the restriction stands for looking at only the Nth order terms coming from the exponentials. Introducing sets of complete momentum states, we can write this as [we suppress the x^+ dependence in $\Phi(x^+, p)$]

$$\begin{aligned} S_c(x^+; p)| = (-ie)^N \sum_{m=0}^{N} {}^NC_m(-1)^m \int \frac{dk}{8\pi}\frac{dp_1}{2\pi}\cdots \\ \times \frac{dp_{N+1}}{2\pi}2\pi\delta(p - p_1 - \cdots - p_{N+1}) \\ \times \Big(\prod_{i=1}^{m}\Phi(p_i)\Big)\Big(\prod_{j=m+2}^{N+1}\Phi(p_j)\Big) \\ \times \langle k + p_1 + \cdots + p_{m+1}|\hat{O}|k \\ + p_1 + \cdots + p_m\rangle, \end{aligned} \tag{62}$$

where we understand that the first product is unity for $m = 0$ while the second factor is unity for $m = N$. Redefining $k \to k - p_1 - \cdots - p_m$ followed by the transformation $p_1 \leftrightarrow p_{m+1}$, Eq. (62) can be written as

$$\begin{aligned} S_c(x^+; p)| = \int \frac{dk}{8\pi}\frac{dp_1}{2\pi}\cdots\frac{dp_{N+1}}{2\pi}2\pi\delta(p - p_1 - \cdots \\ - p_{N+1})\Phi(p_2)\Phi(p_3)\cdots\Phi(p_{N+1}) \\ \times \langle k + p_1|\hat{O}|k\rangle(-ie)^N \sum_{m=0}^{N} {}^NC_m(-1)^m. \end{aligned} \tag{63}$$

The sum in the last line of (63) simply vanishes because

$$\sum_{m=0}^{N} {}^NC_m(-1)^m = (1 - 1)^N = 0. \tag{64}$$

This shows that the contributions coming from the exponentials at any order N vanish when $x^+ = x'^+$ so that the phases do not contribute to the propagator in the coincident limit in spite of the fact that these now involve noncommuting operators. As a result, the propagator (50) can be written in the coincident limit as

$$S_c(x^+; p) = \int \frac{dk}{8\pi}\langle k + p|\hat{O}|k\rangle = \int \frac{dk}{2\pi}\langle k + p|\frac{1}{4}(1 - 2(\hat{\mathcal{O}}_+ + 1)^{-1})|k\rangle, \tag{65}$$

and since the operator $\hat{O}$ (47) is independent of coordinates, it follows that the propagator is independent of the coordinate x^+ in the coincident limit

$$\partial_+S_c(x^+; p) = 0. \tag{66}$$

This is consistent with the Ward identity of the theory and also follows from the equations of motion (42).

Once the coincident limit of the propagator is determined, we can integrate (7) to determine the normalized effective action in the right-handed sector in the following manner. We recall the definitions in (47) which lead, for example, to

$$\frac{d\hat{\mathcal{O}}_{+}}{d\hat{a}_{+}} = ie\hat{\mathcal{O}}_{+}. \tag{67}$$

Let us define

$$\hat{F} = \tfrac{1}{2}\ln\hat{\mathcal{O}}_{+}, \tag{68}$$

in terms of which we can write

$$\begin{aligned} \hat{\mathcal{O}}_{+} &= e^{2\hat{F}}, \\ \hat{\mathcal{O}} &= 1 - 2(\hat{\mathcal{O}}_{+} + 1)^{-1} = 1 - 2(e^{2\hat{F}} + 1)^{-1}. \end{aligned} \tag{69}$$

From the definition in (68) as well as (67), it follows that

$$\frac{d\hat{F}}{d\hat{a}_{+}} = \frac{1}{2\hat{\mathcal{O}}_{+}}\frac{d\hat{\mathcal{O}}_{+}}{d\hat{a}_{+}} = \frac{ie}{2}. \tag{70}$$

Using this, we can now derive the well-defined derivative

$$\begin{aligned} \frac{d\ln\cosh\hat{F}}{d\hat{a}_{+}} &= \frac{ie}{2}(e^{\hat{F}} + e^{-\hat{F}})^{-1}(e^{\hat{F}} - e^{-\hat{F}}) \\ &= \frac{ie}{2}(1 - 2(e^{2\hat{F}} + 1)^{-1}) \\ &= \frac{ie}{2}(1 - 2(\hat{\mathcal{O}}_{+} + 1)^{-1}) = \frac{ie}{2}\hat{\mathcal{O}}. \end{aligned} \tag{71}$$

Using these relations we can write the complete normalized effective action in the right-handed sector as

$$\begin{aligned} \Gamma_{\mathrm{R,eff}} &= -\frac{i}{2}\int\frac{dk}{2\pi}\langle k|\ln\cosh\hat{F} - \ln\cosh\frac{\beta\hat{K}}{4}|k\rangle \\ &= -\frac{i}{2}\int\frac{dk}{2\pi}\langle k|\ln\cosh\Big(\frac{1}{2}\ln\hat{\mathcal{O}}_{+}\Big) - \ln\cosh\frac{\beta\hat{K}}{4}|k\rangle. \end{aligned} \tag{72}$$

We note here that the perturbative expansion for the effective action is much easier to obtain from the propagator in (65).

The thermal part of this effective action has the right (delta function) structure that had already been observed in the perturbative calculation in the right-handed sector [17], which is a consequence of the Ward identity in the theory. (We remind the readers that we are not interested in the zero temperature part of this effective action which, as we have argued, would not correspond to the regularized action.) However, the expansion of this effective action on C_{+} (namely, setting $A_{+(-)} = 0$) does not quite agree with the perturbative result order by order. In fact, the difference already shows up in the quartic effective action. This is indeed very interesting and brings out the power of calculations in the mixed space which we have stressed repeatedly. The perturbative calculation [17,18] which was carried out in momentum space misses out on a class of terms because of some subtlety that is not present in the mixed space. Namely, a class of terms had been set to zero in the perturbative calculation because of the identity

$$\frac{1}{p(p+q)} + \frac{1}{q(p+q)} - \frac{1}{pq} = 0. \tag{73}$$

Although this identity is naively true, it does not hold when principal values are involved, which is the case in the perturbative calculation. The correct identity in this case is

$$\frac{1}{p(p+q)} + \frac{1}{q(p+q)} - \frac{1}{pq} = \pi^2\delta(p)\delta(q), \tag{74}$$

and with this correction, the perturbative effective action in the right-handed sector coincides exactly with the quartic effective action derived from (72). [It is worth pointing out here that the effective action (72) is complete compared with the one proposed in [6] which was not derived there.] We note here that in the leading order of the hard thermal loop approximation, when operators commute, this effective action coincides with that for the $0 + 1$ case for every value of the momentum with the identification $k = 2m$. This is easily seen by comparing (65) in the commuting limit with (30). (The extra factor of $\frac{1}{2}$ is associated with light-cone coordinates.)

The effective action for the left-handed sector can similarly be derived and is given by

$$\Gamma_{\mathrm{L,eff}} = -\frac{i}{2}\int\frac{dk}{2\pi}\langle k|\ln\cosh\Big(\frac{1}{2}\ln\hat{\mathcal{O}}_{-}\Big) - \ln\cosh\frac{\beta\hat{K}}{4}|k\rangle, \tag{75}$$

where k, p should be understood as the light-cone components conjugate to x^{+} (which should be written as k_{+}, p_{+}) and we have identified

$$\begin{aligned} \hat{\mathcal{O}}_{-} &= e^{(ie\hat{a}_{-})/2}e^{(\beta\hat{K})/2}e^{(ie\hat{a}_{-})/2}, \qquad \hat{a}_{-} = \hat{a}_{-(+)} - \hat{a}_{-(-)}, \\ \hat{a}_{-(\pm)} &= \int_{-\infty}^{\infty}dx^{-}\hat{A}_{-(\pm)}(x^{-}). \end{aligned} \tag{76}$$

Once again, the thermal part of this effective action has the right (delta function) structure as in the perturbative calculation, and the thermal part agrees with the perturbative result order by order when restricted to C_{+} [with the modification due to the subtlety discussed in (74)]. The finite temperature effective action for the $1 + 1$ dimensional fermion interacting with an arbitrary Abelian gauge background can, therefore, be obtained from

$$\Gamma_{\mathrm{eff}} = \Gamma_{\mathrm{R,eff}} + \Gamma_{\mathrm{L,eff}}, \tag{77}$$

and the thermal part of (77) leads to the correct perturbative result order by order on the branch C_{+} with the correction (74). Furthermore, since (77) represents the complete effective action and since it is a functional of $(\hat{a}_{\pm(+)} - \hat{a}_{\pm(-)})$ [see (72)–(75)], it can be checked as in

(33) that all the retarded (advanced) amplitudes vanish in this theory. This should be contrasted with the fact that this had been verified explicitly only up to the four-point function in perturbation theory [16].

V. SUMMARY

In summary, we have proposed an alternative method [6] for determining effective actions at finite temperature for fermions interacting with an arbitrary background field. This is done by determining the complete fermion propagator (in the closed time path formalism) directly by using the antiperiodicity condition appropriate at finite temperature. We have illustrated in detail how our proposal works with the examples of the $0 + 1$ dimensional QED as well as the Schwinger model. The next step in this direction would involve determining the effective action for massive QED in $1 + 1$ dimensions at finite temperature. This would lead directly to the finite temperature effective actions for the known solvable examples in four dimensional QED [1,22].

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