Four-dimensional $\mathcal{N} = 1$ super Yang-Mills theory from a matrix model

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We consider a supersymmetric matrix quantum mechanics. This is obtained by adding Myers and mass terms to the dimensional reduction of 4D $\mathcal{N} = 1$ super Yang-Mills theory to one dimension. Using this model we construct 4D $\mathcal{N} = 1$ super Yang-Mills theory in the planar limit by using the Eguchi-Kawai equivalence. This regularization turns out to be free from the sign problem at the regularized level. The same matrix quantum mechanics is also used to provide a nonperturbative formulation of 4D $\mathcal{N} = 1$ super Yang-Mills theory on a noncommutative space.

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I. INTRODUCTION

Supersymmetry is a promising framework for physics beyond the standard model. For this reason it is important to understand its nonperturbative aspects such as confinement and the mechanism of supersymmetry breaking. Usually lattice regularization provides a tool to study field theories in the nonperturbative regime. However, a technical obstacle arises in this type of regularization, namely, it is difficult to keep supersymmetry (although some progress has been achieved for some specific kind of theories, for a review see [1]). In order to avoid this obstacle for large-N supersymmetric Yang-Mills (SYM) theories, we can use another regularization method known as Eguchi-Kawai reduction [2].

Another motivation to study SYM theories in the large-N limit is that they are expected to describe the nonperturbative dynamics of string theory [3–7]. For instance, $(0 + 1)$-dimensional maximally supersymmetric $U(N)$ gauge theory is conjectured to be dual to type IIA superstring theory on a black 0-brane background [7]. This specific example has been studied using Monte Carlo simulation [8–10]; using these numerical techniques the stringy α' corrections can be evaluated [10]. There have been much efforts to study the $0 + 0$-dimensional theory [4] numerically, too. See e.g. [11–14]. Finally, large-N Yang-Mills theories are interesting on their own because they might be solvable analytically [15] while preserving essential features of QCD with $N = 3$ (for a recent review, see [16]).

As previously mentioned, the Eguchi-Kawai equivalence [2] can be used as an alternative method to regularize large-N SYM. The main idea of this method is that large-N gauge theories are equivalent to certain lower dimensional matrix models. Furthermore in this prescription the degrees of freedom of the reduced spaces are embedded in the infinitely large matrices. A UV regularization can be introduced by taking the size of the matrices to be large but finite. This regularization, differently from the lattice one, does not break supersymmetry. With such a motivation, in [17] a nonperturbative formulation of the maximally supersymmetric Yang-Mills in four dimensions was proposed. The authors of [17] have considered a particular solution of the Berenstein-Maldacena-Nastase (BMN) matrix model [18], namely, a set of concentric fuzzy spheres, which has been argued to be stable due to its Bogomol'nyi-Prasad-Sommerfield (BPS) nature. Expanding the BMN matrix model about this background, the 4D $\mathcal{N} = 4$ SYM was recovered through the Eguchi-Kawai equivalence.

In this paper, we provide a nonperturbative formulation of 4D $\mathcal{N} = 1$ (pure) SYM in the planar limit by using the technique introduced in [17]. There are two main motivations to extend the results presented in [17]. First, 4D $\mathcal{N} = 1$ supersymmetric theories are more interesting as a candidate of new physics in the LHC, and it is important to consider the $\mathcal{N} = 1$ (pure) SYM as a simplest example. 4D $\mathcal{N} = 1$ SYM is dynamically richer than 4D $\mathcal{N} = 4$ and given that there is no known gravity dual of 4D $\mathcal{N} = 1$ SYM providing analytical results, numerical simulations are a valuable tool. Even though in principle 4D $\mathcal{N} = 1$ SYM on the lattice can be studied without fine-tuning[1] it is computationally very demanding, and a detailed study is difficult (for recent numerical studies see [19]). On the contrary, our supersymmetric matrix models would require less resources, and allow a better numerical analysis of $\mathcal{N} = 1$ SYM.

Second, several groups are seriously studying 4D $\mathcal{N} = 1$ SYM on the lattice, using conventional computationally demanding numerical techniques. When the results of these studies become available they could be used to further check the validity of the Eguchi-Kawai regularization. After its validity has been further confirmed, the

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[1]In the context of lattice regularization, fine-tuning means adding counterterms in order to restore supersymmetry in the continuum limit.

Eguchi-Kawai construction could be used to analyze field theories with extended supersymmetries [17], which cannot be studied by using the lattice unless we introduce fine-tuning.

In the first part of this paper we formulate the Eguchi-Kawai reduction of 4D $\mathcal{N} = 1$ SYM by using a BMN-like mass-deformed matrix quantum mechanics with four supersymmetries. In the regularization of 4D $\mathcal{N} = 4$ SYM introduced by [17], only 16 out of 32 supercharges are kept unbroken and the restoration of the other 16 supersymmetries is not obvious, although supporting evidence has been found in [17,20]. In the present case, all 4 supercharges are manifestly kept unbroken and hence we expect that 4D $\mathcal{N} = 1$ SYM is recovered in the continuum limit.

In the second part of this paper we consider noncommutative super Yang-Mills theories in four dimensions. These appear, for example, in string theory as effective theories on D-branes with flux. Given that noncommutative Yang-Mills theories have a D-brane origin, they can be regularized by using matrix models as explained in [21]. Super Yang-Mills in flat noncommutative space is obtained by studying the theory in a background satisfying the Heisenberg algebra $[\hat{x}, \hat{y}] = i\theta$, which cannot be realized using finite-N matrices. It turns out that the Heisenberg algebra can be described at finite-N level by considering compact fuzzy manifolds like the fuzzy sphere embedded in flat space. The flat noncommutative space is then recovered as the tangent space to these fuzzy manifolds. One unsatisfactory property of this prescription is that transverse directions are necessary for embedding the compact fuzzy spaces into flat space (for example, if we embed S^2 into $\mathbb{R}^3$ there is one transverse direction). In the field theory description these directions turn into scalar fields, and therefore, only noncommutative gauge theories with scalars can be realized in this way. In the case of supersymmetric models, this implies that it is only possible to regularize theories with extended supersymmetries.[2] In this article we show that by using our matrix model with the background proposed in [17] in an appropriate limit, we can regularize 4D $\mathcal{N} = 1$ noncommutative *pure* super Yang-Mills. Using this construction, the transverse direction becomes an ordinary *commutative* coordinate and as a consequence we recover pure $\mathcal{N} = 1$ SYM with no additional scalars.

This paper is organized as follows. In Sec. II we review the Eguchi-Kawai equivalence. In Sec. II A we explain its deformation, namely, the "quenched" Eguchi-Kawai model, which we then use to formulate 4D $\mathcal{N} = 1$ SYM. In Sec. III we provide a supersymmetric matrix quantum mechanics, and applying the method explained in Sec. II A, we provide the Eguchi-Kawai formulation of 4D $\mathcal{N} = 1$ SYM. In Sec. III C we prove that this regularization does not suffer from the sign problem. In Sec. IV we provide a nonperturbative formulation of 4D $\mathcal{N} = 1$ SYM on noncommutative space. In the appendix we introduce four-dimensional $\mathcal{N} = 1$ SYM on the three-sphere and express it in a form which is convenient for our purpose.

[2]If we use the twisted Eguchi-Kawai model [22], which is written in terms of unitary matrices, we do not need transverse directions. However it is difficult to supersymmetrize it.

II. THE EGUCHI-KAWAI REDUCTION

In this section we review the Eguchi-Kawai equivalence [2]. The equivalence guarantees that D-dimensional $SU(N)$ gauge theory and its one-point reduction are equivalent if the global $(Z_N)^D$ symmetry of the latter is not broken. In the bosonic case, however, this symmetry is broken for $D > 2$ in the 0D theory; it is unbroken only above a critical volume [23]. To cure this problem, deformations of the 0D theory, the quenched [24,25] and twisted [22] Eguchi-Kawai models (QEK, TEK, respectively) were proposed soon after the original one.[3] In these models, deformations are introduced such that $(Z_N)^D$-unbroken backgrounds become stable. However, recently it was found that both TEK [27–29] and QEK [30] fail at very large-N—their deformations cannot stabilize the backgrounds completely. On the other hand, by combining quenched and/or twisted prescriptions with supersymmetry, the background can be stabilized [27,31].

In Sec. II A we review the diagrammatic approach to the quenched Eguchi-Kawai model [25] (for the case of compact spaces, see also[17,32]). First we consider the simplest case, namely, the equivalence between matrix quantum mechanics and the zero-dimensional matrix model, and then we proceed with the Eguchi-Kawai construction of the field theory on S^3 [17].

A. Quenched Eguchi-Kawai model

We consider a matrix quantum mechanics with a mass term,

$$S_{1d} = N \int dt \, \mathrm{Tr}\left(\frac{1}{2}(D_t X_i)^2 - \frac{1}{4}[X_i, X_j]^2 + \frac{m^2}{2} X_i^2\right), \quad (1)$$

where X_i ($i = 1, 2, \cdots, d$) are $N \times N$ traceless Hermitian matrices. The covariant derivative D_t is given by $D_t X_i = \partial_t X_i - i[A, X_i]$. At large-$N$, this model can be reproduced starting from the zero-dimensional model

$$S_{0d} = \frac{2\pi}{\Lambda} \cdot N \, \mathrm{Tr}\left(-\frac{1}{2}[Y, X_i]^2 - \frac{1}{4}[X_i, X_j]^2 + \frac{m^2}{2} X_i^2\right), \quad (2)$$

where Y and X_i are $N \times N$ traceless Hermitian matrices. We embed the (regularized) translation generator into the matrix Y,

[3]Another recent proposal can be found in [26].

$$Y^{b.g.} = \mathrm{diag}(p_1, \cdots, p_N), \qquad p_k = \frac{\Lambda}{N}\left(k - \frac{N}{2}\right). \quad (3)$$

By expanding Y around this background, $Y = Y^{b.g.} + A$, the Feynman rules of the one-dimensional theory are reproduced at large-N, as we will see in the following.

The action can be rewritten as

$$S_{0d} = \frac{2\pi}{\Lambda} \cdot N\left\{\frac{1}{2}\sum_{i,j}|(p_i - p_j)(X_k)_{ij} - i[A, X_k]_{ij}|^2 + \mathrm{Tr}\left(-\frac{1}{4}[X_i, X_j]^2 + \frac{m^2}{2}X_i^2\right)\right\}. \quad (4)$$

We add to it the gauge-fixing and Faddeev-Popov terms $\frac{2\pi}{\Lambda} \cdot N\,\mathrm{Tr}(\frac{1}{2}[Y^{b.g.}, A]^2 - [Y^{b.g.}, b][Y, c])$. Then, the planar diagrams are the same as in the 1D theory up to a normalization factor. For example, a scalar two-loop planar diagram with a quartic interaction (see Fig. 1) is

$$\frac{d(d-1)}{2}\left(\frac{1}{2}\cdot\frac{2\pi N}{\Lambda}\right)\sum_{i,j,k=1}^{N}\frac{(\Lambda/2\pi N)}{(p_i - p_k)^2 + m^2} \times \frac{(\Lambda/2\pi N)}{(p_j - p_k)^2 + m^2} \simeq \frac{d(d-1)}{4}\cdot\frac{2\pi}{\Lambda}\cdot N^2\int_{-\Lambda/2}^{\Lambda/2}\frac{dp}{2\pi}\int_{-\Lambda/2}^{\Lambda/2}\frac{dq}{2\pi} \times \frac{1}{(p^2 + m^2)(q^2 + m^2)}. \quad (5)$$

The essence of this expression is that *the adjoint action of the background matrix can be identified with the derivative* and *the matrix elements of the fluctuations can be identified with the Fourier modes in momentum space*. The corresponding diagram in the 1D theory is

$$\frac{d(d-1)}{4}\cdot \mathrm{Vol}\cdot N^2\int_{-\Lambda/2}^{\Lambda/2}\frac{dp}{2\pi}\int_{-\Lambda/2}^{\Lambda/2}\frac{dq}{2\pi} \times \frac{1}{(p^2 + m^2)(q^2 + m^2)}, \quad (6)$$

where Vol is the spacetime volume, and hence by interpreting Λ and Λ/N to be UV and IR cutoffs, those diagrams agree up to the factor $\frac{\Lambda}{2\pi}\cdot$ Vol. The other planar diagrams also correspond up to the same factor.

The nonplanar diagrams do not have such a correspondence, but in an appropriate limit they are negligible. In the 1D theory, by taking a planar limit they are suppressed by a factor N^{-2}. In the reduced model, they are suppressed if the IR cutoff Λ/N goes to zero. To see this, let us calculate the two-loop nonplanar diagram in Fig. 1, for example. It reads $-\frac{d(d-1)}{4m^4}\frac{\Lambda}{2\pi}$, which is suppressed by a factor $(\Lambda/N)^2$ compared with planar diagrams.

Therefore, by taking the limit

$$N \to \infty, \qquad \Lambda \to \infty, \qquad \frac{\Lambda}{N} \to 0 \quad (7)$$

the 1D model on $\mathbb{R}$ is reproduced from the 0D model.

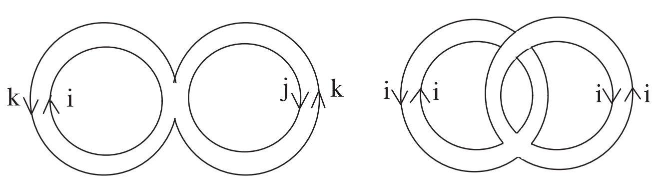

FIG. 1. Two-loop planar and nonplanar diagrams with quartic interaction vertex.

B. Eguchi-Kawai construction of Yang-Mills on S^3

Next let us construct the Yang-Mills theory on three-sphere by using the Eguchi-Kawai equivalence. The essence of QEK is to find a background whose adjoint action can be identified with the spacetime derivative. So, the strategy is *to find a set of three matrices whose adjoint action can be identified with the derivative on* S^3. Such matrices were found in [17,33]. As in the Appendix, we take the radius of the sphere to be $2/\mu$.

We introduce matrices L_i which satisfy the commutation relation of the $SU(2)$ generators,

$$[L_i, L_j] = i\epsilon_{ijk}L_k. \quad (8)$$

Since these matrices cannot be diagonalized simultaneously, we embed them in the following block diagonal form:

$$L_i = \begin{pmatrix} \ddots & & & & \\ & L_i^{[j_{s-1/2}]} & & & \\ & & L_i^{[j_s]} & & \\ & & & L_i^{[j_{s+1/2}]} & \\ & & & & \ddots \end{pmatrix}, \quad (9)$$

where $L_i^{[j_s]}$ is a $(2j_s + 1)\times(2j_s + 1)$ matrix which acts on the spin j_s representation. The size of the matrix N is

$$N = \sum_s (2j_s + 1). \quad (10)$$

We introduce a regularization by restricting the representation space to a limited number of j_s. Furthermore we take the integer s satisfying

$$-\frac{T}{2} \le s \le \frac{T}{2}, \quad (11)$$

where T is an even integer. We introduce another integer $P \ge T/2$ and take j_s to be

$$j_s = \frac{P + s}{2}. \quad (12)$$

The large-N limit is taken in the following way

$$P \to \infty, \qquad T \to \infty, \qquad N \to \infty, \qquad T/P \to 0. \quad (13)$$

By using these matrices we can relate a matrix model to a gauge theory on S^3 as follows. The action of 3D theory can be written as

$$S_{3d} = \frac{N}{\lambda_{3d}}\left(\frac{2}{\mu}\right)^3 \int d\Omega_3 \,\mathrm{Tr}\bigg(-\frac{\mu^2}{4}(\mathcal{L}_i X_j - \mathcal{L}_j X_i)^2 + \frac{\mu}{2}(\mathcal{L}_i X_j - \mathcal{L}_j X_i)[X_i, X_j] - \frac{1}{4}[X_i, X_j]^2 + \frac{\mu^2}{2}X_i^2 - i\mu\epsilon^{ijk}X_iX_jX_k + i\mu^2\epsilon^{ijk}X_i(\mathcal{L}_j X_k)\bigg), \tag{14}$$

where the derivative $\mathcal{L}_i$ is defined by (A10). This can be reproduced from the bosonic three matrix model

$$S = \frac{N}{\lambda}\left(-\frac{1}{4}[X_i, X_j]^2 - i\mu\epsilon_{ijk}X^iX^jX^k + \frac{\mu^2}{2}X_i^2\right), \tag{15}$$

where $\lambda^{-1} = (16\pi^2/\mu^3 NP)\lambda_{3d}^{-1}$ ($16\pi^2/\mu^3 NP$ is a normalization factor analogous to $\Lambda/2\pi$), by expanding the action around a classical solution

$$X_i = -\mu L_i, \tag{16}$$

and identifying $\mathcal{L}_i$ and $X_i^{(3d)}$ with the counterparts in 0D theory as

$$\mathcal{L}_i \to [L_i, \cdot], \qquad X_i^{(3d)} \to X_i^{(0d)} + \mu L_i, \tag{17}$$

and replacing the trace and the integral by trace,

$$\left(\frac{2}{\mu}\right)^3 \int d\Omega_3 \,\mathrm{tr} \to \mathrm{Tr}. \tag{18}$$

The UV and IR momentum cutoffs are given by μT and μ, respectively, and we will take the limit so that

$$\mu \to 0, \qquad \mu T \to \infty. \tag{19}$$

We also require $\mu^2 P \to \infty$ so that spacetime noncommutativity disappears (see Sec. IV).

Finally we would like to add few remarks. First, the background is a classical solution and hence as long as it is stable we do not need to quench it. Second, when we take the large-N limit fixing the IR momentum cutoff μ, to suppress the nonplanar diagrams it is necessary to change the background to $-\mu L_i \otimes \mathbf{1}_k$ and take $k \to \infty$ limit.

III. 4D $\mathcal{N} = 1$ SYM FROM MATRIX QUANTUM MECHANICS

In [17], a regularization of 4D $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ has been proposed by using the BMN matrix model [18]. In this section we will generalize this regularization to the case of $\mathcal{N} = 1$ SYM. We consider the 4-supercharge matrix quantum mechanics given by the dimensional reduction of 4D $\mathcal{N} = 1$ SYM to one dimension. We consider its BMN-like deformation [34] in order for the matrix model to have the matrices (9) as a solution. Then, applying a similar identification to the one introduced in [17], we obtain a regularization of $\mathcal{N} = 1$ SYM on $\mathbb{R} \times S^3$. This regularization keeps all 4 supersymmetries unbroken.

A. BMN-like matrix quantum mechanics

We start by considering the following matrix quantum mechanics:

$$S_0 = \frac{N}{\lambda}\int dt\,\mathrm{Tr}\bigg(\frac{1}{2}(D_t X_i)^2 + \frac{1}{4}[X_i, X_j]^2 - \frac{i}{2}\bar{\psi}\gamma^0 D_t\psi - \frac{1}{2}\bar{\psi}\gamma_i[X_i, \psi]\bigg). \tag{20}$$

Here X_i ($i = 1, 2, 3$) are $N \times N$ traceless Hermitian matrices, the covariant derivative D_t is defined by $D_t = \partial_t - i[A, \cdot]$, γ^μ are gamma matrices in four dimensions, and ψ_α are fermionic matrices with four-component Majorana spinor index α. This matrix model is obtained by dimensional reduction of 4D $\mathcal{N} = 1$ SYM to one dimension, and has 4 supercharges which correspond to 4D $\mathcal{N} = 1$ supersymmetry. We deform it by adding BMN-like terms [34],[4]

$$S = S_0 + S_m, \tag{21}$$

where

$$S_m = \frac{N}{\lambda}\int dt\,\mathrm{Tr}\bigg(\frac{i\beta}{2}\mu\bar{\psi}\gamma^{123}\psi + i\mu\epsilon^{ijk}X_iX_jX_k - \frac{\mu^2}{2}X_i^2\bigg). \tag{22}$$

The additional terms contains a "mass" parameter μ. We also introduced a constant β, which will be fixed later. It is straightforward to see that this action is invariant under the SUSY transformations

$$\delta_\epsilon A = -i\bar{\epsilon}\gamma_0\psi, \qquad \delta_\epsilon X_i = -i\bar{\epsilon}\gamma_i\psi,$$

$$\delta_\epsilon\psi = \bigg((D_t X_i)\gamma^{0i} - \frac{i}{2}\gamma^{ij}[X_i, X_j] + \frac{1}{2}\mu X_i\epsilon^{ijk}\gamma^{jk}\bigg)\epsilon. \tag{23}$$

Here ϵ is a time-dependent parameter

$$\epsilon(t) = e^{-\alpha\mu t\gamma^{0123}}\epsilon_0, \tag{24}$$

where α is a constant which satisfies $\alpha - \beta = 1$, and ϵ_0 is a constant Majorana spinor. Note that $\epsilon(t)$ satisfies the Majorana condition. In fact different choices of α and β are related by a time-dependent field redefinition [34]. As we will see, a specific choice of α and β is convenient to see the correspondence to 4D $\mathcal{N} = 1$ SYM manifestly.

It turns out that this matrix model has the fuzzy sphere solution. To see this we set $\psi = 0$ in the equations of motion

[4] In [34] more general kind of mass deformations to (20) has been studied systematically.

$$-[X_j,[X_j,X_i]]+\frac{3i}{2}\mu\epsilon^{ijk}[X_j,X_k]-\mu^2X_i=0. \tag{25}$$

Hence,

$$X_i=-\mu L_i \tag{26}$$

is a classical solution if L_i satisfies the commutation relation of the $SU(2)$ generators (8). Here we take L_i to be a matrices given in (9) in order to obtain the four-dimensional theory. Furthermore, this solution is invariant under the SUSY transformation.

B. Correspondence to 4D $\mathcal{N}=1$

Now, we consider the correspondence to the $\mathcal{N}=1$ SYM on $\mathbb{R}\times S^3$. By expanding around the solution of (26), $X_i=-\mu L_i+a_i$, the bosonic part of the action becomes

$$\begin{aligned}S_{\text{bosonic}}=\frac{N}{\lambda}\int dt\,\mathrm{Tr}\Big(&\frac{1}{2}(D_ta_i-i\mu[L_i,A_t])^2\\&+\frac{\mu^2}{4}([L_i,a_j]-[L_j,a_i])^2\\&-\frac{\mu}{2}([L_i,a_j]-[L_j,a_i])[a_i,a_j]+\frac{1}{4}[a_i,a_j]^2\\&-\frac{\mu^2}{2}a_i^2+i\mu\epsilon^{ijk}a_ia_ja_k-i\mu^2\epsilon^{ijk}a_i[L_j,a_k]\Big).\end{aligned} \tag{27}$$

We can easily see that, by formally replacing

$$\begin{aligned}&[L_i,\cdot]\to\mathcal{L}_i,\qquad A_t\to A_t(x),\qquad a_i\to X_i(x),\\&\mathrm{Tr}\to(2/\mu)^3\int d\Omega_3\,\mathrm{Tr},\qquad \lambda\to\lambda_{4d},\end{aligned} \tag{28}$$

the matrix model and the field theory can be identified. Similarly, we can identify the fermionic part of the action. The fermionic part of the matrix model can be expressed as

$$\begin{aligned}S_{\text{fermionic}}=\frac{N}{\lambda}\Big(-\frac{i}{2}\Big)\int dt\,\mathrm{Tr}(&\bar\psi\gamma^0D_t\psi\\&+\bar\psi\gamma^i(i\mu[L_i,\psi]-i[a_i,\psi])-\beta\mu\bar\psi\gamma^{123}\psi).\end{aligned} \tag{29}$$

Hence, by taking the parameters α and β of the matrix model as $\alpha=\frac{1}{4}$ and $\beta=-\frac{3}{4}$, the fermionic part of the SYM and matrix model become manifestly equivalent.

Using the replacement (28), we can see the correspondence of the SUSY transformations defined in (23) and (A14). The time dependence of the parameter ϵ is also same for the SYM on $\mathbb{R}\times S^3$ (A16) and the matrix model (24). Furthermore when we take the continuum limit, we have to scale the gauge coupling constant appropriately with the UV momentum cutoff.

Before concluding this section, an important remark is in order. In 4D $\mathcal{N}=1$ SYM, there is a $(Z_N)^4$-unbroken phase [35], that is volume independent. (For related works, see [36,37]). One may think that, because of this volume independence, 4d $\mathcal{N}=1$ SYM is related to the dimensionally reduced model (20). However, the situation is not so simple. In order for the small volume limit of 4D $\mathcal{N}=1$ to be described by (20), the Z_N symmetry must be broken [38]. If the Z_N is not broken, the derivative ∂_μ and the commutator $[A_\mu,\cdot]$ in the covariant derivative give contributions of the same order. As a consequence the Kaluza-Klein excited modes and the zero modes as well give effective masses of the same order. Therefore, even in the small volume limit, we cannot simply truncate the Kaluza-Klein modes. In the original Eguchi-Kawai model, this problem is avoided by using the unitary matrix. However it is difficult to keep supersymmetry unbroken with unitary variables. This is the reason why we have used the technique introduced in [17].

C. Absence of the sign problem

It is easy to see that the regularization described above does not suffer from the notorious sign problem after the Wick rotation. First, the bosonic part of the action is real. Therefore it is sufficient to see that the Pfaffian (or the determinant in the Weyl representation) of the Dirac operator is free from the sign problem.

When $\beta=0$, the determinant is the same as that of the undeformed model and in the nonlattice regularization method there is no sign problem [8]. The proof is a straightforward generalization of that of the zero-dimensional model [12]. For the proof, the Weyl representation is more convenient. First let us briefly summarize the proof in 0D theory. In the Weyl representation, the Dirac operator $M_{\alpha ij,\beta kl}$, which is defined by

$$\bar\psi_{\alpha ji}M_{\alpha ij,\beta kl}\psi_{\beta kl}=\mathrm{Tr}\,\bar\psi\Gamma^\mu[A_\mu,\psi], \tag{30}$$

reads[5]

$$M_{\alpha ij,\beta kl}=\Gamma^\mu_{\alpha\beta}(A_{\mu ik}\delta_{jl}-A_{\mu lj}\delta_{ik}), \tag{31}$$

where A_μ ($\mu=1,\cdots,4$) are Hermitian matrices and ψ, $\bar\psi$ are complex matrices with two-component Weyl indices. Γ^μ can be chosen as

$$\Gamma^\mu=\sigma^\mu(\mu=1,2,3),\qquad \Gamma^4=i\cdot\mathbf{1}_2, \tag{32}$$

where σ^μ are the Pauli matrices. Let $\varphi_{\alpha,ij}$ to be an eigenvector of M of the eigenvalue λ. Then, noticing that the adjoint operators $N_{\mu;ij,kl}\equiv A_{\mu ik}\delta_{jl}-A_{\mu lj}\delta_{ik}$ satisfy $N_{\mu;ji,lk}=-N^*_{\mu;ij,kl}$, and $\sigma^2\Gamma^\mu\sigma^2=-(\Gamma^\mu)^*$, we obtain

[5] Strictly speaking we have to project the $U(1)$ part because it gives zero eigenvalue of the Dirac operator, but we omit it here just for notational simplicity. In the 1D theory at finite temperature it is not necessary.

$$\sigma^2_{\alpha\alpha'} M_{\alpha' ji,\beta' lk} \sigma^2_{\beta'\beta} = (M_{\alpha ij,\beta kl})^*. \tag{33}$$

Therefore,

$$M_{\alpha ij,\beta kl}(\sigma^2\varphi^\dagger)_{\beta kl} = \sigma^2_{\alpha\gamma}(M_{\gamma ji,\beta lk}\varphi_{\beta lk})^* = \lambda^*(\sigma^2\varphi^\dagger)_{\alpha ij}, \tag{34}$$

and hence $(\sigma^2\varphi^\dagger)_{\alpha ij}$ is eigenvector of the eigenvalue λ^*. Note that they are linearly independent and hence the determinant is written as

$$\det M = \prod_i |\lambda_i|^2 \geq 0, \tag{35}$$

where i is a label for pairs of eigenvalues. It is manifestly free from the sign problem.

Next let us consider the 1D theory. First let us consider the case when $\beta = 0$ [8]. After the Wick rotation, the fermionic part of the action is

$$\int dt \, \mathrm{Tr}\left(\bar\psi D_t \psi - \sum_{\mu=1}^{3} \bar\psi \sigma^\mu [X_\mu, \psi]\right). \tag{36}$$

In the momentum cutoff prescription [8], we compactify the time direction with period $1/T$ (T is the temperature) and fix the gauge symmetry so that the gauge field becomes static and diagonal,

$$A(t) = T \cdot \mathrm{diag}(\alpha_1, \cdots, \alpha_N), \qquad -\pi \leq \alpha_i < \pi. \tag{37}$$

Furthermore we introduce the momentum cutoff $\Lambda \in \mathbb{Z}$ such that

$$\begin{aligned} X_\mu(t) &= \sum_{n=-\Lambda}^{\Lambda} \tilde X_\mu(n) e^{i\omega n t}, \\ \psi_\alpha(t) &= \sum_{r=-\Lambda+1/2}^{\Lambda-1/2} \tilde\psi_\alpha(r) e^{i\omega r t}, \\ \bar\psi_\alpha(t) &= \sum_{r=-\Lambda+1/2}^{\Lambda-1/2} \tilde{\bar\psi}_\alpha(r) e^{-i\omega r t}, \end{aligned} \tag{38}$$

where $\omega = 2\pi T$. Here n and r run integer and half-integer values, respectively. Then the fermionic part becomes

$$\tilde{\bar\psi}_{\alpha ji}(p) M_{\alpha ijp,\beta klq} \tilde\psi_{\beta kl}(q), \tag{39}$$

where

$$\begin{aligned} M_{\alpha ijp,\beta klq} &= i(p\omega - T(\alpha_i - \alpha_j))\delta_{\alpha\beta}\delta_{ik}\delta_{jl}\delta_{pq} \\ &\quad - \sum_{\mu=1}^{3} \sigma^\mu_{\alpha\beta}(\tilde X_{\mu ik}(p-q)\delta_{jl} \\ &\quad - \tilde X_{\mu lj}(p-q)\delta_{ik}). \end{aligned} \tag{40}$$

Note that M has momentum indices p and q in this case. Because $(\tilde X_{\mu ij}(p))^* = \tilde X_{\mu ji}(-p)$, the Dirac operator M satisfies

$$\sigma^2_{\alpha\alpha'} M_{\alpha' ji(-p),\beta' lk(-q)} \sigma^2_{\beta'\beta} = (M_{\alpha ijp,\beta klq})^*, \tag{41}$$

and hence if $M\varphi = \lambda\varphi$ then $M(\sigma^2\varphi^\dagger) = \lambda^*(\sigma^2\varphi^\dagger)$. Therefore the determinant of M is always equal to or larger than zero.

For generic values of β, the Dirac operator is shifted by a mass term $\beta\mu \cdot \mathbf{1}$ and the eigenvalues are shifted simply as $\lambda + \beta\mu$, $\lambda^* + \beta\mu = (\lambda + \beta\mu)^*$. Therefore the determinant is equal to or larger than zero also in this case.

The same discussion is applicable in any dimension, as long as the theory is obtained from 4D $\mathcal{N} = 1$, and hence with the momentum cutoff the sign does not appear. Of course in higher dimensions we need to use the lattice regularization and hence the positivity of the determinant is violated. However the sign problem is treatable in the following sense, at least in less than three dimensions. Suppose that one simulates the model by using the phase quenched action $S_{\mathrm{quench}} = S_{\mathrm{bosonic}} - \log|\det M|$, where $\det M$ is the fermion determinant. Then the effect of the phase factor $\det M/|\det M|$ can be taken into account by the reweighting as $\langle \mathcal{O}\rangle = \langle \mathcal{O} \cdot \mathrm{phase}\rangle_q / \langle \mathrm{phase}\rangle_q$, where $\langle\cdot\rangle$ and $\langle\cdot\rangle_q$ represent the expectation value of the original and phase quenched models, respectively. If fluctuation of the phase becomes large, both of the numerator and denominator in the right-hand side become small and the numerical error cause fatal problem. For lower dimensional theories which is obtained from 4D $\mathcal{N} = 1$ SYM, however, as one approaches to the continuum limit, the phase factor goes close to 1 for most of the configurations; see [9] for 1D and [39,40] for 2D. In such a case, the reweighting method works. In this sense the sign problem is treatable. It would be nice if similar property can be seen in the three- and four-dimensional lattices.

Although the sign problem is treatable in the sense explained above, however, at finite cutoff level small sign effect remains and hence in order to calculate the expectation value precisely one has to perform the reweighting procedure. For that, one has to calculate the fermion determinant. One of the advantages of the present method is that the positivity is kept exactly, the reweighting is not necessary and hence by using the rational hybrid Monte Carlo algorithm we do not need to calculate the determinant. This property reduces the computational cost drastically.

IV. 4D $\mathcal{N} = 1$ NONCOMMUTATIVE SYM

In this section we provide a matrix model formulation of 4D $\mathcal{N} = 1$ noncommutative SYM. First we explain how gauge theories on noncommutative space are obtained from the large-N matrix models [21]. Then we discuss the finite-N regularization.

Let us start with a bosonic D-matrix model

$$S = -\frac{1}{4g^2}\sum_{\mu\neq\nu} \mathrm{Tr}[X_\mu, X_\nu]^2. \tag{42}$$

The model has a classical solution[6]

$$X_\mu^{(0)} = \hat{p}_\mu (\mu = 1, \cdots, d),$$
$$X_\mu^{(0)} = 0 (\mu = d + 1, \cdots, D), \tag{43}$$
$$[\hat{p}_\mu, \hat{p}_\nu] = i\theta_{\mu\nu} \cdot \mathbf{1}_N,$$

where $\mathbf{1}_N$ is an $N \times N$ unit matrix. By expanding the action (42) around it we obtain the $U(1)$ noncommutative Yang-Mills theory (NCYM) on the fuzzy space $\mathbb{R}^d$ with $(D - d)$ scalar fields. The construction goes as follows: let us define the "noncommutative coordinate" $\hat{x}^\mu = (\theta^{-1})^{\mu\nu} \hat{p}_\nu$ they satisfy

$$[\hat{x}^\mu, \hat{x}^\nu] = -i(\theta^{-1})^{\mu\nu} \cdot \mathbf{1}_N. \tag{44}$$

This commutation relation is the same as for the coordinates on the fuzzy space $\mathbb{R}^d$ with noncommutativity parameter θ. As a consequence the functions of $\hat{x}$ can be mapped to functions on the fuzzy space $\mathbb{R}^d$. More precisely, we have the following mapping rule:

$$f(\hat{x}) = \sum_k \tilde{f}(k) e^{ik\hat{x}} \leftrightarrow f(x) = \sum_k \tilde{f}(k) e^{ikx},$$
$$f(\hat{x}) g(\hat{x}) \leftrightarrow f(x) \star g(x),$$
$$i[\hat{p}_\mu, \cdot] \leftrightarrow \partial_\mu, \tag{45}$$
$$\mathrm{Tr} \leftrightarrow \frac{\sqrt{\det\theta}}{4\pi^2} \int d^d x,$$

where $\star$ represents the Moyal product,

$$f(x) \star g(x) = f(x) \exp\left(-\frac{i}{2} \overleftarrow{\partial}_\mu (\theta^{-1})^{\mu\nu} \overrightarrow{\partial}_\nu\right) g(x). \tag{46}$$

Using this prescription we obtain $U(1)$ NCYM with coupling constant

$$g_{NC}^2 = 4\pi^2 g^2 / \sqrt{\det\theta}. \tag{47}$$

Similarly, by taking the background to be $A_\mu^{(0)} = \hat{p}_\mu \otimes \mathbf{1}_k (\mu = 1, \cdots, d)$ we obtain $U(k)$ NCYM. The UV cutoff is $\Lambda \sim ((N/k)\sqrt{\det\theta})^{1/d}$, and g_{NC} should be renormalized appropriately.

Next let us combine the above technique with the Eguchi-Kawai prescription. We consider the three matrix model, $D = 3$, and take the background to be

$$X_1^{(0)} = \hat{p}_{n_1} \otimes \mathbf{1}_{n_2}, \qquad X_2^{(0)} = \hat{q}_{n_1} \otimes \mathbf{1}_{n_2},$$
$$X_3^{(0)} = \mathbf{1}_{n_1} \otimes \mathrm{diag}(p_1, \cdots, p_{n_2}), \tag{48}$$

where

$$[\hat{p}_{n_1}, \hat{q}_{n_1}] = -i\theta \cdot \mathbf{1}_{n_1} \tag{49}$$

and

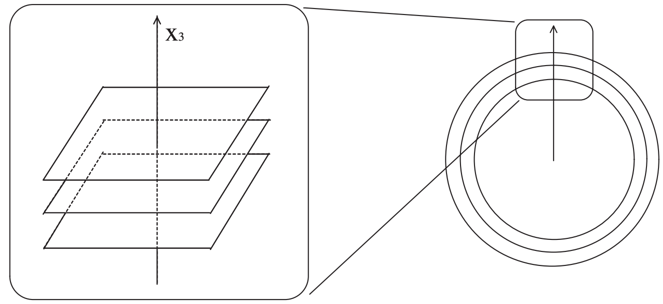

FIG. 2. Fuzzy planes extending to (x_1, x_2)-direction placed along x_3 (left) can be obtained by zooming in the north pole of a set of concentric fuzzy spheres (right).

$$p_k = \frac{\Lambda}{n_2}\left(k - \frac{n_2}{2}\right). \tag{50}$$

Intuitively, the fuzzy planes extending into the (x_1, x_2)-direction are located at each value of x_3 (Fig. 2, left). Then, in the large-N limit (n_1, $n_2 \to \infty$) we obtain two noncommutative directions using the previous construction and one ordinary (commutative) direction by the Eguchi-Kawai prescription. Note that the gauge group is $U(\infty)$.

To realize this configuration at finite-N, we can use the background (9). First let us consider the single fuzzy sphere of spin j, described by $(2j + 1) \times (2j + 1)$ matrices. By zooming in the north pole, i.e. by looking only close to $L_3 = j$, we have

$$[\mu L_1, \mu L_2] \simeq -i\mu^2 j. \tag{51}$$

Hence, we can identify μL_1 and μL_2 to be $\hat{p}$ and $\hat{q}$ with the noncommutative parameter

$$\theta = \mu^2 j. \tag{52}$$

The tangent space at the north pole, which is identified with the noncommutative plane, is located at

$$x_3 = \mu j. \tag{53}$$

In order to obtain the three-dimensional noncommutative flat space described in (48) we must consider the tangent spaces for the whole family of concentric fuzzy spheres defined in (9). These planes must be required to have the same noncommutative parameter θ, but will be placed at different positions $x_3 = \mu j_s$. In this description, the distance between the neighboring spheres μ turns into the infrared momentum cutoff along the commutative direction. Therefore, we have to take the following limit

$$\mu^2 j_s \to \theta, \qquad \mu \to 0. \tag{54}$$

Let us now consider how to impose this limit in the background (9). We first define the infinitesimal parameter ϵ as $\mu = \epsilon$. Then, $j_0 = P$ will behave as

$$P \sim \epsilon^{-2}. \tag{55}$$

[6]This solution cannot be realized at finite N.

In order for the noncommutative parameter to coincide, the maximum and minimum of $\mu^2 j_s$ must go to same value

$$\mu^2\left(P \pm \frac{T}{2}\right) \to \mu^2 P. \tag{56}$$

Hence, T must scale as

$$T \sim \epsilon^\gamma, \tag{57}$$

with $\gamma > -2$. We must also send the distance μT between the first and last tangent planes, which corresponds to the UV cutoff in the Eguchi-Kawai reduction, to infinity,

$$\Lambda \sim \mu T \to \infty. \tag{58}$$

From the previous relation we obtain $\gamma < -1$. In summary, by sending T and P to infinity in the following way:

$$\mu \sim \epsilon, \qquad P \sim \epsilon^{-2}, \qquad T \sim \epsilon^\gamma, \qquad \epsilon \to 0, \tag{59}$$

where $-2 < \gamma < -1$, we obtain the 3D (Euclidean) noncommutative space from three matrices.

Similarly, we can obtain $(1 + 3)$-dimensional $\mathcal{N} = 1$ noncommutative super Yang-Mills by applying this procedure to the BMN-like matrix model.

Stability of the background

For the bosonic models, noncommutative backgrounds which lead to the ordinary flat noncommutative space are unstable [27,29,31]. Such an instability was originally argued in [41] from the point of view of UV/IR mixing [42]. The argument in [27,31] can be applied to the present case as well and hence the 4D bosonic NCYM cannot be constructed in this way, as also expected from the NCYM calculation [41].[7]

In supersymmetric models this type of instability does not seem to exist. Given that the concentric fuzzy sphere background is supersymmetric, we can expect it to be stable in the limit discussed above at least at zero-temperature. It would be interesting to study the stability explicitly by using the Monte Carlo simulation.[8] Given that noncommutative spaces in bosonic models are unstable, as mentioned previously, we expect the existence of a critical temperature above which the noncommutative space destabilizes.

V. CONCLUSION AND DISCUSSIONS

In this article we have introduced a BMN-like supersymmetric deformation of the 4-supercharge matrix quantum mechanics which can be obtained from 4D $\mathcal{N} = 1$ SYM through dimensional reduction. By using it, we provided nonperturbative formulation of 4D $\mathcal{N} = 1$ planar SYM and 4D $\mathcal{N} = 1$ noncommutative SYM. These models can be studied numerically by using the nonlattice simulation techniques of supersymmetric matrix quantum mechanics [8].[9] An important application which we have in mind is the analysis of the finite temperature phase structure of $\mathcal{N} = 1$ SYM. In theories that have a gravity dual, as $\mathcal{N} = 4$ SYM, it is possible to infer the existence of a deconfinement phase transition at strong coupling. This is just the usual transition from pure anti-de Sitter space to a black hole that takes place on the gravity side. Using the formulation of [17], we may reproduce the transition from gauge theory. In the case of $\mathcal{N} = 1$ SYM the gravity dual is not known and as a consequence it is difficult to study the phase structure of the theory analytically. Our nonperturbative formulation allow us to study the finite temperature theory numerically and to investigate the presence of a deconfinement phase transition. However a possible subtlety is that the background (9) , which is necessary for the Eguchi-Kawai equivalence, may be destabilized before the phase transition takes place. Also, it is important to confirm that the Eguchi-Kawai construction works at strong coupling. For that, it is important to compare with the lattice calculation.

Another interesting direction is to use the formulation to obtain insights into nonsupersymmetric theories. In [45], a large-N nonsupersymmetric gauge theory with a quark in the two-index representation has been discussed. This model is interesting because it can be regarded as a certain large-N limit of one-flavor QCD, in the sense that it reduces to the ordinary one at $N = 3$. It was found that this theory can be embedded into 4D $\mathcal{N} = 1$ SYM, and a class of interesting quantities such as the fermion condensate take the same value as in the counterpart in SYM at large-N. Our formulation would be used to obtain insights into the nonsupersymmetric theory through this equivalence. In this context, it is interesting to study the nonsupersymmetric theory itself, not necessarily with one flavor, by using the Eguchi-Kawai equivalence. For such systems, two kinds of the large-N reductions—the one on S^3 [17] discussed in this paper and the usual one with unitary variables studied in [37]—are applicable. In this case a possible disadvantage of the former formulation is that it is necessary to put many fuzzy spheres on top of each other in order to stabilize the background against the repulsive force coming from many fermions. It would be nice if we could study the system by using the Eguchi-Kawai equivalence.

Another interesting direction is to formulate large-N field theories on more complicated spacetime using the

[7]Recently it has been claimed that by using Eguchi-Kawai model with double trace deformation [26] 4D bosonic NCYM can be defined [43]. If so then the deformation should eliminate UV/IR mixing somehow. It would be interesting to study this point in detail.

[8]The stability of fuzzy sphere in zero-dimensional supersymmetric matrix model has been studied by using Monte Carlo simulation in [44].

[9]In one dimension lattice simulation also works, that is, supersymmetry is restored in the continuum limit [9].

Eguchi-Kawai equivalence. For example, from the discussion in Sec. II B, one can easily see that, by restricting the spin of $SU(2)$ generators in (9) to be integer values, one obtains super Yang-Mills theory on $\mathbb{R}P^3$. By noticing the similarity of the ordinary QEK and Taylor's T-dual prescription for matrix models [46], a generalization of Taylor's work to nontrivial manifolds [47] may lead to other examples of Eguchi-Kawai construction for more complicated spaces. (Note that the original construction [17] has been obtained in this way). Generalization to 4D $\mathcal{N} = 2$ is also possible. For that purpose, we have to introduce a mass deformation which preserves eight supercharges. Such a deformation has been discussed in [34].

Furthermore the BMN-like matrix model has several applications. For example, it could be used to numerically confirm the existence of the conjectured "commuting matrix phase" in the BMN matrix model proposed in [48,49]. (Originally such a phase has been conjectured in 4D $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ [50]). Confirming the existence of this phase is important because it would enable us to study $\mathcal{N} = 4$ SYM in the strong coupling regime. This model can also be used to study the stability of the fuzzy sphere solution in a supersymmetric setup.[10] This will provide further intuition about the dynamic of the fuzzy sphere solution for the BMN matrix model.

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APPENDIX: $\mathcal{N} = 1$ SYM ON $\mathbb{R} \times S^3$

In this section, we write down the action for $\mathcal{N} = 1$ SYM on $\mathbb{R} \times S^3$ in a form convenient for our purpose [17] (a detailed discussion of SYM on curved spaces can be found in [52]). We take the radius of the sphere to be $2/\mu$. The action of $U(N)$ SYM is given by

$$S = -\frac{N}{\lambda_{4d}} \int dt \int_{S^3} d^3x \sqrt{-g(x)}\, \mathrm{Tr}\left(\frac{1}{4} F_{ab}^2 + \frac{i}{2} \bar{\psi} \gamma^a D_a \psi\right), \tag{A1}$$

where λ_{4d} is the 't Hooft coupling constant, $g_{\mu\nu}(x)$ is the metric and $g(x)$ is its determinant. The field strength is

$$F_{\mu\nu} = \partial_\mu A_\mu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \tag{A2}$$

and D_a is the covariant derivative defined by

$$D_a \psi = \partial_a \psi - i[A_a, \psi] + \frac{1}{4} \omega_{abc} \gamma^{bc} \psi \tag{A3}$$

for adjoint fermions. The Greek indices μ, ν refer to the Einstein frame and the Latin indices to the local Lorentz frame.

The sphere part of this geometry has the group structure of $SU(2)$. Given this group structure, there exists a right-invariant 1-form dgg^{-1} and dual Killing vectors $\mathcal{L}_i$, satisfying the commutation relation

$$[\mathcal{L}_i, \mathcal{L}_j] = i\epsilon_{ijk} \mathcal{L}_k. \tag{A4}$$

Using the coordinates (θ, ψ, φ) defined by $g = e^{-i\varphi\sigma_3/2} e^{-i\theta\sigma_2/2} e^{-i\psi\sigma_3/2}$, the vielbein E^i can be expressed as

$$\begin{aligned} E^1 &= \frac{1}{\mu}(-\sin\varphi d\theta + \sin\theta \cos\varphi d\psi), \\ E^2 &= \frac{1}{\mu}(\cos\varphi d\theta + \sin\theta \sin\varphi d\psi), \\ E^3 &= \frac{1}{\mu}(d\varphi + \cos\theta d\psi). \end{aligned} \tag{A5}$$

In these coordinates the metric is given by

$$ds^2 = \frac{1}{\mu^2}[d\theta^2 + \sin^2\theta d\varphi^2 + (d\psi + \cos\theta d\varphi)^2]. \tag{A6}$$

The spin connection ω_{abc} can be read off from the Maurer-Cartan equation,

$$dE^i - \omega^i{}_{jk} E^j \wedge E^k = 0, \tag{A7}$$

$$\omega_{ijk} = \frac{\mu}{2} \epsilon_{ijk}. \tag{A8}$$

And the Killing vectors are given by

$$\mathcal{L}_i = -\frac{i}{\mu} E_i^M \partial_M, \tag{A9}$$

where

$$\begin{aligned} \mathcal{L}_1 &= -i\left(-\sin\varphi \partial_\theta - \cot\theta \cos\varphi \partial_\varphi + \frac{\cos\varphi}{\sin\theta} \partial_\psi\right), \\ \mathcal{L}_2 &= -i\left(\cos\varphi \partial_\theta - \cot\theta \sin\varphi \partial_\varphi + \frac{\sin\varphi}{\sin\theta} \partial_\psi\right), \\ \mathcal{L}_3 &= -i\partial_\varphi. \end{aligned} \tag{A10}$$

The Killing vectors represent a complete basis for the tangent space on S^3. Furthermore given that the vielbeins are defined everywhere on S^3, the indices i can be used as a label for the vector fields and 1-forms.[11]

By using the Killing vectors $\mathcal{L}_i$, the bosonic part of the action can be rewritten as [17]

[10] Fuzzy sphere stability in the bosonic matrix quantum mechanics has been studied numerically in [51].

[11] This property is necessary in order to identify this index with the one in the matrix model [53].

$$S_{\rm boson}=\left(\frac{2}{\mu}\right)^3\frac{N}{\lambda_{4d}}\int dt\int d\Omega_3\,{\rm Tr}\Big(\frac{1}{2}(D_tX_i-\mu\mathcal{L}_iA_t)^2+\frac{\mu^2}{4}(\mathcal{L}_iX_j-\mathcal{L}_jX_i)^2-\frac{\mu}{2}(\mathcal{L}_iX_j-\mathcal{L}_jX_i)\times[X_i,X_j]+\frac{1}{4}[X_i,X_j]^2-\frac{\mu^2}{2}X_i^2+i\mu\epsilon^{ijk}X_iX_jX_k-i\mu^2\epsilon^{ijk}X_i(\mathcal{L}_jX_k)\Big), \tag{A11}$$

where X_i is defined in such a way that the 1-form of the gauge field on S^3 take the form $A=X_iE^i$, and $d\Omega_3$ is the volume form of the unit three-sphere. The fermionic part can be expressed as

$$S_{\rm fermion}=\frac{N}{\lambda_{4d}}\left(-\frac{i}{2}\right)\left(\frac{2}{\mu}\right)^3\int dt\int d\Omega_3\,{\rm Tr}\Big(\bar\psi\gamma^0D_0\psi+\bar\psi\gamma^i(i\mu\mathcal{L}_i\psi-i[X_i,\psi])+\frac{3}{4}\mu\bar\psi\gamma^{123}\psi\Big). \tag{A12}$$

The SUSY transformations of $\mathcal{N}=1$ SYM on curved background are given by

$$\delta A_a=i\bar\psi\gamma_a\epsilon, \tag{A13}$$

$$\delta\psi=\frac{1}{2}F^{ab}\gamma_{ab}\epsilon. \tag{A14}$$

The parameter ϵ is related to the isometry of the geometry. In the case of $\mathbb{R}\times S^3$, ϵ must satisfy

$$\nabla_a\epsilon=\frac{\mu}{4}\gamma_a\gamma^{123}\epsilon, \tag{A15}$$

which corresponds to the Killing spinor equation in supergravity. Actually, given two spinors ϵ and ζ satisfying (A15), $\bar\zeta\gamma^\mu\epsilon$ is a Killing vector. The solution to (A15) is given by

$$\epsilon=e^{-(1/4)\mu t\gamma^{0123}}\epsilon_0, \tag{A16}$$

where ϵ_0 is a constant.

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