

The $E(2)$ particle

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(Received 2 September 2009; published 18 December 2009)

Recently it has been advocated [A. G. Cohen and S. L. Glashow, Phys. Rev. Lett. **97**, 021601 (2006)] that for describing nature within the minimal symmetry requirement, certain subgroups of the Lorentz group may play a fundamental role. One such group is $E(2)$ which induces a Lie algebraic noncommutative spacetime [M. M. Sheikh-Jabbari and A. Tureanu, Phys. Rev. Lett. **101**, 261601 (2008); arXiv:0811.3670] where translation invariance is not fully maintained. We have constructed a consistent structure of noncommutative phase space for this system, and furthermore we have studied an appropriate point particle action on it. Interestingly, the Einstein dispersion relation $p^2 = m^2$ remains intact. The model is constructed by exploiting a dual canonical phase space following the scheme developed by us earlier [S. Ghosh and P. Pal, Phys. Rev. D **75**, 105021 (2007)].

DOI: [10.1103/PhysRevD.80.125021](https://doi.org/10.1103/PhysRevD.80.125021)

PACS numbers: 11.30.Cp, 03.30.+p, 11.10.Nx

I. INTRODUCTION AND MOTIVATION

The symmetry principle and its realization in nature has played a fundamental role throughout the development of physics. One of the key players in this regard is the Poincaré group, the connected component of the Lorentz group along with spacetime translations. The Poincaré group is identified as the symmetry of nature by the special theory of relativity. It is the isometry group of the $3 + 1$ -dimensional Minkowski space, the arena of relativistic classical and quantum field theories. However, in recent years, small observed violations of the discrete symmetries, C (charge conjugation), P (parity), and T (time reversal) at high energies, as well as the theoretical possibility of violation of Lorentz symmetry, again at high energy, have given rise to a new paradigm: an appropriate subgroup of the Lorentz group, together with spacetime translations, might be sufficient to explain the (so far) observed nature. The underlying criterion is that the subgroup together with C , P , and T will generate the full Lorentz group. In this scheme, the departures from the discrete symmetries and (predicted) Lorentz invariance get connected. This is the principal idea of very special relativity (VSR) by Cohen and Glashow [1], (for later works see [2]), who identify the subgroups as $T(2)$ (isomorphic to two-dimensional translations), $E(2)$ (isomorphic to 3-parameter Euclidean motion), $HOM(2)$, (isomorphic to 3-parameter orientation preserving transformations), and lastly $SIM(2)$ (isomorphic to the 4-parameter similitude group).¹

¹An echo of this idea is present in the recently postulated Horava-Lifschitz gravity [3], which also gives up the full diffeomorphism symmetry in a quantum gravity model

In attempting to construct a quantum field theory based on the above VSR subgroups, Sheikh-Jabbari and Tureanu [4] noticed a problem: all the above proper subgroups allow only one-dimensional representation and hence cannot represent the nature faithfully. However, the authors of [4] provide an ingenious resolution of the “representation problem”: generalize the normal products of operators as deformed or twisted coproducts [5] in terms of which the commutation relations of the (Lorentz) symmetry generators remain intact along with their (physically realized) higher dimensional representation. The reduction in symmetry in going from full Lorentz group to one of the VSR subgroups can be achieved [4] by the “Drinfeld twist” [5]. Finally, a specific VSR subgroup will induce a specific Drinfeld twist, which in turn is identified with a particular noncommutative (NC) spacetime structure. The circle is completed by the said VSR subgroup being the isometry of the NC spacetime or stated simply the NC structure is invariant under one of the VSR subgroups.

In the present article, our interest lies in studying the behavior of the simplest nontrivial dynamical system, that of a relativistic point particle, in NC spacetimes admitted by one of the VSR subgroups. We choose the NC spacetime associated with $E(2)$ [4] which has a Lie algebraic form of noncommutativity,

$$[x^-, x^i] = ilx^i, \quad (1)$$

where $x^- = (t - x^3)/2$, $i = 1, 2$, and l is the numerical NC parameter. It is interesting to note that the $E(3)$ particle, following our terminology, has been studied earlier in [6]. It should be mentioned at the outset that strictly speaking, this NC spacetime does not conform to the VSR principle

since it does not have the full translation invariance. In fact the only NC spacetime that is allowed by the VSR principle is $[x^\mu, x^\nu] = i\theta^{\mu\nu}$ with constant $\theta^{\mu\nu}$ as asserted by [4]. But we still chose (1) because the NC structure, being of operatorial (Lie algebraic) form, construction of a particle model living on it is more interesting and involved. There are instances where particular NC phase space structures are compatible with modified energy-momentum dispersion relations, as, for example, in the likes of doubly special relativity (DSR, not to be confused with VSR) models [7–10] (in this connection see also [11]). Interestingly the present analysis shows that the special theory of relativity relation ($p^2 = m^2$) remains intact for $E(2)$, although the result is by no means obvious. We will closely follow the formalism developed in our earlier work [10] since both the κ -Minkowski NC spacetime in DSR [7,8,10] and the NC spacetime (1) considered here are Lie algebraic in nature.

At this time let us put our work in the proper perspective. We are working in a classical Hamiltonian framework where the phase space plays the fundamental role and it is imperative that all the Jacobi identities between basic variables are satisfied. In the corresponding quantum theory the latter leads to the associativity property of the operators which is a must. In the present context we will show that the NC bracket (1) along with the validity of the Jacobi identity requires the $\{x^\mu, p^\nu\}$ bracket to be non-canonical, and thus p^μ no longer behaves as the translation operator. One can try to construct a translation generator, but it will not satisfy all the Jacobi identities and so its role in a Hamiltonian scheme is not very clear. However, questions concerning the validity of the Jacobi identity become much more severe in the quantum theory due to operator ordering ambiguities and regularization problems. Jacobi identity violation and nonassociative operators are discussed in quantum mechanics problems [12] and quantum field theory problems [13]. Also Jacobi identity constraint can lead to important physical consequences [14].

In recent years high energy physics in NC spacetime received a strong impetus after the work of Seiberg and Witten [15], who demonstrated that certain low energy limits of open string theory are dual to an NC gauge theory where the NC parameters are represented by two form background fields having constant values. In general, NC spacetimes of the operatorial form (Snyder spacetime [16] being the earliest example) are not very common mainly because it is difficult to construct a covariant phase space NC structure keeping in mind the restrictions imposed by Jacobi identity. The Lie algebraic forms of NC spacetime ($[x^\mu, x^\nu] = iC_p^{\mu\nu}x^\rho$) have appeared in various contexts such as in fuzzy spheres [17], κ -Minkowski spacetime in DSR [7], of course the new forms given in [4] for VSR [1] (studied in the present paper), etc. Examples of operator forms of NC phase space with mixed algebra appear in [14,16,18].

Whenever a new NC spacetime is proposed, it is a challenging task to ascertain how the dynamics is affected by the NC nature. This started with the simplest and most studied NC space, the Moyal plane, which appeared as an effective configuration space for low energy (below Landau level) charged particles moving in a plane with a perpendicular constant magnetic field [15]. For obvious reasons, the dynamical model construction based on the operatorial form of NC turns out to be more involved. We successfully implemented this program in [10] for the κ -Minkowski case where we demonstrated that the particular NC spacetime is compatible with a modified dispersion (mass-energy) relation [7,19]. This is precisely the aim of the present note. We have constructed a point particle model, valid in the lowest nontrivial order in l , that lives in the NC spacetime (1). An important observation, as already mentioned, is that the mass-energy relation remains the unchanged Einstein relation.

II. FORMALISM

Let us quickly recapitulate the scheme we employed [10] in constructing relativistic particle models for the κ -Minkowski form in the Magueijo-Smolin base [19]. It is in principle possible, by way of Darboux's theorem, to construct an invertible map that takes us from the NC variables (or phase space) to canonical variables (or phase space) and back. However, in practice, for specific NC structures it can be difficult to derive this map explicitly. If one is able to derive such a map, it becomes a very convenient tool to construct dynamical models in NC phase space simply by starting from a known form of (canonical) action and then exploiting the map to reexpress the action in NC variables and subsequently work out the dynamics. In the κ -Minkowski case mentioned above, the exact map (valid to all orders of the NC parameter) was available [10,19] and we used it successfully to construct DSR particle models.

Returning to the NC spacetime (1), we will follow the same principle as in [10]. Unfortunately it is extremely difficult to construct the exact Darboux map, and we restrict ourselves to the lowest nontrivial order in the NC parameter l .

III. THE PRESENT NC MODEL

We will not consider the quantization of the particle model in the present paper. Hence the commutators are to be interpreted as classical Poisson brackets. We start by rewriting the NC phase space² in a manifestly covariant form,

²Some sectors of this algebra are quite close to the ones studied in [8], but a closer look reveals that they are distinct in nature.

$$\begin{aligned} \{x^\mu, x^\nu\} &= l[(\eta^\mu x^\nu - \eta^\nu x^\mu) - (\alpha^\mu x^\nu - \alpha^\nu x^\mu) \\ &\quad - ((\eta x) + (\alpha x))(\eta^\mu \alpha^\nu - \eta^\nu \alpha^\mu)], \\ \{x^\mu, p^\nu\} &= -g^{\mu\nu} - l[(\eta - \alpha)^\mu p^\nu + (\eta p)\alpha^\mu(\eta - \alpha)^\nu], \\ \{p^\mu, p^\nu\} &= 0. \end{aligned} \quad (2)$$

For $l = 0$ the algebra reduces to the canonical one. We use the short form $(ab) = a^\mu b_\mu$. We have introduced two constant unit vectors, a timelike one η_μ and a spacelike one α_μ ,

$$\begin{aligned} \eta^\mu &= (1, 0, 0, 0); & \alpha^\mu &= (0, 0, 0, 1); & \eta^2 &= 1; \\ \alpha^2 &= -1; & \alpha^\mu \eta_\mu &= (\alpha \eta) = 0. \end{aligned} \quad (3)$$

It is important to note that in (2) we have provided the full NC phase space which did not appear in [4] or elsewhere. We have derived (2) by demanding validity of the Jacobi identity for all possible combinations of phase space degrees of freedom. It is quite obvious that the mixed Jacobi identity for x^μ, x^ν, p^λ will never be satisfied with a canonical $\{x^\mu, p^\nu\} = -g^{\mu\nu}$ Poisson bracket. Note that for the present algebra, all the Jacobi identities are exactly satisfied. This also means that p^μ is no longer the correct translation generator. We consider the Jacobi identities to be sacred, especially as their validity is directly connected with the associativity of operators upon quantization.

IV. CANONICAL VARIABLES, SYMMETRY GENERATORS, AND $E(2)$ INVARIANCE OF THE NC PHASE SPACE ALGEBRA

In order to compute the Darboux map, corresponding to the NC bracket system (2), let us define a set of canonical phase space variables X^μ, P^μ ,

$$\{X^\mu, P^\nu\} = -g^{\mu\nu}; \quad \{X^\mu, X^\nu\} = \{P^\mu, P^\nu\} = 0. \quad (4)$$

After a long computation we obtain the $O(l)$ Darboux map,

$$\begin{aligned} p^\mu &= P^\mu; \\ x^\mu &= X^\mu + l[(XP)(\eta - \alpha)^\mu + (\eta P)\{(\eta X) - (\alpha X)\}\alpha^\mu]. \end{aligned} \quad (5)$$

The inverse map is the following:

$$X^\mu = x^\mu - l[(\eta p)\{(\eta x) - (\alpha x)\}\alpha^\mu + (px)(\eta - \alpha)^\mu]. \quad (6)$$

We follow our earlier prescription [10] where the Poincaré algebra remains undeformed. Hence the generators are first constructed in the canonical phase space where they have the conventional structure and automatically satisfy the canonical Poincaré algebra. This leads us to the Lorentz (rotation) generator,

$$\begin{aligned} J^{\mu\nu} &= X^\mu P^\nu - X^\nu P^\mu \\ &= [x^\mu - l\{(\eta p)\{(\eta x) - (\alpha x)\}\alpha^\mu \\ &\quad + ((px)(\eta - \alpha)^\mu)\}]p^\nu - (\mu \rightleftharpoons \nu). \end{aligned} \quad (7)$$

In the second line $J^{\mu\nu}$ is expressed in terms of NC variables using the inverse map (6), and expectedly it contains extra l -dependent operator terms besides the canonical structure. With the latter expression one needs to use the NC brackets (2).

It is interesting to note that the transformation of p^μ still remains canonical,

$$\{J^{\mu\nu}, p^\lambda\} = -g^{\mu\lambda}p^\nu + g^{\nu\lambda}p^\mu; \quad \{J^{\mu\nu}, p^2\} = 0. \quad (8)$$

This observation is crucial because it shows that there is no need to alter the energy-momentum dispersion relation, and the special theory dispersion,

$$p^2 = m^2, \quad (9)$$

is Lorentz invariant and can serve equally well in the presently studied NC space. This is unlike the case of DSR [7,10,19] where a modification is needed in the dispersion relation to make it invariant in the NC case.

However, the transformation rule for x^μ undergoes drastic changes,

$$\begin{aligned} \{J^{\mu\nu}, x^\lambda\} &= (-g^{\mu\lambda}x^\nu + g^{\nu\lambda}x^\mu) + l[\{(\eta x) - (\alpha x)\}p^\mu \eta^\nu \alpha^\lambda \\ &\quad + (\eta p)x^\mu(\eta - \alpha)^\nu \alpha^\lambda - (xp)(\eta - \alpha)^\mu g^{\nu\lambda} \\ &\quad - (\eta p)\{(\eta) - (\alpha x)\}\alpha^\mu g^{\nu\lambda}] - (\mu \rightleftharpoons \nu). \end{aligned} \quad (10)$$

First of all we need to ensure that the relevant configuration space variables behave correctly under $J_z \equiv J^{12}$:

$$\{J^{12}, x^\lambda\} = x^1 g^{2\lambda} - x^2 g^{1\lambda}, \quad (11)$$

where we have used (10). This shows that x^i behaves as a vector. Also from (10), $\{J^{12}, x^0\} = \{J^{12}, x^3\} = 0$, we find that x^\pm is invariant under J^{12} . These are the same as the transformations given in [4].

Furthermore, in our formalism, the Poincaré generators in (7) are expressed in terms of the canonical variables (X^μ, P^μ) . Hence, by construction they will obey the canonical Poincaré algebra. This means the $E(2)$ generators T_1, T_2, J_z constructed out of the Poincaré generators (see below) will satisfy the canonical $E(2)$ algebra,

$$\{T_1, T_2\} = 0, \quad \{J_z, T_1\} = T_2, \quad \{J_z, T_2\} = T_1.$$

We have also seen that under Poincaré transformations, with these generators, the momentum p^μ transforms canonically (8) whereas the position x^μ fails to do so, as shown in (10).

Before proceeding with specific model building we need to ensure that, in our prescribed scheme, the NC algebra (2) is $E(2)$ invariant. In other words this means that we have to check the stability of the NC symplectic structure (2) under $E(2)$ transformations. In general, for a set of generators (of

symmetry transformations) J^a and a generic symplectic structure $\{A, B\} = C$, one needs to check the identity below:

$$\delta^{J^a}\{A, B\} = \delta^{J^a}C; \quad \delta^{J^a}A = \{J^a, A\}, \quad (12)$$

and explicitly the left-hand side of (12) means

$$\delta^{J^a}\{A, B\} = \{J^a, \{A, B\}\} = \{\delta^{J^a}A, B\} + \{A, \delta^{J^a}B\}. \quad (13)$$

Now notice that the above equations (12) and (13) can be rewritten as

$$\{\{J^a, A\}, B\} + \{\{B, J^a\}, A\} + \{\{A, B\}, J^a\} = 0, \quad (14)$$

which is nothing but the Jacobi identity concerning the three operators J^a, A, B . Recall that our basic phase space variables satisfy all the Jacobi identities. Hence it is clear that the composite operators, constructed out of these basic variables, will also obey the Jacobi identities.³

Returning to the problem at hand, one can consider the Poincaré generators (translation and Lorentz rotation) in place of J^a and replace A, B by the phase space variables x^μ, p^ν and the Jacobi identities will be preserved. In the particular case of $E(2)$ the set of generators consists of $T_1 = K_x + J_y, T_2 = K_y J_x, J_z$ where $J_i = \frac{1}{2}\epsilon_{ijk}J_{jk}$ and $K_i = J_{0i}$ with $i = x, y, z$ are, respectively, generators of rotations and boosts. As we have argued just now, since Jacobi identities individually for $J_{\mu\nu}$ and P_μ will be satisfied, any combinations of them, in particular, the $E(2)$ generators defined above, will also respect the Jacobi identities. Indeed the computations are straightforward and one can explicitly check (which we have done) that the Jacobi identities are satisfied for $E(2)$ generators in place of generic operators J^a . This means that the stability of the noncommutative (phase space) bracket structure that we have proposed in (2) is stable under $E(2)$ transformations.

The careful reader is probably worried (and rightly so) since it is understood that all the Poincaré symmetry transformations are not valid for VSR. The quick example that is given is that translation symmetry in i directions (x and y directions) is lost since the NC bracket is $\{x^-, x^i\} = lx^i$ as given in (1). This assertion is true provided one considers the translation generator \mathcal{T}^μ that satisfies $\{\mathcal{T}^\mu, x^\nu\} = g^{\mu\nu}$ between the translation generators $(Tr)^\mu$ and x^μ . However, as we have emphasized before [see below (3)], it is not possible to satisfy all the Jacobi identities between x^μ and \mathcal{T}^μ with $\{x^\mu, x^\nu\}$ being noncommutative but $\{\mathcal{T}^\mu, x^\nu\} = g^{\mu\nu}$ being canonical. On the other hand, it is enough (at least for the present purpose) to have the NC algebra $\{x^-, x^i\} = lx^i$ as a necessary property of the spacetime and validity of all the phase space Jacobi identities as a

³Indeed, violations of Jacobi identities can arise for composite operators in the quantum theory due to operator ordering and regularizing problems (in quantum field theory, see, for example, [12–14]).

necessary requirement to construct a consistent point particle action, as shown in the next section. It is important to point out that in the Hamiltonian framework the full phase space has to be considered. If one was concerned only with the coordinate space, then, by itself, the coordinate Jacobi identity $\{x^\mu, \{x^\nu, x^\lambda\}\} + \text{cyclic terms} = 0$ even with the NC bracket $\{x^-, x^i\} = lx^i$ as given in (1). The need to modify the $\{x^\mu, p^\nu\}$ bracket as in (2) comes when one has to satisfy the mixed Jacobi identity $\{x^\mu, \{x^\nu, p^\lambda\}\} + \text{cyclic terms} = 0$.

Thus finally we have the important result that the NC symplectic structure proposed in [4] transforms covariantly under the generators of $E(2)$. We stress that there is no clash between our formalism and the idea of VSR (regarding it not being invariant under all translations) simply because the momenta p_μ in (2) are not the translation generators because of the additional terms in the $\{x^\mu, p^\nu\}$ commutator in (2). Indeed, for the translation generators \mathcal{T}^i the NC bracket (2) ceases to be invariant.

V. POINT PARTICLE MODEL AND FIRST ORDER ACTION

Now we are faced with the problem of writing a suitable action in the NC phase space that will be consistent with the NC brackets (2) and correct dispersion relation (9). It becomes clear at once how useful the Darboux construction is because as in the case of Lorentz generators (7) we can once again start with well-known relativistic canonical action for a massive particle,

$$L = \dot{X}^\mu P_\mu - \frac{\lambda}{2}(P^2 - m^2). \quad (15)$$

(The procedure is the same as that followed by us in [10].) Now we must go over to the NC variables [via the inverse map (6)], thus obtaining the cherished form of the action,

$$L = (p\dot{x}) - l[\{(\eta p) - (\alpha p)\}(x\dot{p}) + (\eta p)(\alpha p)\{(\dot{x}\eta) - (\dot{x}\alpha)\} + \{(\eta x) - (\alpha x)\}(\alpha p)(\eta\dot{p})] - \frac{\lambda}{2}(p^2 - m^2). \quad (16)$$

This is a first order action with a modified symplectic structure but unmodified dispersion relation. In order to check whether the NC phase space (2) is recovered, it is straightforward to perform the Hamiltonian constraint analysis of Dirac [20], the salient points of which are briefly described below.

VI. DIRAC CONSTRAINT ANALYSIS

In the Hamiltonian formulation of the constrained system [20], any relation between dynamical variables not involving the time derivative is considered as a constraint. Constraints can appear from the construction of the canonically conjugate momenta (known as the primary con-

straint) or they can appear from demanding time invariance of the constraints (secondary constraint).

Once the full set of constraints is in hand they are classified as first class constraint (FCC) or second class constraint (SCC) according to whether the constraint Poisson bracket algebra is closed or not, respectively. The presence of constraints indicates a redundancy of degrees of freedom (DOFs) that is that not all the DOFs are independent. FCCs signal local gauge invariances in the system. If FCCs are present, there are two ways of dealing with them (in the quantum case). Either one keeps all the DOFs but imposes the FCCs by restricting the set of physical states to those satisfying $(\text{FCC})|\text{state}\rangle = 0$, or one is allowed to choose further constraints, known as gauge fixing conditions, so that these together with the FCCs turn into an SCC set and these will also give rise to Dirac brackets that we presently discuss. In the case of SCCs, say, for $\text{SCC}_1, \text{SCC}_2$ with $[\text{SCC}_1, \text{SCC}_2] = c$ where c is not another constraint, to proceed as before with $(\text{SCC})|\text{state}\rangle = 0$ one reaches an inconsistency because in $\langle \text{state} | [\text{SCC}_1, \text{SCC}_2] | \text{state} \rangle = \langle \text{state} | c | \text{state} \rangle$ the left-hand side = 0 but the right-hand side $\neq 0$. For consistent imposition of the SCCs one defines the Dirac brackets between two generic variables A and B ,

$$\{A, B\}_{\text{DB}} = \{A, B\} - \{A, \text{SCC}_i\} \{\text{SCC}_i, \text{SCC}_j\}^{-1} \{\text{SCC}_j, B\}, \quad (17)$$

where SCC_i are a set of SCC and $\{\text{SCC}_i, \text{SCC}_j\}$ is the constraint matrix. For SCCs this matrix is invertible, and since $\{A, \text{SCC}_i\}_{\text{DB}} = \{\text{SCC}_i, A\}_{\text{DB}} = 0$ for all A , one can implement $\text{SCC}_i = 0$ strongly meaning that some of the DOFs can be removed thereby reducing the number of DOFs in the system but one must use the Dirac brackets. Hence, SCCs induce a change in the symplectic structure, and subsequently one quantizes the Dirac brackets. The same principle is valid for the gauge fixed FCC system mentioned before.

VII. RECOVERING THE NC PHASE SPACE

In the present model (16), canonical momenta for x^μ and p^μ ,

$$\pi_x^\mu = \frac{\partial L}{\partial \dot{x}^\mu}; \quad \pi_p^\mu = \frac{\partial L}{\partial \dot{p}^\mu},$$

yield the constraints

$$\psi_1^\mu = \pi_x^\mu - p^\mu + l(\eta p - \alpha p)p^\mu + l(\alpha p)(\eta p)(\eta - \alpha)^\mu \approx 0, \quad (18)$$

$$\psi_2^\mu = \pi_p^\mu + l\{(\eta p) - (\alpha p)\}x^\mu + l(\alpha p)\{(\eta x) - (\alpha x)\}\eta^\mu \approx 0. \quad (19)$$

The set of constraints ψ_a^μ , $a = 1, 2$ turns out to be SCC, and the constraint bracket matrix $\{\psi_a^\mu, \psi_b^\nu\}$ is computed

below:

$$\{\psi_a^\mu, \psi_b^\nu\} = \begin{pmatrix} 0 & B^{\mu\nu} \\ -B^{\nu\mu} & C^{\mu\nu} \end{pmatrix},$$

$$B^{\mu\nu} = g^{\mu\nu} - l(\eta - \alpha)^\nu p^\mu - l(\eta p)(\eta - \alpha)^\mu \alpha^\nu,$$

$$C^{\mu\nu} = l(\eta - \alpha)^\mu x^\nu - (\eta - \alpha)^\nu x^\mu + l\{(\eta x) - (\alpha x)\}(\alpha^\mu \eta^\nu - \alpha^\nu \eta^\mu). \quad (20)$$

The nonvanishing inverse of the above matrix is

$$(\{\psi_a^\nu, \psi_b^\rho\})_{ab}^{-1} = \begin{pmatrix} D^{\mu\nu} & E^{\mu\nu} \\ -E^{\nu\mu} & 0 \end{pmatrix},$$

$$D^{\mu\nu} = l(\eta - \alpha)^\mu x^\nu - (\eta - \alpha)^\nu x^\mu + \{(\eta x) - (\alpha x)\}(\alpha^\mu \eta^\nu - \alpha^\nu \eta^\mu),$$

$$E^{\mu\nu} = -g^{\mu\nu} - l(\eta - \alpha)^\mu p^\nu - l(\eta p)(\eta - \alpha)^\nu \alpha^\mu. \quad (21)$$

Using the constraints (18) and (19), the inverse of the constraint matrix (21), and the definition of the Dirac brackets (17), it is easy to convince oneself that, indeed, the NC phase space algebra (2) is recovered.

VIII. NAMBU-GOTO ACTION

The remaining task is to construct a Nambu-Goto-like point particle action for the present model. This means that we need to eliminate the momenta p^μ and the multiplier field λ from the first order action. To that end, we first compute the variational equations of motion from the action,

$$\dot{x}_\mu + l[\eta_\mu \{(\eta x) - (\alpha x)\}(\alpha \dot{p}) - (x \dot{p})] + \alpha_\mu \{(\dot{x} p) - (\eta p)((\eta \dot{x}) - (\alpha \dot{x})) - (\eta \dot{p})((\eta x) - (\alpha x))\} + x_\mu \{(\eta \dot{p}) - (\alpha \dot{p})\} = \lambda p_\mu, \quad (22)$$

$$\dot{p}_\mu - l[(\eta p)(\alpha \dot{p})(\eta_\mu - \alpha_\mu) + \{(\eta \dot{p}) - (\alpha \dot{p})\}p_\mu] = 0. \quad (23)$$

Rewriting (23) as $\dot{p}_\sigma G^{\sigma\mu} = 0$, one can check to $O(l)$ the inverse of $G^{\sigma\mu}$ exists, and so $\dot{p}_\mu = 0$ is a consistent solution. Putting this back in (22) and after squaring we obtain

$$\lambda = \frac{\sqrt{\dot{x}^2}}{m} - l \left[\frac{(\eta \dot{x})(\alpha \dot{x})\{(\eta \dot{x}) - (\alpha \dot{x})\}}{\dot{x}^2} + \{(\eta \dot{x}) - (\alpha \dot{x})\} \right], \quad (24)$$

where we have exploited the mass-shell constraint and considered only terms of $O(l)$. To the same order we also obtain

$$(p \dot{x}) = m\sqrt{\dot{x}^2}. \quad (25)$$

Collecting all the terms we obtain the cherished Nambu-

Goto form of the Lagrangian for the $E(2)$ particle to order l :

$$L = m\sqrt{\dot{x}^2} - lm^2\{(\eta\dot{x}) - (\alpha\dot{x})\}\left[1 + \frac{(\eta\dot{x})(\alpha\dot{x})}{\dot{x}^2}\right]. \quad (26)$$

Clearly for $l = 0$ the action reduces to the conventional Nambu-Goto action for a relativistic massive particle. The first order action (16) as well as its (classically) equivalent Nambu-Goto form (26) are major results of the present work.

IX. RECOVERING THE NC PHASE SPACE (ONCE AGAIN)

It is an interesting problem to analyze this particular Nambu-Goto Lagrangian (26) given its involved time-derivative structure. This will act as an internal consistency check as well. Note that although formally it does not contain higher time-derivative terms, the peculiar nature of the terms forces us to treat this model as a higher derivative one. We will follow the prescription of [21] where one replaces selectively some time derivatives by other new auxiliary degrees of freedom (thus rendering the system with a smaller number of time derivatives) and imposes new constraints so that the model physically remains unchanged.

Let us define $\dot{x}_\mu = p_\mu$ and rewrite the Lagrangian accordingly,

$$\begin{aligned} L = & \Pi_\mu^{(x)}(\dot{x}_\mu - p_\mu) + \Pi_\mu^{(p)}p_\mu - \frac{\lambda}{2}(\Pi^{(p)2} - m^2) \\ & - l[(\alpha p)(\eta p)\{(\eta\dot{x}) - (\alpha\dot{x})\} \\ & + (x\dot{p})\{(\alpha p) - (\eta p)\} + (\eta\dot{p})(\alpha p)\{(\eta x) - (\alpha x)\}]. \end{aligned} \quad (27)$$

Notice that $\Pi_\mu^{(x)}$ acts as a Lagrange multiplier enforcing the above identification as a constraint. Integrating out λ and $\Pi_\mu^{(p)}$ from the Lagrangian equations of motion, we find

$$p_\mu - \lambda\Pi_\mu^{(p)} = 0; \quad \lambda = \frac{\sqrt{P^2}}{m}; \quad \Pi_\mu^{(p)} = \frac{m p_\mu}{\sqrt{P^2}}. \quad (28)$$

Substituting these back into (28) in the $\lambda = 1$ gauge, one recovers the Nambu-Goto Lagrangian (26) showing the

equivalence between the Nambu-Goto and first order form. This demonstration is relevant because in the formalism [21] applied here it is clear that construction of the first order form from the higher order form is not unique. It is expected that if one chooses different ways in reducing the higher order form to first order form the resulting actions including the constraint structure will be different but there will be explicit relations connecting the sets of variables.

Constraints obtained from the first order action

$$\begin{aligned} \Psi_\mu^{(1)} = & P_\mu^{(x)} - \Pi_\mu^{(x)} + l[(\eta p)(\alpha p)(\eta_\mu - \alpha_\mu) \\ & + \{(\eta p) - (\alpha p)\}p_\mu], \end{aligned} \quad (29)$$

$$\begin{aligned} \Psi_\mu^{(2)} = & P_\mu^{(p)} + l[\{(\eta p) - (\alpha p)\}x_\mu \\ & + \{(\eta x) - (\alpha x)\}(\alpha p)\eta_\mu], \end{aligned} \quad (30)$$

where $P_\mu^{(x)} = \frac{\partial L}{\partial \dot{x}_\mu}$, $P_\mu^{(p)} = \frac{\partial L}{\partial \dot{p}_\mu}$ are identical to the ones obtained previously (18) and (19), ensuring that the same NC phase space (2) will reappear.

X. CONCLUSION

In this paper we have focused on a particular VSR subgroup $E(2)$ of the Lorentz group and the induced noncommutative spacetime with a Lie algebraic form [4]. We have constructed a relativistic point particle model that lives in this noncommutative phase space and enjoys an undistorted energy-momentum dispersion relation. Our results are restricted to the first nontrivial order in l —the noncommutative parameter.

An important task that remains is to extend the model to all orders in l . In principle this should be possible due to the Darboux theorem. Another interesting area will be to study the solutions of the equations of motion for the “free” $E(2)$ particle. (Actually the single particle theory may not be free in the conventional sense.) It might also be possible to introduce external gauge interactions by way of minimal couplings.

ACKNOWLEDGMENTS

It is a pleasure to thank Anca Tureanu for helpful correspondence at the early stage of our work.

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