Towards a solution of the strong *CP* problem by compact extra dimensions

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We propose a way towards a dynamical solution of the strong *CP* in models with compact extra dimensions. To this aim we consider a one dimensional toy model for QCD, which contains a vacuum angle and a strong *CP*-like problem. We further consider a higher dimensional theory, which has a trivial vacuum structure and which reproduces the perturbative properties of the toy model in the low-energy limit. In the weak coupling regime, where our computations are valid, we show that the vacuum structure of the low-energy action is still trivial and the strong *CP* problem does not arise. Also, no axionlike particles are generated in this setup.

PACS numbers: 11.10.Kk, 11.27.+d

I. INTRODUCTION

Quantum chromodynamics has a complicated vacuum structure [1]. Because of the nontrivial group of mappings from the coordinate space to the gauge group $\pi^3(SU(3)) = \mathbb{Z}$, there is an infinite number of perturbative vacua $|n\rangle$ which are not connected by smooth gauge transformations. Physical passage between these vacua is possible at an exponentially suppressed rate. One can further define physical vacua, that is to say, vacua between which transition is not possible. These are called θ vacua and are expressed as a discrete Fourier transform of the *n* vacua, $|\theta\rangle = \sum_{n=-\infty}^{\infty} \times e^{in\theta}|n\rangle$. The physics depends on which vacua we are in, so θ becomes a new parameter of the theory. Equivalently this may be expressed by a new effective term in the Lagrangian of the form $i\theta q(x)$, where $q(x)$ is the topological charge density.[1] The presence of this term implies that $\langle q(x)\rangle$ does not vanish and is an odd function of θ [2]. The nonvanishing of $\langle q(x)\rangle$ leads in turn to the *CP* violation and can be measured experimentally. In principle θ can take any value in the interval $[0, 2\pi)$, but according to the measurement of the neutron dipole moment, it turns out to be extremely small, $\theta < 2 \times 10^{-10}$. The fact that this value is unnaturally small is known as the strong *CP* problem.

Different models were proposed to explain the smallness of θ. Some scenarios [3] lead to a larger probability of living in a universe with small θ, but most of the models rely on the addition of some dynamical properties to θ. A possible way is to promote the parameter θ to a field, the axion, which will relax dynamically to zero [4]. This solution is however, in its minimal setup, ruled out by the nondetection of this axion particle [5]. More complicated models exist and can avoid these bounds; some of these also use extra dimensions and allow the axion to propagate in the bulk [6].

Though both the *CP* problem and its axion solution have their origin in the symmetry properties of the system (existence of large gauge transformation for the first, and Peccei-Quinn symmetry for the second), one should be careful with the high-energy extensions of the theory. In the case where a high-energy effect breaks some of the symmetries, the situation may change. This, for example, happens in the case of the global Peccei-Quinn symmetry. It is usually considered that it should be broken by higher order operators arising from quantum gravity corrections [7]. The contributions from these operators scale like a power of $f_a/M_{\rm Pl}$, where f_a is the axion coupling constant. Because of the astrophysical bounds from supernova cooling by light particles, f_a has to be large and the ratio $f_a/M_{\rm Pl}$ cannot be small enough, leading to corrections to the potential that spoils the solution of the strong *CP* problem. These problems are avoided in invisible axion models [8], where the Peccei-Quinn symmetry is a consequence of gauge invariance, and thus survives the quantum gravity effects. If the high-energy theory does not violate the Peccei-Quinn symmetry, the solution to the *CP* problem is unaffected. On the other hand, if the high-energy theory breaks the invariance against large gauge transformation, this may remove the *CP* problem by itself. This is the essence of the solution of the *CP* problem proposed in this article.

Extra dimensions can provide a solution to the strong *CP* problem without light axion particles [9–11]. As we will see, the solution in this setup is also dynamical, but does not imply the existence of some light particle or any additional degree of freedom. The basic idea is that the group of mappings of a higher dimensional space to the gauge group $[\pi^d(SU(3)), d \leq 4]$ is trivial[2] and θ vacua cannot be defined. This type of solution was already pro-

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[1]$q(x) = \frac{1}{32\pi^2}\tilde{F}^{\mu\nu}F_{\mu\nu}(x).$

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[2]Except $\pi^4(SU(3)) = \mathbb{Z}_2$, which should not be an issue for our discussion.

posed some time ago with compact extra dimensions [9] or with infinite ones [10]. It was however pointed out in [11] that the solution might be more complicated than expected from the simple topological argument. The presence of a suitable extra dimension removes the degeneracy of the $|n\rangle$ vacua, thus preventing the definition of θ vacua. This solution is only formal, and the strong CP problem may reappear when we require that the observable effects of the extra dimensions are small. In this limit it can be expected that the usual QCD be recovered and the strong CP problem reappear in some way.[3] To show that the strong CP problem actually disappears, we should not only get a trivial vacuum structure but also show that the physical consequences of a nonvanishing θ also does disappear—that is to say, the topological charge should vanish on average ($\langle q \rangle = 0$).

To our best knowledge, no complete setup where fields are explicitly localized on a brane and where the strong CP problem is solved has been constructed till now. Solving the strong CP problem in real QCD using extra dimensions is difficult. The full QCD is complicated and the localization of non-Abelian theories on a brane is nontrivial [12]. In particular the gauge theories in more than four dimensions are generically not renormalizable [while $SU(N)$ gauge theories might however be nonperturbatively renormalizable in five or even six dimensions, see [13]]. Thus, an eventual extra-dimensional extension of QCD might need a UV completion, that is to say will itself have to be thought of as an effective theory coming as a low-energy limit of some renormalizable theory (for instance string theory). However, certain important features of the solution of the strong CP problem can be addressed in a simplified model. We will consider here the simplest possible model where a strong CP-like problem occurs and can be solved by the addition of an extra dimension. Keeping this model as simple as possible will enable us to perform the relevant computations explicitly, retaining all the important topological properties.

The simplest model which contains a QCD-like θ angle is the Abelian Higgs model in $1 + 1$ dimensions. This model has the same complicated vacuum structure as QCD [$\pi^1(U(1)) = \mathbb{Z}$], and its effective action also contains a θ parameter. This model was already studied many times as a toy model for QCD. For instance the QCD $U(1)$ problem was first solved in the Abelian Higgs model [14]. This solution could then be mapped to QCD [15]. Note that the Abelian Higgs model was also successfully used as a toy model for the electroweak baryogenesis [16,17]. Our calculations could also be relevant to such issues.

We then consider a higher dimensional theory, which reproduces the perturbative properties of the Abelian Higgs

model as a low-energy action. In this latter theory, the spatial dimensions have the topology of a sphere, so that the vacuum structure is trivial [$\pi^2(U(1)) = 0$]. At low energy, particles are localized on the equator (which will be the brane) and the extra dimension extends towards the poles. To simplify the computations we use in practice a geometrically simpler realization of this topology, and consider a pancake (the sphere is flattened to two disks).

This article is structured as follows. We first review the relevant properties of the Abelian Higgs model in $1 + 1$ dimensions in Sec. II. In Sec. III, we consider the Abelian Higgs model in $2 + 1$ dimensions and localize the fields with the help of a warp factor. We show that the $1 + 1$ dimensional theory is correctly recovered as low-energy effective action and discuss the validity of the classical Kaluza-Klein decomposition in some detail. In Sec. IV, we discuss the nonlinear sector of the theory. The $2 + 1$ dimensional equivalent of the $1 + 1$ dimensional vacuum structure is presented. We show that the degeneracy of the $1 + 1$ dimensional $|n\rangle$ vacua is lifted and this new structure is interpreted as a potential for the topological charge density. Finally the evolution of the topological charge density is considered. At least in the weak coupling regime where our calculations are valid, it relaxes very fast to zero, which solves the strong CP problem. We conclude in Sec. [5] and discuss the extension to more realistic models.

II. TOY MODEL

Toy models to address the strong CP problem in the framework of extra dimensions were already considered. In [11], $2 + 1$ dimensional electrodynamics was localized on a one dimensional brane. When two spatial dimensions form a sphere, the topological charge density $\langle q(x) \rangle$ is shown to get the effective action of a harmonic oscillator with a small frequency. This kind of dynamics does not completely solve the θ problem since $\langle q(x) \rangle$ is slowly oscillating in time. Furthermore, electrodynamics in $1 + 1$ dimensions is not really adapted to our purposes. It is a trivial theory and contains a strong CP-like problem only when charged particles are added. In [18] the effective low dimensional action from [11] was reconsidered with charged fermions, and the strong CP problem was addressed but without reference to some extra-dimensional scenario. A simple model that contains θ vacua and a strong CP-like problem is the Abelian Higgs model in $1 + 1$ dimensions. This is the model that we will consider, and before discussing its embedding into a higher dimensional space, we recall some of its properties [19].

A. Basic properties of the Abelian Higgs model in $1 + 1$ dimensions

The action for the Abelian Higgs model in $1 + 1$ dimensions reads

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[3]This is indeed the case in [11] when the extra dimension is taken to be an orbifold.

$$S = \int dx dt \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - V(\phi) + \frac{1}{2} |D_\mu \phi|^2 \right), \quad (1)$$

where $D_\mu = \partial_\mu - i\tilde{e} A_\mu$, and $\phi = (H + \tilde{v}) e^{i\sigma}$ is a complex scalar field with the symmetry breaking potential

$$V(\phi) = \frac{\tilde{\lambda}}{4} (|\phi|^2 - \tilde{v}^2)^2. \quad (2)$$

Note that we use tilde everywhere to denote $1 + 1$ dimensional variables. We will now discuss the perturbative properties of this action. We want to identify the degrees of freedom and derive their spectrum and interactions. As we will later compare the effective action of the $2 + 1$ dimensional Abelian Higgs model localized on the boundary of the disk to the $1 + 1$ dimensional Abelian Higgs model, we will directly consider the space dimension to be a circle of length $2\pi R$. We always consider $1/R$ to be much smaller that the particle masses. To extract the physical degrees of freedom in the simplest way, we use the unitary gauge ($\sigma = 0$) and perform the spontaneous symmetry breaking. We replace the coordinate $x \to R\theta$ and decompose the fields in partial waves

$$A_\mu = \frac{1}{\sqrt{2\pi R}} \sum_n e^{in\theta} A_\mu^n, \qquad H = \tilde{v} + \frac{1}{\sqrt{2\pi R}} \sum_n e^{in\theta} H^n, \quad (3)$$

with $(A^n)^* = A^{-n}$ and $(H^n)^* = H^{-n}$, so that the fields A_μ, H are real. Substituting the partial wave expansion (3) into the action (1) we get

$$S = \int dt \sum_n \left[\frac{1}{2} |\dot{H}^n|^2 - \frac{1}{2} \left(2\tilde{\lambda}\tilde{v}^2 + \frac{n^2}{R^2} \right) |H^n|^2 \right.$$
$$- \frac{\tilde{\lambda}\tilde{v}}{\sqrt{2\pi R}} \sum_m H^n H^m H^{-n-m}$$
$$\left. - \frac{\tilde{\lambda}}{8\pi R} \sum_{m,k} H^n H^m H^k H^{-n-m-k} \right] + S_{GF}. \quad (4)$$

We see that the physical spectrum contains a scalar field (Higgs) with the mass $m_H^2 = 2\tilde{\lambda}\tilde{v}^2$ and its dispersion relation is, as expected, $\omega_n^2 = m_H^2 + n^2/R^2$. The gauge field action reads

$$S_{GF} = \int dt \sum_n \left[\frac{1}{2} |\dot{A}_1^n|^2 - \frac{\tilde{e}^2 \tilde{v}^2}{2} |A_1^n|^2 \right.$$
$$+ \frac{1}{2} \left(\tilde{e}^2 \tilde{v}^2 + \frac{n^2}{R^2} \right) |A_0^n|^2 + \frac{in}{2R} (\dot{A}_1^n A_0^{-n} - \dot{A}_1^{-n} A_0^n)$$
$$+ \frac{\tilde{v}\tilde{e}^2}{\sqrt{2\pi R}} \sum_m (A_0^n A_0^m - A_1^n A_1^m) H^{-n-m}$$
$$\left. + \frac{\tilde{e}^2}{4\pi R} \sum_{m,k} (A_0^n A_0^k - A_1^n A_1^k) H^m H^{-n-m-k} \right]. \quad (5)$$

The field A_0 is not dynamical and can be integrated out

using its equations of motion,

$$A_0^n \left(\tilde{e}^2 \tilde{v}^2 + \frac{n^2}{R^2} \right) + \frac{in}{R} \dot{A}_1^n + \frac{2\tilde{e}^2}{\sqrt{2\pi R}} \tilde{v} \sum_k A_0^k H^{n-k}$$
$$+ \frac{\tilde{e}^2}{2\pi R} \sum_{m,k} A_0^k H^m H^{n-m-k} = 0. \quad (6)$$

Inserting Eq. (6) into the gauge field action (5) and rescaling[4]

$$A_1^n \to \sqrt{\frac{\tilde{e}^2 \tilde{v}^2 + n^2/R^2}{\tilde{e}^2 \tilde{v}^2}} A_1^n, \quad (7)$$

we are left with [up to terms of order $\mathcal{O}((\tilde{e}\tilde{v}R)^{-2})$]

$$S_{GF} = \int dt \sum_n \left[\frac{1}{2} |\dot{A}_1^n|^2 - \frac{1}{2} \left(\tilde{e}^2 \tilde{v}^2 + \frac{n^2}{R^2} \right) |A_1^n|^2 \right.$$
$$- \frac{\tilde{v}\tilde{e}^2}{\sqrt{2\pi R}} \sum_m A_1^n A_1^m H^{-n-m}$$
$$- \frac{\tilde{e}^2}{4\pi R} \sum_{m,k} A_1^n A_1^k H^m H^{-n-m-k}$$
$$\left. + \text{ higher order interactions} \right]. \quad (8)$$

This is the action of a scalar field with the mass $m_W = \tilde{e}\tilde{v}$. Integrating out A_0 leads to some complications, such as $\mathcal{O}((\tilde{e}\tilde{v}R)^{-2})$ corrections in the interactions' terms, which we will consider small as we are interested in the large R limit. In the following we will mainly perform semiclassical calculations. To ensure that they are valid, we will consider only small couplings. Consequently, higher order interactions in (8) are small and not relevant to our discussion.[5] Note that the couplings in the Lagrangian (1) are dimensionful. A weak coupling criterion means that

$$\frac{\tilde{\lambda}}{m_H^2} \sim \frac{\tilde{e}^2}{m_W^2} \ll 1 \Leftrightarrow \tilde{v} \gg 1. \quad (9)$$

Of course the QCD coupling is large and this toy model is not realistic in this sense. However, the strong *CP* problem does not rely on the fact that QCD is strongly coupled and we may very well use a simple toy model which enables simple perturbative calculations.

As emphasized in the introduction, the Abelian Higgs model has almost identical nonperturbative properties as QCD. We list them in the following and, when it happens, point out where they differ from QCD.

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[4]After replacing A_0, the form of the kinetic term for A_1 is not canonical anymore and A_1 has to be rescaled.
[5]They are not needed to compare with the effective action for localized $2 + 1$ dimensional fields since we will make similar approximations there.

B. Vacuum structure

In short, the ensemble of vacuum configurations is the ensemble of mappings from the space S^1 to the gauge group $U(1)$. This ensemble can be divided into equivalence classes identifying vacua that can be related by a smooth gauge configuration. Mathematically this is represented by the homotopy group $\pi^1(U(1)) = \mathbb{Z}$. This means that there is a discrete infinity of classes that cannot be related by a smooth gauge configuration. An element of the class n is for instance

$$\phi^{(n)} = \tilde{v}\exp\left(\frac{inx}{R}\right), \qquad A_1^{(n)} = \frac{n}{\tilde{e}R}, \qquad A_0 = 0. \quad (10)$$

These equivalence classes can be distinguished by the Chern-Simons number:

$$N_{CS} = \frac{\tilde{e}}{2\pi}\int dx A_1(x). \quad (11)$$

It takes the value n when applied on a vacuum state of the equivalence class n. The transition from one equivalence class to another can formally be achieved by a discontinuous gauge transformation. An example of the gauge transformation that changes the Chern-Simons number by n is

$$U^n = \exp\left(\frac{inx}{R}\right). \quad (12)$$

C. Sphaleron

Physically, the transition between two vacua needs to go through a set of nonvacuum configurations that form an energy barrier (see Fig. 1). For instance, the set of static field configurations

$$\phi^{cl} = \tilde{v}e^{(ix\tau/R)}[\cos(\pi\tau) + i\sin(\pi\tau)\tanh(m_H x \sin(\pi\tau))],$$
$$A_1^{cl} = \frac{\tau}{\tilde{e}R}, \quad (13)$$

forms a path that goes from vacuum $n = 0$ at $\tau = 0$ to vacuum $n = 1$ at $\tau = 1$, minimizing the energy of the intermediate configurations. The configuration of maximal energy

$$E_{sph} = \tfrac{2}{3}m_H\tilde{v}^2 \quad (14)$$

E

E_{sph}

Sphaleron

Instanton
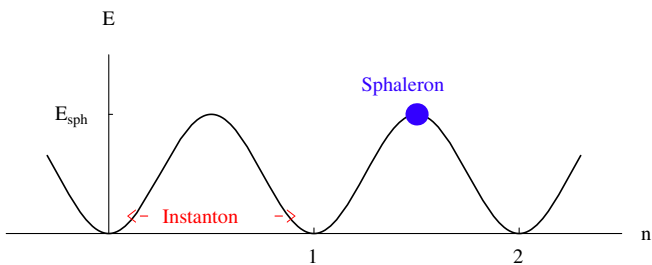
1   2   n

FIG. 1 (color online). Sketch of the vacuum energy as a function of the winding number n.

is reached at $\tau = \frac{1}{2}$, and is called the sphaleron. It is relevant for the high temperature behavior of the theory. Thermal fluctuations can reach the required energy E_{sph} and the system may pass classically between vacua. Although all the previous and further discussed properties of the Abelian Higgs model exactly match QCD, we should point out that, strictly speaking, the sphaleron does not exist in QCD. The corresponding configurations in QCD contain a free scale parameter, and their potential energy can be made infinitely small in the limit of large configuration size. However, at large scale, kinetic energy along the path grows and the transition rate is still suppressed. This small difference will not be of importance for our discussion.

D. Instanton

At small or vanishing temperature, the system can also tunnel from one vacuum to another. In quantum field theory, tunneling is represented by instantons, which are solutions of the classical equations of motion in Euclidean space-time. The set of configurations (13) can serve as Ansatz to compute the Euclidean action, which gives the leading information to compute the tunneling rate.

The proper instanton in this model, i.e. the field configuration that minimizes the Euclidean action and describes the tunneling between the states $|0\rangle$ and $|n\rangle$, is the Nielsen-Olesen vortex [20] with winding number

$$Q = \Delta N_{CS} = \frac{\tilde{e}}{4\pi}\int \varepsilon_{\mu\nu}F^{\mu\nu}d^2x = n. \quad (15)$$

Parametrizing the Euclidean space-time in polar coordinates (r, θ), the field configuration reads

$$\phi(r, \theta) = e^{in\theta}f(r), \quad (16)$$

$$A^i(r, \theta) = \varepsilon^{ij}\hat{r}^j A(r), \quad (17)$$

where $\hat{r}$ is the unit vector $\hat{r} = (\cos\theta, \sin\theta)$ and ε^{ij} the completely antisymmetric tensor with $\varepsilon^{01} = 1$. The functions A and f have to satisfy the following limits:

$$f(r)\overset{r\to 0}{\to}cr^{|n|}, \qquad f(r)\overset{r\to\infty}{\to}\tilde{v},$$
$$A(r)\overset{r\to 0}{\to}0, \qquad A(r)\overset{r\to\infty}{\to} -\frac{n}{\tilde{e}r}. \quad (18)$$

Introducing dimensionless variables

$$r\to\frac{r}{\sqrt{\lambda}v}, \qquad A\to\frac{\sqrt{\lambda}v}{e}A, \qquad f\to\frac{\sqrt{\lambda}v}{e}f, \quad (19)$$

the classical action of the instanton is

$$S_{\text{cl}} = \pi \tilde{v}^2 \mu^2 \int_0^\infty r dr \left\{ \left(A'(r) + \frac{A(r)}{r} \right)^2 + f'(r)^2 \right.$$

$$\left. + f(r)^2 \left(A(r) - \frac{1}{r} \right)^2 + \frac{\mu^2}{2} \left(f^2(r) - \frac{1}{\mu^2} \right)^2 \right\}$$

$$= \pi \tilde{v}^2 \mu^2 B(\mu), \qquad (20)$$

where $\mu^2 = \tilde{\lambda}/\tilde{e}^2$ and the function $B(\mu)$ is of order one, and depends weakly on its argument.

E. θ vacua and topological charge

We shall consider the nonlocal gauge transformation U^1 that changes the Chern-Simons number by one. It commutes with the Hamiltonian and is unitary. The operator U^1 can therefore be diagonalized simultaneously with the Hamiltonian and must have eigenvalues of modulus 1. The θ vacua are defined as superpositions of the $|n\rangle$ states which are at the same time an eigenvector of U^1 with eigenvalue $e^{i\theta}$:

$$|\theta\rangle = \sum_n e^{in\theta} |n\rangle. \qquad (21)$$

A physical transition between the θ vacua is not possible. They form different sectors with different physical properties and θ is a parameter of the theory. The topological charge density $q = \varepsilon_{\mu\nu} F^{\mu\nu}$ in a θ vacuum does not vanish on average [2]:

$$\langle q \rangle = 8\pi K e^{-S_{\text{cl}}} \sin\theta, \qquad (22)$$

with S_{cl} the instanton action and the factor $K \sim m_H$ taking quantum corrections into account.

III. 2 + 1 DIMENSIONAL ABELIAN HIGGS MODEL ON A DISK

We will now recreate the $1 + 1$ dimensional Abelian Higgs model as the low-energy effective theory for a $2 + 1$ dimensional model. The perturbative properties of our low-energy theory has to resemble closely the $1 + 1$ dimensional Abelian Higgs model, but hopefully have a different vacuum structure. We expect that this is possible since, as $\pi_2(U(1)) = 0$, no θ vacua exist in the original $2 + 1$ dimensional theory.

As discussed in the introduction, we will suppose that two spatial dimensions are the surface of a pancake, where the low-energy fields live on its boundary and the extra dimension extends from the boundary towards the center. This pancake can be adroitly sliced into two disks. By symmetry we can expand the fields into functions that are odd or even with respect to pancake flipping. In the following we will only consider one disk where the fields can have either Neumann or Dirichlet boundary conditions on its boundary.

In $2 + 1$ dimensions, we consider a complex Higgs field ϕ and an Abelian gauge field. The space-time is a disk, which we parametrize with polar coordinates (t, θ, r). We

will also introduce a warp factor $\Delta(r)$ to localize the fields [12,21]. Note that we do not want to consider gravity here and the warp factor will be thought as coming from the coupling to some external classical field, which multiplies the whole action,

$$S = \int dr d\theta dt \sqrt{g} \Delta(r) \left[-\frac{1}{4} g^{AB} g^{CD} F_{AC} F_{BD} \right.$$

$$\left. + \frac{1}{2} g^{AB} (D_A \phi)^* (D_B \phi) - V(\phi) \right], \qquad (23)$$

with the Higgs potential $V(\phi) = \frac{\lambda}{4} (|\phi|^2 - v^2)^2$ and $D_M = \partial_M - ieA_M$. The metric of the disk is

$$g_{MN} = \begin{pmatrix} 1 & & \\ & -r^2 & \\ & & -1 \end{pmatrix},$$

and the volume element $\sqrt{g} = r$. The action can be rewritten more explicitly as

$$S = \int dr d\theta dt \Delta \left\{ \frac{1}{2} \left[\frac{1}{r} F_{0\theta}^2 + r F_{0r}^2 - \frac{1}{r} F_{\theta r}^2 \right] \right.$$

$$+ \frac{1}{2} \left[r |D_0 \phi|^2 - r |D_r \phi|^2 - \frac{1}{r} |D_\theta \phi|^2 \right.$$

$$\left. \left. - r \frac{\lambda}{2} (|\phi|^2 - v^2)^2 \right] \right\}. \qquad (24)$$

In the following, we will use

$$\Delta(r) = e^{-2M|R-r|}. \qquad (25)$$

The exact form of Δ is merely chosen to simplify the algebra. It is only needed that $\int dr \Delta(r) < \infty$ and that $\Delta(r)$ decreases from the brane ($r = R$) towards the extra dimension. In the following we will consider the following relation between the parameters: $M \gg m_H \sim m_W \gg 1/R$.

A. Linearized theory and spectrum

We decompose the Higgs field in polar coordinates as $\phi = (v + H)e^{i\sigma}$, and retain the quadratic part of the total (gauge field + Higgs) Lagrangian;

$$\mathcal{L}_0 = \frac{1}{2} \Delta(r) \left[\frac{1}{r} F_{0\theta}^2 + r F_{0r}^2 - \frac{1}{r} F_{\theta r}^2 \right]$$

$$+ \frac{1}{2} \Delta v^2 \left[r(\partial_0 \sigma - eA_0)^2 - r(\partial_r \sigma - eA_r)^2 \right.$$

$$\left. - \frac{1}{r} (\partial_\theta \sigma - eA_\theta)^2 \right] - r \Delta \frac{m_H^2}{2} H^2$$

$$+ \frac{\Delta}{2} \left[r(\partial_0 H)^2 - r(\partial_r H)^2 - \frac{1}{r} (\partial_\theta H)^2 \right], \qquad (26)$$

with $m_H^2 = 2\lambda v^2$. The linearized equations of motion read (with the rescaling $\sigma \to e\sigma$)

$$-\partial_0(\partial_0\sigma - A_0) + \frac{1}{r\Delta}\partial_r r\Delta(\partial_r\sigma - A_r)$$
$$+ \frac{1}{r^2}\partial_\theta(\partial_\theta\sigma - A_\theta) = 0, \tag{27}$$

$$\frac{1}{r\Delta}\partial_r r\Delta(\partial_0 A_r - \partial_r A_0) + \frac{1}{r^2}\partial_\theta(\partial_0 A_\theta - \partial_\theta A_0)$$
$$+ m_W^2(A_0 - \partial_0\sigma) = 0, \tag{28}$$

$$-\partial_0(\partial_0 A_\theta - \partial_\theta A_0) + \frac{r}{\Delta}\partial_r \frac{\Delta}{r}(\partial_r A_\theta - \partial_\theta A_r)$$
$$- m_W^2(A_\theta - \partial_\theta\sigma) = 0, \tag{29}$$

$$-\partial_0(\partial_0 A_r - \partial_r A_0) - \frac{1}{r^2}\partial_\theta(\partial_r A_\theta - \partial_\theta A_r)$$
$$- m_W^2(A_r - \partial_r\sigma) = 0, \tag{30}$$

$$\left(\frac{1}{r\Delta}\partial_r \Delta r\partial_r + \frac{1}{r^2}\partial_\theta^2 - \partial_0^2 - m_H^2\right)H = 0. \tag{31}$$

Note that the phase σ and the gauge fields A_M decouple from the physical Higgs H. These two sectors are analyzed separately.

1. Higgs sector

After Fourier transform

$$H(t, \theta, r) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \sum_n e^{in\theta} h^n(r), \tag{32}$$

the Higgs field h^n satisfies the following equation of motion:

$$\left(\frac{1}{r\Delta}\partial_r \Delta r\partial_r - \frac{n^2}{r^2} - m_H^2 + \omega^2\right)h^n(r) = 0. \tag{33}$$

The explicit solution of Eq. (33), which satisfies the boundary conditions $h^n(r) \propto r^n$ at $r \to 0$, is given in terms of the Laguerre function:

$$h^n(r) \propto e^{-r(M+\Omega_H)}(r)^n L_{-(M/2\Omega_H)-n-1/2}^{2n}(2r\Omega_H), \tag{34}$$

with $\Omega_H = \sqrt{M^2 + m_H^2 - \omega^2}$. We should further impose that either $h'(r)$ (symmetric mode) or $h(r)$ (antisymmetric mode) vanishes at $r = R$. For $n = 0$, the lowest mode is symmetric $h_0^0 = $ const and has the energy $\omega_0 = m_H$.

For $n \neq 0$, there is a low-energy symmetric mode that satisfies, in the limit $M \gg m_H$, the usual dispersion relation $\omega_0^n = \sqrt{m_H^2 + n^2/R^2} + \mathcal{O}(e^{-2MR})$ (see Appendix A), so that we have a Higgs particle on the brane, with mass m_H and, compared to (4), a correct dispersion relation.

The next mode is antisymmetric and has $\omega_1 \gtrsim M$. Further modes have higher energy and build a discrete spectrum (labeled with k) which satisfies $\omega_{k+1} - \omega_k = \frac{\pi}{2R}$ at large k (see Appendix A).

2. Gauge sector

After Fourier transform

$$A_N(t, \theta, r) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \sum_n e^{in\theta} a_N^n(r), \tag{35}$$

the equations of motion for σ, a_0^n, a_θ^n, a_r^n are

$$i\omega(-i\omega\sigma - a_0^n) + \frac{1}{\Delta r}\partial_r \Delta r(\partial_r\sigma - a_r^n)$$
$$+ \frac{in}{r^2}(in\sigma - a_\theta^n) = 0, \tag{36}$$

$$\frac{-1}{\Delta r}\partial_r(\Delta r(i\omega a_r^n + \partial_r a_0^n)) + \frac{1}{r^2}(n\omega a_\theta^n + n^2 a_0^n)$$
$$+ (a_0^n + i\omega\sigma)m_W^2 = 0, \tag{37}$$

$$\omega^2 a_\theta^n + n\omega a_0^n + \frac{r}{\Delta}\partial_r \frac{\Delta}{r}(\partial_r a_\theta^n - ina_r^n)$$
$$- (a_\theta^n - in\sigma)m_W^2 = 0, \tag{38}$$

$$\omega^2 a_r^n - i\omega\partial_r a_0^n - \frac{in}{r^2}\partial_r a_\theta^n - \frac{n^2}{r^2}a_r^n - m_W^2(a_r^n - \partial_r\sigma) = 0, \tag{39}$$

with $m_W^2 = e^2 v^2$ the W mass. For the continuity of the fields we require that a_θ^n, a_r^n vanish at the origin for all n, except for a_0^0 which might be constant. Note that these four equations are not independent in the general case. A possible resolution is to work in the unitary gauge ($\sigma = 0$). We extract a_0^n from Eq. (36) and replace it in Eqs. (38) and (39):

$$\left(-\frac{r}{\Delta}\partial_r \frac{\Delta}{r}\partial_r - \omega^2 + \frac{n^2}{r^2} + m_W^2\right)a_\theta^n - \frac{2in}{r}a_r^n = 0, \tag{40}$$

$$\left(-\partial_r \frac{1}{r\Delta}\partial_r r\Delta - \omega^2 + \frac{n^2}{r^2} + m_W^2\right)a_r^n + \frac{2in}{r^3}a_\theta^n = 0. \tag{41}$$

Equation (37) for a_0^n is not independent and can be dropped. The analysis is slightly subtle and we have to treat all possible cases separately.

Consider first $n = 0$. Equations (40) and (41) decouple and read (for $r \leq R$)

$$(w^2 - m_W^2)a_\theta^n(r) + \left(\frac{2Mr - 1}{r}\right)(a_\theta^n)'(r) + (a_\theta^n)''(r) = 0, \tag{42}$$

$$\left(-m_W^2 + w^2 - \frac{1}{r^2}\right)a_r^n(r) + \left(\frac{2Mr + 1}{r}\right)(a_r^n)'(r)$$
$$+ (a_r^n)''(r) - 2M\delta(r - R)a_r^n = 0. \tag{44}$$

The presence of the δ function in Eq. (44) changes the boundary condition on the brane for the symmetric mode to $(a_r^n)'(R - \varepsilon) = -Ma_r^n(R) = -(a_r^n)'(R + \varepsilon)$, which makes

the lowest energy mode for a_r^n heavy ($\omega > M$). For a_θ^n, we have the general solution

$$a_\theta^n \propto e^{-r(M+\Omega_W)} r^2 L_{1/2((M/\Omega_W)-3)}^2 (2r\Omega_W), \quad (45)$$

with $\Omega_W = \sqrt{M^2 + m_W^2 - \omega^2}$. The lowest energy mode satisfies $(a_\theta^n)'(R) = 0$ and is light, $\omega_0^0 = m_W + O(e^{-2MR})$, and the next mode has energy of the order M.

We shall now consider the case $n \neq 0$. Equations (40) and (41) are coupled and should in general be solved numerically. However if we notice that the mode we are looking for lives on the brane and therefore should have

$$F_{r\theta} = -in a_r^n(r) + \partial_r a_\theta^n(r) = 0, \quad (46)$$

Eqs. (40) and (41) reduce to (for $r \leq R$)

$$\left(-m_W^2 + w^2 - \frac{n^2}{r^2}\right) a_\theta^n(r) + \left(2M + \frac{1}{r}\right)(a_\theta^n)'(r) \\ + (a_\theta^n)''(r) = 0. \quad (47)$$

This is similar to the equation for the Higgs mode, with m_H replaced by m_W. The modes are therefore similar to the Higgs ones,

$$a_\theta^n \propto e^{-r(M+\Omega_W)} r^n L_{-(M/2\Omega_W)-n-1/2}^{2n}(2r\Omega_W), \quad (48)$$

and $a_r^n = -\frac{i}{n}\partial_r a_\theta^n(r)$ have to be continuous at the poles and on the brane. The lowest energy mode satisfies $(a_\theta^n)'(R) = a_r^n(R) = 0$ and has energy $\omega_0^n = \sqrt{m_W^2 + n^2/R^2} + \mathcal{O}(e^{-2MR})$. As for the Higgs, the next mode has energy $\omega_1 > M$. Like for $n = 0$, no other low-energy mode is found in Eqs. (40) and (41).

To summarize, up to corrections $\mathcal{O}(e^{-2MR}) \ll 1$ the low-energy spectrum and the dispersion relations exactly match the usual $1 + 1$ dimensional Abelian Higgs model.

B. Effective action

We can now build the effective $1 + 1$ dimensional action for the Kaluza-Klein (KK) modes. To simplify the algebra, we will remove all the $\mathcal{O}(e^{-2MR})$ suppressed terms. We work in the unitary gauge and use the following decomposition for the fields:

$$\phi = v + \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \sum_{n=-\infty}^{\infty} e^{in\theta} h_k^n(r) H_k^n(t),$$

$$A_\theta = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \sum_{n=-\infty}^{\infty} e^{in\theta} a_{\theta,k}^n(r) A_k^n(t),$$

$$A_r = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \sum_{n=-\infty}^{\infty} e^{in\theta} a_{r,k}^n(r) A_k^n(t), \quad (49)$$

$$A_0 = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \sum_{n=-\infty}^{\infty} e^{in\theta} a_{0,k}^n(r) A_k^n(t),$$

with k labeling different KK modes, and n labeling the

angular momentum. Note that we have to impose that the fields are real, that is to say $(A^n)^* = A^{-n}$, $(H^n)^* = H^{-n}$ and $(a_M^n)^* = a_M^{-n}$, $(h^n)^* = h^{-n}$.

To get the $1 + 1$ dimensional action for the whole tower of the KK modes, we substitute the KK expansion (49) in the Lagrangian (33), and integrate over the extra-dimension coordinate r. In the following, we will consider only the $k = 0$ mode in the KK expansion (49) to get the low-energy effective theory. We will also neglect the back-reaction of heavy Kaluza-Klein modes on the low-energy action, supposing that these corrections are suppressed. This assumption will be checked in the next section.

1. Gauge kinetic term

Combining the relations (36) and (46) we get

$$a_0^n = \frac{\omega_n^2 - m_W^2}{n\omega_n} a_\theta^n(r). \quad (50)$$

The expansion (49) is inserted in the gauge kinetic term

$$S_{GF} = \int dr d\theta dt \Delta \frac{1}{2}\left[\frac{1}{r}F_{0\theta}^2 + rF_{0r}^2 - \frac{1}{r}F_{\theta r}^2\right]. \quad (51)$$

We use the formulas (50) and (46), separate the t and r dependencies, and get after some straightforward algebra (keeping only the $k = 0$ mode)

$$S_{GF} = \int dt \sum_n \frac{I}{2}\left(|\dot{A}^n|^2 - \frac{(\omega_n^4 - m_W^4)}{\omega_n^2}|A^n|^2\right), \quad (52)$$

with

$$I = \int dr \frac{\Delta}{r}\left(|a_\theta^n|^2 + \frac{r^2}{n^2}|(a_\theta^n)'(r)|^2\right). \quad (53)$$

The dot means derivative with respect to time and the prime derivative with respect to r. We fix the normalization of a_θ^n such that $I = 1$, which, with the dispersion relation $\omega_n^2 = m_W^2 + \frac{n^2}{R^2}$, leads to

$$S_{GF} = \int dt \sum_n \frac{1}{2}\left(|\dot{A}^n|^2 - \frac{(n^2/R^2)^2}{m_W^2 + n^2/R^2}|A^n|^2\right). \quad (54)$$

2. Higgs kinetic term

We shall now consider the Higgs kinetic part of the action,

$$S_D = \int dr d\theta dt \Delta \frac{1}{2}\left[r|D_0\phi|^2 - r|D_r\phi|^2 - \frac{1}{r}|D_\theta\phi|^2\right]. \quad (55)$$

Again, we insert the KK expansion (49), use the relations (50) and (46), and get

$$S_D = \int dt \sum_n \left(\frac{1}{2} m_W^2 |A^n|^2 I_a + \frac{1}{2} |\dot{H}^n|^2 \int dr |h^n(r)|^2 \right.$$

$$\left. - \frac{n^2}{2R^2} |H^n|^2 I_h \right) + S_D^{\text{int}}, \tag{56}$$

with

$$I_a = \int dr \frac{\Delta}{r} \left[r^2 \frac{(\omega_n^2 - m_W^2)^2}{n^2 \omega_n^2} |a_\theta^n|^2 \right.$$

$$\left. - \frac{r^2}{n^2} |(a_\theta^n)'(r)|^2 - |a_\theta^n|^2 \right], \tag{57}$$

$$I_h = \frac{R^2}{n^2} \int dr \Delta r \left(|(h^n)'(r)|^2 + \frac{n^2}{2r^2} |h^n(r)|^2 \right), \tag{58}$$

and we normalize h^n such that

$$\int dr \Delta r |h^n(r)|^2 = 1. \tag{59}$$

The interactions between gauge field and Higgs read

$$S_D^{\text{int}} = \int dt \left(-\frac{e^2}{2\pi} \sum_{n,k,m} A^n A^m H^k H^{-n-m-k} I_b \right.$$

$$\left. - \frac{e^2 v}{\sqrt{2\pi}} \sum_{n,m} A^n A^m H^{-n-m} I_c \right), \tag{60}$$

with

$$I_b = \int dr \frac{r\Delta}{2} h^k h^{-n-m-k} \left(-\frac{n}{R^2 \omega_n} \frac{m}{R^2 \omega_m} a_\theta^n a_\theta^m \right.$$

$$\left. + \frac{1}{nm} (a_\theta^n)'(a_\theta^m)' + \frac{1}{r^2} a_\theta^n a_\theta^m \right), \tag{61}$$

$$I_c = \int dr \frac{r\Delta}{2} h^{-n-m} \left(-\frac{n}{R^2 \omega_n} \frac{m}{R^2 \omega_m} a_\theta^n a_\theta^m \right.$$

$$\left. + \frac{1}{nm} (a_\theta^n)'(a_\theta^m)' + \frac{1}{r^2} a_\theta^n a_\theta^m \right). \tag{62}$$

3. Higgs potential

The remaining Higgs potential is

$$S_V = \int dr d\theta dt \left[-r\Delta \frac{\lambda}{4} (|\phi|^2 - v^2)^2 \right], \tag{63}$$

which is expanded to

$$S_V = \int dt \left(-\frac{m_H^2}{2} \sum_n |H^n|^2 - \frac{\lambda v}{\sqrt{2\pi}} \sum_{n,m} H^n H^m H^{-n-m} I_d \right.$$

$$\left. - \frac{\lambda}{8\pi} \sum_{n,m,k} H^n H^m H^k H^{-n-m-k} I_e \right), \tag{64}$$

with

$$I_d = \int \Delta r h^n h^m h^{-n-m}, \qquad I_e = \int \Delta r h^n h^m h^k h^{-n-m-k}. \tag{65}$$

4. Low-energy theory

In Appendix B we compute that

$$I_a = -1 + \frac{n^2}{R^2 \omega^2}, \qquad I_h = 1, \tag{66}$$

and to $\mathcal{O}((MR)^{-1}, (m_H R)^{-2})$, we have

$$I_b = \frac{M}{R}, \qquad I_c = \sqrt{\frac{M}{R}}, \qquad I_d = \sqrt{\frac{M}{R}}, \qquad I_e = \frac{M}{R}, \tag{67}$$

which gives us

$$S_{\text{eff}} = \int dt \sum_n \frac{1}{2} \left[|\dot{A}_0^n(t)|^2 - \left(m_W^2 + \frac{n^2}{R^2} \right) |A_0^n(t)|^2 \right.$$

$$\left. + |\dot{H}_0^n(t)|^2 - \left(m_H^2 + \frac{n^2}{R^2} \right) |H_0^n(t)|^2 \right] + S_{\text{eff}}^{\text{int}}, \tag{68}$$

and up to $\mathcal{O}((MR)^{-1}, (m_H R)^{-2})$,

$$S_{\text{eff}}^{\text{int}} = \int dt \left[\sum_{m,n} \left(-\frac{\lambda v \sqrt{M}}{\sqrt{2\pi R}} H^n H^m H^{-n-m} \right. \right.$$

$$\left. - \frac{e^2 v \sqrt{M}}{\sqrt{2\pi R}} A^n A^m H^{-n-m} \right)$$

$$+ \sum_{m,n,k} \left(-\frac{\lambda M}{8\pi R} H^n H^m H^k H^{-n-m-k} \right.$$

$$\left. \left. - \frac{e^2 M}{4\pi R} A^n A^m H^k H^{-m-n-k} \right) \right]. \tag{69}$$

Compared to (4) and (8), we see that up to at least $\mathcal{O}((MR)^{-1}, (m_W R)^{-2})$ we get the same interactions as in the $1 + 1$ dimensional Abelian Higgs with the identifications

$$\tilde{e}^2 = e^2 M, \qquad \tilde{v}^2 = v^2/M, \qquad \tilde{\lambda} = \lambda M. \tag{70}$$

Note that these identifications are just what we would expect from dimensional analysis. We conclude that the theory (23) successfully reproduces the $1 + 1$ dimensional Higgs model on the boundary of the disk as its low-energy limit.[6]

C. Validity of the Kaluza-Klein expansion

Our computations are purely classical, and up to now we have not taken into account the corrections to the effective action coming from interactions with heavy modes. This

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[6]Note that the $\mathcal{O}((m_W R)^{-2})$ corrections not displayed here match the one of (8) in the large M limit.

goes beyond the scope of this article, and we will only describe here a region of the coupling parameter space where our semiclassical computations can be trusted.

The first problem that occurs in most warped scenarios is that the couplings become large far from the brane. For instance, this can be seen if we rescale the gauge field like $eA_M \rightarrow A_M$. The gauge kinetic term becomes $\frac{\Delta}{e^2} F_{MN} F^{MN}$, and we see that the effective charge is $\frac{e}{\sqrt{\Delta}}$. As $\Delta(r)$ becomes very small far from the brane, the effective charge becomes very large. This leads to large quantum corrections and large effective couplings between modes which probe the region with small Δ. This problem first concerns very energetic modes that live in the bulk and can probe the region with small Δ but having strongly coupled heavy modes may also lead to large corrections to the low-energy interactions.

1. Behavior of the theory at very high energies

We will compute here the effective interactions between heavy modes and show under which conditions interactions between light and heavy modes do not affect the low-energy theory. We only show explicit calculations for the Higgs modes. The gauge fields case is similar.

If we compute the effective theory for all the KK modes, we get the terms in the action of the form (64) and (65)

$$-\frac{\tilde{\lambda}^{n,m,p,-n-m-p}_{k_1,k_2,k_3,k_4}}{8\pi R} H^n_{k_1} H^m_{k_2} H^p_{k_3} H^{-n-m-p}_{k_4}.$$

The quartic coupling between KK modes is given by

$$\tilde{\lambda}^{n_1,n_2,n_3,n_4}_{k_1,k_2,k_3,k_4} = \lambda \int r\Delta\left(\prod_{i=1}^{4} h^{n_i}_{k_i}(r)\right) dr. \tag{71}$$

For KK modes with very high energy $\omega^n_k \gg M$, this integral can be approximated as (see Appendix C)

$$\tilde{\lambda}^{n_1,n_2,n_3,n_4}_{k_1,k_2,k_3,k_4} \sim \frac{\lambda}{R} e^{2MR} \frac{\prod_{i=1}^{4} \sqrt{\omega^{n_i}_{k_i}}}{\sum_{i=1}^{4} (\omega^{n_i}_{k_i})^2}. \tag{72}$$

It can be shown numerically that this formula fits well for all $k_i \geq 1$. We shall now study what the consequences are on these large couplings.

2. Corrections to the renormalizable couplings

The simplest correction to the effective action is the appearance of a nonrenormalizable $\lambda^l_6 H^6_l$ coupling, which comes at tree level from the following diagram.
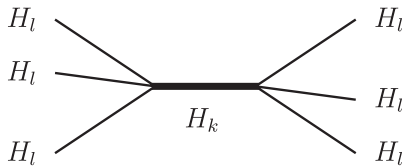
The contribution from heavy modes will be of the order

$$\tilde{\lambda}^l_6 \sim \sum_k \tilde{\lambda}^2_{l,l,l,k} \frac{1}{\omega^2_k}. \tag{73}$$

For heavy KK modes, $l \geq 1$, we can use Eq. (72) and the relation $\omega_k \sim \frac{\pi k}{2R}$, which leads to

$$\tilde{\lambda}^l_6 \sim \lambda^2 e^{4MR} \sum_k \frac{l^3}{k(k^2 + 3l^2)^2}. \tag{74}$$

The sum converges fast but the factor e^{4MR} is very large. If we want our classical calculations to be valid, we have to impose a strong restriction on the $2 + 1$ dimensional couplings λ, e^2, so that $\tilde{\lambda}_6$ and other corrections are small; that is,

$$\lambda, e^2 \ll M e^{-2MR}. \tag{75}$$

This assumption will be considered in the following; it enables performing semiclassical computations. Although the assumption (75) seems very restrictive, there is no obvious reason that the further results are only valid under this constraint. We suppose that they are also valid under more general assumptions, but we will not prove it here. The extension to larger couplings requires lattice computations [22].

IV. NONLINEAR SOLUTIONS

After discussing the perturbative spectrum of the theory we study solutions of the nonlinear equations of motion. In the usual $2 + 1$ dimensional Abelian Higgs model—not localized on a brane—neither a sphaleron nor instanton exists. There only exists solitons— Nielsen-Olesen vortices in two spatial dimensions. The vortices possess the property that, if we consider a closed curve C surrounding the vortex center, the following relation is satisfied:

$$\frac{e}{2\pi} \oint_C \mathbf{A} \cdot \mathbf{dl} = n, \tag{76}$$

where n is an integer called topological charge of the vortex.

Processes discussed in [11,23] can happen: starting from the three dimensional vacuum, a vortex-antivortex pair is created near the south pole. The vortex can move across the equator up to the north pole. The bosonic configuration of the final state looks on the equator (brane) like the n vacua. Indeed using the relation (76), the Chern-Simons number is

$$N_{CS} = \frac{e}{2\pi} \int dx A_1 = n. \tag{77}$$

From the point of view of someone living on the brane, a topological transition (indistinguishable from an instanton transition) occurred and the vacua changed (from $|0\rangle$ to $|1\rangle$). The difference with the pure two dimensional theory is that the end point is not a *vacuum* state. It contains two

vortices, which represent a large energy cost for the system.

We shall now see what happens when the warp factor is introduced and the fields are localized. The main effect of the warp factor is to damp the action of field configurations localized far from the brane. We therefore expect that, at the classical level, the vortex is light close to the pole and heavier when crossing the brane.

The $1 + 1$ dimensional $|n\rangle$ vacua with $n \neq 0$ are not vacua anymore in the higher dimensional theory. They become resonances with the energy of two vortices with topological charge n and some suppressed decay rate.

A. Vortex at the center of the disk

With the warp factor, the vortex is repulsed from the brane and a stable solution of the equations of motion can only be found at the center of the disk. In this section we study the energy of the vortex at the center of the disk and find its dependency on the parameters of the model and on its topological charge n. The vortex has the same properties as the $1 + 1$ dimensional instanton, that is to say $\phi(r = 0) = 0$, rotational symmetry, and winding number $Q = n$. It is also parametrized by Eqs. (16) and (17) but the functions $f(r)$, $A(r)$ are deformed by the presence of the warp factor. The energy of the $A(r)$, $f(r)$ field configuration is

$$E = \pi v^2 \mu^2 \int dr \left\{ \frac{\Delta}{r} A'^2 + \Delta \left[r f'^2 + \frac{f^2}{r}(n - A)^2 \right. \right.$$
$$\left. \left. + r \frac{\mu^2}{2}(f^2 - \mu^{-2})^2 \right] \right\}, \quad (77)$$

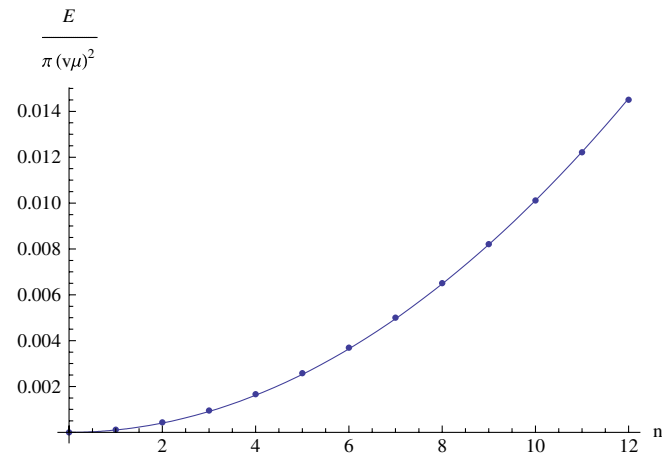with $\mu^2 = \frac{\lambda}{e^2}$ and their equation of motion reading

FIG. 2 (color online). Vortex energy $\frac{E}{\pi v^2 \mu^2}$ as a function of the topological charge n, fitted with $0.000\,101 n^2$. Other parameters are fixed to $\bar{M} = 2$, $\bar{R} = 3$.
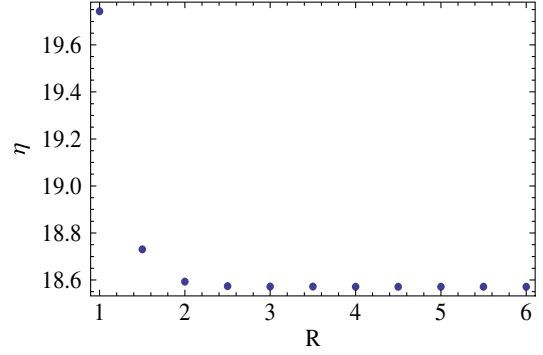
FIG. 3 (color online). Parameter η as a function of the disk radius $\bar{R}$. Other parameters are $n = 1$, $\bar{M} = 4$.

$$-\frac{r}{\Delta} \partial_r \Delta \frac{A'}{r} - f^2(n - A) = 0,$$
$$-\frac{r}{\Delta} \partial_r \Delta r f' + f(n - A)^2 + r^2 \mu^2 (f^2 - \mu^{-2}) f = 0. \quad (78)$$

In the following, we will solve numerically Eq. (78) with boundary conditions (18) as in [17]. There are three dimensionless parameters, one for the gauge-Higgs fields μ, one for the warp factor $\bar{M} = \frac{M}{\sqrt{\lambda} v}$, and one for the size of the disk $\bar{R} = \sqrt{\lambda} v R$. The dependencies of the energy with respect to $\bar{M}$, $\bar{R}$, and the topological charge n will be studied. For the sake of simplicity we will set $\mu = 1$.

The dependency on the topological charge is a parabola $E \propto n^2$ for fixed $\bar{M}$, $\bar{R}$. An example of this is shown in Fig. 2. We checked that for $\bar{M} = 0..10$, $\bar{R} = 1..6$, $n = 0..10$, the discrepancy between the fit $E \propto n^2$ and the points is much less that 0.1%

The dependence on the radius size $\bar{R}$ is $E \propto \exp(-2\bar{M}\bar{R})$, with some deviation from this behavior at small values of $\bar{R}$, In Fig. 3, we plot the value of
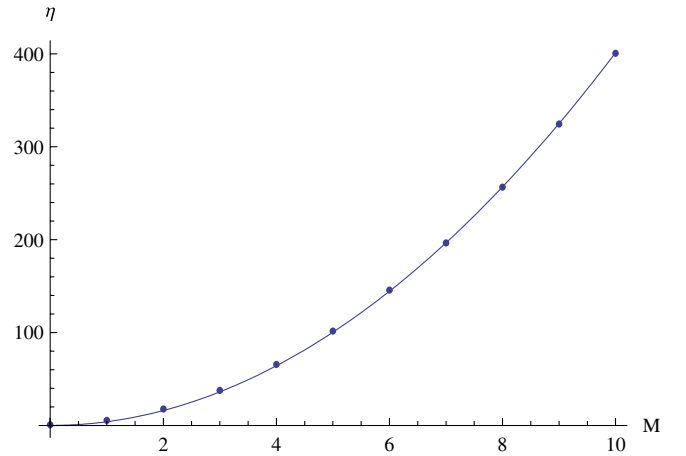
FIG. 4 (color online). Parameter η as a function of the brane scale $\bar{M}$, fitted with $4.013\bar{M}^2$. Discrepancies from points to the fits only occur for very small $\bar{M}$.

$$\eta = \frac{E}{\pi v^2 \mu^2 n^2 \exp(-2\bar{M}\bar{R})}. \tag{79}$$

We will now extract the $\bar{M}$ dependency. We use of the previous dependencies and plot η for $\bar{M} = 0..10$. In Fig. 4, η is fitted with values of $n = 0..9$, and the large $\bar{R}$ limit is used. For large $\bar{M}$ the dependency is again a parabola $E \propto \bar{M}^2$, and deviation from this occurs at a small value of $\bar{M}$.

To summarize, within the relevant region ($\bar{M} \gg 1 \gg 1/\bar{R}$) we have the following vortex energy:

$$E \simeq 4\pi v^2 \mu^2 \bar{M}^2 n^2 e^{-2\bar{M}\bar{R}}. \tag{80}$$

Coming back to dimensionful parameters, this takes the form

$$E \simeq 4\pi \frac{M^2}{e^2} n^2 e^{-2MR}. \tag{81}$$

B. Vortex sitting on the brane: Sphaleron

The other case of interest is when a vortex of topological charge one is located on the brane, that is to say $\phi(R, \bar{\theta}) = 0$ at some $\theta = \bar{\theta}$; this is the $2 + 1$ dimensional analog of the $1 + 1$ dimensional sphaleron. The presence of the brane breaks the rotational symmetry of the vortex. The exact solution is found by integrating numerically the two dimensional equations arising from the variation of the energy functional. Details of the numerical procedure are given in Appendix D.

The results of the numerical simulation are shown in Fig. 5. We only consider the case $m_H R \gg 1$. As long as $\bar{M} \gtrsim 1$, the $2 + 1$ dimensional sphaleron energy exactly matches the $1 + 1$ dimensional one. Note that if the KK mass is larger than the sphaleron negative mode $\omega_- \sim$
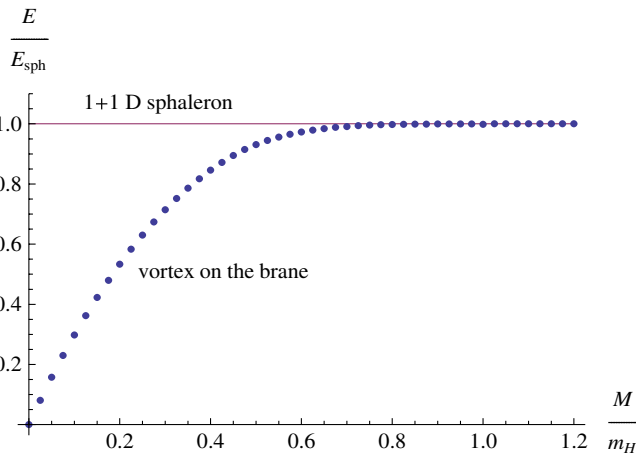
FIG. 5 (color online). Energy of the different topological solutions as a function of the brane mass scale (for $\frac{m_H}{m_w} = 1$). The energy is scaled so that the $1 + 1$ dimensional sphaleron has energy 1. The energy of the vortex sitting on the brane meets the $1 + 1$ dimensional sphaleron energy for large M. The accuracy of the results is estimated to be better than 0.1%.
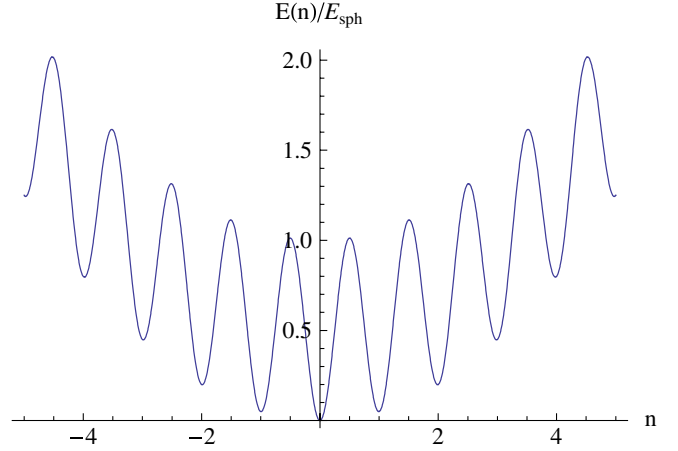
FIG. 6 (color online). Sketch of the minimal energy for a configuration with winding number n. The value in local minima corresponds to the energy of two vortices at the center of the disk, and the maxima contain the contribution of the sphaleron energy.

$0.8 m_H$, the properties of the sphaleron are expected to be part of the low-energy theory and match the $1 + 1$ dimensional result (14).

C. Movement of the global mode in the potential— solution to the strong *CP* problem

The topological charge can be seen as a field [2]; in the $1 + 1$ dimensional Abelian Higgs model it has a periodic potential (see Fig. 1). The addition of a compact extra dimension breaks this periodicity (see Fig. 6).

We shall compare the different scales in our model. As usual we consider the inequality $M \gg m_H \sim m_W \gg 1/R$ together with the $1 + 1$ dimensional weak coupling requirement $\tilde{v} \gg 1$ and the $2 + 1$ one (75). The sphaleron mass

$$\frac{m_{\text{sph}}}{m_H} \approx \tilde{v}^2 \gg 1 \tag{82}$$

is much larger than the particle mass in the weak coupling regime and much larger than the vortex mass,[7]

$$\frac{m_{\text{sph}}}{m_{\text{vortex}}} \approx \frac{m_H m_w^2}{M^3 e^{-2MR}} \gg 1. \tag{83}$$

At first sight, the energy of the vortex $E_v(n)$, and therefore the energy $E_n = 2E_v(n)$ of the state $|n\rangle$, seems very small. However if we consider that the couplings (electric charge

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[7] In fact, the full set of requirements for the parameters is the following: $m_H^2/v^2 M \ll e^{-2MR} \ll m_H^3/M^3 \ll 1$. The first inequality is the weak coupling of the $2 + 1$ dimensional theory the second is (83), and the third says that the KK modes are heavierthan the localized ones.

here) have to be tiny to be in the regime where our classical approximations are valid, we have[8]

$$\frac{m_{\text{vortex}}}{m_H} = \frac{M^2}{e^2 m_H e^{2MR}} \gg \frac{M}{m_H} \gg 1. \qquad (84)$$

The latter relation shows that vortex pairs can decay to particles and the system relaxes to the $n = 0$ vacuum. In this case the strong CP problem is solved. As we see from Fig. 6, at low energy the system will relax dynamically to the state $n = 0$ by tunneling. If the system is in the ground state, the $1 + 1$ dimensional instanton transitions which make the average of the topological density ($\langle q \rangle \neq 0$) nonvanishing are not anymore instanton transitions and can occur only if a large energy is available in the system. At low energy, for example, for the measurement of the neutron dipole moment, a topological transition cannot occur; the topological density vanishes ($\langle q \rangle = 0$ since already $\langle |q| \rangle = 0$), and therefore θ is effectively zero.

In the strong coupling regime, if the mass of the vortex happens to be too small, the states $|n\rangle$ will be almost degenerate and we expect that the strong CP problem will reappear at some point. Indeed when the $|n\rangle$ are almost degenerate, the real vacuum will be a superposition of several $|n\rangle$. Several different superpositions of these $|n\rangle$ states with different phases and slightly higher energy will exist, and we expect to have some sort of discrete strong CP problem there. The precise analysis of these states is left for future work.

V. CONCLUSION

Extra dimensions might give an appealing solution to the strong CP problem. The solution relies on a global field dynamic and does not imply the existence of a new particle. Unlike for the axion solution, it cannot be ruled out by astrophysical observations. However, the processes with change of the topological number (and, at the same time, fermion number) proceed differently in the proposed setup.

In this paper the $2 + 1$ dimensional toy model was studied, which at low energy reduces to the $1 + 1$ Abelian Higgs model, often used to model some QCD properties. If the inequalities $m_{\text{sph}} \gg m_{\text{vortex}} \gg m_H, m_W$ are satisfied, the $1 + 1$ dimensional $|n\rangle$ vacua become, for $n \neq 0$, resonances in the $2 + 1$ dimensional theory. These resonances have a high energy and decay reasonably fast to particles. The system relaxes to the $|n = 0\rangle$ vacuum, which is the true vacuum of the $2 + 1$ dimensional theory. The vacuum structure is trivial and the strong CP problem is solved in this case. We showed that these inequalities (82)–(84) are realized in the weak coupling regime (75). We suppose that these relations will remain true in the strong coupling regime. This seems reasonable for the relations

(82) and (83), while for (84) it seems to fail, since we used the weak coupling assumption. However in the strong coupling case, the quantum corrections are large and, unlike the classical action, are not suppressed by the warp factor at the vortex center. Indeed quantum corrections are expected to be of order $\hbar$ independently of the normalization of the action. It would be interesting to study the strong coupling regime with lattice simulations to check whether this large extra-dimension solution to the strong CP problem also works at large couplings.

We choose in this paper a very simple geometry, which allows for simple analysis but which has the annoying property that the size of the universe is just 2π larger than the extra dimension. This restricts the semiclassical calculations to very weak couplings. A way to avoid strong coupling problems would be to consider a more complicated geometry, which allows for unrelated universe and extra-dimension size. We could then have a very large or infinite space and a small enough extra dimension. The setup will be more complicated and may not enable a simple analysis of the nonlinear solutions. However, as we have shown here, it seems that the localization of the fields on a brane can be done independently of the bulk geometry. It is indeed an interesting result on its own that an Abelian Higgs can be localized on the boundary of a disk.

Finally, we may speculate how the obtained results transfer to the real world QCD. We expect that the degeneracy of the n vacua will be lifted by the extra dimension and the new $|n\rangle$ states (or resonances) will have an energy monotonically increasing with n. The tunneling between the n and $n - 1$ will still be suppressed by the instanton (or, strictly speaking, bounce) action. At very high energy, these bounce transitions are possible and will look like normal instanton transitions. If we can arrange the geometry so that the states (or resonances) $|n \neq 0\rangle$ have sufficiently large energy, bounces will not contribute in a measurement of the neutron dipole moment. The topological charge density will be zero and there will not be any observable effect of a θ angle, so that the strong CP problem will be solved.

APPENDIX A: SPECTRUM OF KALUZA-KLEIN MODES

1. Higgs dispersion relation

We derive the dispersion relation for the light mode of the Higgs. To this aim, we have to impose the boundary

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[8]In fact, even stronger inequality is true, $m_{\text{vortex}} \gg M$.

condition on the brane, $h'(R) = 0$. Suppose that the low-energy mode satisfies $\omega \sim m_H \ll M$. We can then use the asymptotic expansion for the large argument of the Laguerre function:

$$L_\nu^\lambda(z) \overset{z \to \infty}{\to} \sin(\nu\pi)\Gamma(\lambda + \nu + 1)z^{-\lambda-\nu-1}$$
$$\times e^z\left(1 + \frac{(1 + \nu)(1 + \lambda + \nu)}{z} + \mathcal{O}(z^{-2})\right). \quad (A1)$$

In this limit, Eq. (34) becomes

$$h_0^n(r) \propto e^{-(\delta\omega^2 r/2M)}(2rM)^{-(\delta\omega^2/4M^2)}$$
$$\times \left(1 - \frac{n^2}{2Mr} + \mathcal{O}(M^{-2})\right), \quad (A2)$$

with $\delta\omega^2 = \omega^2 - m_H^2$. Imposing $h_0'(R) = 0$ gives $\delta\omega^2 = \frac{n^2}{R^2}$.

2. Gauge field dispersion relation

The computation is the same as for the Higgs, the function $a_\theta^n(r)$ being the same as $h_0^n(r)$ replacing m_H by m_W. Imposing $a_\theta'(R) = 0$ implies $\omega^2 = m_w^2 + \frac{n^2}{R^2}$. Note that from Eqs. (50) and (46) this implies $a_r^n(R) = 0$ and $a_0'(R) = 0$, which are the correct boundary conditions for these fields.

3. High-energy modes for the Higgs

Using the expansion (A1), we see that for $\omega \gg M$,

$$h^n(r) \sim e^{ir\omega}. \quad (A3)$$

One of the requirements $h(R) = 0$ or $h'(R) = 0$ are satisfied if $R\omega = k\pi/2$, which gives $\omega_k - \omega_{k-1} = \frac{\pi}{2R}$.

APPENDIX B: COUPLINGS IN THE EFFECTIVE ACTION

In this section, we compute the parameters I. We compute exactly the parameters related to the quadratic part of the action, first I_a. We start from the normalization condition for a_θ^n (53), integrate $|(a_\theta^n)'(r)|^2$ by part, use the equation of motion (40) for a_θ^n, and use the relation (46),

$$1 = \int dr \frac{\Delta}{r}\left(a_\theta^n a_\theta^{-n} + \frac{r^2}{n^2}(a_\theta^n)'(r)(a_\theta^{-n})'(r)\right)$$
$$= \frac{\omega_n^2 - m_W^2}{n^2}\int drr\Delta|a_\theta^n(r)|^2.$$

From which we get

$$\int drr\Delta|a_\theta^n(r)|^2 = \frac{n^2}{\omega_n^2 - m_W^2}. \quad (B1)$$

This, together with the normalization condition (53), gives us

$$I_a = -1 + \frac{n^2}{\omega_n^2 - m_W^2} \cdot \frac{(\omega_n^2 - m_W^2)^2}{n^2\omega_n^2} = -1 + \frac{n^2}{R^2\omega_n^2}. \quad (B2)$$

To calculate I_h, we integrate $|(h^n)'|^2$ by part and use the equation of motion (33)

$$I_h = \frac{R^2}{n^2}\int drr\Delta\left((h^n)'(r)(h^{-n})'(r) + \frac{n^2}{2r^2}|h^n(r)|^2\right)$$
$$= \frac{R^2}{n^2}\int drr\Delta(\omega_n^2 - m_H^2)|h^n(r)|^2,$$

which, using the normalization condition (59) and the dispersion relation, gives $I_h = 1$.

The other parameters I_b, I_c, I_d, I_e are computed to $\mathcal{O}((MR)^{-1}, (m_H R)^{-2})$ using the asymptotic expansion (A1) for the mode functions. Indeed because of the presence of the warp-factor and the fact that the modes a_θ^n, h^n are roughly constant, the integrals $I_{b,c,d,e}$ are dominated by values close to the brane $(r = R)$. At these points, the argument of the Laguerre functions is large and the asymptotic expansion can be used. We first compute I_d, I_e (65); to this aim, we first have to normalize $h^n(r)$. From (A2), and imposing $\int drr\Delta|h^n(r)|^2 = 1$, we get

$$h^n = e^{M(R-r)-r\Omega_H}\frac{r^n(2\Omega)^{n+(1/2)-(M/2\Omega_H)}}{\sqrt{2R}\Gamma(n + \frac{1}{2} - \frac{M}{2\Omega_H})}$$
$$\times L_{-n-(1/2)-(M/2\Omega_H)}^{2n}(2r\Omega_H)$$
$$\overset{\Omega_H r \gg 1}{\to} \sqrt{\frac{M}{R}}\left(\frac{R}{r}\right)^{-(n^2/(4R^2M^2))}e^{-(r-R)(n^2/2MR^2)}$$
$$\times \left(1 + \mathcal{O}\left(\frac{1}{MR}\right)\right). \quad (B3)$$

Some straightforward algebra gives

$$I_d = \int dr\Delta rh^n h^m h^{-n-m} = \sqrt{\frac{M}{R}}\left(1 + \mathcal{O}\left(\frac{1}{MR}\right)\right), \quad (B4)$$

$$I_e = \int dr\Delta rh^n h^m h^k h^{-n-m-k} = \frac{M}{R}\left(1 + \mathcal{O}\left(\frac{1}{MR}\right)\right). \quad (B5)$$

Note that these integrals, at least at leading order, do not depend on the values of n, m, k.

To compute the remaining I_b, I_c, we need the asymptotic expansion for $a_\theta^n(r)$. It only differs from $h^n(r)$ by the normalization condition. As $a_\theta^n(r)$ is constant near $r = R$, we can neglect the term $|(a_\theta^n)'(r)|^2$ in the normalization condition (53), which becomes $\int dr\frac{\Delta}{r}|a_\theta^n|^2 = 1$. This implies that

$$a_\theta^n = \sqrt{MR}\left(\frac{R}{r}\right)^{-(n^2/4R^2M^2)}e^{-(r-R)(n^2/2MR^2)}\left(1 + \mathcal{O}\left(\frac{1}{MR}\right)\right). \quad (B6)$$

Using (B3) and (B6) and performing some straightforward algebra gives

$$I_b = \int dr \frac{r\Delta}{2} h^k h^{-n-m-k} \left(-\frac{n}{R^2 \omega_n} \frac{m}{R^2 \omega_m} a_\theta^n a_\theta^m + \frac{1}{nm} \right.$$
$$\times (a_\theta^n)'(a_\theta^m)' + \frac{1}{r^2} a_\theta^n a_\theta^m \right)$$
$$= \int dr \frac{r\Delta}{2} h^k h^{-n-m-k} \frac{1}{r^2} a_\theta^n a_\theta^m \left(1 + \mathcal{O}\left(\frac{1}{m_W^2 R^2}\right) \right)$$
$$= \frac{M}{R} \left(1 + \mathcal{O}\left(\frac{1}{MR}, \frac{1}{m_W^2 R^2}\right) \right)$$

and

$$I_c = \int dr \frac{r\Delta}{2} h^{-n-m} \left(-\frac{n}{R^2 \omega_n} \frac{m}{R^2 \omega_m} a_\theta^n a_\theta^m + \frac{1}{nm} (a_\theta^n)' \right.$$
$$\times (a_\theta^m)' + \frac{1}{r^2} a_\theta^n a_\theta^m \right)$$
$$= \int dr \frac{r\Delta}{2} h^{-n-m} \frac{1}{r^2} a_\theta^n a_\theta^m \left(1 + \mathcal{O}\left(\frac{1}{m_W^2 R^2}\right) \right)$$
$$= \sqrt{\frac{M}{R}} \left(1 + \mathcal{O}\left(\frac{1}{MR}, \frac{1}{m_W^2 R^2}\right) \right).$$

APPENDIX C: COUPLINGS AT VERY HIGH ENERGY

We already computed the effective coupling $\lambda = \lambda_{0,0,0,0} = M\tilde{\lambda}$ in the low-energy effective action (69), but we are now interested in couplings $\lambda_{k_1,k_2,k_3,k_4}$, with k_i large.

At very high energy ($\omega_n^k \gg M$), the wave functions $h_k^n(r)$ are large near the center of the disk ($r = 0$). The integrals for the effective couplings will be saturated by values close to $r = 0$; therefore we make use of the small argument expansion of the Laguerre functions

$$L_\nu^\lambda(z) \overset{z \ll 1}{\to} \frac{\Gamma(\nu + \lambda + 1)}{\Gamma(\nu + 1)} \left(\frac{1}{\Gamma(\lambda + 1)} - \frac{\nu z}{\Gamma(\lambda + 2)} + \dots \right).$$
(C1)

This expansion is introduced in the wave function h_k^n:

$$h_k^n(r) \overset{rM \ll 1}{\to} e^{+M(R-r)-r(\Omega_H)} \frac{r^n (2\omega)^{n+(1/2)-(M/2\Omega_H)}}{\sqrt{2R} \Gamma(2n+1) \Gamma(-n - \frac{M}{2\Omega_H} + \frac{1}{2})}.$$
(C2)

While computing the effective action, we will identify

$$\frac{\lambda_{k_1,k_2,k_3,k_4}}{8\pi R} = \lambda \int r\Delta dr \prod_{i=1}^{4} h_{k_i}^{n_i}(r)$$
$$\approx \frac{2\lambda e^{2MR}}{\pi^2 R^2} \frac{\prod_i \sqrt{\omega_i}}{\sum_i \omega_i^2} \gamma(n_1, n_2, n_3, n_4), \quad \text{(C3)}$$

with $\gamma(0, 0, 0, 0) = 1$ and $\gamma(n_1, n_2, n_3, n_4)$ of order unity.

APPENDIX D: VORTEX SITTING ON THE BRANE, NUMERICAL ANALYSIS

To simplify slightly the analysis, we notice that the vortex solution has a size of a few $1/m_H$ which is in our assumptions $M \gg m_H \gg 1/R$ much smaller than the radius of the disk. Furthermore what happens away from the brane is damped by the warp factor. We can therefore consider that the disk looks like a half plane in the region where the vortex lies. The vortex on the brane still possesses one discrete symmetry: the reflection along the perpendicular to the brane passing through its center. We will therefore solve the equations on a quarter of a plane. To get a better resolution of the instanton center, we parametrize the quarter of a plane in polar coordinates $r \in]0, \infty[, \theta \in [0, \pi/2]$.

We choose the $A_0 = 0$ gauge and use the remaining time independent gauge freedom to cancel the radial component of the vector field[9]: $A_r = 0$. We also use dimensionless variables as in (19). The energy functional reads

$$E[H, \sigma, A_\theta] = v^2 \mu^2 \int r dr d\theta e^{-2Mr|\sin\theta|} \left(\frac{1}{2}\left(\partial_r A + \frac{A}{r} \right)^2 \right.$$
$$+ \frac{1}{2}(\partial_r f)^2 + \frac{1}{2r^2}(\partial_\theta f)^2 + \frac{f^2}{2}\left[(\partial_r \sigma)^2 \right.$$
$$+ \left(\frac{1}{r}\partial_\theta \sigma - A \right)^2 \right] + \frac{\mu^2}{4}\left(f^2 - \frac{1}{\mu^2} \right)^2 \right).$$
(D1)

Numerically, we minimize the energy functional using as a starting configuration the usual vortex configuration $A(r), f(r)$ from (16) and (17).

$$A(r, \theta) = A(r), \qquad f(r, \theta) = f(r), \qquad \sigma(r, \theta) = \theta.$$
(D2)

The convergence to the exact solution is achieved by introducing an artificial time dependency and solving Hamiltonian equations:

$$\frac{d}{dt}A_\theta = -\frac{\delta E}{\delta A_\theta}, \qquad \frac{d}{dt}H = -\frac{\delta E}{\delta H}, \qquad \frac{d}{dt}\sigma = -\frac{\delta E}{\delta \sigma}.$$
(D3)

Note that an upper bound on the vortex energy can be

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[9]We are dealing with a static (time independent) configuration.

found integrating the energy functional (D1) with the usual vortex configuration in a flat $2 + 1$ dimensional space (16) and (17).

To avoid trivial divergences at $r = 0$, we start the integration at $r = \varepsilon \sim 10^{-5}$ and consider a finite radius R to end the integration.[10] As we use polar coordinates to describe the Higgs field, we should also arrange the boundary conditions to avoid H vanishing near $r = \varepsilon$ and rescale the field σ such that it is single valued at $r = 0$, $\sigma(r, \theta) \to \sqrt{r}\sigma(r, \theta)$; this allows correctly fixing the value of σ so that the action remains finite[11] at $r \to 0$. Boundary conditions are set on the four edges:

$$A_\theta(\varepsilon, \theta) = a(\varepsilon) \sim 0, \qquad H(\varepsilon, \theta) = f(\varepsilon) \sim 0,$$
$$\sigma(\varepsilon, \theta) = \theta\sqrt{\varepsilon} \sim 0, \tag{D4}$$

$$\partial_\theta A_\theta(r, 0) = 0, \qquad \partial_\theta H(r, 0) = 0, \qquad \sigma(r, 0) = 0, \tag{D5}$$

$$A_\theta(R, \theta) + R\partial_r A_\theta(R, \theta) = 0, \qquad H(R, \theta) = \frac{1}{\mu}, \tag{D6}$$
$$\sigma(r, \theta) - 2R\partial_r\sigma(R, \theta) = 0,$$

$$\partial_\theta A_\theta\left(r, \frac{\pi}{2}\right) = 0, \qquad \partial_\theta H\left(r, \frac{\pi}{2}\right) = 0, \qquad \sigma\left(r, \frac{\pi}{2}\right) = \frac{\pi}{2}\sqrt{r}. \tag{D7}$$

The conditions at $r = 0$ and $r = R$ are chosen to get a finite action, and the conditions at $\theta = 0$ and $\theta = \pi/2$ are dictated by the discrete symmetries and the topological charge.

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[10] In dimensionless units we take $\bar{R} = 5$. We checked that this does not introduce sizable errors.

[11] The action near $r = 0$ contains a term proportional to $r^3(\partial_r\sigma)^2$, which is finite if $\sigma \sim r^a$, with $a > -1/2$; that is to say $\sqrt{r}\sigma = \bar{\sigma} \to 0$ as $r \to 0$.

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