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Relativistic theory of infinite statistics fields

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Infinite statistics in which all representations of the symmetric group can occur is known as a special case of quon theory. However, the validity of relativistic quon theories is still in doubt. In this paper we prove that there exists a relativistic quantum field theory which allows interactions involving infinite statistics particles. We also give some consistency analysis of this theory, such as conservation of statistics and Feynman rules.

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I. INTRODUCTION

In conventional quantum theory the identical particles always obey Bose-Einstein statistics or Fermi-Dirac statistics, which are characterized by commutation or anticommutation relations, respectively. This restriction in fact requires a symmetrization postulate that all particles should be in a symmetric state or an antisymmetric state [1]. Without such a postulate, new approaches to particle statistics with small violations of Bose or Fermi statistics are allowed. One famous approach is called quon theory [2] in which the annihilation and creation operators obey the q-deformed commutation relation, $a_k a_l^\dagger - q a_l^\dagger a_k = \delta_{kl}$, $-1 \le q \le 1$. There exist three special cases in quon theory, Bose statistics for $q = +1$, Fermi statistics for $q = -1$, and infinite statistics [3] for $q = 0$.

Infinite statistics with $a_k a_l^\dagger = \delta_{kl}$ involves no commutation relation between two annihilation or creation operators. The quantum states are orthogonal under any permutation of the identical particles, so it allows all representations of the symmetric group to occur. Furthermore, the loss of local commutativity also implies violation of locality, which is an important character of quantum gravity. By virtue of these properties, infinite statistics has been applied to many subjects, such as black hole statistics [4–6], dark energy quanta [7–11], large N matrix theory [12–14], and holography principle [15,16]. Many of these applications involve discussions in the relativistic case.

Unfortunately, the validity of relativistic theory obeying infinite statistics is still in doubt. Greenberg has showed that the infinite statistics theory is valid in the nonrelativistic case. This theory can also have relativistic kinematics. Cluster decomposition and the CPT theorem still hold for free fields [2]. However, there are two difficulties for infinite statistics to have a consistent relativistic theory. First, the physical observables do not commute at spacelike separation. This is bad news for a relativistic theory which requires Lorentz invariance for any physical scattering process (the time ordering of the operator product in the S matrix is not Lorentz invariant). Second, by requiring that the energies of systems that are widely spacelike separated should be additive, Greenberg shows that the conservation of statistics in a relativistic theory limits that $q = \pm 1$, which means it must be a Bose or Fermi case [17]. The $q = 0$ case for infinite statistics has been excluded.

In this paper we prove the existence of interacting relativistic field theory obeying infinite statistics by solving the two difficulties above. First, we directly analyze the Lorentz invariance of the S matrix from the infinitesimal Lorentz transformations on S. The loss of local commutativity does not destroy the invariance. In fact we find that the infinite statistics theory obeys a much weaker locality condition which is also sufficient for Lorentz invariance of the S matrix. Second, we show that this field theory obeys conservation of statistics rule by requiring some special form of the interaction Hamiltonian. In addition, we expect that the characteristic feature of conventional Feynman rules still hold in this new theory.

This paper is organized as follows. In Sec. II, we introduce the elementary ingredients of infinite statistics in nonrelativistic case. In Sec. III, we prove the Lorentz invariance of the S matrix. In Sec. IV, we discuss the condition that the energies are additive for product states, and show that the conservation of statistics still holds. Section V discusses the Feynman rules and provides some simple examples. General conclusions are given in Sec. VI.

II. INFINITE STATISTICS

The basic algebra of infinite statistics is

$$a_k a_l^\dagger = \delta_{kl}, \tag{1}$$

where the operator a_k annihilates the vacuum

$$a_k |0\rangle = 0. \tag{2}$$

This relation determines a Fock-state representation in a linear vector space. The m-particle state is constructed as

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$$|\phi_m\rangle = (a_{k_1}^\dagger)^{m_1}(a_{k_2}^\dagger)^{m_2}\ldots(a_{k_j}^\dagger)^{m_j}|0\rangle \tag{3}$$

with $m_1 + m_2 + \ldots + m_j = m$, where we have $a_{k_i}^\dagger \neq a_{k_{i+1}}^\dagger$. Such states have positive norms and the normalization factor equals one. Since there is no commutation relation between two annihilation or creation operators, the states created by any permutations of creation operators are orthogonal. That's why it is also called quantum Boltzmann statistics.

One can define a set of number operators $\hat{n}_i$ such that

$$\hat{n}_i|\phi_m\rangle = m_i|\phi_m\rangle, \qquad [\hat{n}_i, a_j] = -\delta_{ij}a_j. \tag{4}$$

Then the total number operator is $N = \sum_i \hat{n}_i$, and the energy operator is given by $E = \sum_i \epsilon_i \hat{n}_i$, where ϵ_i is the single particle energy. The explicit form of $\hat{n}_i$ was given by Greenberg [3]

$$\hat{n}_i = a_i^\dagger a_i + \sum_k a_k^\dagger a_i^\dagger a_i a_k + \sum_{k_1,k_2} a_{k_1}^\dagger a_{k_2}^\dagger a_i^\dagger a_i a_{k_2} a_{k_1} + \cdots + \sum_{k_1,k_2,\ldots,k_s} a_{k_1}^\dagger a_{k_2}^\dagger \cdots a_{k_s}^\dagger a_i^\dagger a_i a_{k_s} \cdots a_{k_2} a_{k_1} + \cdots, \tag{5}$$

which is obviously a nonlocal operator. One can easily check that this definition obeys Eq. (4).

III. THE LORENTZ INVARIANCE OF THE S MATRIX

It is not difficult to construct infinite statistics fields that transform irreducibly under the Lorentz group. In momentum space, the annihilation field $\psi_l^+(x)$ and creation field $\psi_l^-(x)$ with mass M are

$$\psi_l^{+(n)}(x) = \sum_{\sigma n}(2\pi)^{-3/2}\int d^3p\, u_l^{(n)}(\boldsymbol{p},\sigma)e^{ip\cdot x}a_{\boldsymbol{p}}^{(n)}(\sigma), \tag{6}$$

$$\psi_l^{-(n)}(x) = \sum_{\sigma n}(2\pi)^{-3/2}\int d^3p\, v_l^{(n)}(\boldsymbol{p},\sigma)e^{-ip\cdot x}a_{\boldsymbol{p}}^{\dagger(n)}(\sigma), \tag{7}$$

where p^μ denotes four-momentum (here we use the conventional notation from [18] that $p^\mu = (p^0, \boldsymbol{p})$ and $p^0 \equiv \sqrt{\boldsymbol{p}^2 + M^2}$), σ labels spin z components (or helicity for massless particles), and the superscript (n) labels particle species $a_{\boldsymbol{p}}^{(n)}(\sigma)a_{\boldsymbol{p}'}^{\dagger(m)}(\sigma') = \delta(nm)\delta(\sigma\sigma')\delta^3(\boldsymbol{p}-\boldsymbol{p}')$. With these fields we will be able to construct the interaction density as [19]

$$\mathcal{H}(x) = \sum_{N,M}\sum_{n_1'\cdots n_N'}\sum_{n_1\cdots n_M}\sum_{l_1'\cdots l_N'}\sum_{l_1\cdots l_M} g_{l_1'\cdots l_N',l_1\cdots l_M}^{(n_1'\cdots n_N',n_1\cdots n_M)} \times \psi_{l_1'}^{-(n_1')}(x)\cdots\psi_{l_N'}^{-(n_N')}(x)\psi_{l_1}^{+(n_1)}(x)\cdots\psi_{l_M}^{+(n_M)}(x). \tag{8}$$

In conventional local quantum field theory (LQFT), we usually construct $\mathcal{H}(x)$ out of a linear combination $\psi(x) = \kappa\psi^+(x) + \lambda\psi^{-c}(x)$, where c denotes the antiparticle. By the requirement of relativistic microcausality ($[\psi(x), \psi^\dagger(y)]_\mp = 0$ for spacelike $x - y$), we always have $\kappa = \lambda$. However, in a theory based on infinite statistics this local commutativity does not hold. *So we cannot determine the relationship between κ and λ, and the basic field in this theory should be $\psi^+(x)$ and $\psi^-(x)$.* Moreover, from Eq. (5) and other operator definitions such as charge operators one may guess that a general operator formulation is defined as [3]

$$\mathcal{A}(\mathcal{O}) = \sum_{m=0}^{\infty}\sum_{n_1,\ldots,n_m,}\sum_{\sigma_1,\ldots,\sigma_m,}^{3}\int d^3k_1\cdots d^3k_m a_{\boldsymbol{k}_1}^{\dagger(n_1)}(\sigma_1) \times\cdots a_{\boldsymbol{k}_m}^{\dagger(n_m)}(\sigma_m)\mathcal{O}a_{\boldsymbol{k}_m}^{(n_m)}(\sigma_m)\cdots a_{\boldsymbol{k}_1}^{(n_1)}(\sigma_1). \tag{9}$$

We will see that this definition is important in the next section's discussion.

With above operator definitions, we can see the Lorentz invariance of the S matrix. One traditional condition comes from the Dyson series for the S operator

$$\begin{aligned} S &= T\left\{\exp\left(-i\int_{-\infty}^{\infty} dt V(t)\right)\right\} \\ &= 1 + \sum_{N=1}^{\infty}\frac{(-i)^N}{N!} \times \int d^4x_1\cdots d^4x_N T\{\mathcal{H}(x_1)\cdots\mathcal{H}(x_N)\}, \end{aligned} \tag{10}$$

where $V(t)$ is the interaction term $H = H_0 + V$ and $T\{\}$ denotes the time-ordered product. Since the time ordering of two spacetime points x_1, x_2 is invariant unless $x_1 - x_2$ is spacelike, the standard sufficient condition that makes S Lorentz invariant is that the $\mathcal{H}(x)$ all commute at spacelike separations

$$[\mathcal{H}(x), \mathcal{H}(x')] = 0 \quad \text{for } (x - x')^2 \geq 0. \tag{11}$$

Now we compute $[\mathcal{H}(x), \mathcal{H}(x')]$ under infinite statistics [20]. First, we write the interaction Hamiltonian density as a polynomial $\mathcal{H}(x) = \sum_i g_\alpha \mathcal{H}_\alpha(x)$; each term $\mathcal{H}_\alpha$ is a product of definite numbers of annihilation fields and creation fields. Then we have

$$\begin{aligned} [\mathcal{H}(x), \mathcal{H}(x')] = &\sum_\alpha g_\alpha^2[\mathcal{H}_\alpha(x), \mathcal{H}_\alpha(x')] \\ &+ \sum_{\alpha<\beta} g_\alpha g_\beta([\mathcal{H}_\alpha(x), \mathcal{H}_\beta(x')] \\ &- [\mathcal{H}_\alpha(x'), \mathcal{H}_\beta(x)]). \end{aligned} \tag{12}$$

By using Eqs. (6)–(8), we have

$$[\mathcal{H}_\alpha(x), \mathcal{H}_\alpha(x')] \sim \int d^3p_1 \cdots d^3p_j d^3p'_1 \cdots d^3p'_j [\cdots] a^\dagger_{\boldsymbol{p}_1} \cdots a^\dagger_{\boldsymbol{p}_i} (a_{\boldsymbol{p}_{i+1}} \cdots a_{\boldsymbol{p}_j} a^\dagger_{\boldsymbol{p}'_1} \cdots a^\dagger_{\boldsymbol{p}'_i}) \times a_{\boldsymbol{p}'_{i+1}} \cdots a_{\boldsymbol{p}'_j} [e^{i[-(p_1+\cdots+p_i)+(p_{i+1}+\cdots+p_j)]x + i[-(p'_1+\cdots+p'_i)+(p'_{i+1}+\cdots+p'_j)]x'} - (x \leftrightarrow x')], \quad (13)$$

$$\begin{aligned}[\mathcal{H}_\alpha(x), \mathcal{H}_\beta(x')] - [\mathcal{H}_\alpha(x'), \mathcal{H}_\beta(x)] \\ \sim \int d^3p_1 \cdots d^3p_j d^3p'_1 \cdots d^3p'_l [\cdots] a^\dagger_{\boldsymbol{p}_1} \cdots a^\dagger_{\boldsymbol{p}_i} (a_{\boldsymbol{p}_{i+1}} \cdots a_{\boldsymbol{p}_j} a^\dagger_{\boldsymbol{p}'_1} \cdots a^\dagger_{\boldsymbol{p}'_k}) \\ \times a_{\boldsymbol{p}'_{k+1}} \cdots a_{\boldsymbol{p}'_l} [e^{i[-(p_1+\cdots+p_i)+(p_{i+1}+\cdots+p_j)]x + i[-(p'_1+\cdots+p'_k)+(p'_{k+1}+\cdots+p'_l)]x'} - (x \leftrightarrow x')] \\ + \int d^3p_1 \cdots d^3p_l d^3p'_1 \cdots d^3p'_j [\cdots] a^\dagger_{\boldsymbol{p}_1} \cdots a^\dagger_{\boldsymbol{p}_k} (a_{\boldsymbol{p}_{k+1}} \cdots a_{\boldsymbol{p}_l} a^\dagger_{\boldsymbol{p}'_1} \cdots a^\dagger_{\boldsymbol{p}'_i}) \\ \times a_{\boldsymbol{p}'_{i+1}} \cdots a_{\boldsymbol{p}'_j} [e^{i[-(p_1+\cdots+p_k)+(p_{k+1}+\cdots+p_l)]x + i[-(p'_1+\cdots+p'_i)+(p'_{i+1}+\cdots+p'_j)]x'} - (x \leftrightarrow x')], \quad (14)\end{aligned}$$

where $[\cdots]$ denotes the product of u, v, and π factors; j, l denote the total numbers of fields in $\mathcal{H}_\alpha$, $\mathcal{H}_\beta$; while i, k denote the numbers of creation fields. Since the elements of the S matrix are the matrix elements of the S operator between free-particle states

$$S_{\boldsymbol{p}'_1\boldsymbol{p}'_2\cdots,\boldsymbol{p}_1\boldsymbol{p}_2\cdots} = \langle 0| \cdots a_{\boldsymbol{p}'_2} a_{\boldsymbol{p}'_1} (S) a^\dagger_{\boldsymbol{p}_1} a^\dagger_{\boldsymbol{p}_2} \cdots |0\rangle, \quad (15)$$

the condition Eq. (11) becomes

$$0 = \langle\beta| \int d^4x_1 \cdots d^4x_{i-1} d^4x_{i+2} \cdots d^4x_n \times T\{\mathcal{H}(x_1) \cdots [\mathcal{H}(x_i), \mathcal{H}(x_{i+1})] \cdots \mathcal{H}(x_n)\} |\alpha\rangle, \quad (16)$$

for $(x_i - x_{i+1})^2 \geq 0$, where $\langle\beta|, |\alpha\rangle$ denote the final state and the initial state. Those annihilation and creation operators in Eqs. (13) and (14) should contract with a's and $a^\dagger$'s in the initial states, final states, and other $\mathcal{H}$s (except $\mathcal{H}(x_i)$ and $\mathcal{H}(x_{i+1})$) in Eq. (16), or they will directly annihilate the vacuum state and get zero. One should note that the S matrix involves a four-momentum conservation relation $S_{\beta\alpha} \sim \delta^4(p_\beta - p_\alpha)$ (see Chapter 3 in [18] for details). So after those annihilation and creation operators in Eqs. (13) and (14) are totally contracted, we find the commutation relation $[\mathcal{H}(x), \mathcal{H}(x')]$ is constituted by terms of the form

$$\sim \int \prod d^3k f(\boldsymbol{k}) [e^{i(\sum p + \sum k)(x-x')} - e^{-i(\sum p + \sum k)(x-x')}], \quad (17)$$

where the terms including k come from self-contractions (contractions do not involve the initial states or final states, such as contractions in $a_{\boldsymbol{p}_{i+1}} \cdots a_{\boldsymbol{p}_j} a^\dagger_{\boldsymbol{p}'_1} \cdots a^\dagger_{\boldsymbol{p}'_i}$), while $\sum p$ is a sum of some particle momenta in the initial or final states, which is, more explicitly, the sum of momentum in Eqs. (13) and (14), $(-(p_1 + \cdots + p_i) + (p_{i+1} + p_j)$ or $-(p_1 + \cdots + p_k) + (p_{k+1} + p_l))$ minus the self-contracted momenta. One can easily see that Eq. (17) is nonzero. As a result, *the interaction density $\mathcal{H}(x)$ will not commute with $\mathcal{H}(x')$ at spacelike separations $x - x'$, which means that this theory cannot be local*.

However, the failure of the above commutation in $T\{\}$ of Eq. (16) does not prohibit the existence of a relativistic field theory. There exists a less restrictive sufficient condition for Lorentz invariance of the S matrix, which directly comes from the infinitesimal Lorentz transformations of S operator (see Chapter 3, page 145 in [18]). This condition is

$$0 = \int d^3x \int d^3y \boldsymbol{x} [\mathcal{H}(\boldsymbol{x}, 0), \mathcal{H}(\boldsymbol{y}, 0)]. \quad (18)$$

We can also put this condition together with the initial states and the final states [21]. According to our analysis presented above, this condition becomes

$$\begin{aligned} 0 &= \int d^3x \boldsymbol{x} \int d^3y \int \prod d^3k f(\boldsymbol{k}) \\ &\quad \times [e^{i(\sum \boldsymbol{p} + \sum \boldsymbol{k})(\boldsymbol{x}-\boldsymbol{y})} - e^{-i(\sum \boldsymbol{p} + \sum \boldsymbol{k})(\boldsymbol{x}-\boldsymbol{y})}] \\ &= \int d^3x \boldsymbol{x} \int \prod d^3k f(\boldsymbol{k}) \delta^3(\sum \boldsymbol{p} + \sum \boldsymbol{k}) \\ &\quad \times [e^{i(\sum \boldsymbol{p} + \sum \boldsymbol{k})\boldsymbol{x}} - e^{-i(\sum \boldsymbol{p} + \sum \boldsymbol{k})\boldsymbol{x}}]. \end{aligned} \quad (19)$$

By integrating over $\boldsymbol{y}$, we can get a Dirac delta function $\delta^3(\sum \boldsymbol{p} + \sum \boldsymbol{k})$. By integrating over k, it requires $\sum \boldsymbol{p} + \sum \boldsymbol{k} = 0$, and then Eq. (19) must always hold because $e^0 - e^{-0} = 0$. More specifically, it follows from the presented derivation that a stronger condition is valid, namely,

$$0 = \int d^3y \boldsymbol{x} [\mathcal{H}(\boldsymbol{x}, 0), \mathcal{H}(\boldsymbol{y}, 0)], \quad (20)$$

which ensures that the integral in Eq. (19) vanishes. *So we conclude that the interacting field theory based on infinite statistics is Lorentz invariant*.

A similar analysis can be applied to the commutation relation $[\mathcal{A}(\mathcal{H}(x)), \mathcal{A}(\mathcal{H}(x'))] = \mathcal{A}(\mathcal{O}(x, x'))$ (see the Appendix for details), in which the interaction density $\mathcal{A}(\mathcal{H})$ is defined in Eq. (9). This commutation can be decomposed into a sum of terms that are similar to

Eqs. (13) and (14). Noting that the momenta of a's and $a^\dagger$'s summed over in these terms [such as $\boldsymbol{p}_i$ in $\int d^3 p_i a^\dagger_{\boldsymbol{p}_i} \cdots \mathcal{O}(x, x') \cdots a_{\boldsymbol{p}_i}$ terms] have no contribution to the momentum conservation relation in $\mathcal{O}$, the final terms after total contraction are still as the form (17). Then by using condition Eq. (18), *we conclude that the theory with interaction density of the form* $\mathcal{A}(\mathcal{H}(x))$ *is also Lorentz invariant*.

IV. CONSERVATION OF STATISTICS

Here we try to impose the condition that the energy should be additive for product states on the interaction $\mathcal{H}(x)$. For subsystems that are widely spacelike separated, the contribution to the energy should be additive if [1]

$$[\mathcal{H}(x), \psi(x')] \to 0, \quad \text{as } x - x' \to \infty \text{ spacelike} \tag{21}$$

for all fields. One can check that this condition is equivalent to

$$(e^{iH_A t}\psi_A) \otimes (e^{iH_B t}\psi_B) = e^{i(H_A+H_B)t}(\psi_A \otimes \psi_B), \tag{22}$$

while subsystems A and B are widely spacelike separated. By using Eq. (21), Greenberg expected that the Hamiltonian operators should be effectively bosonic, which leads to "conservation of statistics" and acquires that $q = \pm 1$ [2,17]. However, we think this restriction is too strong, and we provides a much less restrictive requirement on $\mathcal{H}(x)$, which also leads to conservation of statistics.

In order to satisfy the energy additive condition with $q = 0$, we should replace the density $\mathcal{H}$ with $\mathcal{A}(\mathcal{H})$. Thus Eq. (21) becomes

$$[\mathcal{A}(\mathcal{H}(x)), \psi(x')] \to 0, \quad \text{as } x - x' \to \infty \text{ spacelike}. \tag{23}$$

Noting that the basic fields are $\psi^\pm$ here and $[\mathcal{A}(\mathcal{O}), a_{\boldsymbol{p}}] = -a_{\boldsymbol{p}}\mathcal{O}$, $[\mathcal{A}(\mathcal{O}), a^\dagger_{\boldsymbol{p}}] = \mathcal{O}a^\dagger_{\boldsymbol{p}}$ (see the Appendix for details), we can get

$$\begin{aligned} &[\mathcal{A}(\mathcal{H}(x)), \psi^+(x')] = -\psi^+(x')\mathcal{H}(x), \\ &[\mathcal{A}(\mathcal{H}(x)), \psi^-(x')] = \mathcal{H}(x)\psi^-(x'). \end{aligned} \tag{24}$$

We note that in infinite statistics

$$\psi^{+(n)}(x)\psi^{-(m)}(x') \sim [\cdots]\delta(nm)\Delta_+(x - x'), \tag{25}$$

where the coefficient $[\cdots]$ may contain some derivative times such as $\gamma^\mu\partial_\mu$ for spin $\frac{1}{2}$ particles and $\partial^\mu\partial_\mu$ for spin one particles, while $\Delta_+(x - x') \equiv \frac{1}{(2\pi)^3}\int \frac{d^3p}{2p^0} e^{ip(x-x')}$ is a standard function [22] in which $p^0 \equiv \sqrt{\boldsymbol{p}^2 + M^2}$. Moreover, for $(x - x')^2 \geq 0$

$$\Delta_+(x - x') = \frac{M}{4\pi^2\sqrt{(x - x')^2}} K_1(M\sqrt{(x - x')^2}), \tag{26}$$

in which $K_1(z)$ is modified Bessel (Hankel) function of order 1 possessing for complex z asymptotic $K_1(z) \approx \sqrt{\frac{\pi}{2z}}e^{-z}$. So $\Delta_+(x - x')$ and its derivations are $\to 0$ as $x - x' \to \infty$ spacelike. Then by using Eqs. (24) and (25), we infer that the condition (23) is satisfied if the interaction density $\mathcal{H}(x)$ has the form Eq. (8) with N, $M \geq 1$. Moreover, in order to get condition (22) satisfied for our new definition $\mathcal{A}(H)$, we also need $[\mathcal{A}(\mathcal{H}(x)), \mathcal{A}(\mathcal{H}(x'))] \to 0$, as $x - x' \to \infty$ spacelike; this proof is given in the Appendix. Here we just exclude the terms in $\mathcal{H}$ which are products containing only annihilation fields $\psi^+(x)$ [or creation fields $\psi^-(x)$]. As a result, in a self-consistent relativistic field theory obeying infinite statistics, $\mathcal{A}(\mathcal{H})$ plays the role of a Hamiltonian density other than $\mathcal{H}$. Since the operator definition $\mathcal{A}(\mathcal{H})$ is quite normal in infinite statistics field theory [3], our requirement is much looser than the condition that Hamiltonian operator must be effectively bosonic.

Although the interaction density may not be bosonic, conservation of statistics still holds in our theory. To see this, let us consider the case that infinite statistics fields couple to normal fields (we denote infinite statistics fields by the subscript I and normal statistics fields by the subscript B). According to conventional fields theory and our above discussion, all interactions must involve any number of bosons, an even number of fermions (including zero), at least one annihilation infinite statistics field, and at least one creation infinite statistics field. These three kinds of particles commute with each other, so $\mathcal{A}(\mathcal{O}_I\mathcal{O}'_B) = \mathcal{A}(\mathcal{O}_I)\mathcal{O}'_B$. Since we exclude the terms in $\mathcal{H}$ which are products containing only annihilation fields (or creation fields), then the term $T\{\mathcal{A}(\mathcal{H}(x_1)) \cdots \mathcal{A}(\mathcal{H}(x_N))\}$ in the S operator (10) must have both ψ_I^+ and ψ_I^- fields after $a^{(n)}_{I\boldsymbol{p}}(\sigma)a^{\dagger(m)}_{I\boldsymbol{p}'}(\sigma') = \delta(nm)\delta(\sigma\sigma')\delta^3(\boldsymbol{p} - \boldsymbol{p}')$ contractions. So there must be infinite statistics particles both in the initial and final states, which forbids any process that the in-state obeys infinite statistics (normal statistics) while the out-state obeys normal statistics (infinite statistics). Moreover, since the interaction vertices such as $\mathcal{A}(\psi_I^+\psi_I^{+c})\psi_B$ and $\mathcal{A}(\psi_I^+)(\psi_B^\dagger\psi_B)$ do not exist, we also exclude those virtual processes such as annihilation of a pair of infinite (normal) statistics particles producing a normal (infinite) statistics particle, which also break conservation of statistics. So *we conclude that our theory obeys the conservation of statistics*. Some examples are presented in the next section.

V. FEYNMAN RULES AND EXAMPLES

In order to derive Feynman rules, first we see "Wick's theorem" for infinite statistics fields; by using the relation $a^{(n)}_{I\boldsymbol{p}}(\sigma)a^{\dagger(m)}_{I\boldsymbol{p}'}(\sigma') = \delta(nm)\delta(\sigma\sigma')\delta^3(\boldsymbol{p} - \boldsymbol{p}')$, any product of a set of infinite statistics operators can be finally expressed as a normal product. This looks a bit different from conventional field theory, in which contractions can arise between any creator and annihilator pairs by permuting the operators, while in our theorem contractions can only arise between the neighboring operators. However, by inducing $\mathcal{A}(\mathcal{H})$, we can also realize some permutations. For example, in order to get contraction between a final particle $\langle 0|\cdots a$ and a creation field $\psi^-(y)$ in $\langle 0|\cdots a(\mathcal{A}(\psi^-(x)\psi^+(x))\mathcal{A}(\psi^-(y)\psi^+(y))\cdots|0\rangle$, we can use $\int d^3 p a_{\boldsymbol{p}}^\dagger(\psi^-(x)\psi^+(x))a_{\boldsymbol{p}}(\psi^-(y)\psi^+(y)) = \psi^-(y)(\psi^-(x)\psi^+(x))\psi^+(y)$ to move $\psi^-(y)$ to the left. *So by using the operator definition* (9), *we can get "Wick's theorem" for infinite statistics fields*.

Since the operators cannot be moved arbitrarily, there will be some limits on the Feynman rules. In fact, the step functions $\theta(x)$ do not just appear in propagators, but also affect the external lines. To see this, let us take the Hamiltonian density $\mathcal{A}(\mathcal{H})$ in which $\mathcal{H} = \psi^-\psi^-\psi^+ + \psi^-\psi^+\psi^+$, for example. Then the S operator contains a term $\theta(x-y)(\psi^-(x)\psi^-(x)\psi^+(x))\times(\psi^-(y)\psi^+(y)\psi^+(y)) + \theta(y-x)(\psi^-(y)\psi^-(y)\psi^+(y))\times(\psi^-(x)\psi^+(x)\psi^+(x))$, in which $\psi^-(x)\psi^-(x)$ and $\psi^+(y)\psi^+(y)$ are unexchangeable. If the final state is $a_{\boldsymbol{p}_{1'}}^\dagger a_{\boldsymbol{p}_{2'}}^\dagger|0\rangle$ and the initial state is $a_{\boldsymbol{p}_1}^\dagger a_{\boldsymbol{p}_2}^\dagger|0\rangle$, such term will be a sum of two subgraphs: One has two external lines carrying momenta $p_{1'}$, $p_{2'}$ at x point, two external lines carrying momenta p_1, p_2 at y point, and one internal line $\Delta(x-y)$. While the other has two external lines carrying momenta $p_{1'}$, $p_{2'}$ at y point, two external lines carrying momenta p_1, p_2 at x point, and one internal line $\Delta(y-x)$. The propagator $\Delta(x-y)$ is defined as

$$\Delta_{lm}(x-y) \equiv i\theta(x-y)(\psi_l^+(x)\psi_m^-(y)) = (2\pi)^{-4}\int d^4q \times \frac{-P_{lm}^{(L)}(q)e^{iq(x-y)}}{2\sqrt{\boldsymbol{q}^2+M^2}(q^0-\sqrt{\boldsymbol{q}^2+M^2}+i\epsilon)}, \quad (27)$$

in which $P_{lm}^{(L)}$ is defined in Chapter 6.2 in [18]. Noting that the position related term $e^{iq(x-y)}$ is still the same as in the conventional propagator, we can infer that the Feynman rules for external lines in momentum space (after integrating over the spacetime position x, y) are the same as before. So the contribution of the external lines to the S matrix are the same in the two subgraphs. If we denote $\Delta(q) = (2\pi)^{-4}(-P_{lm}^{(L)}(q))/(2\sqrt{\boldsymbol{q}^2+M^2}(q^0-\sqrt{\boldsymbol{q}^2+M^2}+i\epsilon))$ as the internal line contribution in momentum space, then the total S matrix for this $12\rightarrow 1'2'$ progress contains (external line terms) $\cdot\,(\Delta(q)+\Delta(-q))$ = (external line terms) $\cdot\,\Delta_F(q)$. Especially for spin-0 and spin-1 fields

$$\Delta_F(q) = (2\pi)^{-4}\frac{P_{lm}^{(L)}(q)}{q^2+M^2-i\epsilon} \quad (28)$$

is just the conventional propagator in momentum space. *So we expect that the Feynman rules for our new theory are similar to conventional rules.* This allows us to apply some traditional methods such as renormalization analysis.

Here we give two explicit examples for infinite statistics field interactions. For simple, we take the interaction density $\mathcal{A}(\mathcal{H})$ in which $\mathcal{H}$ is trilinear in a set of scalar fields and consider only tree-level graphs. In this case one basic scattering process type should be $12\rightarrow 1'2'$, i.e. the initial state is $a_{\boldsymbol{p}^1}^\dagger a_{\boldsymbol{p}^2}^\dagger|0\rangle$, while the final state is $a_{\boldsymbol{p}_{1'}}^\dagger a_{\boldsymbol{p}_{2'}}^\dagger|0\rangle$.

First, for the pure infinite statistics interaction, we take $\mathcal{H} = \phi^-\phi^-\phi^+ + \phi^-\phi^+\phi^+$, the Feynman diagrams are presented in Fig. 1. Figure 1(a) describes the term $\theta(x-y)(\phi^-(x)\phi^-(x)\phi^+(x))(\phi^-(y)\phi^+(y)\phi^+(y)) + \theta(y-x)\times(\phi^-(y)\phi^-(y)\phi^+(y))(\phi^-(x)\phi^+(x)\phi^+(x))$. The internal propagator comes from the contraction between the neighboring ϕ^+ and ϕ^-, while the external line terms come from the contractions between the initial (final) states and fields $\phi^\pm$. The S-matrix element is given by

$$S^{(1a)}_{\boldsymbol{p}_{1'}\boldsymbol{p}_{2'},\boldsymbol{p}_1\boldsymbol{p}_2} \sim u^*(\boldsymbol{p}_{2'})u^*(\boldsymbol{p}_{1'})u(\boldsymbol{p}_2)u(\boldsymbol{p}_1) \times\int d^4x\int d^4y e^{-ip_{2'}x}e^{-ip_{1'}x}e^{ip_2y}e^{ip_1y}\Delta(x-y) + u^*(\boldsymbol{p}_{2'})u^*(\boldsymbol{p}_{1'})u(\boldsymbol{p}_2)u(\boldsymbol{p}_1) \times\int d^4x\int d^4y e^{-ip_{2'}y}e^{-ip_{1'}y}e^{ip_2x}e^{ip_1x}\Delta(y-x). \quad (29)$$

Substitute the $\Delta(x-y)$ definition (27) into Eq. (29) to obtain

$$S^{(1a)}_{\boldsymbol{p}_{1'}\boldsymbol{p}_{2'},\boldsymbol{p}_1\boldsymbol{p}_2} \sim u^*(\boldsymbol{p}_{2'})u^*(\boldsymbol{p}_{1'})u(\boldsymbol{p}_2)u(\boldsymbol{p}_1)\int d^4q\delta^4(p_1+p_2-q)\delta^4(p_{1'}+p_{2'}-q) \times\frac{1}{2\sqrt{\boldsymbol{q}^2+M^2}}[(q^0-\sqrt{\boldsymbol{q}^2+M^2}+i\epsilon)^{-1}+(-q^0-\sqrt{\boldsymbol{q}^2+M^2}+i\epsilon)^{-1}] \sim u^*(\boldsymbol{p}_{2'})u^*(\boldsymbol{p}_{1'})u(\boldsymbol{p}_2)u(\boldsymbol{p}_1)\int d^4q\delta^4(p_1+p_2-q)\delta^4(p_{1'}+p_{2'}-q)\times\Delta_F(q) \sim u^*(\boldsymbol{p}_{2'})u^*(\boldsymbol{p}_{1'})u(\boldsymbol{p}_2)u(\boldsymbol{p}_1) \times\delta^4(p_1+p_2-p_{1'}-p_{2'})\Delta_F(p_1+p_2). \quad (30)$$

Figure 1(b) describes the term $\theta(x-y)\int d^3k\times[(\phi^-(x)\phi^+(x)\phi^+(x))a_{\boldsymbol{k}}^\dagger(\phi^-(y)\phi^-(y)\phi^+(y))a_{\boldsymbol{k}} + a_{\boldsymbol{k}}^\dagger\times(\phi^-(x)\phi^+(x)\phi^+(x))a_{\boldsymbol{k}}(\phi^-(y)\phi^-(y)\phi^+(y))] + x\leftrightarrow y$. By using Eq. (1) this term is just $\theta(x-y)[(\phi^-(x)\phi^+(x))\times(\phi^-(y)\phi^-(y)\phi^+(y)\phi^+(x))+(\phi^-(y)\phi^-(x)\phi^+(x)\phi^+(x))\times(\phi^-(y)\phi^+(y))] + x\leftrightarrow y$. After a similar calculation, we

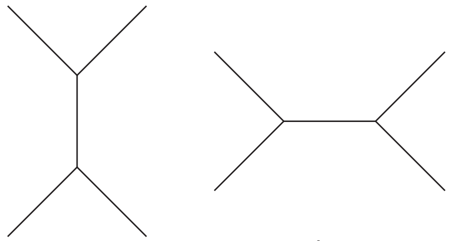

FIG. 1. Pure infinite statistical interactions.

can get

$$S^{(1b)}_{\boldsymbol{p}_{1'}\boldsymbol{p}_{2'},\boldsymbol{p}_1\boldsymbol{p}_2} \sim u^*(\boldsymbol{p}_{2'})u^*(\boldsymbol{p}_{1'})u(\boldsymbol{p}_2)u(\boldsymbol{p}_1) \times \delta^4(p_1 + p_2 - p_{1'} - p_{2'})\Delta_F(p_1 - p'_1). \tag{31}$$

Second, for the case that infinite statistics fields couple to a bosonic field, we take $\mathcal{H} = \phi_I^- \phi_I^+ \phi_B$ (we denote infinite statistics fields by the subscript I and bosonic fields by the subscript B), the Feynman diagrams are presented in Fig. 2. Figure 2(a) and 2(b) describe the term $\theta(x-y) \times (\phi_B(x)\phi_I^-(x)\phi_I^+(x))(\phi_I^-(y)\phi_I^+(y)\phi_B(y)) + x \leftrightarrow y$. The propagator comes from the contraction between infinite statistics fields ϕ_I^+ and ϕ_I^-. Take particles 1, $1'$ the bosonic particles, we get

$$\begin{aligned} S^{(2a,2b)}_{\boldsymbol{p}_{1'}\boldsymbol{p}_{2'},\boldsymbol{p}_1\boldsymbol{p}_2} \sim{}& u_I^*(\boldsymbol{p}_{2'})u_B^*(\boldsymbol{p}_{1'})u_I(\boldsymbol{p}_2)u_B(\boldsymbol{p}_1) \\ &\times \delta^4(p_1 + p_2 - p_{1'} - p_{2'})\Delta_{FI}(p_1 + p_2) \\ &+ u_I^*(\boldsymbol{p}_{2'})u_B^*(\boldsymbol{p}_{1'})u_I(\boldsymbol{p}_2)u_B(\boldsymbol{p}_1) \\ &\times \delta^4(p_1 + p_2 - p_{1'} - p_{2'})\Delta_{FI}(p_1 - p_{2'}). \end{aligned} \tag{32}$$

Figure 2(c) describes the term $\theta(x-y) \times [\phi_B^+(x),\phi_B^-(y)]\int d^3k[a_{I\boldsymbol{k}}^\dagger(\phi_I^-(x)\phi_I^+(x))a_{I\boldsymbol{k}}(\phi_I^-(y)\phi_I^+(y)) + (\phi_I^-(x)\phi_I^+(x))a_{I\boldsymbol{k}}^\dagger(\phi_I^-(y)\phi_I^+(y))a_{I\boldsymbol{k}}] + x \leftrightarrow y$, i.e. the term $\theta(x-y)[\phi_B^+(x), \phi_B^-(y)][\phi_I^-(y)\phi_I^-(x)\phi_I^+(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^-(y)\phi_I^+(y)\phi_I^+(x)] + x \leftrightarrow y$. The propagator comes from the contraction between bosonic fields $\phi_B(x)$ and $\phi_B(y)$. In this case

$$S^{(2c)}_{\boldsymbol{p}_{1'}\boldsymbol{p}_{2'},\boldsymbol{p}_1\boldsymbol{p}_2} \sim u_I^*(\boldsymbol{p}_{2'})u_I^*(\boldsymbol{p}_{1'})u_I(\boldsymbol{p}_2)u_I(\boldsymbol{p}_1) \times \delta^4(p_1 + p_2 - p_{1'} - p_{2'})\Delta_{FB}(p_1 - p'_1). \tag{33}$$

The graphs in Fig. 1 fully describe the tree-level scattering amplitudes $12 \to 1'2'$ involving pure infinite statistics fields. Fig. 2 describes $12 \to 1'2'$ involving both infinite statistics fields and bosonic fields. We see that those processes breaking the conservation of statistics as presented in Fig. 3 are excluded, as we have shown in Sec. IV.

Although the Feynman rules for our new theory seem similar to those conventional rules. The real (physical) processes are a bit different, this is due to the nonlocal property of infinite statistics. For example, let us take the interaction density $\mathcal{H}$ is trilinear in a set of scalar fields, and consider tree-level process $512 \to 51'2'$. In standard LQFT, the particles 12 and $1'2'$ are connected by scattering interaction. While the particle 5 propagates free. So the process $512 \to 51'2'$ is totally the same as $12 \to 1'2'$ and is trivial. However in the theory based on infinite statistics, the interaction density $\mathcal{A}(\mathcal{H})$ has a form (9) with infinite terms. So all the basic terms $\mathcal{T}$ in Fig. 1 and 2 have infinite copies such as $\mathcal{T}$, $\int d^3ka_{\boldsymbol{k}}^\dagger\mathcal{T}a_{\boldsymbol{k}}, \cdots$. For example, the time product $T\{\mathcal{A}(\mathcal{H}(x))\mathcal{A}(\mathcal{H}(y))\}$ in the S matrix has a term $\theta(x-y)\int d^3k_1d^3k_2a_{\boldsymbol{k}1}^\dagger \times (\phi^-(x)\phi^-(x)\phi^+(x))a_{\boldsymbol{k}_1}a_{\boldsymbol{k}_2}^\dagger(\phi^-(y)\phi^+(y)\phi^+(y))a_{\boldsymbol{k}_2} + x \leftrightarrow y$, i.e. the term $\theta(x-y)\int d^3ka_{\boldsymbol{k}}^\dagger(\phi^-(x)\phi^-(x)\phi^+(x)) \times (\phi^-(y)\phi^+(y)\phi^+(y))a_{\boldsymbol{k}} + x \leftrightarrow y$. We can see that this is just the $\int d^3ka_{\boldsymbol{k}}^\dagger\mathcal{T}a_{\boldsymbol{k}}$ copy of term $\mathcal{T}$ in Fig. 1(a). This term corresponds to the process $512 \to 51'2'$, in which the particles 12 and $1'2'$ are still connected by the local term $\mathcal{T}$, while the particles 5 in the initial state and the final state also take part in the scattering process and they are connected by the nonlocal operator $a^\dagger$, a in $\int a^\dagger\mathcal{T}a$. *So we show that the scattering processes* $\cdots i \cdots 512 \to \cdots i \cdots 51'2'$ *are nontrivial in infinite statistics theory.* However the S-matrix elements are still the same $S_{\cdots\boldsymbol{p}_i\cdots\boldsymbol{p}_5\boldsymbol{p}_{1'}\boldsymbol{p}_{2'},\cdots\boldsymbol{p}_i\cdots\boldsymbol{p}_5\boldsymbol{p}_1\boldsymbol{p}_2} = S_{\boldsymbol{p}_{1'}\boldsymbol{p}_{2'},\boldsymbol{p}_1\boldsymbol{p}_2}$. This is why we say that the local scattering processes such as $12 \to 1'2'$ are basic.

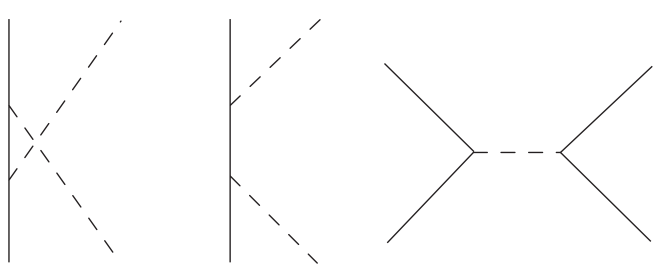

FIG. 2. Interactions between Bose and infinite statistics. The solid lines represent infinite statistical particles, and the dashed lines represent bosons.

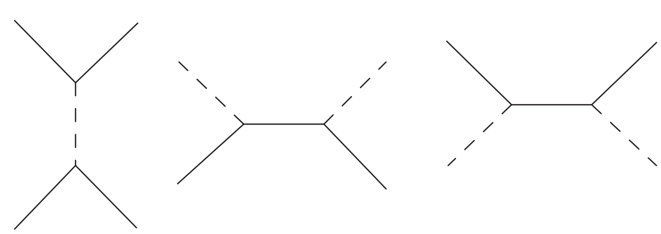

FIG. 3. Interactions violating the conservation of statistics rules. The solid lines represent infinite statistical particles, and the dashed lines represent bosons.

VI. CONCLUSIONS

For a relativistic quantum field theory, the interaction density $\mathcal{H}(x)$ is always built out of both annihilation the field $\psi^+(x)$ [Eq. (6)] and the creation field $\psi^-(x)$ [Eq. (7)]. The space of both functions with complex $a_{\boldsymbol{p}}^{(n)}$, $a_{\boldsymbol{p}}^{\dagger(n)}$ shares various unitary irreducible representations of the Poincaré group. In standard quantum field theory the operators $a_{\boldsymbol{p}}^{(n)}$, $a_{\boldsymbol{p}}^{\dagger(n)}$ obey commutation relations $[a_{\boldsymbol{p}}^{(n)}(\sigma), a_{\boldsymbol{p}'}^{\dagger(m)}(\sigma')]_{\mp} = \delta(nm)\delta(\sigma\sigma')\delta^3(\boldsymbol{p}-\boldsymbol{p}')$, $[a_{\boldsymbol{p}}^{(n)}, a_{\boldsymbol{p}'}^{(m)}]_{\mp} = 0$ and $[a_{\boldsymbol{p}}^{\dagger(n)}, a_{\boldsymbol{p}'}^{\dagger(m)}]_{\mp} = 0$. These local commutativities lead to a locality (causality) condition $[\mathcal{H}(x), \mathcal{H}(x')] = 0$ for $(x-x')^2 \geq 0$, which is sufficient for Lorentz invariance of the S matrix.

While in quantum field theory of infinite statistics fields, the basic algebra is $a_{\boldsymbol{p}}^{(n)}(\sigma)a_{\boldsymbol{p}'}^{\dagger(m)}(\sigma') = \delta(nm)\delta(\sigma\sigma')\delta^3(\boldsymbol{p}-\boldsymbol{p}')$. In this paper, we have proved that this relation also leads to a weaker locality condition $0 = \int d^3y\boldsymbol{x}[\mathcal{H}(\boldsymbol{x},0), \mathcal{H}(\boldsymbol{y},0)]$. Since the weakest sufficient condition for Lorentz invariance of the S matrix is given in [18] that $0 = \int d^3x \int d^3y\boldsymbol{x}[\mathcal{H}(\boldsymbol{x},0), \mathcal{H}(\boldsymbol{y},0)]$. We conclude that the interaction field theory based on infinite statistics is Lorentz invariant.

By applying the condition that the energies are additive for product states, we also showed that this theory obeys conservation of statistics with selected form of interaction Hamiltonian [see $\mathcal{A}(\mathcal{H})$ defined in Eq. (9) in which $\mathcal{H}$ has the form (8) with $N, M \geq 1$]. For all the above reasons, we conclude that the relativistic quantum field theory can involve infinite statistics particles. Since we have showed that the Feynman rules of infinite statistics field are similar to conventional Feynman rules (although unlike the standard LQFT, some physical processes such as $\cdots i \cdots 512 \rightarrow \cdots i \cdots 51'2'$ are nontrivial due to the nonlocal interaction form), some traditional methods such as renormalization analysis can also be extended to our theory.

ACKNOWLEDGMENTS

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APPENDIX

In this Appendix, we first calculate the commutation relation $[\mathcal{A}(\mathcal{H}(x)), \mathcal{A}(\mathcal{H}(x'))]$. Here we also consider one single species of particle for simple. As we have shown in Sec. III, this commutation can be decomposed into a sum of $g_\alpha g_\beta[\mathcal{A}(\mathcal{H}_\alpha(x)), \mathcal{A}(\mathcal{H}_\beta(x'))]$, in which $\mathcal{H}_\alpha$, $\mathcal{H}_\beta$ are products of definite numbers of annihilation fields and creation fields. Here we denote the total numbers of fields in $\mathcal{H}_\alpha$, $\mathcal{H}_\beta$ are j, l, while the numbers of creation fields are i, k. Then we only need to calculate the relation $[\mathcal{A}(a_{\boldsymbol{p}_1}^\dagger \cdots a_{\boldsymbol{p}_i}^\dagger a_{\boldsymbol{p}_{i+1}} \cdots a_{\boldsymbol{p}_j}), \mathcal{A}(a_{\boldsymbol{p}'_1}^\dagger \cdots a_{\boldsymbol{p}'_k}^\dagger a_{\boldsymbol{p}'_{k+1}} \cdots a_{\boldsymbol{p}'_l})]$.

First, we write the product $\mathcal{A}(a_{\boldsymbol{p}_1}^\dagger \cdots a_{\boldsymbol{p}_i}^\dagger a_{\boldsymbol{p}_{i+1}} \cdots a_{\boldsymbol{p}_j}) \cdot \mathcal{A}(a_{\boldsymbol{p}'_1}^\dagger \cdots a_{\boldsymbol{p}'_k}^\dagger a_{\boldsymbol{p}'_{k+1}} \cdots a_{\boldsymbol{p}'_l})$ as

$$\begin{aligned}\mathcal{A}(a_{\boldsymbol{p}_1}^\dagger \cdots a_{\boldsymbol{p}_i}^\dagger a_{\boldsymbol{p}_{i+1}} \cdots a_{\boldsymbol{p}_j}) \cdot \mathcal{A}(a_{\boldsymbol{p}'_1}^\dagger \cdots a_{\boldsymbol{p}'_k}^\dagger a_{\boldsymbol{p}'_{k+1}} \cdots a_{\boldsymbol{p}'_l}) \\ = \mathcal{O}_{11} + \mathcal{O}_{12} + \mathcal{O}_{13} + \ldots + \mathcal{O}_{1n} + \ldots \mathcal{O}_{21} + \mathcal{O}_{22} + \mathcal{O}_{23} + \ldots + \mathcal{O}_{2n} + \ldots \mathcal{O}_{31} + \mathcal{O}_{32} + \mathcal{O}_{33} + \ldots + \mathcal{O}_{3n} \\ + \ldots \vdots \mathcal{O}_{m1} + \mathcal{O}_{m2} + \mathcal{O}_{m3} + \ldots + \mathcal{O}_{mn} + \ldots\ldots,\end{aligned} \tag{A1}$$

in which $\mathcal{O}_{mn}$ is defined as the product of the m-th term in $\mathcal{A}(\mathcal{H}_\alpha)$ and the n-th term in $\mathcal{A}(\mathcal{H}_\beta)$

$$\begin{aligned}\mathcal{O}_{mn} \equiv \Big(\int \prod d^3q a_{\boldsymbol{q}_1}^\dagger \cdots a_{\boldsymbol{q}_{m-1}}^\dagger (a_{\boldsymbol{p}_1}^\dagger \cdots a_{\boldsymbol{p}_i}^\dagger a_{\boldsymbol{p}_{i+1}} \cdots a_{\boldsymbol{p}_j}) a_{\boldsymbol{q}_{m-1}} \cdots a_{\boldsymbol{q}_1}\Big) \\ \cdot \Big(\int \prod d^3q' a_{\boldsymbol{q}'_1}^\dagger \cdots a_{\boldsymbol{q}'_{n-1}}^\dagger (a_{\boldsymbol{p}'_1}^\dagger \cdots a_{\boldsymbol{p}'_k}^\dagger a_{\boldsymbol{p}'_{k+1}} \cdots a_{\boldsymbol{p}'_l}) a_{\boldsymbol{q}'_{n-1}} \cdots a_{\boldsymbol{q}'_1}\Big).\end{aligned} \tag{A2}$$

It Is not difficult to find that $\mathcal{O}_{mn}$ has a recursion relation

$$\mathcal{O}_{(m+1)(n+1)} = \int d^3q a_{\boldsymbol{q}}^\dagger \mathcal{O}_{mn} a_{\boldsymbol{q}}. \tag{A3}$$

Thus we can denote $\mathcal{O}_{mn} + \mathcal{O}_{(m+1)(n+1)} + \mathcal{O}_{(m+2)(n+2)} + \cdots + \mathcal{O}_{(m+s)(n+s)} + \cdots$ by $\mathcal{A}(\mathcal{O}_{mn})$. Then the product (A1) can be simplified as

$$\begin{aligned}\mathcal{A}(a_{\boldsymbol{p}_1}^\dagger \cdots a_{\boldsymbol{p}_i}^\dagger a_{\boldsymbol{p}_{i+1}} \cdots a_{\boldsymbol{p}_j}) \cdot \mathcal{A}(a_{\boldsymbol{p}'_1}^\dagger \cdots a_{\boldsymbol{p}'_k}^\dagger a_{\boldsymbol{p}'_{k+1}} \cdots a_{\boldsymbol{p}'_l}) \\ = \mathcal{A}(\mathcal{O}_{11}) + \mathcal{A}(\mathcal{O}_{12}) + \mathcal{A}(\mathcal{O}_{13}) + \cdots + \mathcal{A}(\mathcal{O}_{1n}) + \cdots \mathcal{A}(\mathcal{O}_{21}) + \mathcal{A}(\mathcal{O}_{31}) + \mathcal{A}(\mathcal{O}_{41}) + \cdots + \mathcal{A}(\mathcal{O}_{m1}) + \cdots.\end{aligned} \tag{A4}$$

We can also define $\mathcal{O}'_{mn}$ as the product of the m-th term in $\mathcal{A}(\mathcal{H}_\beta)$ and the n-th term in $\mathcal{A}(\mathcal{H}_\alpha)$. Then the commutation becomes

$$\begin{aligned}&[\mathcal{A}(a^\dagger_{\boldsymbol{p}_1}\cdots a^\dagger_{\boldsymbol{p}_i}a_{\boldsymbol{p}_{i+1}}\cdots a_{\boldsymbol{p}_j}),\mathcal{A}(a^\dagger_{\boldsymbol{p}'_1}\cdots a^\dagger_{\boldsymbol{p}'_k}a_{\boldsymbol{p}'_{k+1}}\cdots a_{\boldsymbol{p}'_l})]\\&=\mathcal{A}(\mathcal{O}_{11})+\mathcal{A}(\mathcal{O}_{12})+\mathcal{A}(\mathcal{O}_{13})+\cdots+\mathcal{A}(\mathcal{O}_{1n})+\cdots\mathcal{A}(\mathcal{O}_{21})+\mathcal{A}(\mathcal{O}_{31})+\mathcal{A}(\mathcal{O}_{41})+\cdots+\mathcal{A}(\mathcal{O}_{m1})\\&\quad+\cdots-\mathcal{A}(\mathcal{O}'_{11})-\mathcal{A}(\mathcal{O}'_{12})-\mathcal{A}(\mathcal{O}'_{13})-\cdots-\mathcal{A}(\mathcal{O}'_{1n})-\cdots-\mathcal{A}(\mathcal{O}'_{21})-\mathcal{A}(\mathcal{O}'_{31})\\&\quad-\mathcal{A}(\mathcal{O}'_{41})-\cdots-\mathcal{A}(\mathcal{O}'_{m1})-\cdots=\mathcal{A}((\mathcal{O}_{11}+\cdots+\mathcal{O}_{1(j-i+1)}+\mathcal{O}_{21}+\cdots+\mathcal{O}_{(k+1)1})\\&\quad-(\mathcal{O}'_{11}+\cdots+\mathcal{O}'_{1(l-k+1)}+\mathcal{O}'_{21}+\cdots+\mathcal{O}'_{(i+1)1})).\end{aligned}\tag{A5}$$

All the other terms are canceled by the relation

$$\mathcal{A}(\mathcal{O}_{1(j-i+m)})-\mathcal{A}(\mathcal{O}'_{(i+m)1})=0,\qquad\mathcal{A}(\mathcal{O}_{(k+m)1})-\mathcal{A}(\mathcal{O}'_{1(l-k+m)})=0,\qquad m\geq1.\tag{A6}$$

$\mathcal{O}_{mn}$ (or $\mathcal{O}'_{mn}$) in the remaining $j+l+2$ terms are some permutations of $(a^\dagger_{\boldsymbol{p}_1}\cdots a^\dagger_{\boldsymbol{p}_i}a_{\boldsymbol{p}_{i+1}}\cdots a_{\boldsymbol{p}_j})\cdot(a^\dagger_{\boldsymbol{p}'_1}\cdots a^\dagger_{\boldsymbol{p}'_k}a_{\boldsymbol{p}'_{k+1}}\cdots a_{\boldsymbol{p}'_l})$. Moreover, if the interaction density $\mathcal{H}(x)$ is defined in Eq. (8) with $N, M\geq 1$, then by using Eq. (A5) each term in $[\mathcal{A}(\mathcal{H}(x)),\mathcal{A}(\mathcal{H}(x'))]$ involves $\psi^+(x)\psi^-(x')$ [or $\psi^+(x')\psi^-(x)$]. So $[\mathcal{A}(\mathcal{H}(x)),\mathcal{A}(\mathcal{H}(x'))]\to 0$, as $x-x'\to\infty$ spacelike, which is a necessary condition for Eq. (22).

In order to get Eq. (24), we also have to calculate $[\mathcal{A}(\mathcal{O}),a_{\boldsymbol{p}}]$ and $[\mathcal{A}(\mathcal{O}),a^\dagger_{\boldsymbol{p}}]$. Let us see these explicitly

$$\begin{aligned}[\mathcal{A}(\mathcal{O}),a_{\boldsymbol{p}}]&=[\mathcal{O}+\int d^3q a^\dagger_{\boldsymbol{q}}\mathcal{O}a_{\boldsymbol{q}}+\cdots,a_{\boldsymbol{p}}]\\&=\mathcal{O}a_{\boldsymbol{p}}+\int d^3q a^\dagger_{\boldsymbol{q}}\mathcal{O}a_{\boldsymbol{q}}a_{\boldsymbol{p}}+\cdots-(a_{\boldsymbol{p}}\mathcal{O}+\mathcal{O}a_{\boldsymbol{p}}+\int d^3q a^\dagger_{\boldsymbol{q}}\mathcal{O}a_{\boldsymbol{q}}a_{\boldsymbol{p}}+\cdots)=-a_{\boldsymbol{p}}\mathcal{O},\end{aligned}\tag{A7}$$

$$\begin{aligned}[\mathcal{A}(\mathcal{O}),a^\dagger_{\boldsymbol{p}}]&=[\mathcal{O}+\int d^3q a^\dagger_{\boldsymbol{q}}\mathcal{O}a_{\boldsymbol{q}}+\cdots,a^\dagger_{\boldsymbol{p}}]\\&=\mathcal{O}a^\dagger_{\boldsymbol{p}}+a^\dagger_{\boldsymbol{p}}\mathcal{O}+a^\dagger_{\boldsymbol{p}}\int d^3q a^\dagger_{\boldsymbol{q}}\mathcal{O}a_{\boldsymbol{q}}+\cdots-(a^\dagger_{\boldsymbol{p}}\mathcal{O}+a^\dagger_{\boldsymbol{p}}\int d^3q a^\dagger_{\boldsymbol{q}}\mathcal{O}a_{\boldsymbol{q}}+\cdots)=\mathcal{O}a^\dagger_{\boldsymbol{p}}.\end{aligned}\tag{A8}$$

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[19] See Chapter 5.1 in [18] for details.
[20] Here we consider one spin-0 field for simple. Since other cases can be treated as involving some intrinsic indices (i.e. adding some delta functions such as $\delta(mn)$), they will not affect our results.
[21] More explicitly, the correct formulation should be $\langle\beta|f(H,H_0)(\int d^3x\int d^3y\boldsymbol{x}[\mathcal{H}(\boldsymbol{x},0),\mathcal{H}(\boldsymbol{y},0)])\times g(H,H_0)|\alpha\rangle$, where $f(H,H_0)$, $g(H,H_0)$ are both polynomials of H, H_0. Since these Hamiltonians will not change the momentum conservation relation in $[\mathcal{H}(\boldsymbol{x},0),\mathcal{H}(\boldsymbol{y},0)]$ while acting on the initial or final state, we conclude that this will not affect our results.
[22] One can see Chapter 5 in [18] for more details. Since we have proved the existence of relativistic infinite statistics field theory, those analyses based on the Lorentz group are still valid.