

Conformality lost

David B. Kaplan, Jong-Wan Lee, and Dam T. Son

Institute for Nuclear Theory, University of Washington, Seattle, Washington 98195-1550, USA

Mikhail A. Stephanov

Department of Physics, University of Illinois, Chicago, Illinois 60607-7059, USA

(Received 11 August 2009; published 3 December 2009)

We consider zero-temperature transitions from conformal to nonconformal phases in quantum theories. We argue that there are three generic mechanisms for the loss of conformality in any number of dimensions: (i) fixed point goes to zero coupling, (ii) fixed point runs off to infinite coupling, or (iii) an IR fixed point annihilates with a UV fixed point and they both disappear into the complex plane. We give both relativistic and nonrelativistic examples of the last case in various dimensions and show that the critical behavior of the mass gap behaves similarly to the correlation length in the finite temperature Berezinskii-Kosterlitz-Thouless (BKT) phase transition in two dimensions, $\xi \sim \exp(c/|T - T_c|^{1/2})$. We speculate that the chiral phase transition in QCD at large number of fermion flavors belongs to this universality class, and attempt to identify the UV fixed point that annihilates with the Banks-Zaks fixed point at the lower end of the conformal window.

DOI: 10.1103/PhysRevD.80.125005

PACS numbers: 11.15.-q, 05.10.Cc, 11.30.Rd

I. INTRODUCTION

The renormalization group (RG) underlies our understanding of second-order phase transitions, with critical points being identified with fixed points of the appropriate RG equation [1]. Near the phase transition the characteristic energy or momentum scale m (the inverse correlation length) goes to zero as $m \sim |\alpha - \alpha_*|^p$, where α is a parameter that can vary continuously, and $\alpha = \alpha_*$ is the location of the critical point.

In this paper, we argue that there is wide class of phase transitions in which the correlation length behaves very differently, vanishing exponentially on one side of the phase transition, while being strictly zero on the other side

$$m \sim \Lambda_{\text{UV}} \theta(\alpha_* - \alpha) \exp\left(-\frac{c}{\sqrt{\alpha_* - \alpha}}\right), \quad c > 0. \quad (1)$$

This peculiar behavior—where all derivatives of the correlation length with respect to α vanish at the critical point—has been observed before in the Berezinskii-Kosterlitz-Thouless (BKT) phase transition in two dimensions [2]; therefore we will refer to Eq. (1) as “BKT scaling.” The BKT transition is a classical phase transition in two dimensions that can be described in terms of vortex condensation. It arises due to the competition between the entropy of a single vortex and the binding energy of a pair of vortices, both of which scale as $\log R$, R being the size of the system. While this transition is peculiar to two dimensions, we will show that the mechanism underlying BKT scaling from an RG point of view is far more general, and is one of three generic behaviors that can occur when a system in any dimension makes a transition from a conformal to a nonconformal phase. In particular, as we will show, it follows when an IR fixed point of the system

merges with a UV fixed point. In this language it is easy to see why BKT scaling can be found in a wide variety of systems.

The basic mechanism can be illustrated with a simple model with a dimensionless coupling g depending on an external parameter α , for which the β -function takes the form [Fig. 1(a)]

$$\beta(g; \alpha) = \frac{\partial g}{\partial t} = (\alpha - \alpha_*) - (g - g_*)^2, \quad (2)$$

where $t = \ln \mu$, μ being the renormalization scale. For $(\alpha - \alpha_*) > 0$, the fixed points for this system (zeros of β) are given by

$$g_{\pm} = g_* \pm \sqrt{\alpha - \alpha_*}, \quad (3)$$

where g_- , g_+ correspond to IR and UV fixed points, respectively, each describing a conformal phase of the theory.¹ As α decreases, these two fixed points approach each other until they merge at $g_{\pm} = g_*$ for $\alpha = \alpha_*$. For $\alpha < \alpha_*$ the solutions to $\beta = 0$ are complex, and the theory no longer has a conformal phase.

To see that fixed point merger generically gives rise to BKT scaling, consider the case where α is slightly below α_* , and that at a UV scale Λ_{UV} the coupling takes an initial value $g_{\text{UV}} < g_*$. On scaling to the IR, the coupling then flows to larger values, lingering near $g = g_*$ where the β -function is small, and then blowing up quickly, defining an intrinsic IR scale Λ_{IR} , which is insensitive to the initial value g_{UV} . This behavior is displayed in Fig. 1(b). The scale Λ_{IR} will characterize the longest correlation lengths in this theory, and can be computed by integrating Eq. (2):

¹By IR and UV fixed points we mean zeros of the β -function which are attractive or repulsive in the IR, respectively.

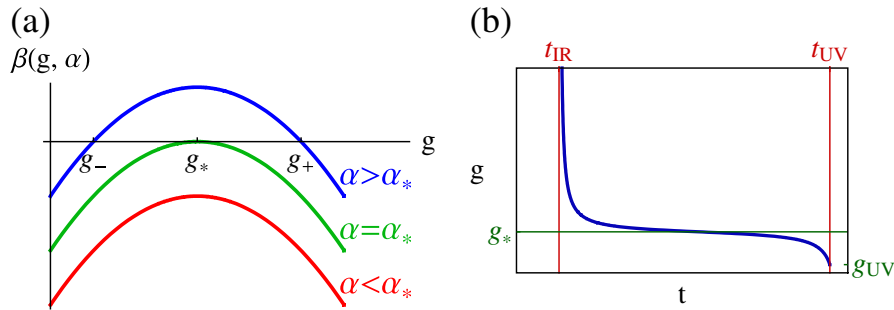


FIG. 1 (color online). (a) A toy β -function. For $\alpha > \alpha_*$ there are fixed points at g_{\pm} which are UV- and IR-stable respectively; these fixed points merge at g_* for $\alpha = \alpha_*$, and disappear for $\alpha < \alpha_*$; (b) The RG flow of the coupling g as a function of $t = \ln\mu$ in the nonconformal phase, with $(t_{UV} - t_{IR}) \propto 1/\sqrt{\alpha_* - \alpha}$.

$$\frac{\Lambda_{IR}}{\Lambda_{UV}} = \exp[t_{IR} - t_{UV}] = \exp\left[\int_{g_{UV}}^{g_{IR}} \frac{dg}{\beta(g; \alpha)}\right] \approx e^{-\pi/\sqrt{\alpha_* - \alpha}}, \quad (4)$$

where we have assumed $|g_{IR,UV} - g_*| \gg |\alpha - \alpha_*|^{1/2}$.

How general is this mechanism of fixed point annihilation? Suppose a system has a nontrivial IR fixed point at $g = \bar{g}(\alpha)$ whose location depends continuously on a parameter α , and that at a critical value $\alpha = \alpha_*$ there is a phase transition where conformality is lost. At this phase transition, the β -function must somehow lose a zero. This can come about in three ways:

- (A) \bar{g} can decrease until it merges with the trivial fixed point at $g = 0$, giving rise to a trivial, asymptotically unconfined theory;
- (B) \bar{g} can run off to infinite coupling and disappear;
- (C) \bar{g} can merge with a UV fixed point, as in our toy model, giving rise to BKT scaling.

Examples of scenarios (A) and (B) are afforded by supersymmetric QCD (SQCD). At large number of colors N_c , the parameter $x \equiv N_f/N_c$ may be treated as continuous, where N_f is the number of quark flavors. It has been shown by Seiberg [3,4] that SQCD is conformal in the window $3/2 \leq x \leq 3$. For x just below 3, the theory has a Banks-Zaks fixed point at weak coupling [5]; approaching $x = 3$ from below, this fixed point merges with the trivial fixed point at $g = 0$, and for $x > 3$ the theory is in the asymptotically unconfined “free electric phase.” This is an example of mechanism “A” above. In contrast, at the lower end of the conformal window at $x = 3/2$, SQCD goes from a strongly coupled conformal theory when $x \geq 3/2$ to a “free magnetic phase” when $x \leq 3/2$. In the free magnetic phase, the Coulomb force between charges takes the form $e^2 \ln(\Lambda r)/r^2$ where Λ is associated with the Landau pole of the dual magnetic theory. The log behavior of the coupling can be explained by a β -function which is negative and approaches zero as $\beta \sim -1/g$ for large g . Thus it

appears that conformality in the electric description is lost via mechanism (B). [Yet, since in the dual magnetic theory conformality is lost via mechanism (A), it would appear that scenarios (A) and (B) can describe the same physics in terms of different degrees of freedom.]

In this paper we give several examples of theories which exhibit the mechanism (C) of fixed point merger and BKT scaling. Following our RG analysis of the original BKT transition, we analyze the quantum mechanical example of a $1/r^2$ potential in d dimensions, which can be solved nonperturbatively and which exhibits the phenomenon of fixed point merger. We show how this analysis has many parallels in the AdS/CFT correspondence [6–8], and that loss of conformality via fixed point merger is analogous (if not holographically dual) to the instability of AdS space at the Breitenlohner-Freedman (BF) bound [9].

Our next example is a relativistic theory of gauged fermions confined to a defect. Here a perturbative analysis near $d = 2$ dimensions reveals fixed point merger and BKT scaling. A rainbow approximation to the gap equation gives qualitatively similar results.

One of the motivations for this paper is to understand the chiral phase transition that happens in (nonsupersymmetric) large- N_c QCD when the number of flavors of massless fermions N_f varies. As with SQCD, we know there exists a conformal window for QCD in the parameter $x = N_f/N_c$ where the upper end occurs at $x_* = 11/2$, near which the Banks-Zaks calculation is perturbative and reliable. For decreasing x conformality must eventually be lost, since for small x chiral symmetry breaking is expected. We speculate that the phase transition at this lower boundary of the conformal window occurs due to fixed point merger. This suggestion is not new: it has been advocated before by Gies and Jaeckel based on the results from the functional RG approach [10]. If this picture is correct, then near the transition the chiral condensate must exhibit BKT scaling. Incidentally, this exponential behavior is also typically found when one solves the gap equation obtained from (an unsystematic) truncation of the Schwinger-Dyson hier-

archy (see, e.g., [11–16] for further references).² *A priori*, the relationship between the RG picture of merging fixed points and the gap equation is not obvious; however our analysis of relativistic defect fermions yields the same result in the regime where both approaches are reliable.

If QCD does indeed exhibit BKT scaling, then our arguments suggest that within the conformal window there exists another theory, QCD*, which is defined at the UV fixed point. We conclude with speculations about this theory.

II. THE BKT PHASE TRANSITION

The BKT phase transition [17,18] is due to the deconfinement of vortices in the XY model at a critical temperature T_c , above which the theory is conformal. The behavior of the correlation length Eq. (1) below the phase transition can be understood from the appropriate RG equation [19]. We can exploit the equivalence between the XY model and the zero temperature sine-Gordon model in 1 + 1 dimensions:

$$L = \frac{T}{2} (\partial_\mu \phi)^2 - 2z \cos \phi, \quad (5)$$

where T corresponds to the temperature of the XY model in units of the spin coupling. Near the phase transition, it is useful to use the variables $u = 1 - 1/8\pi T$ and $v = 2z/T\Lambda^2$ —where Λ is the UV cutoff associated with the vortex core—in terms of which the perturbative β -functions are

$$\beta_u = -2v^2, \quad \beta_v = -2uv. \quad (6)$$

Changing variables to $v + u = \tau$ and $v - u = 2w$, one sees that τw invariant under RG flow, and the running of τ is governed by

$$\beta(\tau; w\tau) = \mu \frac{d\tau}{d\mu} = -2w\tau - \tau^2. \quad (7)$$

This β -function has exactly the quadratic form of our toy model Eq. (2), with the substitution

$$(\alpha - \alpha_*) \rightarrow -2w\tau, \quad (g - g_*) \rightarrow \tau. \quad (8)$$

However, this β -function is only valid for small τ and w , so the region about $\tau = 0$ is excluded for fixed $w\tau$, as shown in Fig. 2. Because of the excluded region, the physics for the BKT model is slightly different than for the toy model: in the nonconformal phase ($w\tau > 0$), instead of starting from the left of $\tau_* = 0$ in the UV and flowing to the right in the IR, the system starts at the top of the hill just to the right of τ_* and flows to the right in the IR. While it may appear that this requires a fine-tuned initial condition for τ , that is not the case in terms of the u and v variables. Starting the flow near $\tau = 0$ gives a factor of 1/2 in the exponent for

²In this context, BKT scaling is sometimes called “Miransky scaling.”

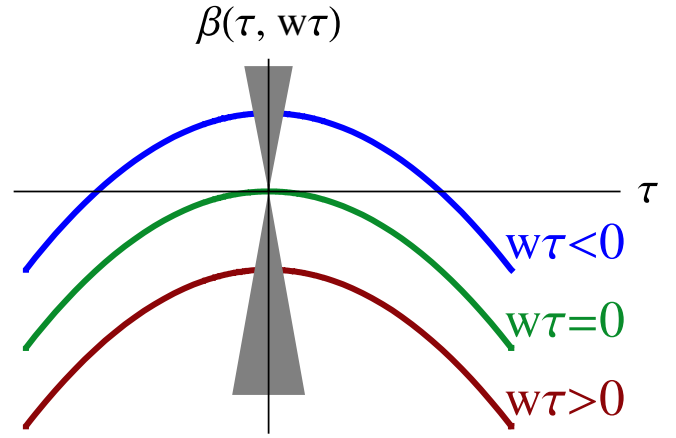


FIG. 2 (color online). The function $\beta(\tau)$ in the vicinity of $\tau = 0$ for the BKT transition Eq. (7); the gray region is outside the realm of validity of the calculation.

the correlation length relative to the expression Eq. (4):

$$\xi_{\text{BKT}} \Lambda \simeq e^{\pi/(2\sqrt{2w\tau})}. \quad (9)$$

The critical temperature is found by solving $w\tau = 0$; expanding about $T = T_c$ yields the familiar BKT result

$$\xi_{\text{BKT}} \Lambda \simeq e^{b'/|T-T_c|^{1/2}}, \quad (10)$$

where b' is a nonuniversal number that can be expressed in terms of z and Λ .

III. A NONRELATIVISTIC EXAMPLE: QUANTUM MECHANICS IN $1/r^2$ POTENTIAL

It is well known that the solutions for a quantum particle in a potential

$$V(r) = \alpha/r^2 \quad (11)$$

possess conformal symmetry when the potential is repulsive or weakly attractive ($\alpha > \alpha_*$), but that for sufficiently attractive potential ($\alpha < \alpha_*$), conformality is lost and the potential has discrete bound states.³ For a range of α , the zero-energy, s -wave solution to the Schrödinger equation for two particles with mass $m = 1$ in d dimensions interacting via the potential $V(r)$ is given by

$$\psi = c_- r^{\nu_-} + c_+ r^{\nu_+}, \quad \nu_{\pm} = -\frac{(d-2)}{2} \pm \sqrt{\alpha - \alpha_*},$$

$$\alpha_* \equiv -\frac{(d-2)^2}{4}. \quad (12)$$

This solution is valid for α in the range

$$\alpha_* \leq \alpha \leq \alpha_* + 1; \quad (13)$$

³There is a vast literature on the $1/r^2$ potential. For textbook treatment, see Ref. [20]; for an early reference, see [21]; for relatively recent RG treatments see [22–26] and references therein.

for $\alpha < \alpha_*$ the above solution becomes complex, and the Hamiltonian does not have ground state, while for $\alpha > \alpha_* + 1$ then $\nu_- < -d/2$ and the r^{ν_-} solution is not normalizable near $r \sim 0$. Within the range Eq. (13), if either c_+ or c_- vanish, then the solution is scale-invariant. Solutions for which both c_+ and c_- are nonzero define an intrinsic length scale, $L \equiv (c_+/c_-)^{1/(\nu_- - \nu_+)}$ and therefore do not exhibit conformal invariance; however in this case the solution always approaches $c_+ r^{\nu_+}$ for large r (since $\nu_+ \geq \nu_-$) and so we can identify the $c_- = 0$ solution with an IR attractive fixed point and the $c_+ = 0$ solution with a UV fixed point, in a manner we can make precise. Arranging to have one of these solutions or the other requires different boundary conditions at the origin, so we see that the theory is actually not well defined with the potential Eq. (11), but that it must be augmented by a δ function at the origin which controls the boundary condition at $r = 0$:

$$V(r) = \alpha/r^2 - g\delta^d(r). \quad (14)$$

We will show that the coupling g obeys an RG equation analogous to our toy model Eq. (2), and that the two conformal solutions $c_- = 0$ and $c_+ = 0$ will correspond to two different fixed points of the coupling g . As α approaches α_* from above, we will show that the two fixed points merge, $g_{\pm} = g_*$, at a value for g_* which we will compute. For $\alpha < \alpha_*$ a UV cutoff must be imposed on the theory in order to have a ground state and an IR scale emerges which is related to the UV cutoff through the BKT scaling formula Eq. (4). We show this in two different ways: first we perform a nonperturbative analysis, and then we use Feynman diagrams in a perturbative calculation in $2 + \epsilon$ dimensions. Both calculations shed light on the relativistic example we provide later, and on our conjecture about the behavior of QCD as a function of the number of flavors.

A. nonperturbative calculation

1. The exact wavefunction and energy

To solve the Schrödinger equation exactly for two-particle scattering via a $1/r^2$ potential in d dimensions we need to regulate the singularity at $r = 0$. We choose to do so by considering the potential

$$V(r) = \begin{cases} \alpha/r^2, & r > r_0, \\ -g/r_0^2, & r < r_0, \end{cases} \quad (15)$$

where r_0^{-1} will serve as the cutoff Λ_{UV} .

At low energy, there is a region $r_0 < r \ll 1/\sqrt{E}$ where the $E\psi$ term in the Schrödinger equation can be neglected, and for $\alpha > \alpha_*$ we find the solution Eq. (12)

$$\psi = c_- r^{\nu_-} + c_+ r^{\nu_+}, \quad (16)$$

with the ratio c_+/c_- given in terms of Bessel functions as

$$\frac{c_+}{c_-} = -r_0^{(\nu_- - \nu_+)} \frac{\gamma + \nu_-}{\gamma + \nu_+}, \quad \gamma \equiv \left[\frac{\sqrt{g} J_{d/2}(\sqrt{g})}{J_{(d-2)/2}(\sqrt{g})} \right]. \quad (17)$$

The quantity (c_+/c_-) is a dimensionful quantity characterizing this solution; by requiring that it does not change as we change the UV cutoff r_0 , we arrive at the exact β -function for γ (defining RG time $t = -\ln r_0$):

$$\begin{aligned} \beta_\gamma &= \frac{\partial \gamma}{\partial t} = -(\gamma + \nu_+)(\gamma + \nu_-) \\ &= (\alpha - \alpha_*) - (\gamma - \gamma_*)^2, \end{aligned} \quad (18)$$

with

$$\begin{aligned} \alpha_* &= -\left(\frac{d-2}{2}\right)^2, & \gamma_* &= \frac{d-2}{2}, \\ \gamma_{\pm} &= -\nu_{\mp} = \frac{d-2}{2} \pm \sqrt{\alpha - \alpha_*}. \end{aligned} \quad (19)$$

We recognize this to be the same β -function as our toy model Eq. (2) with fixed points at γ_{\pm} ; referring to Eq. (17) we see that the IR fixed point corresponds to $\gamma = \gamma_-$ and $c_- = 0$, while the UV fixed point is associated with $\gamma = \gamma_+$ and $c_+ = 0$.

For general d and $\alpha < \alpha_*$, scaling solutions do not exist; physical quantities, such as the bound state energy, depend on the UV cutoff. Motivated by the discussion of coupling constant flow in our toy model, we know that physical quantities will be insensitive to the value of γ (the UV coupling) so long as $\gamma < \gamma_*$, as seen in Fig. 1(b). So we take $\gamma \rightarrow -\infty$, which is reached in the limit of a hard-core repulsive potential for $r < r_0$, $g \rightarrow -\infty$. The ground state wave function is then described by the Bessel function $\psi(r) = r^{-(d-2)/2} K_{i\eta}(kr)$ with $\eta = \sqrt{\alpha_* - \alpha}$ and the boundary condition $\psi(r_0) = 0$. For small real η we can solve for k and find the binding energy

$$B = k^2 = \frac{1}{r_0^2} \exp\left(-\frac{2\pi}{\sqrt{\alpha_* - \alpha}} + O(1)\right), \quad (20)$$

Note that this scale for the binding energy is easily attained from the RG analysis as

$$B \simeq \Lambda_{\text{IR}}^2 = \left(\frac{1}{r_0} e^{\int_{-\infty}^{\gamma} d\gamma/\beta_\gamma}\right)^2 = \frac{1}{r_0^2} e^{-2\pi/\sqrt{\alpha_* - \alpha}}. \quad (21)$$

If one takes γ to be arbitrarily close to γ_* in the UV, and then takes $\alpha \rightarrow \alpha_*$ then the binding energy goes to zero, but the exponent is only half as large,

$$B \sim \frac{1}{r_0^2} \exp\left(-\frac{\pi}{\sqrt{\alpha_* - \alpha}}\right), \quad (22)$$

recalling the result for the BKT transition.

2. Onset of the Efimov effect

Although the β -function in Eq. (18) takes the same form as the toy β -function in Eqs. (2) and (17), implies that the coupling g is a multivalued function of γ , with $|\gamma| \rightarrow \infty$ identified with the zeros of $J_{(d-2)/2}(\sqrt{g})$. Therefore for $\alpha < \alpha_*$ our RG equation actually describes limit cycle behavior: as γ runs from $-\infty$ to $+\infty$ in RG period $T = -\int d\gamma/\beta_\gamma = \pi/\sqrt{\alpha_* - \alpha}$, g runs from one Bessel function zero to the next. It follows that there is not just one IR scale defined by this RG flow, as in Fig. 1(b), but an infinite number of such scales, each successively smaller than the previous by a factor of $\exp[-\pi/\sqrt{\alpha_* - \alpha}]$. This behavior can explain the Efimov effect in 3-body bound states.

The classic Efimov effect [27] concerns the system of three identical bosons. When the scattering length between two bosons becomes large, the three-body system develops a series of ever shallower bound states. This occurs because the three particles interact via an α/r^2 potential for $r \gg r_0$, where r_0 is the two-body effective range, and α a fixed number satisfying $\alpha < \alpha_*$. These systems require a 3-body interaction, and the renormalization of this interaction exhibits the limit-cycle behavior discussed above. The infinite tower of IR scales is associated with the infinite number of ‘‘Efimov states’’ below threshold, exhibiting a geometric spectrum [26]. Such states have been observed in systems of trapped atoms tuned to a Feshbach resonance.

Three degenerate bosons tuned to infinite scattering length (so-called ‘‘unitary bosons’’) do not have a variable α parameter; in order to see a transition very similar to what happens at $\alpha = \alpha_*$ we need a case when the Efimov effect appears as one changes a tunable parameter. This is realized by nonrelativistic fermions at unitarity with different masses for two spin components, M (heavy) and m (light). The Efimov effect occurs in the p -wave channel for two heavy and one light fermions if $M/m > 13.6$ [28].

It is known that for $8.6 < M/m < 13.6$ one can additionally fine tune the three-body interaction to resonance [29]. From our point of view, the two theories with and without fine-tuning in the three-body channel correspond to the UV and IR fixed points. When $M/m \rightarrow 13.6$, the two fixed points approach each other: the difference between theories with and without 3-body fine-tuning becomes smaller and smaller. Finally when $M/m > 13.6$, the fixed point completely disappears, and an energy scale appears in the problem: the ground state energy of the three-body bound state.

3. Operator anomalous dimensions at the IR and UV fixed points

We can gain insight about the two fixed points by looking at the dimension of the operators. Let us consider the two-particle operator $\psi\psi$. According to the operator/state correspondence developed in Ref. [30], one can find dimensions of this operator by putting two particles in a harmonic potential. The Hamiltonian of the system is given

by

$$H = -\frac{1}{2}\nabla_1^2 - \frac{1}{2}\nabla_2^2 + V(|\mathbf{r}_1 - \mathbf{r}_2|) + \frac{1}{2}\omega^2(r_1^2 + r_2^2). \quad (23)$$

In terms of the center of mass coordinate \mathbf{R} and relative coordinate \mathbf{r} , the Hamiltonian can be rewritten as $H = H_R + H_r$ where the ground state energy of H_R equals $d\omega/2$ and

$$H_r = -\nabla_r^2 + V(r) + \frac{1}{4}\omega^2 r^2, \quad (24)$$

where the potential is given in Eq. (15). The ground state wave functions and energies for this Hamiltonian for g tuned to one of the fixed points g_\pm is easily seen to equal

$$\psi_\pm = e^{-\omega r^2/4} r^{\nu_\pm}, \quad E_r^\pm = \left(\frac{d}{2} + \nu_\pm\right)\omega \quad (25)$$

in the limit $r_0 \rightarrow 0$. (Recall that the fixed points g_\pm correspond to solutions $\psi = r^{\nu_\pm}$ in the absence of the harmonic potential). Therefore the total ground state energy is

$$E^\pm = (d + \nu_\pm)\omega, \quad (26)$$

and so the scaling dimensions of the two-particle operator $\psi\psi$ are

$$\Delta_\pm = (d + \nu_\pm) = \frac{d+2}{2} \pm \sqrt{\alpha - \alpha_*}, \quad (27)$$

where Δ_+ and Δ_- are the operator dimension at the IR and UV fixed points, respectively. We emphasize the fact that

$$(\Delta_+ + \Delta_-) = d + 2 \quad (28)$$

in any spatial dimension d and for any in the range $\alpha_* \leq \alpha < (\alpha_* + 1)$ in Eq. (13), with $\Delta_+ = \Delta_- = (d+2)/2$ at $\alpha = \alpha_*$. Note that $(d+2)$ is the scaling dimension of a nonrelativistic Lagrange density, since time has twice the scaling dimension as space; we return to this below, when we discuss the AdS/CFT correspondence.

B. The renormalization group: ϵ expansion

When we consider relativistic quantum field theories a nonperturbative solution will not be available, and we will have to rely on either perturbation theory, or a truncation of the Schwinger-Dyson equations. It is therefore instructive to examine a perturbative analysis of the $1/r^2$ potential. We have seen that the doubly degenerate fixed point at the phase transition occurs at coupling $\gamma = \gamma_* = (d-2)/2$; we therefore start with the action in $d = 2 + \epsilon$ spatial dimensions, where perturbation theory can correctly describe the phase transition. In order to facilitate the use of Feynman diagrams, we write the theory in second quantized form with a contact interaction,

$$\begin{aligned}
 S = & \int dt d^d \mathbf{x} \left(i \psi^\dagger \partial_t \psi - \frac{|\nabla \psi|^2}{2} + \pi \frac{g}{4} \psi^\dagger \psi^\dagger \psi \psi \right) \\
 & - \int dt d^d \mathbf{x} d^d \mathbf{y} \psi^\dagger(t, \mathbf{x}) \psi^\dagger(t, \mathbf{y}) \\
 & \times \frac{\alpha}{|\mathbf{x} - \mathbf{y}|^2} \psi(t, \mathbf{y}) \psi(t, \mathbf{x}), \quad (29)
 \end{aligned}$$

where the factor of π in the contact interaction is chosen for future convenience. The Feynman rules are as follows:

(i) Propagator

$$\frac{i}{\omega - \mathbf{p}^2/2}, \quad (30)$$

(ii) Contact vertex

$$i\pi g \mu^{-\epsilon}, \quad (31)$$

(iii) ‘‘Meson exchange’’

$$\frac{2\pi i \alpha}{\epsilon} \frac{1}{|\mathbf{q}|^\epsilon}. \quad (32)$$

Note the unusual $1/\epsilon$ pole in the ‘‘meson propagator.’’ It arises because the Fourier transform of a $1/r^2$ interaction is log divergent in $d = 2$ dimensions.

It is easy to see that α does not get renormalized (as one would expect, being the strength of a nonlocal interaction); however the coupling g runs. From the above Feynman rules, the β -function for g arises from the sum of the tree graph and the one-loop graph shown in Fig. 3, with the result

$$\begin{aligned}
 \beta(g; \alpha) &= \frac{\partial g}{\partial t} = \epsilon g - \frac{g^2}{2} + 2\alpha \\
 &= 2\left(\alpha + \frac{\epsilon^2}{4}\right) - \frac{1}{2}(g - \epsilon)^2, \quad (33)
 \end{aligned}$$

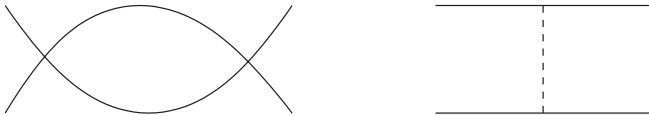


FIG. 3. Two diagrams contributing to the β -function in Eq. (33). Note that the second diagram is a tree diagram.

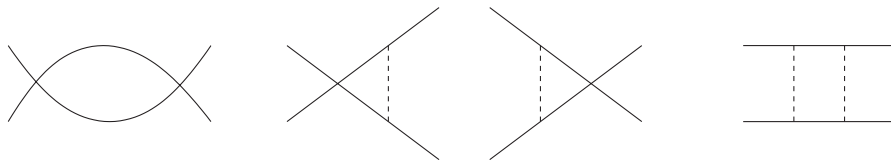


FIG. 4. Diagrams contributing to the perturbative β -function for \tilde{g} .

which we recognize to be equivalent to our toy model, up to an unimportant rescaling of g by 2, with

$$g_* = \epsilon, \quad \alpha_* = -\frac{\epsilon^2}{4}, \quad (34)$$

Note that our perturbative expansion is justified for small ϵ , but α_* coincides with the exact result in Eq. (19). For $\alpha > \alpha_*$ the β -function has two zeros: $g_{\pm} = g_* \pm 2\sqrt{\alpha - \alpha_*}$. At $\alpha = 0$, $g_- = 0$ is the IR stable fixed point, corresponding to a noninteracting theory—for the generic short-ranged potential, low-energy scattering is trivial; $g_+ = 2\epsilon$ corresponds to a fine-tuned potential with a bound state at threshold (i.e., an infinite scattering length). As one decreases α the two fixed points approach each other, merging at $g_{\pm} = g_*$ when $\alpha = \alpha_*$.

For $\alpha < \alpha_*$ the potential requires a cutoff and has a bound state; we can estimate the size of the bound state to be given by the correlation length $\xi = \Lambda_{\text{IR}}^{-1}$ in Eq. (4); this gives a binding energy $B \sim \Lambda_{\text{IR}}^2$, or

$$B \sim \Lambda_{\text{UV}}^2 \exp\left(-\frac{2\pi}{\sqrt{\alpha_* - \alpha}}\right). \quad (35)$$

Note that this formula is independent of ϵ and therefore appears to be independent of dimension. In fact the above estimate is verified in the nonperturbative calculation of the previous section.

An unusual feature of our calculation of the β -function (Fig. 3) is the contribution from a tree graph. We close this section by noting that a more conventional calculation is obtained by making the following change of variable:

$$g = \tilde{g} - \frac{2\alpha}{\epsilon}. \quad (36)$$

Then the RG equation becomes

$$\frac{\partial \tilde{g}}{\partial t} = \epsilon \tilde{g} - \frac{\tilde{g}^2}{2} + 2\tilde{g}\left(\frac{\alpha}{\epsilon}\right) - \frac{1}{2}\left(\frac{2\alpha}{\epsilon}\right)^2. \quad (37)$$

The first term on the right-hand side comes from the engineering dimension. The other terms come from the diagrams as in Fig. 4. All diagrams now have loops. This is a more natural approach from the point of view of Wilsonian RG, where one looks at the logarithm in the momentum integral instead of the $1/\epsilon$ poles. But we emphasize that the two RG equations lead to the same physical consequences.

1. Summary of the QM example

Before proceeding, we summarize our findings.

- (i) The theory has two fixed points, IR and UV, when $\alpha > \alpha_* = -(d-2)^2/4$.
- (ii) When there are two fixed points, the dimensions of the scalar operators at the IR and UV fixed points are Δ_+ and Δ_- , and they satisfy $\Delta_+ + \Delta_- = d + 2$.
- (iii) When $\alpha < \alpha_*$, the fixed points do not exist and the theory develops a bound state energy which scales as $\Lambda_{UV}^2 \exp(-2\pi/\sqrt{\alpha_* - \alpha})$.

IV. A HOLOGRAPHIC PERSPECTIVE

As far as we know, there is no simple holographic dual description of quantum mechanics with $1/r^2$ potentials, nor of the field theoretical models considered later in this paper. However, holography provides an interpretation of conformality loss which turns out to be very useful in developing our intuition about such phase transitions: the loss of conformality can be associated with the violation of the Breitenlohner-Freedman (BF) bound.

A. The conformal phase: pair of theories

In our RG discussion, for $\alpha > \alpha_*$ there are two CFTs that merge into one at $\alpha = \alpha_*$. This situation is reminiscent of what occurs in holography [31]: a higher dimensional theory containing a scalar field ϕ with mass m^2 in the interval $-d^2/4 < m^2 < -d^2/4 + 1$ corresponds to *two* different boundary theories in which the dimensions of the operator O dual to ϕ have two different values

$$\Delta_{\pm} = \frac{d}{2} \pm \frac{1}{2} \sqrt{d^2 + 4m^2} \equiv \frac{d}{2} \pm \nu. \quad (38)$$

The main point here is that in the asymptotics of the scalar field near the boundary $z = 0$ of the AdS space, $\phi(z) = c_- z^{\Delta_-} + c_+ z^{\Delta_+}$, one can interpret c_- as the source coupled to O , and c_+ as its expectation value, and vice versa.

Instead of repeating the discussion in Ref. [31], we illustrate its main points in a simple model. In this model, one sees that the theory with $[O] = \Delta_-$ can be obtained from the theory with $[O] = \Delta_+$ by adding to the Lagrangian a term O^2 with a fine-tuned coefficient (in other words, we will go “against the RG flow,” cf. Ref. [32] where one follows the RG flow from the UV fixed point to the IR fixed point).

Consider a massive scalar field in AdS_{d+1} space. We use Euclidean signature in this subsection, so the metric is

$$ds^2 = \frac{R^2}{z^2} (dz^2 + dx^\mu dx^\mu). \quad (39)$$

We will set the radius of the AdS space $R = 1$. The action for the scalar field $\phi = \phi(z, x)$ is

$$\begin{aligned} S &= \frac{1}{2} \int dz d^d x \sqrt{g} (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2) \\ &\quad - \frac{1}{\epsilon^{\Delta_+}} \int d^d x J(x) \phi(\epsilon, x) \\ &= \frac{1}{2} \int dz d^d x \frac{1}{z^{d+1}} [z^2 (\partial_z \phi)^2 + z^2 (\partial_\mu \phi)^2 + m^2 \phi^2] \\ &\quad - \frac{1}{\epsilon^{\Delta_+}} \int d^d x J(x) \phi(\epsilon, x). \end{aligned} \quad (40)$$

In our model, this action is taken as the definition of the CFT. This CFT “lives” on the boundary in the sense the external source J couples only to the field at some small $z = \epsilon$, with $1/\epsilon$ playing the role of the momentum UV cutoff. The operator $O(x)$ that J couples to is defined as $O(x) = \epsilon^{-\Delta_+} \phi(\epsilon, x)$. The extra power of ϵ is chosen so that subsequent results have a regular $\epsilon \rightarrow 0$ limit.

We assume a large N parameter so that one can use the saddle point approximation, in which ϕ satisfies the field equation

$$\phi'' - \frac{d-1}{z} \phi' - q^2 \phi - \frac{m^2}{z^2} \phi + \epsilon^{\Delta_- - 1} J \delta(z - \epsilon) = 0, \quad (41)$$

where we have changed to momentum space. We assume $q\epsilon \ll 1$, i.e., q is much smaller than the UV cutoff. To completely specify the solution we impose two boundary conditions. Near $z = 0$ there are two possible solutions to this equation, $\phi \sim z^{\Delta_{\pm}}$ where Δ_{\pm} are defined in Eq. (38). We require that

$$\phi = c_0 z^{\Delta_+}, \quad z \rightarrow 0, \quad (42)$$

i.e., we require ϕ to follow the most regular asymptotic behavior at small z . We leave the boundary condition at $z \rightarrow \infty$ for later discussion. Equation (42) is valid for $z < \epsilon$, but due to the insertion of a source at $z = \epsilon$, ϕ contains both asymptotics once z is larger than ϵ ,

$$\phi = c_+ z^{\Delta_+} + c_- z^{\Delta_-}, \quad \epsilon < z \ll q^{-1}. \quad (43)$$

Clearly, c_- is proportional to the source J . Matching boundary conditions one finds

$$c_- = \frac{J}{\Delta_+ - \Delta_-}. \quad (44)$$

From the point of view of the interior region $z > \epsilon$, Eqs. (43) and (44) effectively fix the boundary condition near $z = \epsilon$. Here we obtain a key ingredient of the AdS/CFT prescription: the coefficient in front of the z^{Δ_-} part of the field is the source coupled the operator ϕ .

The coefficients c_0 in Eq. (42) and c_+ in Eq. (43) can be determined only after the boundary condition at $z = \infty$ is fixed. We can relate the expectation value of O with c_+ :

$$\langle O \rangle_J = \frac{\phi(\epsilon)}{\epsilon^{\Delta_+}} = \frac{J}{(\Delta_+ - \Delta_-) \epsilon^{\Delta_+ - \Delta_-}} + c_+. \quad (45)$$

Therefore, up to a singular contribution, the expectation value of O is related to c_+ . The two-point function $\langle OO \rangle$ is then

$$\langle OO \rangle = \frac{\partial}{\partial J} \langle O \rangle|_J = \frac{J}{2\nu\epsilon^{2\nu}} + \frac{\partial c_+}{\partial J}. \quad (46)$$

Let us now impose the boundary condition at $z \rightarrow \infty$. To ensure finiteness of the action, it is sufficient to require

$$\phi(z, x) \rightarrow 0, \quad z \rightarrow \infty. \quad (47)$$

The saddle point solution is now completely determined

$$\phi(z) = \begin{cases} D\epsilon^{-\nu} K_\nu(q\epsilon) z^{d/2} I_\nu(qz), & z < \epsilon, \\ D\epsilon^{-\nu} I_\nu(q\epsilon) z^{d/2} K_\nu(qz), & z > \epsilon, \end{cases} \quad (48)$$

with $D = J\epsilon^{-\nu}$. This solution corresponds to

$$c_- = \frac{J}{2\nu}, \quad c_+ = -\frac{\Gamma(1-\nu)}{\nu^2 \Gamma(\nu) 2^{1+2\nu}} J q^{2\nu}. \quad (49)$$

The two-point function $\langle OO \rangle$ is proportional to $q^{2\nu}$, consistent with dimension of O being $\Delta_+ = d/2 + \nu$.

Now we turn on a deformation O^2 with a coefficient that will be fine-tuned to get another conformal field theory. The action is now

$$S = \frac{1}{2} \int dz d^d x \frac{1}{z^{d+1}} [z^2 (\partial_z \phi)^2 + z^2 (\partial_\mu \phi)^2 + m^2 \phi^2] - \int d^d x \left[\frac{\lambda}{2\epsilon^d} \phi^2(\epsilon) + J \frac{\phi(\epsilon)}{\epsilon^{\Delta_-}} \right]. \quad (50)$$

Let us first set $J = 0$. The field equation is

$$-\phi'' + \frac{d-1}{z} \phi' + \frac{m^2}{z^2} \phi + q^2 \phi - \frac{\lambda}{\epsilon} \delta(z-\epsilon) \phi = 0. \quad (51)$$

One can integrate this equation from $z = 0$ to larger z . For $z < \epsilon$, ϕ is purely z^{Δ_+} , and for $z > \epsilon$ it becomes a mixture of z^{Δ_+} and z^{Δ_-} , the relative weight of which depends on λ . The most interesting value of λ is when ϕ is purely z^{Δ_-} for $z > \epsilon$. This happens when λ is fine-tuned to the critical value

$$\lambda = \Delta_+ - \Delta_-. \quad (52)$$

There is a quantum-mechanical interpretation of this fine-tuning. If one identifies z as the radial coordinate r of a two-dimensional space, then Eq. (51) is the radial Schrödinger equation for the wave function $\psi = z^{-d/2} \phi$ of a particle moving in a potential which is a sum of a $1/r^2$ piece and a delta-shell piece,

$$V(r) = \frac{\nu^2}{r^2} - \frac{\lambda}{\epsilon} \delta(r-\epsilon). \quad (53)$$

with $-q^2$ playing the role of the energy. The value (52) corresponds to the case when the potential has a zero-energy bound state.

Let λ be fine-tuned to this value, and turn on the source J . Using the asymptotics $\phi \sim z^{\Delta_+}$ for $z < \epsilon$ and integrating the field equations passed $z = \epsilon$, we find that for $z > \epsilon$, the coefficient c_+ is now proportional to J :

$$c_+ = -\frac{J}{\Delta_+ - \Delta_-}. \quad (54)$$

The expectation value for O is now related to c_- ,

$$\langle O \rangle_J = c_-. \quad (55)$$

The assignment of source and expectation value is reverse to the case $\lambda = 0$. If one imposes the boundary condition $\phi(z) \rightarrow 0$ when $z \rightarrow \infty$, then the solution to Eq. (51) is given by Eq. (48), but now

$$D = \frac{J\epsilon^\nu}{1 - 2\nu I_\nu(q\epsilon) K_\nu(q\epsilon)}. \quad (56)$$

The solution corresponds to

$$c_+ = -\frac{J}{2\nu}, \quad c_- = \frac{2^{2\nu-1} \Gamma(\nu)}{\Gamma(1-\nu)} \frac{J}{q^{2\nu}}. \quad (57)$$

In particular $\langle OO \rangle \sim q^{-2\nu}$, corresponding to $[O] = \Delta_- = d/2 - \nu$.

Thus, in this simple holographic model, the UV stable fixed point of the CFT with the fine-tuned O^2 interaction corresponds to the same bulk theory, but with the opposite assignment for the source and the expectation value.

B. Below the Breitenlohner-Freedman bound

Here we speculate on the fate of the bulk theory with a scalar with m^2 below the Breitenlohner-Freedman (BF) bound $-d^2/4$. The most interesting case is when m^2 is only slightly below the BF bound, where the boundary theory is approximately conformal over a large energy range. The dual bulk description should involve a space-time that is approximately AdS, cutoff both at the UV and the IR by the respective ‘‘walls.’’

First, for the set up with a scalar m^2 below the BF bound, there must be an UV cutoff in the theory. For example, the theory with a dual description can arise as a low-energy limit of another theory whose UV is free of any instability. Let us model that by imposing a hard cutoff on the AdS space, and impose a boundary condition on the scalar ϕ at the cutoff. The precise form of the boundary condition is not important, for definiteness we take it to be Dirichlet: $\phi(z_{UV}) = 0$.

One expect that a IR scale will be generated by the condensation of ϕ . We model that scale very roughly by another, IR, cutoff at z_{IR} , and impose another Dirichlet boundary condition there.

Now let us look at the field equation for ϕ ,

$$\phi'' - \frac{d-1}{z} \phi' - \frac{m^2}{z^2} \phi - q^2 \phi = 0. \quad (58)$$

Changing variables to $\phi = z^{(d-1)/2}\psi$, this equation becomes

$$-\psi'' + \frac{m^2 + (d^2 - 1)/4}{z^2}\psi = -q^2\psi. \quad (59)$$

This equation, with the boundary condition on at z_{IR} and z_{UV} , gives us an infinite tower of particles. The mass square of the particles in this tower is the eigenstate of a particle in a one-dimensional potential which is α/r^2 enclosed between two infinite walls at z_{UV} and z_{IR} . The condition of absence of tachyon is equivalent to the condition that the potential does not contain a negative-energy eigenstate. This requires the interval between the two cutoffs is not too large,

$$\ln \frac{z_{\text{IR}}}{z_{\text{UV}}} < \frac{\pi}{\sqrt{m_{\text{BF}}^2 - m^2}}. \quad (60)$$

In a more realistic setup where the scale z_{IR} appears dynamically, one can expect that it appears at the scale required for preventing a tachyon,

$$z_{\text{IR}} \sim z_{\text{UV}} \exp\left(\frac{\pi}{\sqrt{m_{\text{BF}}^2 - m^2}}\right). \quad (61)$$

We expect that this situation is rather generic in holography. It would be interesting to construct an explicit solution in string theory which exhibits the BKT scaling.

V. A RELATIVISTIC EXAMPLE: DEFECT QFT

In this section we consider a relativistic quantum field theory that exhibits the phenomenon of fixed point annihilation. The example resembles QCD with large number of flavors, but the phase transition occurs in the regime of weak coupling.

We consider a theory of a fermion living on a d -dimensional membrane, and interacting through a $SU(N_c)$ gauge field that lives in $(3 + 1)$ dimensions. We shall assume that the $SU(N_c)$ gauge coupling does not run; it can be easily accomplished by taking the gauge field to be part of a conformal field theory, say, the $\mathcal{N} = 4$ super-Yang-Mills theory. The most interesting case is $d = 3 (= 2 + 1)$, which was analyzed in Refs. [33,34] using the Schwinger-Dyson approximation and gauge/gravity duality. Here we shall take $d = 2 + \epsilon$ (i.e., $(1 + \epsilon) + 1$) to take advantage of a small parameter $\epsilon \ll 1$.

The action is

$$S = \int d^d x (i\bar{\psi}\gamma^\mu\partial_\mu\psi + g\bar{\psi}\gamma^\mu\psi A_\mu) - \frac{1}{4} \int d^4 x F_{\mu\nu}^a F_{\mu\nu}^a + \dots, \quad (62)$$

and we assume there is a UV cutoff Λ . The d -dimensional photon propagator is obtained by integrating out the 4d propagator over transverse directions,

$$D_{\mu\nu}(q) = \int \frac{d^{2-\epsilon}q_\perp}{(2\pi)^{2-\epsilon}} \frac{-i}{q^2} \left(g_{\mu\nu} - (1 - \xi) \frac{q_\mu q_\nu}{q^2} \right), \quad (63)$$

where $q^2 = q_\parallel^2 - q_\perp^2$. For small ϵ , the result is

$$D_{\mu\nu}(q) = \frac{ig_{\mu\nu}}{2\pi\epsilon} \left(\frac{1}{(-q^2)^{\epsilon/2}} - \frac{1}{\Lambda^\epsilon} \right). \quad (64)$$

Note that the dependence on the gauge parameter ξ disappears in the small ϵ limit (if one assumes $\xi \sim 1$).

A. Schwinger-Dyson treatment (rainbow approximation)

Before going to the RG treatment, we review how the chiral phase transition is found using the gap equation. This treatment is very similar to that used in QCD [11,14]. The lowest-order gap equation is (Fig. 5)

$$-i\Sigma(p) = -g^2 \int \frac{d^d q}{(2\pi)^d} D_{\mu\nu}(p-q) \gamma^\mu t^a \times \frac{\gamma^\nu q_\nu + \Sigma(q)}{q^2 - \Sigma^2(q) + i\epsilon} \gamma^\nu t^a. \quad (65)$$

Inserting the photon propagator (63) and performing a Wick rotation, the equation becomes, for small ϵ ,

$$\Sigma(p) = \frac{g^2 C_A}{4\pi^3 \epsilon} \int d^d q \frac{1}{|p-q|^\epsilon} \frac{\Sigma(q)}{q^2 + \Sigma^2(q)}, \quad (66)$$

$$C_A \equiv \frac{N_c^2 - 1}{2N_c},$$

where the integral is taken in Euclidean space. It will become clear later that the dominant contribution to the integral comes from the regions $p \ll q$ and $p \gg q$, with $p \sim q$ giving a subleading contribution. Changing variables to

$$x = \ln \frac{p}{m}, \quad y = \ln \frac{q}{m}, \quad (67)$$

where $m = \Sigma(0)$ will be the mass gap, the Schwinger-Dyson equation becomes

$$\Sigma(x) = \frac{g^2 C_A}{2\pi^2 \epsilon} \left[\int_0^x dy [e^{-\epsilon(x-y)} - e^{-\epsilon(x_m-y)}] \Sigma(y) + \int_x^{x_m} dy [1 - e^{-\epsilon(x_m-y)}] \Sigma(y) \right], \quad (68)$$

where $x_m = \ln(\Lambda/m)$ and Λ is the UV cutoff. Differentiating Eq. (68) over x , we find

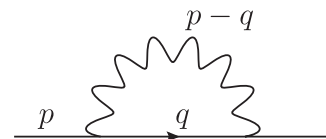


FIG. 5. The one-loop graph that contributes to the gap equation.

$$\Sigma'(x) = -\frac{g^2 C_A}{2\pi^2} \int_0^x dy e^{-\epsilon(x-y)} \Sigma(y), \quad (69)$$

$$\Sigma''(x) = \frac{g^2 C_A \epsilon}{2\pi^2} \int_0^x dy e^{-\epsilon(x-y)} \Sigma(y) - \frac{g^2 C_A}{2\pi^2} \Sigma(x), \quad (70)$$

from which we find that Σ satisfies the differential equation

$$\Sigma''(x) + \epsilon \Sigma'(x) + \frac{g^2 C_A}{2\pi^2} \Sigma(x) = 0, \quad (71)$$

with boundary conditions

$$\Sigma'(0) = 0, \quad \Sigma(x_m) = 0. \quad (72)$$

The solution to the equation is

$$\Sigma(x) = m e^{-\epsilon x/2} \frac{\cos(\kappa x - \delta)}{\cos \delta}, \quad (73)$$

with

$$\kappa = \sqrt{\frac{g^2 C_A}{2\pi^2} - \frac{\epsilon^2}{4}}. \quad (74)$$

When ϵ and κ are small, Σ varies slowly on the logarithmic scale, which validates the assumption that the integral in Eq. (66) is dominated by regions where p and q are very different.

The boundary conditions imply

$$\tan \delta = \frac{\epsilon}{2\kappa}, \quad \cos(\kappa x_m - \delta) = 0, \quad (75)$$

from which one finds

$$x_m = \frac{1}{\kappa} \left[\left(n + \frac{1}{2} \right) + \arctan \frac{\epsilon}{2\kappa} \right], \quad (76)$$

where n is an integer. The solution with $n = 0$ corresponds to the biggest gap and is favored energetically. The dynamically generated mass gap is

$$m \sim \Lambda \exp \left[-\frac{1}{\kappa} \left(\frac{\pi}{2} + \arctan \frac{\epsilon}{2\kappa} \right) \right], \quad (77)$$

$$\kappa = \sqrt{\frac{g^2 C_A}{2\pi^2} - \frac{\epsilon^2}{4}}.$$

So we find that there is a phase transition occurring at

$$g_*^2 = \frac{\pi^2 \epsilon^2}{2C_A}, \quad (78)$$

and the critical behavior of the gap near $g = g_*$ conforms with BKT scaling.

B. RG treatment: beyond the rainbow

The RG equation can be written in a way very similar to the RG equation for the QM example with $1/r^2$ potential. One introduces an extra four-fermi interaction into the Lagrangian

$$S = \int d^d x (i \bar{\psi} \gamma^\mu \partial_\mu \psi + g \bar{\psi} \gamma^\mu \psi A_\mu - \frac{c}{2} (\bar{\psi} \gamma^\mu t^a \psi)^2) - \frac{1}{4} \int d^4 x F_{\mu\nu}^a F_{\mu\nu}^a + \dots \quad (79)$$

The tree level one-gluon exchange contains a $1/\epsilon$ factor from the gluon propagator (64) and contributes to the beta function for c :

$$\beta(c) = \epsilon c - \frac{N_c}{2\pi} c^2 - \frac{g^2}{2\pi}. \quad (80)$$

The phase transition occurs at $g = g_*$ where $\beta(c)$ has a double zero,

$$g_*^2 = \frac{\pi^2 \epsilon^2}{N_c}. \quad (81)$$

When $g > g_*$, we need to solve the RG equation,

$$\frac{\partial c}{\partial \ln \mu} = \beta(c), \quad (82)$$

with the boundary condition that the bare four-fermi coupling is zero at the UV cutoff, $g(\Lambda) = 0$. The solution is

$$c(\mu) = \frac{\pi \epsilon}{N} + \frac{2\pi}{N} \kappa \tan \left[\kappa \ln \frac{\Lambda}{\mu} - \delta \right], \quad (83)$$

$$\kappa = \sqrt{\frac{g^2 N_c}{4\pi^2} - \frac{\epsilon^2}{4}},$$

where $\delta = \arctan(\epsilon/2\kappa)$. The coupling constant becomes infinite at

$$m = \Lambda \exp \left[-\frac{1}{\kappa} \left(\frac{\pi}{2} + \delta \right) \right]. \quad (84)$$

We find that in the limit $N_c \rightarrow \infty$, the result from the RG approach coincide with what is obtained from the Schwinger-Dyson approach. However, for finite N_c the results of the two approaches are different. This is not unexpected, since the RG sums up a wider class of diagrams than the gap equation.

For $g < g_*$, there are two zeros of the β -functions

$$c_\pm = \frac{1}{N_c} (\pi \epsilon \mp \sqrt{\pi^2 \epsilon^2 - g^2 N_c}). \quad (85)$$

On the other hand, the scaling dimension of operator $\bar{\psi} \psi$ is

$$\Delta_\pm[\bar{\psi} \psi] = 1 + \epsilon - \frac{N_c}{2\pi} c_\pm = 1 + \frac{\epsilon}{2} \pm \sqrt{\pi^2 \epsilon^2 - g^2 N_c}. \quad (86)$$

We find that

$$\Delta_+ + \Delta_- = 2 + \epsilon = d. \quad (87)$$

up to possible corrections of order ϵ^2 .

C. Summary of the relativistic example

The lessons we learn from the relativistic examples are very similar to the nonrelativistic example:

- (i) $g < g_*$: two fixed points. The dimensions of the operator $\bar{\psi}\psi$ at the two fixed points satisfy $\Delta_+ + \Delta_- = d$.
- (ii) $g > g_*$: no fixed points, gap formation, BKT scaling.

VI. QCD AT LARGE N_c AND N_f

We now turn our attention to the most interesting, and most difficult, example; the chiral phase transition in QCD with large N_c and N_f . Denote $x = N_f/N_c$. We consider the Veneziano limit $N_c \rightarrow \infty$, $N_f \rightarrow \infty$, x fixed, and denote the (rescaled) 't Hooft coupling as

$$a_s = \frac{g^2 N_c}{(4\pi)^2}. \quad (88)$$

The beta function of QCD in this regime is

$$\beta(a_s) = -\frac{2}{3}[(11 - 2x)a_s^2 + (34 - 13x)a_s^3 + \dots]. \quad (89)$$

For $x < 11/2$ the theory is asymptotically free. If x is slightly below $11/2$ the beta function has a nontrivial zero which is still in the perturbative regime:

$$a_{s*} = \frac{2}{75}(11 - 2x). \quad (90)$$

This is the Banks-Zaks (BZ) fixed point [5]. In the IR, the theory is an interacting CFT. This fixed point moves to strong coupling as one makes $11/2 - x \sim 1$. For small x , $x \ll 1$, we believe that the theory has chiral symmetry breaking and a confinement scale. It is natural to assume that there is a critical N_f/N_c ratio x_{crit} at which the chiral condensate goes to zero.⁴

If the picture emerging from the previous examples also holds for QCD, then conformality is lost when the BZ fixed point annihilates with another UV fixed point. Therefore, we predict that when x is slightly larger than x_{crit} , QCD has an UV fixed point, in addition to the IR fixed point (and the free UV fixed point).

The situation is illustrated in Fig. 6. The UV fixed point called QCD*, is a different CFT compared to the usual IR fixed point, for example, the dimension of the operator $\bar{\psi}\psi$ should be different between the two fixed points.

What is the nature of QCD*? It could be that the β -function for the QCD gauge coupling simply has an unstable fixed point at strong coupling. However, this picture implicitly assumes that the set of relevant operators in QCD* consists of just kinetic terms for the gauge fields and fermions, as is the case at weak coupling. At strong coupling other operators could be relevant as well, and

⁴One assumes that some UV scale, e.g., the scale Λ_{QCD} defined from one-loop running, is fixed when x is changed.

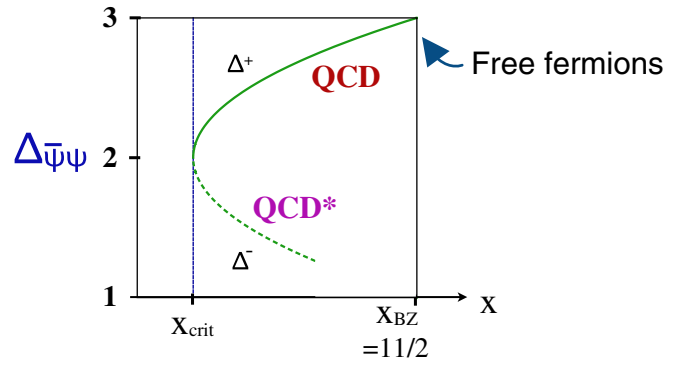


FIG. 6 (color online). A possible picture for the QCD chiral phase transition in N_f/N_c . The lines denote the dependence of the dimension of the chiral condensate $\bar{\psi}\psi$ at the fixed point. The solid line is the IR fixed point, and the dashed line is the UV fixed point.

guided by our defect QFT example of Sec. V, it is natural to consider the possibility that a chirally symmetric four-fermion operator is relevant in QCD*

$$\mathcal{L}_{\text{QCD}^*} = \mathcal{L}_{\text{QCD}} - c(\bar{\psi}\gamma^\mu t^a \psi)^2 \quad (91)$$

and that the unstable fixed point exists at some value $\{g_*, c_*\}$ in the two-dimensional space of couplings. By analogy with Sec. V, we then expect that the beta function for c contains linear, quadratic and constant terms,

$$\beta(c) = \gamma_1 c - \gamma_2 N_c c^2 + \gamma_0 g^2, \quad (92)$$

where the linear $\gamma_1 c$ term is due to the anomalous dimension of the four-fermi operator, the quadratic c^2 term is due to the one-loop graph involving two four-fermi vertices, and the constant g^2 term is due to, e.g., one-gluon exchange graph.⁵ This is essentially the picture advocated by Gies and Jaeckel [10].⁶ The constants $\gamma_0, \gamma_1, \gamma_2$ depend on $x = N_f/N_c$; and x_{crit} is where $\beta(c)$ has a double zero.

We do not know where in x the fixed point QCD* exists. A particularly interesting possibility is that QCD* exists at weak coupling, say near $x = 11/2$. As described subsequently, we find many theories similar to QCD* in the perturbative regime, but none of them possess the full chiral symmetry of QCD, and hence cannot be QCD*. The possible “phase diagram” is illustrated in Fig. 6, the line corresponding to QCD* does not continue to the vicinity of $x = 11/2$.

In the rest of this section, we shall be looking for perturbative UV fixed points that flow to the BZ fixed

⁵The constant term can be of a different power of g , but it does not affect the argument.

⁶Beta functions containing three terms of the almost the same origin arise in orbifolds of $\mathcal{N} = 4$ super-Yang-Mills theory for the coefficients of double-traced operators [35]. The constant term also appears in the running for the Landau liquid parameters in the RG treatment of color superconductivity [36].

point. Models A and C below have been considered in Ref. [37].

A. Model A

As the first step, we try to find a fixed point when *one* operator, $\bar{\psi}\psi$, has dimension different from its dimension at the BZ fixed point. From AdS/CFT experience, we expect that operator dimensions at the two fixed points satisfy

$$\Delta_+ + \Delta_- = d = 4. \quad (93)$$

At the BZ fixed point $\Delta_+ \approx 3$, therefore at the QCD* fixed point, $\Delta_- \approx 1$, which means that this operator is almost a free scalar. This suggests that we look for QCD* in the following Lagrangian, which will be called model A,

$$\mathcal{L}_{\text{model A}} = \mathcal{L}_{\text{QCD}} + \frac{1}{2}(\partial_\mu \phi)^2 - \frac{y}{\sqrt{2}}\bar{\psi}\psi\phi - \frac{\lambda}{24}\phi^4. \quad (94)$$

It is convenient to define the rescaled couplings,

$$a_s = \frac{g^2 N_c}{(4\pi)^2}, \quad a_y = \frac{y^2 N_c N_f}{(4\pi)^2}, \quad \hat{\lambda} = \frac{\lambda N_c N_f}{(4\pi)^2}. \quad (95)$$

These constants will be found to be $O(N_c^0)$ at the fixed point. In this regime, the beta functions are

$$\beta_{a_s} = -\frac{2}{3}[(11 - 2x)a_s^2 + (34 - 13x)a_s^3], \quad (96)$$

$$\beta_{a_y} = -6a_s a_y + 2a_y^2, \quad (97)$$

$$\beta_{\hat{\lambda}} = -12a_y^2 + 4a_y \hat{\lambda}. \quad (98)$$

The fixed point for the gauge coupling, a_{s^*} , is the same as at the BZ fixed point to leading order in $1/N_c$. The fixed point for the Yukawa and four-scalar couplings are

$$a_{y^*} = 3a_{s^*}, \quad \hat{\lambda} = 3a_{y^*} = 9a_{s^*}. \quad (99)$$

Thus, model A has a perturbative fixed point.

We can compute the dimension of the scalar operators at the fixed point,

$$\Delta[\bar{\psi}\psi]_{\text{BZ}} = 3 - 3a_{s^*}, \quad (100a)$$

$$\Delta[\phi]_{\text{model A}} = 1 + a_{y^*}. \quad (100b)$$

Notice that at in the model-A fixed point, the operator $\bar{\psi}\psi$ is replaced by the operator ϕ . We see here that

$$\Delta[\bar{\psi}\psi]_{\text{BZ}} + \Delta[\phi]_{\text{model A}} = 4, \quad (101)$$

which coincides with our expectation (93). From the QM intuition, we may expect that when $\Delta_+ = \Delta_- = 2$, the BZ fixed point and model A become identical.

According to our expectation, the new fixed point should be an UV fixed point, and that there exist a deformation of this fixed point that leads to the BZ fixed point. The

deformation is provided by the scalar mass term $m^2\phi^2$. This deformation is relevant if $\Delta[\phi]$ is less than 2. If such perturbation is present, the scalar ϕ decouples below a certain energy scale, leaving the theory to be in the BZ fixed point, as expected.

It might seem that the way model-A Lagrangian was introduced, with an extra scalar field and Yukawa interaction, is very different from the way done in Eq. (91). It seems that if we want to change the dimension of $\bar{\psi}\psi$, then one should introduce a four-fermi interaction into the QCD Lagrangian:

$$\mathcal{L}_{\text{model A}} = \mathcal{L}_{\text{QCD}} + c(\bar{\psi}\psi)^2. \quad (102)$$

We argue here, however, that the two forms of the Lagrangian are just two different representations of the same fixed point; with Eq. (94) being the weak-coupling representation near the upper end of the conformal window ($x = 11/2$), and Eq. (102) being the more useful representation near the lower end ($x = x_{\text{crit}}$). Indeed, the propagator of ϕ at the IR fixed point is $q^{2\Delta_\phi - 4}$ where Δ_ϕ is the dimension of ϕ ; near the lower end of the conformal window $\Delta = 2$, therefore the scalar propagator is almost momentum independent, and the scalar-mediated interaction between fermions becomes pointlike. This is similar to the equivalence between Nambu–Jona-Lasinio and Yukawa models in dimensions between 2 and 4 [38,39].

B. Model B

Model B is similar to model A, except there are now two scalar fields,

$$\mathcal{L} = \mathcal{L}_{\text{QCD}} + \frac{1}{2}(\partial_\mu \phi_1)^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 - \frac{y}{\sqrt{2}}\bar{\psi}(\phi_1 + i\gamma^5 \phi_2)\psi - \frac{\lambda}{24}(\phi_1^2 + \phi_2^2)^2. \quad (103)$$

The Lagrangian preserves vector $SU(N_f)$ and axial $U(1)_A$ (more precisely, the nonanomalous discrete subgroup of it). The behavior of model B is exactly like in model A: there is a fixed point for y and λ ; and the running of g is not altered in large N_c, N_f regime.

C. Model C

Both model A and B preserves only a small subset of the $SU(N_f) \times SU(N_f)$ chiral symmetry of QCD. The simplest Lagrangian which preserves chiral symmetry is

$$\mathcal{L} = \mathcal{L}_{\text{QCD}} - y(\bar{\psi}t^A\psi\phi^A + i\bar{\psi}t^A\gamma^5\psi\pi^A) + \text{Tr}\partial^\mu\Phi^\dagger\partial_\mu\Phi - \lambda_1(\text{Tr}\Phi^\dagger\Phi)^2 - \lambda_2\text{Tr}(\Phi^\dagger\Phi)^2, \quad (104)$$

where $\Phi = (\phi^A + i\pi^A)t^A$, $A = 1, \dots, N_f^2$ or $A = 0, \dots, N_f^2 - 1$ are flavor Gell-Mann matrices, normalized so that $\text{Tr}(t^A t^B) = \frac{1}{2}\delta^{AB}$. We need to find the fixed point

of this theory. This fixed point should have only one IR unstable direction corresponding to the mass term for Φ .

The RG equations for g and y can be read out from Refs. [40–42]. We need the two-loop beta function for the gauge coupling (as the one-loop contribution has a small coefficient near $x = 11/2$), but for the Yukawa and scalar couplings one-loop beta function suffices. Moreover, the two-loop beta function for g and the one-loop beta function for y does not contain scalar self-couplings, thus one can first solve for the fixed point for g and y , and then look for the fixed points of scalar couplings. In this model we define a_y as

$$a_y = \frac{y^2 N_c}{(4\pi)^2}. \quad (105)$$

The beta functions are

$$\beta_{a_s} = -\frac{2}{3}(11 - 2x)a_s^2 - \frac{2}{3}(34 - 13x)a_s^3 - 2x^2 a_s^2 a_y, \quad (106)$$

$$\beta_{a_y} = -6a_s a_y + 2(1 + x)a_y^2. \quad (107)$$

We work around $x = 11/2$. At fixed a_s , there is a zero of β_y , but there is no fixed point of both beta functions. Therefore model C does not have a perturbative fixed point.

D. Model D

The difference between model C and models A, B is that we introduce $O(N^2)$ scalars into model C, while there are only $O(1)$ scalars in models A, B. As a result, the beta function for the gauge coupling changes, and there is no longer a fixed point.

Our last model, model D, interpolates between models B and C. We introduce couplings to $2M^2$ scalars that preserve a $SU(M) \times SU(M) \times SU(k)$ subgroup of the chiral symmetry group, with $M = N_f/k$,

$$\mathcal{L} = \mathcal{L}_{\text{QCD}} - y \bar{\psi}_i^\alpha t_{\alpha\beta}^A (\phi^A + i\gamma^5 \pi^A) \psi_i^\beta + \text{scalar terms}, \quad (108)$$

where α, β runs $1 \dots M$, i runs $1 \dots k$, A runs $1 \dots M^2$. Model B corresponds to $k = N_f$ and model C to $k = 1$. We redefine a_y to be

$$a_y = \frac{y^2 k N_c}{(4\pi)^2}. \quad (109)$$

The beta functions are now

$$\beta_{a_s} = -2a_s^2 \left[\frac{11 - 2x}{3} + \frac{34 - 13x}{3} a_s + \frac{x^2}{k^2} a_y \right]. \quad (110)$$

$$\beta_{a_y} = 2a_y \left[-3a_s + \left(1 + \frac{x}{k^2} \right) a_y \right]. \quad (111)$$

For x slightly below $11/2$, the model has a fixed point for any integer k larger than 1,

$$\begin{aligned} a_{s^*} &= \frac{2k^2 + 11}{25k^2 - 44} \cdot \frac{11 - 2x}{3}, \\ a_{y^*} &= \frac{2k^2}{25k^2 - 44} (11 - 2x), \end{aligned} \quad (112)$$

but there is no perturbative fixed point for $k = 1$ (model C). (The fixed point values for the four-scalar couplings can also be found.)

Therefore, there exist theories that preserve part of the chiral symmetry of QCD and flow to QCD by a relevant deformation, but we have not succeeded in finding a theory that plays the role of QCD* in the perturbative regime. This does not mean QCD* does not exist; in fact we will give arguments, largely based on holography, that it does exist in the nonperturbative region.

1. Operator dimensions in model D

The dimension of the scalar operators ϕ^A, π^A in model D is

$$\Delta[\phi]_{\text{model D}} = 1 + a_{y^*}|_{\text{model D}}. \quad (113)$$

On the other hand the dimension of $\bar{\psi}\psi$ at the BZ fixed point is given in Eq. (100a). Taking the sum of the dimensions, we find

$$\Delta[\bar{\psi}\psi]_{\text{BZ}} + \Delta[\phi]_{\text{model D}} = 4 + \frac{88}{25(25k^2 - 44)} (11 - 2x). \quad (114)$$

So, the rule $\Delta_+ + \Delta_- = 4$ is broken in model D when $k \sim O(1)$. When $k \gg 1$, the equation can be written in the suggestive form

$$\Delta_+ + \Delta_- = 4 + \frac{88}{625} \frac{n_\phi}{N_f^2} (11 - 2x), \quad (115)$$

where $n_\phi = 2M^2$ is the number of scalars. We see that the violation of the rule $\Delta_+ + \Delta_- = 4$ occurs when the number of scalars is of the same order as the number of color degrees of freedom, N_c^2 .

Recall that the rule $\Delta_+ + \Delta_- = 4$ can be understood from AdS/CFT correspondence: Δ is related to the mass square m^2 of the bulk scalar by the equations $\Delta(\Delta - d) = m^2 R^2$. How do we understand the fact that this rule is violated when there are $O(N^2)$ scalars? In fact, it is easy to come up with a mechanism leading to this effect within holography. Recall the AdS radius R is determined by the cosmological constant. Changing the boundary condition for the scalar field alters the vacuum energy (Casimir energy) associated with that scalar field [43]. The change in the vacuum energy is small for one scalar, but becomes of order one for $O(N^2)$ scalars. Thus we have

$$\Delta_{\pm} = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 R_{\pm}^2} \quad (116)$$

and, if $R_+/R_- = 1 + O(n_{\phi}/N^2)$, then $\Delta_+ + \Delta_- = 4 + O(n_{\phi}/N_c^2)$, which is exactly what is found. We can construct an holographic model where the deviation of $\Delta_+ + \Delta_-$ from 4 can be explicitly computed (see Appendix). Interestingly, in this simple model $\Delta_+ + \Delta_- > 4$, as in model D.

The holographic model also lends support to the hypothesis that model C, with full chiral symmetry, exists in strong coupling. Indeed, the reason model C does not exist at weak coupling, from the holographic point of view, is that flipping the boundary condition for $2N_f^2$ scalars from close to z^3 to close to z^1 is too much a disruption for the AdS geometry (for example, in terms of the change of the cosmological constant). However, when both Δ_+ and Δ_- are close to 2, the change of the vacuum energy is parametrically small in $\Delta_+ - \Delta_-$ [see Eq. (A11)], and flipping the boundary condition from z^{Δ_+} to z^{Δ_-} is no longer a large disruption. Hence, the theory where all fermion bilinears have dimension Δ_- , i.e., QCD*, should exist near the merger point. However, arguments based on holographic models can only be taken as suggestive at this moment.

VII. CONCLUSION

There have recently been several lattice studies seeking to find the boundaries of the conformal window in QCD [44–47] and in other QCD-like theories [48–55]. Interest in the phase transition between conformal and nonconformal theories is motivated in part by the invocation of approximate conformal symmetry in numerous theories for physics beyond the standard model.

In this paper we have investigated the nature of such a phase transition, and we suggest that there is a wide class of theories where it is due to the merger and annihilation of fixed points. Several explicit examples of this phenomenon were given, and we speculate that this mechanism is also responsible for the chiral phase transition in QCD in the large N_c , large N_f regime, at some critical value for N_f/N_c . We show that this mechanism leads to the BKT scaling behavior of the chiral condensate at the phase transition, and also implies the existence of the conformal theory QCD* which annihilates with QCD at the lower end of the conformal window. We tried, unsuccessfully, to construct QCD* in the perturbative regime, and argued that it should exist in the nonperturbative regime. It would be interesting to search for evidence for QCD* on the lattice.

The models considered in the last section of our paper, in an attempt to find the UV fixed point of QCD, may be of interest in their own right. For example, these models may be used to explicitly realize the “unhiggs” [56], which

behaves at high energies as a field with noninteger scaling dimension.

The picture of the chiral phase transition realizes walking technicolor when N_f/N_c is only slightly below the phase transition. In the holographic interpretation, conformality is lost when the mass squared of a bulk scalar drops below the BF bound. A naive application of AdS/CFT rules implies that the dimension of the operator $\bar{\psi}\psi$ is equal to 2 at the phase transition. This feature is explicit in the holographic model considered in the Appendix. This conclusion is, interestingly, in agreement with result from the Schwinger-Dyson approach, and also with Ref. [57]. The result illustrates that the dimension of the fermion bilinear on the IR stable branch cannot approach the unitarity bound used in Ref. [58] for estimating the maximal extension of the conformal window.

ACKNOWLEDGMENTS

The authors thank Ken Intriligator, Andreas Karch, and Igor Klebanov for discussions, and Brian Mattern for comments on the manuscript. This work is supported, in part, by U.S. DOE grants No. DE-FG02-00ER41132 (D. B. K., J. W. L., D. T. S.) and DE-FG02-01ER41195 (MAS).

APPENDIX: CASIMIR EFFECT IN AdS₅

We consider a holographic model for a UV-IR pair of conformal theories. Each theory has a set of n_{ϕ} scalar operators that all have the scaling dimension $1 < \Delta_- < 2$ in one theory and $2 < \Delta_+ < 3$ in the other. Both theories are described by the same holographic dual

$$S_5 = \frac{1}{2\kappa^2} \int_M d^5x \sqrt{-g} \left(\mathcal{R} - V_0 - \frac{1}{2} \sum_{i=1}^{n_{\phi}} ((\partial\phi_i)^2 + m^2\phi_i^2) \right). \quad (A1)$$

The two different theories correspond to the two choices of the boundary conditions on the scalar fields ϕ_i . The solution of the classical equations of motion is given by all $\phi_i = 0$ and AdS₅ metric:

$$ds^2 = R_0^2 z^{-2} (dz^2 + dx^2 - dt^2), \quad (A2)$$

with

$$R_0^2 = -12/V_0. \quad (A3)$$

The loop expansion is controlled by dimensionless parameter $\kappa^2 R_0^{-3} \sim N_c^2$, where N_c is the number of colors in the dual theory. One-loop contribution is not negligible in the $N_c \rightarrow \infty$ limit if n_{ϕ} is also large, i.e., $n_{\phi} \sim N_c^2$. For simplicity we assume $n_{\phi}/N_c^2 \ll 1$ and compute $\Delta_+ + \Delta_- - 4$ to leading order in n_{ϕ}/N_c^2 . We use the technique of Ref. [43].⁷ The one-loop contribution of the scalar fields

⁷For earlier calculations of the Casimir energy in AdS space see Refs. [59,60].

depends on the boundary condition. This contribution shifts the vacuum energy V_0 to V_{\pm} . Consequently, the AdS₅ curvature radius R_0 becomes at one loop

$$R_{\pm}^2 = -12/V_{\pm}. \quad (\text{A4})$$

It is convenient to measure lengths in units of R_0 , i.e., $R_0 = 1$, $V_0 = -12$, etc.

The calculation of the vacuum energy is easier to perform after the Wick rotation $t \rightarrow -ix_4$. The correction to the vacuum energy is equal to

$$\int d^5 x_E \sqrt{g_E} (V_{\pm} - V_0) = 2\kappa^2 \frac{n_{\phi}}{2} \log \det_{\pm} [(-\nabla_E^2 + m^2) \sqrt{g_E}], \quad (\text{A5})$$

where index E denotes objects defined using the Wick rotated metric $ds_E^2 = (dz^2 + dx^2 + dx_4^2)$. The expression in the right-hand side (rhs) of Eq. (A6) is formal, since it is UV divergent. The derivative with respect to m^2 eliminates some but not all divergences:

$$\int d^5 x_E \sqrt{g_E} \frac{d}{dm^2} V_{\pm} = \kappa^2 n_{\phi} \text{tr}_{\pm} \{ [(-\nabla_E^2 + m^2) \sqrt{g_E}]^{-1} \sqrt{g_E} \}. \quad (\text{A6})$$

The kernel of the operator $[(-\nabla_E^2 + m^2) \sqrt{g_E}]^{-1}$ is the Green's function defined by the following equation

$$[-\partial_z z^{-3} \partial_z + Q^2 z^{-3} + m^2 z^{-5}] G_{\pm\nu}(z, z'; Q) = \delta(z - z'). \quad (\text{A7})$$

where $\pm\nu$ refer to two different boundary conditions at $z = 0$: $G_{\pm\nu} \sim z^{2\pm\nu}$, where $\nu = \sqrt{4 + m^2}$, and $Q^2 = q_E^2$. The solution to Eq. (A7) is

$$G_{\pm\nu}(z, z'; Q) = z^2 z'^2 I_{\pm\nu}(Qz) K_{\pm\nu}(Qz') \theta(z' - z) + (z \leftrightarrow z'). \quad (\text{A8})$$

Collecting results so far, we find

$$\frac{d}{dm^2} V_{\pm} = \kappa^2 n_{\phi} \int \frac{d^4 q_E}{(2\pi)^4} G_{\pm\nu}(z, z; Q). \quad (\text{A9})$$

The right-hand side of Eq. (A9) is still UV divergent.

However, the difference $V_+ - V_-$ is finite:

$$\begin{aligned} & \frac{1}{\kappa^2 n_{\phi}} \frac{d}{dm^2} (V_+ - V_-) \\ &= \frac{1}{8\pi^2} \int_0^{\infty} dQ Q^3 z^4 (I_{\nu}(Qz) K_{\nu}(Qz) - (\nu \leftrightarrow -\nu)) \\ &= -\frac{1}{8\pi^2} \frac{2 \sin \nu \pi}{\pi} \int_0^{\infty} dx x^3 K_{\nu}^2(x) = -\frac{1}{12\pi^2} \nu(1 - \nu^2). \end{aligned} \quad (\text{A10})$$

Using the fact that both boundary conditions are the same at $\nu = 0$, and thus $V_+ = V_-$ at $\nu = 0$, as well as $d\nu^2 = dm^2$, we can write

$$\begin{aligned} \frac{1}{\kappa^2 n_{\phi}} (V_+ - V_-) &= -\frac{1}{12\pi^2} \int_0^{\nu^2} d\tilde{\nu}^2 \tilde{\nu} (1 - \tilde{\nu}^2) \\ &= -\frac{1}{6\pi^2} \left(\frac{\nu^3}{3} - \frac{\nu^5}{5} \right) < 0. \end{aligned} \quad (\text{A11})$$

(Note that $0 < \nu < 1$.)

Using Eqs. (A3) and (A4), and in the regime $|R_{\pm} - R_0| \ll R_0 = 1$, we can write

$$R_+ - R_- = \frac{1}{6} (V_+ - V_-). \quad (\text{A12})$$

The one-loop corrected scaling dimensions of the scalar operators become

$$\Delta_{\pm} = 2 \pm \sqrt{4 + m^2 R_{\pm}^2}. \quad (\text{A13})$$

and thus

$$\begin{aligned} \Delta_+ + \Delta_- - 4 &= \frac{m^2}{\nu} (R_+ - R_-) \\ &= \frac{\kappa^2 n_{\phi}}{36\pi^2} \nu^2 (4 - \nu^2) \left(\frac{1}{3} - \frac{\nu^2}{5} \right) > 0. \end{aligned} \quad (\text{A14})$$

Since $\kappa^2 \sim N_c^2$ the deviation of $\Delta_+ + \Delta_-$ from 4 is $O(n_{\phi}/N_c^2)$. At the point of merger $\nu \rightarrow 0$, $\Delta_+ + \Delta_- = 4$, hence $\Delta_+ = \Delta_- = 2$, with no correction of order $O(n_{\phi}/N_c^2)$.

[1] K. G. Wilson and J. B. Kogut, Phys. Rep. **12**, 75 (1974).
 [2] J. Kosterlitz, J. Phys. C **7**, 1046 (1974).
 [3] N. Seiberg, Nucl. Phys. **B435**, 129 (1995).
 [4] K. A. Intriligator and N. Seiberg, Nucl. Phys. B, Proc. Suppl. **45BC**, 1 (1996).
 [5] T. Banks and A. Zaks, Nucl. Phys. **B196**, 189 (1982).
 [6] J. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998).
 [7] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998).

[8] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).
 [9] P. Breitenlohner and D. Z. Freedman, Ann. Phys. (N.Y.) **144**, 249 (1982).
 [10] H. Gies and J. Jaeckel, Eur. Phys. J. C **46**, 433 (2006).
 [11] V. A. Miransky, Nuovo Cimento Soc. Ital. Fis. A **90**, 149 (1985).
 [12] B. Holdom, Phys. Lett. B **213**, 365 (1988).
 [13] B. Holdom, Phys. Rev. Lett. **62**, 997 (1989).

- [14] T. Appelquist, J. Terning, and L. C. R. Wijewardhana, Phys. Rev. Lett. **77**, 1214 (1996).
- [15] T. Appelquist, A. Ratnaweera, J. Terning, and L. C. R. Wijewardhana, Phys. Rev. D **58**, 105017 (1998).
- [16] V. A. Miransky, arXiv:hep-ph/0003137.
- [17] V. L. Berezinskii, Zh. Eksp. Teor. Fiz. **59**, 907 (1970).
- [18] J. M. Kosterlitz and D. J. Thouless, J. Phys. C **6**, 1181 (1973).
- [19] J. Zinn-Justin, arXiv:hep-ph/0005272.
- [20] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-relativistic Theory* (Pergamon Press, Oxford, 1977), 3rd ed.
- [21] K. M. Case, Phys. Rev. **80**, 797 (1950).
- [22] S. R. Beane, P. F. Bedaque, L. Childress, A. Kryjevski, J. McGuire, and U. van Kolck, Phys. Rev. A **64**, 042103 (2001).
- [23] T. Barford and M. C. Birse, Phys. Rev. C **67**, 064006 (2003).
- [24] M. Bawin and S. A. Coon, Phys. Rev. A **67**, 042712 (2003).
- [25] T. Barford and M. C. Birse, J. Phys. A **38**, 697 (2005).
- [26] H. W. Hammer and R. Higa, Eur. Phys. J. A **37**, 193 (2008).
- [27] V. Efimov, Phys. Lett. B **33**, 563 (1970).
- [28] V. Efimov, Nucl. Phys. **A210**, 157 (1973).
- [29] Y. Nishida, D. T. Son, and S. Tan, Phys. Rev. Lett. **100**, 090405 (2008).
- [30] Y. Nishida and D. T. Son, Phys. Rev. D **76**, 086004 (2007).
- [31] I. R. Klebanov and E. Witten, Nucl. Phys. **B556**, 89 (1999).
- [32] S. S. Gubser and I. R. Klebanov, Nucl. Phys. **B656**, 23 (2003).
- [33] E. V. Gorbar, V. P. Gusynin, and V. A. Miransky, Phys. Rev. D **64**, 105028 (2001).
- [34] S.-J. Rey, *Strings 2007 Conference* (2007), unpublished.
- [35] A. Dymarsky, I. R. Klebanov, and R. Roiban, J. High Energy Phys. **08** (2005) 011.
- [36] D. T. Son, Phys. Rev. D **59**, 094019 (1999).
- [37] H. Terao and A. Tsuchiya, arXiv:0704.3659.
- [38] K. G. Wilson, Phys. Rev. D **7**, 2911 (1973).
- [39] J. Zinn-Justin, Nucl. Phys. **B367**, 105 (1991).
- [40] M. E. Machacek and M. T. Vaughn, Nucl. Phys. **B222**, 83 (1983).
- [41] M. E. Machacek and M. T. Vaughn, Nucl. Phys. **B236**, 221 (1984).
- [42] M. E. Machacek and M. T. Vaughn, Nucl. Phys. **B249**, 70 (1985).
- [43] S. S. Gubser and I. Mitra, Phys. Rev. D **67**, 064018 (2003).
- [44] T. Appelquist, G. T. Fleming, and E. T. Neil, Phys. Rev. Lett. **100**, 171607 (2008).
- [45] T. Appelquist, G. T. Fleming, and E. T. Neil, Phys. Rev. D **79**, 076010 (2009).
- [46] A. Deuzeman, E. Pallante, M. P. Lombardo, and E. Pallante, arXiv:0810.3117.
- [47] A. Deuzeman, M. P. Lombardo, and E. Pallante, arXiv:0904.4662.
- [48] F. Sannino and K. Tuominen, Phys. Rev. D **71**, 051901 (2005).
- [49] D. D. Dietrich and F. Sannino, Phys. Rev. D **75**, 085018 (2007).
- [50] S. Catterall and F. Sannino, Phys. Rev. D **76**, 034504 (2007).
- [51] S. Catterall, J. Giedt, F. Sannino, and J. Schneible, J. High Energy Phys. **11** (2008) 009.
- [52] Y. Shamir, B. Svetitsky, and T. DeGrand, Phys. Rev. D **78**, 031502 (2008).
- [53] Z. Fodor, K. Holland, J. Kuti, D. N{o}gr{a}di, and C. Schroeder, arXiv:0809.4888.
- [54] T. DeGrand, Y. Shamir, and B. Svetitsky, Phys. Rev. D **79**, 034501 (2009).
- [55] A. J. Hietanen, J. Rantaharju, K. Rummukainen, and K. Tuominen, J. High Energy Phys. **05** (2009) 025.
- [56] D. Stancato and J. Terning, arXiv:0807.3961.
- [57] A. G. Cohen and H. Georgi, Nucl. Phys. **B314**, 7 (1989).
- [58] T. A. Rytov and F. Sannino, Phys. Rev. D **78**, 065001 (2008).
- [59] J. Garriga, O. Pujolas, and T. Tanaka, Nucl. Phys. **B605**, 192 (2001).
- [60] W. D. Goldberger and I. Z. Rothstein, Phys. Lett. B **491**, 339 (2000).