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Construction of a topological charge on fuzzy $S^2 \times S^2$ via a Ginsparg-Wilson relation

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We construct a topological charge of gauge field configurations on a fuzzy $S^2 \times S^2$ by using a Dirac operator satisfying the Ginsparg-Wilson relation. The topological charge defined on the fuzzy $S^2 \times S^2$ can be interpreted as a noncommutative (or matrix) generalization of the 2nd Chern character on $S^2 \times S^2$. We further calculate the number of chiral zero modes of the Dirac operator in topologically nontrivial gauge configurations. Generalizations of our formulation to fuzzy $(S^2)^k$ are also discussed.

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I. INTRODUCTION

Noncommutative geometry [1] appears naturally in string theory [2–4], and is also encoded in the matrix model formulations of the string theory [5,6]. In the superstring theory, the size of six dimensions is expected to become tiny and the ten-dimensional spacetime becomes compactified to four dimensions. Then the number of massless fermions, in particular, the number of generations in the four-dimensional spacetime is given by the topology of the six-dimensional compactified space. Then, if the size of the compactified space is as small as the Planck scale, its coordinates may become noncommutative and we will need to generalize the notion of topology to noncommutative spaces.

In ordinary spaces, the topological charge of gauge field configurations can be provided by the index of the Dirac operator, i.e., the difference of the numbers of chiral zero modes, via the index theorem [7]. Generalizations of the index theorem to noncommutative spaces are, however, mostly formulated in spaces with an infinite size, and it is widely believed that topological charges cannot be defined in a system with finite degrees of freedom.

The situation is similar to the lattice gauge theories, where the theory is defined on a finite number of lattice points and the total degrees of freedom are finite. There a problem to properly define the chiral symmetry and the index theorem arises due to the doubling problem [8]. The problem has been solved successfully by introducing Dirac operators satisfying a Ginsparg-Wilson (GW) relation [9]. While all the gauge field configurations are continuously connected and there seems to be no room for defining separate topological sectors in such systems with finite degrees of freedom, the configuration space becomes disconnected by introducing the admissibility condition, and the various topological sectors can then be realized [10].

In a previous paper [11], we have proposed to use the GW relation to define a topological charge and to classify the gauge field configurations in noncommutative spaces with finite degrees of freedom. We have provided a general prescription to construct a GW Dirac operator with a coupling to background gauge fields. As a concrete example, a GW Dirac operator on the fuzzy S^2 was given. (See also [12] for an earlier construction of the GW Dirac operator on fuzzy S^2 without the background gauge field.)[1]

In this paper, we further apply the proposal in Ref. [11] to fuzzy $S^2 \times S^2$. We first construct a GW Dirac operator on fuzzy $S^2 \times S^2$.[2] Owing to the GW relation, the topological charge is given by the index of the Dirac operator. We then study the commutative limit of the topological charge. It becomes a sum of the 2nd Chern character on $S^2 \times S^2$ and the 1st Chern character. We also investigate the chiral zero modes of the Dirac operator for some specific gauge field backgrounds and confirm that the index of the Dirac operator takes the consistent values. We finally generalize our formulation to fuzzy $(S^2)^k$.

The paper is organized as follows. After briefly reviewing the GW relation on fuzzy S^2 in Sec. II, we construct a GW Dirac operator on fuzzy $S^2 \times S^2$ in Sec. III. In Sec. IV, we calculate the commutative limit of the topological charge. We then study the chiral zero modes of the Dirac operator for the free case in Sec. V A, and for the monopole backgrounds in Sec. V B. Here we also introduce a projected topological charge that gives correct values for topologically nontrivial gauge field configurations.

[1]In the case of noncommutative tori, the gauge fields are represented by unitary matrices of Wilson lines and a GW Dirac operator can be constructed similarly to the lattice gauge theory. It was given in [13] and analyzed in [14]. For constructions of the GW Dirac operators in gauge field backgrounds with nontrivial topology, see [15,16] for fuzzy S^2 and [17] for noncommutative tori.

[2]A Dirac operator on fuzzy $S^2 \times S^2$ without the GW relation was given in [18]. Dynamics of gauge theory on fuzzy $S^2 \times S^2$ was studied in [19].

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Generalizations of our formulation to fuzzy $(S^2)^k$ are given in Sec. VI. Section VII is devoted to conclusions and discussions. In Appendices A and B, we give detailed calculations of the commutative limit. In Appendix C, a full spectrum of the Dirac operator for the free case is obtained. A calculation of the topological charge for a modified Dirac operator is given in Appendix D.

II. BRIEF REVIEW OF GW RELATION ON FUZZY S^2

We first briefly review the Ginsparg-Wilson (GW) relation on a fuzzy S^2, following the prescription given in Ref. [11].

Noncommutative coordinates of fuzzy S^2 are given by $x_i = \mu L_i$, where μ is a noncommutative parameter, and L_i is the n-dimensional irreducible representation matrix of the $SU(2)$ algebra. Then we have the relation $(x_i)^2 = \mu^2 \frac{n^2-1}{4} \mathbf{1}_n = \rho^2 \mathbf{1}_n$, where $\rho = \mu\sqrt{(n^2-1)/4}$ expresses the radius of the S^2. The commutative limit is taken by $\mu \to 0$, $n \to \infty$ with ρ fixed.

In our formulation of the GW relation, we first define two chirality operators as

$$\Gamma_X = a(\sigma_i L_i^R - \tfrac{1}{2})_X, \tag{2.1}$$

$$\hat{\Gamma}_X = \frac{H_X}{\sqrt{H_X^2}}, \qquad H_X = a\left(\sigma_i A_i + \frac{1}{2}\right)_X, \tag{2.2}$$

with covariant coordinates

$$(A_i)_X = (L_i + \rho a_i)_X. \tag{2.3}$$

The subscript $X = 1, 2$ will be used for labeling each S^2 of $S^2 \times S^2$ in the following sections, and it can be ignored in the present section. The superscript R in L_i^R means that this operator acts from the right on matrices, while the other operators without the superscript R act from the left. The number $a = 2/n$ serves as a noncommutative analog of the lattice spacing, and σ_i is the Pauli matrix. The matrices a_i in (2.3) represent the gauge field, and the gauge transformation for the covariant coordinate is given by $A_i \to UA_iU^\dagger$. The fermionic fields ψ on which these chiral operators act are in the fundamental representation of the gauge group, and the gauge transformation is given by $\psi \to U\psi$. Hence, both $\Gamma_X\psi$ and $\hat{\Gamma}_X\psi$ transform covariantly as $\Gamma_X\psi \to U\Gamma_X\psi$ and $\hat{\Gamma}_X\psi \to U\hat{\Gamma}_X\psi$. $U(n)$ gauge symmetry can be realized by taking $L_i = L_i \otimes \mathbf{1}$ and $a_i = a_i^a T^a$, where T^a's are the generators of $U(n)$ and a_i^a's are functions of the coordinates L_i.

From the definitions (2.1) and (2.2), the chirality operators satisfy the relations

$$(\Gamma_X)^\dagger = \Gamma_X, \qquad (\hat{\Gamma}_X)^\dagger = \hat{\Gamma}_X, \qquad (\Gamma_X)^2 = (\hat{\Gamma}_X)^2 = 1. \tag{2.4}$$

One can also show that in the commutative limit, both Γ_X and $\hat{\Gamma}_X$ become the same chirality operator $\gamma_X = (n_i\sigma_i)_X$ on a commutative S^2, where $(n_i)_X = (x_i)_X/\rho$ is a unit vector on S^2.

We next define a GW Dirac operator by

$$(D_{\rm GW})_X = -a^{-1}(\Gamma - \hat{\Gamma})_X. \tag{2.5}$$

It satisfies the GW relation

$$(\Gamma D_{\rm GW} + D_{\rm GW}\hat{\Gamma})_X = 0. \tag{2.6}$$

Hence, the index, i.e., the difference of the numbers of the chiral zero modes, is given by the trace of the chirality operators as

$$\text{index}\,((D_{\rm GW})_X) = \tfrac{1}{2}\mathcal{T}r[\Gamma + \hat{\Gamma}]_X. \tag{2.7}$$

Here $\mathcal{T}r$ is the trace in the whole configuration space, that is, over the spinorial index, the gauge group index, and the matrix space representing the coordinates. Since the definition of $\hat{\Gamma}_X$ depends on the gauge field backgrounds, the right-hand side (rhs) of (2.7) gives a noncommutative generalization of the topological charge. Thus, Eq. (2.7) gives an index theorem on fuzzy S^2.

In the commutative limit, the Dirac operator (2.5) becomes

$$(D_{\rm GW})_X \to D'_X = (\sigma_i(\mathcal{L}_i + \rho P_{ij}a_j) + 1)_X, \tag{2.8}$$

where $\mathcal{L}_i = -i\epsilon_{ijk}x_j\partial_k$'s are the derivative operators along the Killing vectors on S^2, and $P_{ij} = \delta_{ij} - n_in_j$ is the projection operator on the tangential directions on S^2. The tangential components of the gauge field a_i represent the gauge field on S^2 while the normal component becomes a scalar field $\phi = n_ia_i$. Because of the GW relation, the Dirac operator is not coupled to the scalar field, since such a coupling would violate the chiral symmetry on S^2 and contradict with the GW relation.

The commutative limit of the topological charge, the rhs of (2.7), is shown to become [11,20]

$$\frac{1}{2}\mathcal{T}r[\Gamma + \hat{\Gamma}]_X \to \rho^2\left(\int \frac{d\Omega}{4\pi}\,\text{tr}(\epsilon_{ijk}n_kF_{ij})\right)_X, \tag{2.9}$$

where tr is the trace over the gauge group. The field strength F_{ij} is defined as $F_{ij} = \partial_i a'_j - \partial_j a'_i - i[a'_i, a'_j]$, where a'_i is the tangential components of the gauge field, given as $a'_i = \epsilon_{ijk}n_ja_k$. This is the integral of the 1st Chern character on a commutative S^2.

In order to construct topologically nontrivial configurations, we need a bit more modification [15,16,20,21]. Consider, for instance, $U(2)$ gauge theory on the fuzzy S^2. Then some gauge field configurations a_i break the $U(2)$ gauge symmetry to $U(1) \times U(1)$. They correspond to nontrivial elements of $\Pi_2(SU(2)/U(1))$ and physically to the 't Hooft-Polyakov–type monopoles. A topological charge can be also constructed by modifying the index theorem (by inserting a projection operator), and it correctly reproduces the topological charge of such configurations. This issue is discussed later in Sec. V B for the case of fuzzy $S^2 \times S^2$.

III. GW RELATION ON FUZZY $S^2 \times S^2$

We now construct a GW Dirac operator and the corresponding topological charge on fuzzy $S^2 \times S^2$.

As in fuzzy S^2, we first define two chirality operators as

$$\Gamma = \Gamma_1 \Gamma_2, \tag{3.1}$$

$$\hat{\Gamma} = \frac{\{\hat{\Gamma}_1, \hat{\Gamma}_2\}}{\sqrt{\{\hat{\Gamma}_1, \hat{\Gamma}_2\}^2}}, \tag{3.2}$$

where Γ_X and $\hat{\Gamma}_X$ with $X = 1, 2$ are the chirality operators on each fuzzy S^2 labeled by X. They are given in (2.1) and (2.2). For simplicity, we take the radii of the two spheres equal. Note that while the index i of the gauge field $(a_i)_X$ refers to each S^2 labeled by X, the gauge field depends on the coordinates of both S^2's, $(L_i)_1$ and $(L_i)_2$.

From (2.1) and (2.2), one has

$$[\Gamma_1, \Gamma_2] = [\Gamma_1, \hat{\Gamma}_2] = [\hat{\Gamma}_1, \Gamma_2] = 0. \tag{3.3}$$

One can also show from (2.4) that

$$\{\hat{\Gamma}_1, \hat{\Gamma}_2\}^2 = 4 + [\hat{\Gamma}_1, \hat{\Gamma}_2]^2, \tag{3.4}$$

where the second term is of order $\mathcal{O}(n^{-4})$, as is shown below (A8).

From the relation of the chirality operator (2.4) on each sphere, the chirality operators (3.1) and (3.2) on $S^2 \times S^2$ also satisfy the same relations

$$(\Gamma)^\dagger = \Gamma, \qquad (\hat{\Gamma})^\dagger = \hat{\Gamma}, \qquad (\Gamma)^2 = (\hat{\Gamma})^2 = 1. \tag{3.5}$$

One can also show that in the commutative limit, both operators, Γ and $\hat{\Gamma}$, become the same chirality operator $\gamma = \gamma_1 \gamma_2$ on a commutative $S^2 \times S^2$. The second term of (3.4) does not contribute to the commutative limit of (3.2) because of the $\mathcal{O}(n^{-4})$ behavior. It should be, however, noted that this term is relevant in calculating the commutative limit of the topological charge.

We then define a GW Dirac operator as

$$D_{\rm GW} = -a^{-1}(\Gamma - \hat{\Gamma}), \tag{3.6}$$

which satisfies the GW relation

$$\Gamma D_{\rm GW} + D_{\rm GW} \hat{\Gamma} = 0 \tag{3.7}$$

and the index theorem

$$\text{index}\,(D_{\rm GW}) = \tfrac{1}{2}\mathcal{T}r[\Gamma + \hat{\Gamma}], \tag{3.8}$$

where $\mathcal{T}r$ is the trace over the whole configuration space, that is, over the spinorial indices of both spheres, the gauge group index, and the matrix space spanned by polynomials of the coordinates $(L_i)_1$ and $(L_i)_2$.

The commutative limit of the Dirac operator can be similarly obtained. Using the relation

$$\Gamma_1\Gamma_2 - \hat{\Gamma}_1\hat{\Gamma}_2 = \tfrac{1}{2}[(\Gamma_1 - \hat{\Gamma}_1)(\Gamma_2 + \hat{\Gamma}_2) + (\Gamma_1 + \hat{\Gamma}_1)(\Gamma_2 - \hat{\Gamma}_2)], \tag{3.9}$$

and (2.8), one can show that in the commutative limit the GW Dirac operator (3.6) becomes

$$D_{\rm GW} \to D'_1 \gamma_2 + \gamma_1 D'_2, \tag{3.10}$$

where D'_X and γ_X are Dirac and chirality operators on each S^2. This is not exactly the same as the ordinary Dirac operator on a commutative $S^2 \times S^2$,[3] but we will show later that the Dirac operator (3.6) suffices to define a topological charge on fuzzy $S^2 \times S^2$.

Our formulation has the following nice properties. First, it is manifestly covariant under the gauge transformation

$$(A_i)_X \to U(A_i)_X U^\dagger \tag{3.11}$$

for both $X = 1, 2$ with a common U, which is a general unitary matrix depending on the coordinates of both spheres, $(L_i)_1$ and $(L_i)_2$. Second, the GW relation assures the topological property of the index and the topological charge. Finally, the formulation has manifest $SO(3) \times SO(3)$ Poincaré invariance on $S^2 \times S^2$. Because of these properties, the commutative limit of the topological charge we have defined should become a sum of the 1st and the 2nd Chern characters on $S^2 \times S^2$. This is what we will show in the next section.

IV. COMMUTATIVE LIMIT OF THE TOPOLOGICAL CHARGE

In this section, we calculate the commutative limit of the topological charge defined in the rhs of (3.8). As we discussed at the end of the previous section, the result should be a linear combination of a constant, the 1st Chern character and the 2nd Chern character.

$\mathcal{T}r[\Gamma]$ is easily calculated as

$$\mathcal{T}r[\Gamma] = 4n^2\,\text{tr}(\mathbf{1}), \tag{4.1}$$

where tr is the trace over the gauge group space.

On the contrary, the evaluation of $\mathcal{T}r[\hat{\Gamma}]$ is more involved. As we show in Appendix A, by expanding it in the gauge fields, it becomes a sum of five terms if we take terms up to order n^{-4}:

[3] Taking the planar limit at the north pole $(n_i)_{X=1} = (n_i)_{X=2} = \delta_{i,3}$, the four-dimensional gamma matrices become $\gamma_1 = (\sigma_1)_{X=1}(\sigma_3)_{X=2}$, $\gamma_2 = (\sigma_2)_{X=1}(\sigma_3)_{X=2}$, $\gamma_3 = (\sigma_3)_{X=1}(\sigma_1)_{X=2}$, $\gamma_4 = (\sigma_3)_{X=1}(\sigma_2)_{X=2}$, and they do not satisfy the $SO(4)$ Clifford algebra. However, if one multiplies the GW Dirac operator (3.6) by Γ_1 from the left in the definition, for instance, then in the commutative limit, the gamma matrices are multiplied by $(\sigma_3)_{X=1}$ from the left, giving $\tilde{\gamma}_1 = i(\sigma_2)_{X=1}(\sigma_3)_{X=2}$, $\tilde{\gamma}_2 = -i(\sigma_1)_{X=1}(\sigma_3)_{X=2}$, $\tilde{\gamma}_3 = (\sigma_1)_{X=2}$, $\tilde{\gamma}_4 = (\sigma_2)_{X=2}$, which satisfy $SO(2,2)$ Clifford algebra.

$$\mathcal{T}r[\hat{\Gamma}] = \mathcal{T}r\left[\sum_{i=1}^{5} G_i + \mathcal{O}(n^{-5})\right]. \tag{4.2}$$

The terms of order $\mathcal{O}(n^{-5})$ vanish in the commutative limit, since the trace $\mathcal{T}r$ gives a contribution of order n^4. Each term is given by

$$G_1 = \alpha_1\alpha_2, \tag{4.3}$$

$$G_2 = \tfrac{1}{2}(\{\alpha_1, \zeta_2^{(1)} + \zeta_2^{(2)}\} + \{\alpha_2, \zeta_1^{(1)} + \zeta_1^{(2)}\}), \tag{4.4}$$

$$G_3 = \tfrac{1}{2}(\{\alpha_1, \zeta_2^{(3)}\} + \{\alpha_2, \zeta_1^{(3)}\}), \tag{4.5}$$

$$G_4 = \tfrac{1}{2}(\{\zeta_1^{(1)} + \zeta_1^{(2)}, \zeta_2^{(1)} + \zeta_2^{(2)}\}), \tag{4.6}$$

$$G_5 = -\tfrac{1}{8}\alpha_1\alpha_2([\alpha_1, \zeta_2^{(1)}] - [\alpha_2, \zeta_1^{(1)}] + [\zeta_1^{(1)}, \zeta_2^{(1)}])^2, \tag{4.7}$$

where α_X and $\zeta_X^{(i)}$ are zeroth and ith order in the gauge field $(a_i)_X$, and are defined by (A2) and (A4)–(A6). The last term G_5 comes from the denominator of (3.2). Contrary to the commutative limit of the chirality operators or the Dirac operator, we should take care of the order $\mathcal{O}(n^{-4})$ term from the denominator.

The first term $\mathcal{T}r[G_1]$ becomes a constant

$$\mathcal{T}r[G_1] = 4n^2\,\mathrm{tr}(\mathbf{1}). \tag{4.8}$$

It is the same as (4.1). The commutative limit of $\mathcal{T}r[G_2]$ can be calculated as in (2.9) for the fuzzy S^2, and gives terms proportional to the 1st Chern character on each sphere:

$$\mathcal{T}r[G_2] \to 2n \cdot 2\rho^2 \int \frac{d\Omega_1}{4\pi}\frac{d\Omega_2}{4\pi}\,\mathrm{tr}(\epsilon_{abc}n_cF_{ab} + \epsilon_{ijk}n_kF_{ij}). \tag{4.9}$$

The indices a, b, and c refer to the first S^2, while the indices i, j, and k refer to the second S^2. Note, however, that the field strength, $F_{ab}(\Omega_1, \Omega_2)$ and $F_{ij}(\Omega_1, \Omega_2)$, can depend on the coordinates of both S^2. In this sense, (4.9) represents a generalized 1st Chern character defined on a commutative $S^2 \times S^2$. Since (4.9) is of order n, the subleading order terms in n^{-1} in G_2 give a finite contribution. The commutative limit of $\mathcal{T}r[G_3]$ also gives a finite contribution. Since these terms vanish for the configurations that will be discussed later, we do not write these terms explicitly in this paper. We will study topological charges for more general configurations in a separate paper.

The commutative limit of $\mathcal{T}r[G_4]$ can also be calculated as in (2.9) and becomes

$$\mathcal{T}r[G_4] \to (2\rho^2)^2 \int \frac{d\Omega_1}{4\pi}\frac{d\Omega_2}{4\pi}\,\mathrm{tr}(\epsilon_{abc}n_cF_{ab}\epsilon_{ijk}n_kF_{ij}). \tag{4.10}$$

Remarkably, as we show in Appendix B, the commutative limit of $\mathcal{T}r[G_5]$ becomes

$$\mathcal{T}r[G_5] \to -8\rho^4 \int \frac{d\Omega_1}{4\pi}\frac{d\Omega_2}{4\pi}\,\mathrm{tr}(\epsilon_{abc}n_c\epsilon_{ijk}n_kF_{ai}F_{bj}). \tag{4.11}$$

Note that the field strengths with indices from different spheres, F_{ai} and F_{bj}, arise here. Combining these two terms we obtain

$$\mathcal{T}r[G_4 + G_5] \to 4\rho^4 \int \frac{d\Omega_1}{4\pi}\frac{d\Omega_2}{4\pi}\epsilon_{abc}n_c\epsilon_{ijk}n_k \times \mathrm{tr}(F_{ab}F_{ij} - F_{ai}F_{bj} + F_{aj}F_{bi}). \tag{4.12}$$

This gives an integral of the 2nd Chern character on a commutative $S^2 \times S^2$.

To summarize, the commutative limit of the topological charge on $S^2 \times S^2$ becomes

$$\frac{1}{2}\mathcal{T}r[\Gamma + \hat{\Gamma}] \to 4n^2\,\mathrm{tr}(\mathbf{1}) + 2n\rho^2 \int \frac{d\Omega_1}{4\pi}\frac{d\Omega_2}{4\pi}\mathrm{tr}(\epsilon_{abc}n_cF_{ab} + \epsilon_{ijk}n_kF_{ij}) + 2\rho^4 \int \frac{d\Omega_1}{4\pi}\frac{d\Omega_2}{4\pi}\epsilon_{abc}n_c\epsilon_{ijk}n_k \times \mathrm{tr}(F_{ab}F_{ij} - F_{ai}F_{bj} + F_{aj}F_{bi}). \tag{4.13}$$

In the differential forms, it is rewritten as

$$\frac{1}{(2\pi)^2}\int \mathrm{tr}\left(n^2 d\Omega_1 d\Omega_2 + n(d\Omega_1 F_2 + d\Omega_2 F_1) + 2\frac{1}{2!}(F^2)_{12}\right), \tag{4.14}$$

with

$$F_X = \tfrac{1}{2}\rho^2(d\Omega\,\epsilon_{ijk}n_kF_{ij})_X, \tag{4.15}$$

$$(F^2)_{XY} = \frac{2!}{2^2}\rho^4 d\Omega_X d\Omega_Y(\epsilon_{abc}n_c\epsilon_{ijk}n_k(F_{ab}F_{ij} - F_{ai}F_{bj} + F_{aj}F_{bi}))_{XY}. \tag{4.16}$$

Here $d\Omega_X$ is the volume form on each S^2. In the flat limit, F and F^2 become familiar forms on each S^2 and $S^2 \times S^2$, respectively:

$$F_X \to \tfrac{1}{2}(F_{\mu\nu}dx_\mu \wedge dx_\nu)_X, \tag{4.17}$$

$$(F^2)_{XY} \to \frac{1}{2^2}(F_{\mu\nu}F_{\lambda\rho}dx_\mu \wedge dx_\nu \wedge dx_\lambda \wedge dx_\rho)_{XY}. \tag{4.18}$$

The first and the second terms in (4.13) and (4.14) are proportional to n^2 and n, respectively, and they diverge in the commutative (large n) limit. The third term is also twice the 2nd Chern character, and the topological charge we have defined by the GW Dirac operator is different from the index of the ordinary Dirac operator on $S^2 \times S^2$. This is because the Dirac operator is different from the ordinary one as we discussed below (3.10). We will discuss origins of each term in (4.13) and (4.14) by investigating the chiral zero modes in the following sections.

While the topological charge defined this way contains various topological invariants, we can, nevertheless, extract the 2nd Chern character. In order to define a noncommutative analog of the 2nd Chern character on $S^2 \times S^2$, we subtract the extra pieces as follows:

$$\frac{1}{4}\mathcal{T}r[\Gamma + \hat{\Gamma}] - \frac{1}{4n}\mathcal{T}r[\Gamma_1 + \hat{\Gamma}_1] - \frac{1}{4n}\mathcal{T}r[\Gamma_2 + \hat{\Gamma}_2] - \frac{1}{2n^2}\mathcal{T}r[\mathbf{1}]. \tag{4.19}$$

Each term is a topological invariant on fuzzy $S^2 \times S^2$ and is well defined on it.

V. CHIRAL ZERO MODES

In this section, we explicitly calculate the number of chiral zero modes in some specific configurations and compare it with the topological charge in the commutative limit, (4.13) or (4.14). Especially we discuss why the index explicitly depends on the size n of the matrices.

A. Chiral zero modes for the free case

We first investigate the chiral zero modes of the GW Dirac operator for the free case where the gauge field vanishes. Even in the absence of the gauge field, there exist chiral zero modes of the GW Dirac operator and they give the first term of (4.13) or (4.14). We here consider $U(1)$ gauge group, for simplicity.

In the free case, we have a simple relation $[\hat{\Gamma}_1, \hat{\Gamma}_2] = 0$, and the chirality operator (3.2) can be simplified as $\hat{\Gamma} = \hat{\Gamma}_1\hat{\Gamma}_2$. Using the relation (3.9), the GW Dirac operator (3.6) is also simplified as

$$D_{\rm GW} = D_1 + D_2, \tag{5.1}$$

where

$$D_1 = -\tfrac{1}{2}a^{-1}(\Gamma_1 - \hat{\Gamma}_1)(\Gamma_2 + \hat{\Gamma}_2),$$
$$D_2 = -\tfrac{1}{2}a^{-1}(\Gamma_1 + \hat{\Gamma}_1)(\Gamma_2 - \hat{\Gamma}_2). \tag{5.2}$$

Using (3.3), $[\hat{\Gamma}_1, \hat{\Gamma}_2] = 0$, and (2.4), one can easily show the following GW relations for each D_a:

$$\Gamma D_1 + D_1\hat{\Gamma} = 0, \qquad \Gamma D_2 + D_2\hat{\Gamma} = 0, \tag{5.3}$$

where Γ and $\hat{\Gamma}$ are the chirality operators on the fuzzy $S^2 \times S^2$ defined in (3.1) and (3.2). One can also show

$$[D_1, D_2] = 0. \tag{5.4}$$

Now consider states with zero eigenvalues of the Dirac operator $D_{\rm GW}$. The chirality operators can also be diagonalized in this space owing to the GW relation (3.7). Hence, we consider a state $|\psi\rangle$ satisfying

$$D_{\rm GW}|\psi\rangle = 0, \qquad \Gamma|\psi\rangle = \hat{\Gamma}|\psi\rangle = \pm|\psi\rangle. \tag{5.5}$$

Then from (5.3) and (5.4), we have

$$D_{\rm GW}D_a|\psi\rangle = 0, \qquad \Gamma D_a|\psi\rangle = \hat{\Gamma}D_a|\psi\rangle = \mp D_a|\psi\rangle, \tag{5.6}$$

for $a = 1, 2$. Therefore, if either $D_1|\psi\rangle \neq 0$ or $D_2|\psi\rangle \neq 0$ is satisfied, the contributions to the index of $D_{\rm GW}$ cancel each other by $|\psi\rangle$ and $D_a|\psi\rangle$. Thus a chiral zero mode that can contribute to the index must satisfy $D_1|\psi\rangle = 0$ and $D_2|\psi\rangle = 0$. From (5.2), a zero mode of D_1 is given by a zero mode of $\Gamma_1 - \hat{\Gamma}_1$ or a zero mode of $\Gamma_2 + \hat{\Gamma}_2$, owing to $[\Gamma_1 - \hat{\Gamma}_1, \Gamma_2 + \hat{\Gamma}_2] = 0$. Similarly, a zero mode of D_2 is given by that of $\Gamma_1 + \hat{\Gamma}_1$ or $\Gamma_2 - \hat{\Gamma}_2$.

We then study each fuzzy S^2 separately in order to find zero modes of the operators $(\Gamma_X \pm \hat{\Gamma}_X)$. Our formulation has $SO(3)$ Poincaré invariance on each S^2, whose generators are written as

$$(M_i)_X = \left(L_i - L_i^R + \frac{\sigma_i}{2}\right)_X. \tag{5.7}$$

We then consider the eigenstates of the Casimir operator $\sum_i (M_i)_X^2$ as

$$\sum_i (M_i)_X^2|J_X\rangle = J_X(J_X + 1)|J_X\rangle. \tag{5.8}$$

One can show from the $SU(2)$ algebra of (5.7) that the spin J_X takes values $J_X = \frac{1}{2}, \frac{3}{2}, \ldots, n - \frac{1}{2}$. There are some degeneracies in the states $|J_X\rangle$. In addition to the $(2J_X + 1)$-folded degeneracy associated with $(M_3)_X$, the state $|J_X\rangle$ has a two-folded degeneracy for $J_X = \frac{1}{2}, \frac{3}{2}, \ldots, n - \frac{3}{2}$. The highest spin state with $J_X = n - \frac{1}{2}$, however, does not have this two-folded degeneracy. As we show in detail in Appendix C, we can see that the Dirac operator $(\Gamma - \hat{\Gamma})_X$ on each S^2 does not have a zero mode at all in the free case. On the other hand, the operator $(\Gamma + \hat{\Gamma})_X$ does have zero modes in the highest spin states with $J_X = n - \frac{1}{2}$. [See a comment below (C9).] One can also show that $\Gamma_X|J_X = n - \frac{1}{2}\rangle = -\hat{\Gamma}_X|J_X = n - \frac{1}{2}\rangle = -|J_X = n - \frac{1}{2}\rangle$.

Therefore, coming back to the fuzzy $S^2 \times S^2$, the chiral zero modes of the Dirac operator $D_{\rm GW}$ are given by the highest spin states with $J_1 = J_2 = n - \frac{1}{2}$. The chirality defined by an eigenvalue of (3.1) and (3.2) is 1 for all of these states. The degeneracy of these states is $(2J_1 + 1) \times (2J_2 + 1) = 4n^2$, which indeed gives the first term of (4.13).

In the commutative limit, the operator $(\Gamma + \hat{\Gamma})_X$ becomes proportional to the chirality operator on each S^2 and does not have zero modes. In the case of the fuzzy S^2, the highest spin states have nonzero eigenvalues of the GW Dirac operator (2.5) and do not contribute to the index.[4] In

[4]The highest spin states have zero eigenvalues of the Dirac operator with exact chirality [22], but have nonzero eigenvalues of the Dirac operator introduced in [23] and the GW Dirac operator (2.5).

fuzzy $S^2 \times S^2$, however, as we have shown above, these states become zero modes of the Dirac operator (5.1) since it contains the operator $(\Gamma + \hat{\Gamma})_X$. This is the reason why, even for the free case, there is a nonvanishing term in the topological charge defined by the Dirac operator.

B. Monopole configurations and chiral zero modes

In this section, we consider a monopole configuration as topologically nontrivial gauge field configurations. We also introduce a modified index theorem and a topological charge that gives nonvanishing values for such configurations. We then investigate the chiral zero modes of the GW Dirac operator in these backgrounds.

In the case of the fuzzy S^2, we constructed a 't Hooft-Polyakov monopole configuration where the gauge symmetry group $U(2)$ is spontaneously broken down to $U(1) \times U(1)$ [15,16,20,21]. Since the diagonal $U(1)$ is decoupled in the commutative limit, we discuss only the $SU(2)$ part of the gauge group in the following. With the $SU(2)$ gauge group broken down to $U(1)$, this configuration is interpreted as the 't Hooft-Polyakov type monopole containing both of the scalar field with a nonvanishing vev and the monopole gauge field configuration on S^2.

Analogously, we now consider $U(2) \times U(2)$ gauge theory on fuzzy $S^2 \times S^2$. In the presence of the monopole configuration, the gauge symmetry is spontaneously broken from $SU(2) \times SU(2)$ to $U(1) \times U(1)$. The monopole configuration we will investigate is the following:

$$(A_a)_1 = L_a \otimes \mathbf{1}_2 \otimes \mathbf{1}_2 + \mathbf{1}_n \otimes \frac{\tau_a}{2} \otimes \mathbf{1}_2 \doteq \begin{pmatrix} L_a^{(n+1)} & \\ & L_a^{(n-1)} \end{pmatrix} \otimes \mathbf{1}_2, \tag{5.9}$$

$$(A_i)_2 = L_i \otimes \mathbf{1}_2 \otimes \mathbf{1}_2 + \mathbf{1}_n \otimes \mathbf{1}_2 \otimes \frac{\tau_i}{2}, \tag{5.10}$$

where $(A_a)_1$ and $(A_i)_2$ are covariant coordinates of the first and the second sphere. The second and the third factors in the tensor product refer to spin $1/2$ representation of each $SU(2)$ in the $SU(2) \times SU(2)$ gauge group, respectively. The equality $\doteq$ means a unitary equivalence, and we have combined the first two spaces, i.e., matrix space representing the coordinates and the first $SU(2)$ space, into a single matrix representation. $(A_i)_2$ can be similarly written. Each of the configurations describes the 't Hooft-Polyakov type monopole on each S^2, and wraps around the S^2. The normal components of the gauge fields, which are interpreted as two scalar fields on $S^2 \times S^2$, have nonvanishing vev's and break the gauge symmetry.

More generally, we can consider the following type of configurations:

$$(A_a)_1 = \begin{pmatrix} L_a^{(n+m_1)} & \\ & L_a^{(n-m_1)} \end{pmatrix} \otimes \mathbf{1}_2. \tag{5.11}$$

A generalized $(A_i)_2$ can be written similarly. For such configurations, the relation $[(A_a)_1, (A_i)_2] = 0$ is satisfied.

Although they are the noncommutative analogs of topologically nontrivial configurations, the topological charge defined in (3.8) vanishes for these configurations. This can be understood as follows: In the presence of the monopole configurations, the gauge group is spontaneously broken from $SU(2) \times SU(2)$ to $U(1) \times U(1)$. A fermionic field in the fundamental representation of each $SU(2)$ is decomposed into two fermions with opposite electric charges $\pm 1/2$ of each of the unbroken $U(1)$'s, and they cancel the topological charge, or the index of the Dirac operator.

We thus have to modify the index theorem (3.8) to pick up one of the fermions with $\pm 1/2$ electric charges. As is shown in Sec. II D of Ref. [15], we can prove the following index theorem in the projected space:

$$\mathrm{index}(P_1^{(n\pm m_1)} P_2^{(n\pm m_2)} D_{\mathrm{GW}}) = \frac{1}{2}\mathcal{T}r[P_1^{(n\pm m_1)} P_2^{(n\pm m_2)}(\Gamma + \hat{\Gamma})], \tag{5.12}$$

where $P_X^{(n\pm m_X)}$ is the projection operator on the Hilbert space with $n \pm m_X$ dimensions in (5.11). The projection operator is written as

$$P_X^{(n\pm m_X)} = \frac{1}{2}(1 \pm T_X), \tag{5.13}$$

with

$$T_X = \frac{2}{nm_X}\left((A_X)^2 - \frac{n^2 + m_X^2 - 1}{4}\right) = \begin{pmatrix} \mathbf{1}_{n+m_X} & \\ & -\mathbf{1}_{n-m_X} \end{pmatrix}. \tag{5.14}$$

Here we have left out the extra $\mathbf{1}_2$. The operator T_X is interpreted as an electric charge operator of the unbroken $U(1)$ gauge group. Its commutative limit becomes the normalized scalar field as

$$T_X \to 2\phi'_X, \tag{5.15}$$

where $\phi'_X = \phi'^a_X \frac{\tau^a}{2}$ with $\sum_a (\phi'^a_X)^2 = 1$. Without loss of generality, we hereafter consider only the following projection:

$$P_X^{(n+m_X)} \equiv P_X \tag{5.16}$$

with $m_X > 0$.

Following the same calculation that led us to (4.13) in Sec. IV, the commutative limit of the rhs of (5.12) becomes

$$\begin{aligned}
\frac{1}{2}\mathcal{T}r[P_1 P_2(\Gamma + \hat{\Gamma})] \to{}& 4(n + m_1)(n + m_2) + 2(n + m_1)\rho^2 \\
&\times \int \frac{d\Omega_2}{4\pi} \epsilon_{ijk} n_k \,\mathrm{tr}_2(\phi'_2 F_{ij}) + 2(n + m_2)\rho^2 \\
&\times \int \frac{d\Omega_1}{4\pi} \epsilon_{abc} n_c \,\mathrm{tr}_1(\phi'_1 F_{ab}) + 2\rho^4 \int \frac{d\Omega_1}{4\pi}\frac{d\Omega_2}{4\pi} \\
&\times \epsilon_{abc} n_c \epsilon_{ijk} n_k \,\mathrm{tr}_1(\phi'_1 F_{ab})\,\mathrm{tr}_2(\phi'_2 F_{ij}),
\end{aligned} \tag{5.17}$$

where the tr_X stands for the trace over the $SU(2)_X$ gauge group. The monopole configuration (5.11) has the monopole number $(-m_X)$, and its 1st Chern character on each S^2 becomes $(-m_X)$, as is shown below in (5.21). Then, (5.17) becomes

$$\begin{aligned} &4(n+m_1)(n+m_2)+2(n+m_1)(-m_2)\\ &\quad +2(n+m_2)(-m_1)+2(-m_1)(-m_2)\\ &\quad =4n^2+2n(m_1+m_2)+2m_1m_2. \end{aligned} \tag{5.18}$$

In the following, we will calculate the both-hand sides of (5.12) at the matrix level, i.e., before taking the commutative limit, and check that the result agrees with (5.18). We then investigate what kind of chiral zero modes contribute to each term in (5.17).

We first calculate the rhs of (5.12). Because of the relation $[(A_a)_1, (A_i)_2] = 0$, it can be written as

$$\tfrac{1}{2}(\mathcal{T}r_1[P_1\Gamma_1]\mathcal{T}r_2[P_2\Gamma_2]+\mathcal{T}r_1[P_1\hat{\Gamma}_1]\mathcal{T}r_2[P_2\hat{\Gamma}_2]). \tag{5.19}$$

Each factor can be evaluated as

$$\mathcal{T}r_X[P_X\Gamma_X]=-2(n+m_X),\qquad \mathcal{T}r_X[P_X\hat{\Gamma}_X]=2n. \tag{5.20}$$

The operator Γ_X takes its eigenvalue ± 1 in $n \mp 1$ dimensional representation space of the operator $-L_i^R+\sigma_i/2$. By counting the total dimensions of the space, including the space on which PA_i acts, one obtains the first result. The second result is similarly obtained. [See Eqs. (3.34) and (3.36) in [20].] Then one can obtain the monopole charge on each S^2 as

$$\tfrac{1}{2}\mathcal{T}r[P_X(\Gamma_X+\hat{\Gamma}_X)]=-m_X. \tag{5.21}$$

Substituting (5.20) into (5.19), we obtain

$$\tfrac{1}{2}((-2(n+m_1))(-2(n+m_2))+(2n)^2), \tag{5.22}$$

which indeed agrees with the above calculation in the commutative limit (5.18).

We next calculate the left-hand side (lhs) of (5.12) by counting the chiral zero modes of the GW Dirac operator in the monopole backgrounds. The commutativity $[\hat{\Gamma}_1, \hat{\Gamma}_2] = 0$ holds because of the relation $[(A_a)_1, (A_i)_2] = 0$. Then, the chirality operator (3.2) reduces to $\hat{\Gamma} = \hat{\Gamma}_1\hat{\Gamma}_2$, and the GW Dirac operator in the projected space becomes

$$P_1P_2D_{\mathrm{GW}}=P_1P_2D_1+P_1P_2D_2 \tag{5.23}$$

with D_1 and D_2 given in (5.2). The arguments we have given in the free case can be applied to the present case, and it is sufficient to investigate the zero modes of the operators $P_X(\Gamma_X-\hat{\Gamma}_X)$ and $P_X(\Gamma_X+\hat{\Gamma}_X)$ on each fuzzy S^2.

We then classify the states in terms of the Casimir operator of the $SO(3)$ Poincaré symmetry on each S^2. Generators of the $SO(3)$ symmetry are given by

$$(M_i)_X=\left(PA_i-L_i^R+\frac{\sigma_i}{2}\right)_X, \tag{5.24}$$

where A_i's are generalized monopole configurations (5.11). We consider eigenstates of the Casimir operator $\sum_i(M_i)^2$ as in (5.8). As is shown in detail in Sec. III of Ref. [15], in addition to the $(2J_X+1)$-folded degeneracy, the state $|J_X\rangle$ has an extra two-folded degeneracy for $J_X=\frac{m+1}{2},\frac{m+3}{2},\ldots,n+\frac{m-3}{2}$, while the lowest spin sate with $J_X=\frac{m-1}{2}$ and the highest spin state with $J_X=n+\frac{m-1}{2}$ do not have such two-folded degeneracy. The lowest spin states are shown to be zero modes of the operator $P_X(\Gamma-\hat{\Gamma})_X$, while the highest spin states are zero modes of the operator $P_X(\Gamma+\hat{\Gamma})_X$. The other states have nonzero eigenvalues for both of these operators. One can also show that $\Gamma_X|J_X=\frac{m-1}{2}\rangle=\hat{\Gamma}_X|J_X=\frac{m-1}{2}\rangle=-|J_X=\frac{m-1}{2}\rangle$ and that $\Gamma_X|J_X=n+\frac{m-1}{2}\rangle=-\hat{\Gamma}_X|J_X=n+\frac{m-1}{2}\rangle=-|J_X=n+\frac{m-1}{2}\rangle$.

Consequently, coming back to the fuzzy $S^2\times S^2$, the chiral zero modes of the GW Dirac operator $P_1P_2D_{\mathrm{GW}}$ in the monopole background (5.11) are given by the lowest spin states with $J_1=J_2=\frac{m-1}{2}$ and the highest spin states with $J_1=J_2=n+\frac{m-1}{2}$. The chirality defined by an eigenvalue of (3.1) and (3.2) is 1 for all of these states. The index of the Dirac operator $P_1P_2D_{\mathrm{GW}}$ is, therefore, given by counting the degeneracy of these states as

$$m_1m_2+(2n+m_1)(2n+m_2). \tag{5.25}$$

This again agrees with the topological charge in the commutative limit (5.18). Incidentally, the states with $J_1=\frac{m-1}{2}$, $J_2=n+\frac{m-1}{2}$ have nonzero eigenvalues of the operator $P_1P_2D_2$, and hence do not give chiral zero modes of the Dirac operator $P_1P_2D_{\mathrm{GW}}$. Neither do the states with $J_1=n+\frac{m-1}{2}$, $J_2=\frac{m-1}{2}$ contribute to chiral zero modes of the Dirac operator $P_1P_2D_{\mathrm{GW}}$.

Note that the lowest spin states are responsible for the first term of (5.25). This is half of the last term in the rhs of (5.18), and exactly matches with an integral of the 2nd Chern character in the monopole background we are considering. This is reasonable since the lowest spin states correspond to the chiral zero modes of the Dirac operator in the commutative theory. All the other contributions to the zero modes in (5.25), and hence in (5.18) and (5.17), come from the highest spin states, which do not have corresponding chiral zero modes in the commutative theory. Going back to the formula (4.13), we can similarly infer the origins of various terms.

VI. GENERALIZATION TO FUZZY $(S^2)^k$

In this section, we generalize our formulation to fuzzy $(S^2)^k$. As in the fuzzy $S^2\times S^2$, we first define two chirality operators as

$$\Gamma=\Gamma_1\cdots\Gamma_k, \tag{6.1}$$

$$\hat{\Gamma} = \frac{\hat{\Gamma}_1 \cdots \hat{\Gamma}_k + \hat{\Gamma}_k \cdots \hat{\Gamma}_1}{\sqrt{(\hat{\Gamma}_1 \cdots \hat{\Gamma}_k + \hat{\Gamma}_k \cdots \hat{\Gamma}_1)^2}}, \tag{6.2}$$

which satisfy (3.5). As in (3.4), the denominator is written as

$$(\hat{\Gamma}_1 \cdots \hat{\Gamma}_k + \hat{\Gamma}_k \cdots \hat{\Gamma}_1)^2 = 4 + (\hat{\Gamma}_1 \cdots \hat{\Gamma}_k - \hat{\Gamma}_k \cdots \hat{\Gamma}_1)^2. \tag{6.3}$$

The second term is of order $\mathcal{O}(n^{-4})$, since $[\hat{\Gamma}_X, \hat{\Gamma}_Y]$ is of order $\mathcal{O}(n^{-2})$ as is shown below (A8). We then define a GW Dirac operator as in (3.6). It satisfies the GW relation (3.7) and the index theorem (3.8).

Analogously to (3.9), the following relation is satisfied:

$$\begin{aligned}\Gamma_1 \cdots \Gamma_k - \hat{\Gamma}_1 \cdots \hat{\Gamma}_k &= \frac{1}{2^{k-1}} \sum_{n_1, \cdots, n_k = 0,1} \frac{1}{2}(1 - (-1)^{\sum_{X=1}^k n_X}) \\ &\quad \times \prod_{X=1}^k (\Gamma_X + (-1)^{n_X} \hat{\Gamma}_X),\end{aligned} \tag{6.4}$$

where the product respects the ordering of operators from $X = 1$ to $X = k$. The coefficient $(1 - (-1)^{\sum_{X=1}^k n_X})$ ensures the number of operators $(\Gamma - \hat{\Gamma})_X$ in the product to be odd. Since Γ_X and $\hat{\Gamma}_X$ become the same chirality operator in the commutative limit, those terms with smaller number of $(\Gamma - \hat{\Gamma})_X$ in (6.4) are more dominant in the commutative limit.

Then, as in (3.10), the commutative limit of the GW Dirac operator (3.6) becomes

$$\begin{aligned}D_{\rm GW} \to D_1' \gamma_2 \cdots \gamma_k + \gamma_1 D_2' \gamma_3 \cdots \gamma_k + \cdots \\ + \gamma_1 \cdots \gamma_{k-1} D_k',\end{aligned} \tag{6.5}$$

where only the terms with one of the n_X's being 1 in (6.4) contribute. This is a generalized Dirac operator on a commutative $(S^2)^k$. [See the discussion after Eq. (3.10).]

The commutative limit of the topological charge, the rhs of (3.8), gives a generalization of (4.13) and (4.14). We now conjecture the result as follows:

$$\begin{aligned}\frac{1}{2}\mathcal{T}r[\Gamma + \hat{\Gamma}] \to (1 + (-1)^k) 2^{k-1} n^k \,{\rm tr}(\mathbf{1}) \\ + 2^{k-1} \sum_{i=1}^k n^{k-i} C_i.\end{aligned} \tag{6.6}$$

The coefficient $(1 + (-1)^k)$ in the first term represents that this term vanishes when k is odd. This is because the contributions of the two chirality operators cancel for odd k. The integral of the ith Chern character C_i is defined as

$$\begin{aligned}C_i = \frac{1}{(2\pi)^k i!} \int {\rm tr}\bigg[\sum_{1 \le X_1 < \cdots < X_i \le k} \bigg(\prod_{X \notin (X_1 \cdots X_i)} \frac{d\Omega_X}{2} \\ \times (F^i)_{X_1 \cdots X_i}\bigg)\bigg].\end{aligned} \tag{6.7}$$

For instance, $(F)_X$ and $(F^2)_{XY}$ are given in (4.15) and (4.16), and $(F^3)_{XYZ}$ is written as

$$\begin{aligned}&\frac{3!}{2^3}\rho^6 d\Omega_X d\Omega_Y d\Omega_Z (\epsilon_{abc} n_c \epsilon_{ijk} n_k \epsilon_{xyz} n_z (F_{ab} F_{ij} F_{xy} \\ &\quad - F_{ai} F_{bj} F_{xy} + F_{aj} F_{bi} F_{xy} - F_{ab} F_{ix} F_{jy} \\ &\quad + F_{ab} F_{iy} F_{jx} - F_{ax} F_{by} F_{ij} + F_{ay} F_{bx} F_{ij} \\ &\quad + F_{ai} F_{bx} F_{jy} - F_{ai} F_{by} F_{jx} - F_{aj} F_{bx} F_{iy} \\ &\quad + F_{aj} F_{by} F_{ix} - F_{ax} F_{bi} F_{jy} + F_{ay} F_{bi} F_{jx} \\ &\quad + F_{ax} F_{bj} F_{iy} - F_{ay} F_{bj} F_{ix}))_{XYZ},\end{aligned} \tag{6.8}$$

where the indices a, b, and c refer to the sphere X, the indices i, j, and k to the sphere Y, and the indices x, y, and z to the sphere Z. Note, however, that the field strength depends on all of the coordinates, such as $F_{ab}(\Omega_1, \cdots, \Omega_k)$. Only the highest Chern character term in (6.6) is independent of the size n of the matrix. It is important to show the conjecture (6.6) explicitly by taking the commutative limit as we did for the fuzzy $S^2 \times S^2$ in Sec. IV. It needs involved calculations and we will report it in a future publication.

We here demonstrate the justification of (6.6) by considering a topologically nontrivial configuration, i.e., a monopole configuration in $(SU(2))^k$ gauge theory on fuzzy $(S^2)^k$. It is a generalization of (5.11). As in (5.12), we consider the index theorem in the projected space

$$\text{index}\,(P_1 \cdots P_k D_{\rm GW}) = \tfrac{1}{2}\mathcal{T}r[P_1 \cdots P_k(\Gamma + \hat{\Gamma})]. \tag{6.9}$$

If the conjecture (6.6) holds, then as in (5.17), the commutative limit of the rhs of (6.9) becomes

$$\begin{aligned}&\frac{1}{2}\mathcal{T}r[P_1 \cdots P_k(\Gamma + \hat{\Gamma})] \\ &\quad \to (1 + (-1)^k) 2^{k-1} \prod_{X=1}^k (n + m_X) \\ &\quad + 2^{k-1} \sum_{i=1}^k \bigg[\sum_{1 \le X_1 < \cdots < X_i \le k} \bigg(\prod_{X \notin (X_1 \cdots X_i)} (n + m_X) \\ &\quad \times \prod_{X \in (X_1 \cdots X_i)} \rho^2 \bigg(\int \frac{d\Omega}{4\pi} \epsilon_{ijk} n_k \,{\rm tr}(\phi' F_{ij})\bigg)_X\bigg)\bigg].\end{aligned} \tag{6.10}$$

The monopole on each S^2 gives the 1st Chern character $(-m_X)$. Following the same calculation as in (5.18) and (6.10) becomes

$$(1 + (-1)^k)2^{k-1}n^k + (-1)^k 2^{k-1} \sum_{i=1}^{k} n^{k-i} \sum_{1\le X_1<\cdots<X_i\le k} m_{X_1}\cdots m_{X_i}. \quad (6.11)$$

In the following, we will evaluate the both-hand sides of (6.9) at the matrix level, i.e., before taking the commutative limit, and show that the results agree with the conjectured topological charge in the commutative limit (6.11).

Following the same calculations in (5.19) and (5.22), the rhs of (6.9) for the monopole background becomes

$$\frac{1}{2}\Big(\prod_{X=1}^{k} \mathcal{T}r_X[P_X\Gamma_X] + \prod_{X=1}^{k} \mathcal{T}r_X[P_X\hat{\Gamma}_X]\Big) = \frac{1}{2}\Big(\prod_{X=1}^{k}(-2(n+m_X)) + (2n)^k\Big), \quad (6.12)$$

which indeed gives (6.11).

We can also evaluate the lhs of (6.9) by counting the chiral zero modes of the Dirac operator. Denoting each term in (6.4) as D_a with $a = 1, \ldots, 2^{k-1}$, we obtain a generalization of Eq. (5.23). The same arguments we have given in the $S^2 \times S^2$ case hold in the present case: A chiral zero mode of the Dirac operator $P_1\cdots P_k D_{GW}$ must be a simultaneous zero mode of all the operators $P_1\cdots P_k D_a$ with $a = 1, \cdots, 2^{k-1}$. A zero mode of $P_1\cdots P_k D_a$ is given by a zero mode of any of the operators $P_X(\Gamma + \hat{\Gamma})_X$ and $P_X(\Gamma - \hat{\Gamma})_X$ constituting $P_1\cdots P_k D_a$. The lowest spin states with $J_X = \frac{m-1}{2}$ are zero modes of the operator $P_X(\Gamma - \hat{\Gamma})_X$, and the highest spin states with $J_X = n + \frac{m-1}{2}$ are zero modes of the operator $P_X(\Gamma + \hat{\Gamma})_X$. Eventually, we find that the chiral zero modes of the Dirac operator $P_1\cdots P_k D_{GW}$ are given by the states where an even number of J_X's are the highest spin and the remaining J_X's are the lowest spin. The chirality defined by an eigenvalue of (3.1) and (3.2) is 1 for all of these states when k is even, and -1 when k is odd. By counting the number of these states as in (5.25), the index of the Dirac operator $P_1\cdots P_k D_{GW}$ is evaluated as

$$(-1)^k \sum_{i=0,2,\cdots}\Big[\sum_{1\le X_1<\cdots<X_i\le k}\Big(\prod_{X\in(X_1,\cdots,X_i)}(2n+m_X) \times \prod_{X\notin(X_1,\cdots,X_i)} m_X\Big)\Big]. \quad (6.13)$$

This again reproduces the result (6.11). Incidentally, the states with an odd number i of J_X being the highest spin, which we call $J_{X_1}, \ldots, J_{X_i}$, have nonzero eigenvalues of the operator $P_1\cdots P_k D_a$ that is composed of $(\Gamma - \hat{\Gamma})_X$ with $X \in (X_1, \ldots, X_i)$ and $(\Gamma + \hat{\Gamma})_X$ with $X \notin (X_1, \ldots, X_i)$. Those states thus do not contribute to the chiral zero modes of the Dirac operator $P_1\cdots P_k D_{GW}$. We also note that the states with all J_X being the lowest spin are responsible for the term with $i = 0$ in (6.13), giving $\prod_{X=1}^{k}(-m_X)$, which agrees precisely with the kth Chern character of the background gauge fields we are considering. This is reasonable since these states correspond to the chiral zero modes in the commutative theory.

The agreement of (6.12) and (6.13) to (6.11) supports the conjecture (6.10), and hence (6.6).

VII. CONCLUSIONS AND DISCUSSIONS

In this paper, we have constructed a topological charge on the fuzzy $(S^2)^k$ based on a Dirac operator satisfying the GW relation. Our formulation has the manifest gauge invariance and the $SO(3)$ Poincaré invariance on each S^2. Owing to the GW relation, the index theorem is satisfied and accordingly we can construct the topological charge. The commutative limit of the topological charge was evaluated directly for the fuzzy $S^2 \times S^2$, and it becomes a sum of the 1st and the 2nd Chern characters. We then have shown that by combining with other topological invariants we can define a noncommutative generalization of the 2nd Chern character. We also conjectured a form of the commutative limit of the topological charge on fuzzy $(S^2)^k$ for $k > 2$.

We further calculated the chiral zero modes of the Dirac operator for the free case and for the monopole backgrounds, and checked the consistency of our results. The zero modes of the noncommutative GW Dirac operator on fuzzy $(S^2)^k$ consist of the highest spin states and the lowest spin states. The lowest spin states correspond to the zero modes of the commutative Dirac operator. On the other hand, the highest spin states are zero modes of the operator $(\Gamma + \hat{\Gamma})_X$ and do not have the correspondents in the commutative limit. We have indeed found that the chiral zero modes composed of only the lowest spin states give precisely the kth Chern character on $(S^2)^k$.

Some comments are in order. In the definition of $\hat{\Gamma}$ in (3.2), we first normalized both of $\hat{\Gamma}_X$ in (2.2), and then constructed the normalized chirality operator $\hat{\Gamma}$ on $S^2 \times S^2$ in (3.2). Instead, we can directly construct a normalized operator on $S^2 \times S^2$ as

$$\hat{\Gamma}' = \frac{\{H_1, H_2\}}{\sqrt{\{H_1, H_2\}^2}}, \quad (7.1)$$

with H_X defined in (2.2). Defining a Dirac operator as in (3.6), with $\hat{\Gamma}$ replaced by $\hat{\Gamma}'$, the GW relation (3.7) and the index theorem (3.8) are satisfied as well. Moreover, as we show in Appendix D, the commutative limit of the Dirac operator and the topological charge give exactly the same result as (3.10) and (4.13). This agreement indicates that the topological quantities are rigid against slight modifications of the theories.

In this paper, we considered the monopole configurations wrapping around each S^2, but it is more interesting if we can construct configurations wrapping around higher dimensional space. Then the field strengths whose indices mix the different spheres play an important role. It is also interesting, as we have studied for the case of fuzzy S^2 in Ref. [16], to further extend our formulation of the projected index theorem to include more general configurations in the Higgs phase, i.e., when the scalar field takes a nonzero vev.

As we mentioned at the beginning of the Introduction, topological aspects of gauge theory on noncommutative geometry may play an important role in compactified extra dimensional space in string theory. We can pursue these studies further by studying the relation of noncommutative geometry to our world and by investigating dynamics of noncommutative gauge theory. (See also related works [24–26].) Our formulation given in the present paper to define the topological charge and to classify the gauge field configuration space on noncommutative geometry will become useful for these studies.

APPENDIX A: EXPANSION OF $\hat{\Gamma}$ IN THE GAUGE FIELDS

In this Appendix, we expand the chirality operator $\hat{\Gamma}$ in terms of the gauge fields, and provide (4.2).

We first expand the chirality operator $\hat{\Gamma}_X$ on each S^2, defined by (2.2). We decompose H_X into the zeroth and the 1st order in the gauge fields as

$$H_X = \alpha_X + \beta_X, \tag{A1}$$

with

$$\alpha_X = a(\sigma_i L_i + \tfrac{1}{2})_X, \qquad \beta_X = a\rho(\sigma_i a_i)_X. \tag{A2}$$

The operators α_X and β_X are of order $\mathcal{O}(n^0)$ and $\mathcal{O}(n^{-1})$, respectively, since $a = 2/n$ and L_i is of order n. Since $(\alpha_X)^2 = 1$, one has $(H_X)^2 = 1 + \{\alpha_X, \beta_X\} + \beta_X^2$. We then obtain

$$\hat{\Gamma}_X = (\alpha + \zeta^{(1)} + \zeta^{(2)} + \zeta^{(3)} + \mathcal{O}(\beta^4))_X \tag{A3}$$

where $\zeta_X^{(i)}$ is the ith order in β_X and hence in the gauge field $(a_i)_X$. They are written as

$$\zeta_X^{(1)} = \tfrac{1}{2}(\beta - \alpha\beta\alpha)_X, \tag{A4}$$

$$\zeta_X^{(2)} = (-\tfrac{1}{8}(\alpha\beta^2 + \beta\alpha\beta + \beta^2\alpha) + \tfrac{3}{8}\alpha\beta\alpha\beta\alpha)_X, \tag{A5}$$

$$\begin{aligned}\zeta_X^{(3)} = \bigg(&\frac{1}{16}(-\beta^3 + \beta\alpha\beta\alpha\beta + \beta\alpha\beta^2\alpha + \beta^2\alpha\beta\alpha \\ &+ \alpha\beta\alpha\beta^2 + \alpha\beta^2\alpha\beta + \alpha\beta^3\alpha) \\ &- \frac{5}{16}\alpha\beta\alpha\beta\alpha\beta\alpha\bigg)_X,\end{aligned} \tag{A6}$$

The operators α_X and $\zeta_X^{(i)}$ themselves are zeroth and ith order in $1/n$. However, taking the trace over the spinor space with the coordinate matrix space untouched, the operators $\mathrm{tr}_{\sigma_X}(\alpha_X)$ and $\mathrm{tr}_{\sigma_X}(\zeta_X^{(1)})$ become of order n^{-1} and n^{-2}, respectively.

It then follows that

$$\begin{aligned}\{\hat{\Gamma}_1, \hat{\Gamma}_2\} = 2\alpha_1\alpha_2 &+ \{\alpha_1, \zeta_2^{(1)} + \zeta_2^{(2)} + \zeta_2^{(3)}\} \\ &+ \{\alpha_2, \zeta_1^{(1)} + \zeta_1^{(2)} + \zeta_1^{(3)}\} + \{\zeta_1^{(1)} + \zeta_1^{(2)}, \zeta_2^{(1)} + \zeta_2^{(2)}\} \\ &+ \mathcal{O}(n^{-5}).\end{aligned} \tag{A7}$$

While the operators $\{\alpha_1, \zeta_2^{(4)}\}$, $\{\alpha_2, \zeta_1^{(4)}\}$, $\{\zeta_1^{(1)}, \zeta_2^{(3)}\}$, and $\{\zeta_2^{(1)}, \zeta_1^{(3)}\}$ also appear at order n^{-4}, when one considers these terms in $\mathcal{T}r[\hat{\Gamma}]$ in (4.2), one takes a trace like $\mathrm{tr}_{\sigma_X}(\alpha_X)$ and $\mathrm{tr}_{\sigma_X}(\zeta_X^{(1)})$, and these terms become of order $\mathcal{O}(n^{-5})$. One also has

$$[\hat{\Gamma}_1, \hat{\Gamma}_2] = [\alpha_1, \zeta_2^{(1)}] - [\alpha_2, \zeta_1^{(1)}] + [\zeta_1^{(1)}, \zeta_2^{(1)}] + \mathcal{O}(n^{-3}). \tag{A8}$$

Note that (A8) is of order $\mathcal{O}(n^{-2})$, since the leading term $[\alpha_1, \alpha_2]$ vanishes, and the commutators $[\alpha_1, \beta_2]$ and $[\alpha_2, \beta_1]$ are of order $\mathcal{O}(n^{-2})$. This is why the second term in (3.4) is of order $\mathcal{O}(n^{-4})$.

Using the identity (3.4), the chirality operator (3.2) is written as

$$\hat{\Gamma} = \frac{1}{2}\{\hat{\Gamma}_1, \hat{\Gamma}_2\} - \frac{1}{16}\{\hat{\Gamma}_1, \hat{\Gamma}_2\}[\hat{\Gamma}_1, \hat{\Gamma}_2]^2 + \cdots. \tag{A9}$$

Plugging (A7) and (A8) into (A9), we obtain (4.2).

APPENDIX B: COMMUTATIVE LIMIT OF $\mathcal{T}r[G_5]$

In this Appendix, we show Eq. (4.11) by taking the commutative limit of $\mathcal{T}r[G_5]$. Substituting (A4) into (4.7), we obtain

$$G_5 = \sum_{i=1}^{5} K_i \tag{B1}$$

with

$$K_1 = -\frac{1}{32}\alpha_1\alpha_2([\alpha_1, \beta_2] - [\alpha_2, \beta_1])^2, \tag{B2}$$

$$K_2 = -\frac{1}{32}\alpha_1\alpha_2(\alpha_2[\alpha_1, \beta_2]\alpha_2 - \alpha_1[\alpha_2, \beta_1]\alpha_1)^2, \tag{B3}$$

$$\begin{aligned}K_3 = \frac{1}{32}\alpha_1\alpha_2\{&[\alpha_1, \beta_2] - [\alpha_2, \beta_1], \alpha_2[\alpha_1, \beta_2]\alpha_2 \\ &- \alpha_1[\alpha_2, \beta_1]\alpha_1\},\end{aligned} \tag{B4}$$

$$\begin{aligned}K_4 = -\frac{1}{64}\alpha_1\alpha_2&([\alpha_1, \beta_2] - [\alpha_2, \beta_1])[\beta_1, \beta_2] \\ &+ (15 \text{ terms}),\end{aligned} \tag{B5}$$

$$K_5 = -\frac{1}{128}\alpha_1\alpha_2[\beta_1, \beta_2]^2 + (15 \text{ terms}), \tag{B6}$$

where K_1, K_2, and K_3 are 2nd order, K_4 is 3rd order, and K_5 is 4th order in β. In (B5) and (B6), we wrote only a typical term. The remaining 15 terms can be similarly written.

We first calculate the commutative limit of $\mathcal{T}r[K_1]$. Plugging (A2) into (B2), we obtain

$$\mathcal{T}r[K_1] = -\frac{1}{32}a^6\rho^2\mathcal{T}r[(\sigma\cdot L)_1(\sigma\cdot L)_2[(\sigma\cdot L)_1,(\sigma\cdot a)_2]^2 - (\sigma\cdot L)_1(\sigma\cdot L)_2[(\sigma\cdot L)_1,(\sigma\cdot a)_2] \times[(\sigma\cdot L)_2,(\sigma\cdot a)_1] + (1\leftrightarrow 2)], \quad \text{(B7)}$$

where we omitted subleading terms in $1/n$. Taking trace over the spinor space, by using the formula

$$\mathrm{tr}_{\sigma}[\sigma_i\sigma_j\sigma_k] = 2i\epsilon_{ijk}, \quad \text{(B8)}$$

(B7) becomes

$$\tfrac{1}{8}a^6\rho^2\mathcal{T}r'[\epsilon_{abc}L_c\epsilon_{ijk}L_k[L_a, a_i][L_b, a_j] - \epsilon_{abc}L_c\epsilon_{ijk}L_k[L_a, a_i][L_j, a_b] + (1\leftrightarrow 2)], \quad \text{(B9)}$$

where $\mathcal{T}r'$ is the trace over the matrix space and the gauge group space. The indices a, b, and c refer to the first S^2, while the indices i, j, and k refer to the second S^2. Then, the commutative limit of (B9) becomes

$$-2\rho^4\int\frac{d\Omega_1}{4\pi}\frac{d\Omega_2}{4\pi}\mathrm{tr}[\epsilon_{abc}n_c\epsilon_{ijk}n_k(\partial_a a_i\partial_b a_j + \partial_i a_a\partial_j a_b) + 2\partial_a(Pa)_i\partial_i(Pa)_a], \quad \text{(B10)}$$

where $(Pa)_i = P_{ij}a_j$ with $P_{ij} = \delta_{ij} - n_in_j$. (B10) is rewritten as

$$-2\rho^4\int\frac{d\Omega_1}{4\pi}\frac{d\Omega_2}{4\pi} \times\mathrm{tr}[\epsilon_{abc}n_c\epsilon_{ijk}n_k(\partial_a a'_i - \partial_i a'_a)(\partial_b a'_j - \partial_j a'_b)], \quad \text{(B11)}$$

where $a'_i = \epsilon_{ijk}n_ja_k$ is the tangential component of the gauge field.

By using the identity

$$\alpha_1[\alpha_1, \beta_2] = -[\alpha_1, \beta_2]\alpha_1, \quad \text{(B12)}$$

$$\alpha_2[\alpha_2, \beta_1] = -[\alpha_2, \beta_1]\alpha_2, \quad \text{(B13)}$$

(B3) is rewritten as

$$K_2 = -\frac{1}{32}([\alpha_1, \beta_2] - [\alpha_2, \beta_1])^2\alpha_1\alpha_2, \quad \text{(B14)}$$

and (B4) is

$$K_3 = -\frac{1}{32}(\alpha_2[\alpha_1, \beta_2]\alpha_1\alpha_2[\alpha_1, \beta_2]\alpha_2 + [\alpha_1, \beta_2]\alpha_1\alpha_2[\alpha_1, \beta_2] - \alpha_2[\alpha_1, \beta_2][\alpha_2, \beta_1]\alpha_1 - [\alpha_1, \beta_2]\alpha_1\alpha_2[\alpha_2, \beta_1] + (1\leftrightarrow 2)). \quad \text{(B15)}$$

By the same calculation that was done for $\mathcal{T}r[K_1]$, we can show that the commutative limits of $\mathcal{T}r[K_2]$ and $\mathcal{T}r[K_3]$ give the same result (B11) and twice of that, respectively. Therefore, the commutative limit of $\mathcal{T}r(K_1 + K_2 + K_3)$ becomes 4 times of (B11). This gives the 2nd order terms in the gauge field in (4.11).

We next consider $\mathcal{T}r[K_4]$. By substituting (A2) and taking the trace over the spinor space, the first term in K_4, which was presented in (B5), gives

$$\frac{1}{16}a^6\rho^3\mathcal{T}r'[\epsilon_{abc}L_c\epsilon_{ijk}L_k([L_a, a_i] - [L_i, a_a])[a_b, a_j]]. \quad \text{(B16)}$$

Its commutative limit becomes

$$-i\rho^4\int\frac{d\Omega_1}{4\pi}\frac{d\Omega_2}{4\pi}\epsilon_{abc}n_c\epsilon_{ijk}n_k \times\mathrm{tr}((\epsilon_{ade}n_d\partial_e a_i - \epsilon_{ilm}n_l\partial_m a_a)[a_b, a_j]). \quad \text{(B17)}$$

This is rewritten as

$$i\rho^4\int\frac{d\Omega_1}{4\pi}\frac{d\Omega_2}{4\pi}\epsilon_{abc}n_c\epsilon_{ijk}n_k\,\mathrm{tr}((\partial_a a'_i - \partial_i a'_a)[a'_b, a'_j]). \quad \text{(B18)}$$

The remaining 15 terms in (B5) give the same results. Thus, the commutative limit of $\mathcal{T}r[K_4]$ becomes 16 times of (B18). This gives the 3rd order terms in the gauge field in (4.11).

We finally consider $\mathcal{T}r[K_5]$. By substituting (A2) and taking the trace over the spinor space, the first term in K_5, which was presented in (B6), gives

$$\frac{1}{32}a^6\rho^4\mathcal{T}r'[\epsilon_{abc}L_c\epsilon_{ijk}L_k[a_a, a_i][a_b, a_j]]. \quad \text{(B19)}$$

Its commutative limit becomes

$$\frac{1}{2}\rho^4\int\frac{d\Omega_1}{4\pi}\frac{d\Omega_2}{4\pi}\epsilon_{abc}n_c\epsilon_{ijk}n_k\,\mathrm{tr}([a_a, a_i][a_b, a_j]), \quad \text{(B20)}$$

which is rewritten as

$$\frac{1}{2}\rho^4\int\frac{d\Omega_1}{4\pi}\frac{d\Omega_2}{4\pi}\epsilon_{abc}n_c\epsilon_{ijk}n_k\,\mathrm{tr}([a'_a, a'_i][a'_b, a'_j]). \quad \text{(B21)}$$

The remaining 15 terms in (B6) give the same results. Thus, the commutative limit of $\mathcal{T}r[K_5]$ becomes 16 times of (B21). This gives the 4th order terms in the gauge field in (4.11).

Hence we have proved (4.11).

APPENDIX C: SPECTRUM OF THE DIRAC OPERATOR FOR THE FREE CASE

In this Appendix, we calculate the whole spectrum of the GW Dirac operator for the free case. We here consider the $U(1)$ gauge group, for simplicity. For the free case, one has

$$(\Gamma - \hat{\Gamma})_X = -a(\sigma\cdot\tilde{L} + 1)_X, \quad \text{(C1)}$$

$$(\Gamma + \hat{\Gamma})_X = a(\sigma\cdot(L + L^R))_X, \quad \text{(C2)}$$

where $(\tilde{L}_i)_X = (L_i - L_i^R)_X$ is the adjoint operator. Then, the free GW Dirac operator (5.1) is written as

$$D_{\rm GW} = \frac{a}{2}[(\sigma\cdot\tilde{L}+1)_1(\sigma\cdot(L+L^R))_2 + (\sigma\cdot(L+L^R))_1(\sigma\cdot\tilde{L}+1)_2]. \quad (C3)$$

Let us begin with an investigation of each fuzzy S^2. Our formulation has $SO(3)$ Poincaré symmetry on each S^2, whose generator $(M_i)_X$ is given in (5.7). We now write its eigenstates as

$$(M_i)^2_X|J_X, \pm\rangle = J_X(J_X+1)|J_X, \pm\rangle. \quad (C4)$$

Each $|J_X, \pm\rangle$ has $(2J_X+1)$-folded degeneracy associated with $(M_3)_X$. The sign $\pm$ indicates that this state is obtained from the spin l_X state of $(\tilde{L}_i)_X$ as $J_X = l_X \pm \frac{1}{2}$. For $J_X = \frac{1}{2}, \frac{3}{2}, \ldots, n-\frac{3}{2}$, there exist both $|J_X, +\rangle$ and $|J_X, -\rangle$, while for the highest spin $J_X = n - \frac{1}{2}$ there exists only $|J_X, +\rangle$. The state $|J_X, \pm\rangle$ is shown to be an eigenstate of the operator $(\sigma\cdot\tilde{L}+1)_X$ as

$$(\sigma\cdot\tilde{L}+1)_X|J_X, \pm\rangle = \pm(J_X+\tfrac{1}{2})|J_X, \pm\rangle. \quad (C5)$$

Since we have the relation

$$\{\Gamma - \hat{\Gamma}, \Gamma + \hat{\Gamma}\}_X = 0, \quad (C6)$$

and, in particular, for the free case,

$$\{\sigma\cdot\tilde{L}+1, \sigma\cdot(L+L^R)\}_X = 0, \quad (C7)$$

the operator $(\sigma\cdot(L+L^R))_X$ flips the $\pm$ sign as

$$(\sigma\cdot(L+L^R))_X|J_X, \pm\rangle = C_{J_X}|J_X, \mp\rangle \quad (C8)$$

with

$$C_{J_X} = \sqrt{n^2 - \tfrac{1}{4} - J_X(J_X+1)}. \quad (C9)$$

For the highest spin $J_X = n - \frac{1}{2}$, a state $|J_X, -\rangle$ does not exist, and thus $(\sigma\cdot(L+L^R))_X|J_X, +\rangle$ must vanish. Indeed, $C_{J_X} = 0$ in this case, as one can see from (C9).

We now come back to $S^2 \times S^2$. We consider states specified by the spin J_1 and J_2 of each S^2. We will study the following three cases in turn:

$$\begin{aligned} &(a)\ \tfrac{1}{2} \le J_1 \le n-\tfrac{3}{2}, \qquad \tfrac{1}{2} \le J_2 \le n-\tfrac{3}{2} \\ &(b)\ J_1 = n-\tfrac{1}{2}, \qquad \tfrac{1}{2} \le J_2 \le n-\tfrac{3}{2} \\ &(c)\ J_1 = J_2 = n-\tfrac{1}{2}. \end{aligned} \quad (C10)$$

Let us first consider the case (a), where four types of states $|J_1, \pm; J_2, \pm\rangle$ exist. Acting the GW Dirac operator (C3) on these states, we obtain

$$\begin{aligned} D_{\rm GW}(&c_1|J_1,+;J_2,+\rangle + c_2|J_1,+;J_2,-\rangle + c_3|J_1,-;J_2,+\rangle + c_4|J_1,-;J_2,-\rangle) \\ &= (Ac_2 + Bc_3)|J_1,+;J_2,+\rangle + (Ac_1 - Bc_4)|J_1,+;J_2,-\rangle \\ &\quad + (-Ac_4 + Bc_1)|J_1,-;J_2,+\rangle + (-Ac_3 - Bc_2)|J_1,-;J_2,-\rangle, \end{aligned} \quad (C11)$$

with $A = \frac{a}{2}(J_1 + \frac{1}{2})C_{J_2}$ and $B = \frac{a}{2}(J_2 + \frac{1}{2})C_{J_1}$. Diagonalizing $D_{\rm GW}$ in this sector, we obtain the eigenvalues $\pm|A \pm B|$, where two $\pm$ signs need not coincide.

In particular, for $J_1 = J_2$, and hence for $A = B$, there exist two types of zero modes. Their explicit form is given as

$$|1\rangle = \tfrac{1}{2}(|J_1,+;J_2,+\rangle + |J_1,+;J_2,-\rangle - |J_1,-;J_2,+\rangle + |J_1,-;J_2,-\rangle), \quad (C12)$$

$$|2\rangle = \tfrac{1}{2}(|J_1,+;J_2,+\rangle - |J_1,+;J_2,-\rangle + |J_1,-;J_2,+\rangle + |J_1,-;J_2,-\rangle). \quad (C13)$$

We now study their chiralities. The chirality operator (3.1) is rewritten as

$$\begin{aligned} \Gamma = \frac{a^2}{4}[&(\sigma\cdot(L+L^R))_1(\sigma\cdot(L+L^R))_2 + (\sigma\cdot\tilde{L}+1)_1(\sigma\cdot\tilde{L}+1)_2 \\ &- (\sigma\cdot(L+L^R))_1(\sigma\cdot\tilde{L}+1)_2 - (\sigma\cdot\tilde{L}+1)_1(\sigma\cdot(L+L^R))_2]. \end{aligned} \quad (C14)$$

Acting it on the above states, we obtain

$$\Gamma|1\rangle = |2\rangle, \qquad \Gamma|2\rangle = |1\rangle, \quad (C15)$$

where we used $\frac{a^2}{4}[(C_J)^2 + (J+\frac{1}{2})^2] = 1$. We thus have

$$\begin{aligned} \Gamma\frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) &= +\frac{1}{\sqrt{2}}(|1\rangle + |2\rangle), \\ \Gamma\frac{1}{\sqrt{2}}(|1\rangle - |2\rangle) &= -\frac{1}{\sqrt{2}}(|1\rangle - |2\rangle). \end{aligned} \quad (C16)$$

The zero modes in this sector have both chiralities and do not contribute to the index.

We next consider the case (b), where two types of states $|J_1, +; J_2, \pm\rangle$ exist. Acting the GW Dirac operator (C3) on these states, and diagonalizing $D_{\rm GW}$, we obtain the eigenstates as

$$\begin{aligned} D_{\rm GW}&\frac{1}{\sqrt{2}}(|J_1,+,J_2,+\rangle \pm |J_1,+,J_2,-\rangle) \\ &= \pm\frac{a}{2}\left(J_1 + \frac{1}{2}\right)C_{J_2}\frac{1}{\sqrt{2}}(|J_1,+,J_2,+\rangle \pm |J_1,+,J_2,-\rangle). \end{aligned} \quad (C17)$$

There is not a zero mode in this case.

We finally consider the case (c), where only the states $|J_1, +; J_2, +\rangle$ exist. Acting the GW Dirac operator (C3) and the chirality operator (C14) on these states, we obtain

$$D_{\rm GW}|J_1, +; J_2, +\rangle = 0, \tag{C18}$$

$$\Gamma|J_1, +; J_2, +\rangle = +|J_1, +; J_2, +\rangle. \tag{C19}$$

They give chiral zero modes and contribute to the index. Recalling that the state $|J_X, +\rangle$ has the $(2J_X + 1)$-folded degeneracy, the degeneracy of the chiral zero modes is $(2J_1 + 1)(2J_2 + 1) = 4n^2$. This agrees with the first term in (4.13).

Now we have obtained the whole spectrum of the Dirac operator and checked that the chiral zero modes are indeed given by the states that we discussed in Sec. VA.

APPENDIX D: COMMUTATIVE LIMIT IN THE MODIFIED FORMULATION

In this Appendix, we consider the modified formulation given by (7.1), and calculate the commutative limit of the Dirac operator and the topological charge.

By substituting (A1) into (7.1), and expanding it in β and hence in the gauge fields, we obtain

$$\begin{aligned}\hat\Gamma' = {}& \alpha_1\alpha_2 + \left[\frac{1}{4}\{\alpha_2, \beta_1 - \alpha_1\beta_1\alpha_1\} + (1 \leftrightarrow 2)\right] + \frac{1}{32}[7\{\beta_1, \beta_2\} - 5\alpha_1\alpha_2\{\beta_1, \beta_2\}\alpha_1\alpha_2 + 3\{\alpha_1\beta_1\alpha_1, \alpha_2\beta_2\alpha_2\} \\ & - [\{\beta_1, \alpha_2\beta_2\alpha_2\} + \alpha_1\{\beta_1, \beta_2\}\alpha_1 - 3\alpha_1\{\beta_1, \alpha_2\beta_2\alpha_2\}\alpha_1 + \alpha_2\beta_1\alpha_2\beta_2 + \beta_1\alpha_1\beta_2\alpha_1 + \alpha_1\alpha_2\beta_1\alpha_1\alpha_2\beta_2 \\ & + \alpha_2\beta_1\alpha_1\alpha_2\beta_2\alpha_1 + \alpha_1\alpha_2\beta_1\alpha_2\beta_2\alpha_1 + \alpha_2\beta_1\alpha_1\beta_2\alpha_1\alpha_2 + \beta_1\alpha_1\alpha_2\beta_2\alpha_1\alpha_2 + (1 \leftrightarrow 2)] \\ & - [\{\alpha_2, \alpha_1\beta_1^2 + \beta_1^2\alpha_1 + \beta_1\alpha_1\beta_1\} + \alpha_1\beta_1\alpha_2\beta_1 + \beta_1\alpha_2\beta_1\alpha_1 + \beta_1\alpha_1\alpha_2\beta_1 + \alpha_1\alpha_2\beta_1\alpha_2\beta_1\alpha_2 + \alpha_2\beta_1\alpha_2\beta_1\alpha_1\alpha_2 \\ & + \alpha_2\beta_1\alpha_1\alpha_2\beta_1\alpha_2 - 3(\{\alpha_2, \alpha_1\beta_1\alpha_1\beta_1\alpha_1\} + \alpha_1\beta_1\alpha_1\alpha_2\beta_1\alpha_1 + \alpha_1\alpha_2\beta_1\alpha_1\alpha_2\beta_1\alpha_1\alpha_2) + (1 \leftrightarrow 2)]] + \mathcal{O}(\beta^3).\end{aligned} \tag{D1}$$

The first and the second terms in (D1), which are zeroth and 1st order in β, coincide with those of the original formulation, (4.3) and the 1st order terms in (4.4), at the operator level, i.e., before taking the trace. Then, the commutative limit of the Dirac operator $-a^{-1}(\Gamma - \hat\Gamma')$ becomes the same one as the original formulation, (3.10), since the commutative limit of the Dirac operator is affected by $\hat\Gamma'$ only up to order n^{-1}.

We next consider the commutative limit of the topological charge $\frac{1}{2}\mathcal{T}r(\Gamma + \hat\Gamma')$, which is affected by $\hat\Gamma'$ up to order n^{-4}. While $\hat\Gamma'$ and $\hat\Gamma$ differ at $\mathcal{O}(\beta^2)$ at the operator level, the trace of the difference becomes

$$\begin{aligned}\mathcal{T}r[\hat\Gamma' - \hat\Gamma] = {}& \frac{1}{16}\mathcal{T}r[[\alpha_2, \beta_1]\alpha_1\alpha_2[\alpha_2, \beta_1] \\ & - \alpha_1[\alpha_2, \beta_1]\alpha_1\alpha_2[\alpha_2, \beta_1]\alpha_1] \\ & + \mathcal{O}((\beta_2)^2) + \mathcal{O}(\beta_1\beta_2) + \mathcal{O}(\beta^3),\end{aligned} \tag{D2}$$

where we have written only the terms with $(\beta_1)^2$. Since (D2) vanishes in the commutative limit, the commutative limit of the topological charge $\frac{1}{2}\mathcal{T}r(\Gamma + \hat\Gamma')$ becomes the same one as the original formulation, (4.13).

In the original formulation, the commutative limit of $\{\alpha_1, \zeta_2^{(2)}\}$ and $\{\alpha_2, \zeta_1^{(2)}\}$ in (4.4) gave the second order terms in the gauge field in the 1st Chern character. The commutative limit of $\{\zeta_1^{(1)}, \zeta_2^{(1)}\}$ in (4.6) gave the second order terms in the gauge field in (4.10), which is a part of the 2nd Chern character. The commutative limit of $\alpha_1\alpha_2([\alpha_1, \zeta_2^{(1)}] - [\alpha_2, \zeta_1^{(1)}])^2$ in (4.7) gave the second order terms in the gauge field in (4.11). However, in the modified formulation, the corresponding terms are all mixed in the third term in (D1), and it is difficult to perform the same calculations that we have done in the original formulation. While the modified formulation is simpler in the definition since it has normalization procedure only one time, calculations are easier in the original formulation.

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