Spherical structures in conformal gravity and its scalar-tensor extension

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We study spherically symmetric structures in conformal gravity and in a scalar-tensor extension and gain some more insight about these gravitational theories. In both cases we analyze solutions in two systems: perfect fluid solutions and boson stars of a self-interacting complex scalar field. In the purely tensorial (original) theory we find in a certain domain of parameter space finite mass solutions with a linear gravitational potential but without a Newtonian contribution. The scalar-tensor theory exhibits a very rich structure of solutions whose main properties are discussed. Among them, solutions with a finite radial extension, open solutions with a linear potential and logarithmic modifications and also a (scalar-tensor) gravitational soliton. This may also be viewed as a *static* self-gravitating boson star in purely tensorial conformal gravity.

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I. INTRODUCTION

Conformal gravity [1] (CG) was proposed as a possible alternative to Einstein gravity (''GR''), which may supply the proper framework for a solution to some of the most annoying problems of theoretical physics like those of the cosmological constant, the dark matter and the dark energy.

It is therefore very much required to investigate its predictions and consequences as far as possible. Here, we choose to concentrate on localized solutions and to start an investigation of their properties. We take two simple matter sources: perfect fluid and complex scalar field. We find localized solutions for both kinds of sources and present their main features.

The main ingredient of CG is the replacement of the Einstein-Hilbert action with the Weyl action based on the Weyl (or *conformal*) tensor $C_{\kappa\lambda\mu\nu}$ defined as the totally traceless part of the Riemann tensor (we use $R^{\kappa}{}_{\lambda\mu\nu} = \partial_\nu \Gamma^{\kappa}_{\lambda\mu} - \partial_\mu \Gamma^{\kappa}_{\lambda\nu} + \ldots$):

$$C_{\kappa\lambda\mu\nu} = R_{\kappa\lambda\mu\nu} - \frac{1}{2}(g_{\kappa\mu}R_{\lambda\nu} - g_{\kappa\nu}R_{\lambda\mu} + g_{\lambda\nu}R_{\kappa\mu} - g_{\lambda\mu}R_{\kappa\nu}) + \frac{R}{6}(g_{\kappa\mu}g_{\lambda\nu} - g_{\kappa\nu}g_{\lambda\mu}), \tag{1.1}$$

so the gravitational Lagrangian is

$$\mathcal{L}_g = -\frac{1}{2\alpha}C_{\kappa\lambda\mu\nu}C^{\kappa\lambda\mu\nu}, \tag{1.2}$$

where α is a dimensionless positive parameter. The gravitational field equations are formally similar to Einstein equations where the source is the energy-momentum tensor $T_{\mu\nu}$ and in the left-hand-side Bach tensor $W_{\mu\nu}$ replaces the Einstein tensor

$$W_{\mu\nu} = \frac{\alpha}{2}T_{\mu\nu}. \tag{1.3}$$

Bach tensor is defined by

$$W_{\mu\nu} = \tfrac{1}{3}\nabla_\mu\nabla_\nu R - \nabla_\lambda\nabla^\lambda R_{\mu\nu} + \tfrac{1}{6}(R^2 + \nabla_\lambda\nabla^\lambda R - 3R_{\kappa\lambda}R^{\kappa\lambda})g_{\mu\nu} + 2R^{\kappa\lambda}R_{\mu\kappa\nu\lambda} - \tfrac{2}{3}RR_{\mu\nu}. \tag{1.4}$$

Since the Bach tensor is traceless, the energy-momentum tensor must ''comply'' so we will only consider sources with $T^{\mu}_{\mu} = 0$.

The general spherically symmetric line element may be simplified by exploiting the conformal symmetry and has the form [1]

$$ds^2 = B(r)dt^2 - dr^2/B(r) - r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{1.5}$$

The nonvanishing components of the Ricci tensor and the Ricci scalar are

$$R^0_0 = R^r_r = -\frac{B''}{2} - \frac{B'}{r}; \qquad R^\theta_\theta = R^\varphi_\varphi = \frac{1-B}{r^2} - \frac{B'}{r};$$
$$R = \frac{2(1-B)}{r^2} - \frac{4B'}{r} - B'' \tag{1.6}$$

and those of the Bach tensor

$$W^0_0 = -\frac{1}{3r^4} + B^2\left[\frac{1}{3r^4} + \frac{1}{3r^2}\left(\frac{B''}{B} + \left(\frac{B'}{B}\right)^2 - \frac{2}{r}\frac{B'}{B}\right) - \frac{1}{3r} \times \frac{B'B''}{B^2} + \frac{1}{12}\left(\frac{B''}{B}\right)^2 - \frac{1}{6}\frac{B'B'''}{B^2} - \frac{1}{r}\frac{B'''}{B} - \frac{1}{3}\frac{B''''}{B}\right], \tag{1.7}$$

$$W^r_r = -\frac{1}{3r^4} + B^2\left[\frac{1}{3r^4} + \frac{1}{3r^2}\left(\frac{B''}{B} + \left(\frac{B'}{B}\right)^2 - \frac{2}{r}\frac{B'}{B}\right) - \frac{1}{3r}\frac{B'B''}{B^2} + \frac{1}{12}\left(\frac{B''}{B}\right)^2 - \frac{1}{6}\frac{B'B'''}{B^2} + \frac{1}{3r}\frac{B'''}{B}\right], \tag{1.8}$$

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$$W_\theta^\theta = W_\varphi^\varphi$$
$$= \frac{1}{3r^4} - B^2\left[\frac{1}{3r^4} + \frac{1}{3r^2}\left(\frac{B''}{B} + \left(\frac{B'}{B}\right)^2 - \frac{2}{r}\frac{B'}{B}\right) - \frac{1}{3r}\frac{B'B''}{B^2} + \frac{1}{12}\left(\frac{B''}{B}\right)^2 - \frac{1}{6}\frac{B'B'''}{B^2} - \frac{1}{3r}\frac{B'''}{B} - \frac{1}{6}\frac{B''''}{B}\right]. \tag{1.9}$$

A useful property of these components is the following:

$$W_0^0 - W_r^r = -\frac{B(rB)''''}{3r}; \qquad W_r^r + W_\theta^\theta = \frac{B(rB)''''}{6r}. \tag{1.10}$$

In vacuum this is easily integrated to give

$$B(r) = c_0 + c_1 r + c_2/r + \kappa r^2; \qquad c_0^2 = 1 + 3c_1c_2, \tag{1.11}$$

where the relation between the coefficients comes from the $W_r^r = 0$ equation, which is of a third order. In a nonrelativistic fourth-order gravity a similar situation is encountered, namely, the fourth-order ‘‘Poisson equation’’

$$\nabla^2\nabla^2 u = -h, \tag{1.12}$$

where $h(\mathbf{r})$ is the source term. In the spherically symmetric case $\nabla^2\nabla^2 u = (ru)''''/r$ and $u(r)$ is given also by (1.11) without any relation between the parameters. On the other hand, the parameters are related to the source (assumed to extend within $r \leq a$) by

$$c_1 = \frac{1}{2}\int_0^a r^2 h(r)dr; \qquad c_2 = \frac{1}{6}\int_0^a r^4 h(r)dr. \tag{1.13}$$

Since κ is not fixed by the source, the κr^2 term may be considered as a possible background field or in the relativistic context, a cosmological constant contribution.[1] Note also that the volume integral of the matter density (i.e. of $h(r)$) turns up as the coefficient of the *linear* term in the potential rather than the $1/r$ one. It is related to the fact that in this theory the potential of a point particle is linear in accord with the behavior of the Green function. This linear potential enables one to explain galaxy rotation curves without assuming dark matter [1,2].

For the general case of extended sources, we note that since the field equation is of fourth order, special care should be taken with the boundary conditions. It is easy to see that $u'(0)$ and $u'''(0)$ should vanish. The value of $u''(0)$ or $u''(\infty)$ may be free if solutions with a ‘‘cosmological’’ κr^2 term are allowed. If on the other hand the background is assumed to be empty (‘‘flat’’), we may impose further $u''(\infty) = 0$ as well. If the source is localized, the second derivative at the origin is related to the first moment of the matter distribution as

$$u''(0) = \frac{1}{3}\int_0^a rh(r)dr. \tag{1.14}$$

Now let us return to the relativistic field equations with a perfect fluid source described by $T_\nu^\mu = \mathrm{diag}(\rho, -P_r, -P_\perp, -P_\perp)$ (with the additional conformal condition $T_\mu^\mu = 0$). Thanks to (1.10) they reduce to a single very simple field equation

$$\frac{(rB)''''}{r} = -\frac{3\alpha}{2B}(\rho + P_r), \tag{1.15}$$

which has a similar structure to the fourth-order Poisson equation, Eq. (1.12). By comparison we notice that taking $\alpha > 0$ corresponds to gravitational attraction in the weak field limit.

Equation (1.15) should be solved together with the conservation equation

$$P_r' + \frac{1}{r}(3P_r - \rho) + \frac{B'}{2B}(\rho + P_r) = 0 \tag{1.16}$$

and an additional equation of state which relates algebraically ρ, P_r, and $P_\perp$. The regularity of the Bach tensor at the origin introduces an additional boundary condition, $B(0) = 1$ to those of the Poisson case: $B'(0) = B'''(0) = 0$, $B''(\infty) = 2\kappa$.

The inertial mass of such a spherical solution is the ordinary

$$M_I = \int d^3x\sqrt{|g|}T_0^0 = 4\pi\int_0^\infty r^2\rho(r)dr. \tag{1.17}$$

However, since the potential of a point particle in this theory is linear, the gravitational mass is identified as the coefficient of the linear term in the vacuum potential—see (1.13) and (1.15):

$$M_G = 12\pi\int_0^\infty drr^2(\rho(r) + P_r(r))/B(r) = \frac{16\pi}{\alpha}c_1. \tag{1.18}$$

The other parameter, c_2 (the coefficient of the $1/r$ term in the potential) has a dimension of length, which by utilizing Newton’s constant can be converted to a mass. However, we do not have an appropriate dimensionful parameter at our disposal, so we will call c_2 the ‘‘second mass parameter.’’ In terms of the source functions it is given by

$$c_2 = \frac{\alpha}{4}\int_0^\infty drr^4(\rho(r) + P_r(r))/B(r). \tag{1.19}$$

We will see in the following sections that this integral is not always convergent, and whenever it does, it has the wrong sign for an attractive force, causing a non-Newtonian ‘‘near field’’ of such sources. Actually, this problem that ordinary continuous sources do not produce a Newtonian component in CG was noted already by Mannheim and Kazanas [3] (following even earlier studies [4–7] from the 1960’s and 1970’s). Mannheim and Kazanas pointed out a possible solution based on the fact that a highly singular

[1]To be concrete, $R = 4\Lambda = -12\kappa$, so $\kappa > 0$ corresponds to anti-de Sitter (AdS).

source can produce a potential with both $c_1 > 0$ and $c_2 < 0$. Still, when the implications and consequences of CG are analyzed, smooth matter distributions should be considered and studied since they are more widely used to model astrophysical and cosmological sources.

II. SPHERICALLY SYMMETRIC PERFECT FLUID SOLUTIONS

In accord with our objective, which is investigating the properties of self-gravitating solutions in CG, we solved Eqs. (1.15) and (1.16) for a set of matter distributions.

The simplest of all sources is a constant energy density, $\rho(r) = \rho_0$ (for $r \leq a$ and 0 outside), but unlike the Einsteinian case, there are no finite mass solutions of this kind in our case.

The "next to simplest" source is a polytrope—either linear with $P_r = \rho/n$ or nonlinear (and anisotropic) with $P_r = P_0(\rho/3P_0)^\gamma$, where n, γ, and P_0 are all positive constants. The parameter P_0 is indeed the central value of the pressure (if $P_r(0)$ is finite). Note that the special value $\gamma = 1$ gives only the $n = 3$ case of the linear relation, which corresponds to isotropic radiation. The other values of n cannot be obtained as a limit of the nonlinear polytrope.

Next we move to general polytropes, that is, density and pressure related by

$$\rho = 3P_0(P_r/P_0)^{1+A}, \tag{2.1}$$

where for convenience we parametrize the polytropic index by $1/\gamma = 1 + A$. The construction of regular solutions for $r \in [0, \infty]$ requires the boundary conditions

$$B(0) = 1, \qquad B'(0) = 0, \qquad B'''(0) = 0, \qquad P_r(0) = P_0. \tag{2.2}$$

The fifth boundary condition was fixed by imposing the value $B''(\infty)$, which is related to the free "cosmological constant parameter" κ [see (1.11)]. The numerical results further indicate that the solutions behave asymptotically according to

$$B = \kappa r^2 + B_1 r + B_0 + \ldots, \qquad P_r \propto r^{-p}, \tag{2.3}$$

where the constant p depends on κ and on A.

We will discuss separately the solutions available for vanishing and nonvanishing κ, that is $B''(\infty) = 0$ and $B''(\infty) \neq 0$.

A. Solutions with $\kappa = 0$

By examining the conservation equation, Eq. (1.16), we obtain the physically acceptable decay of the function $P_r(r)$ in terms of the parameter A. It turns out that solutions with an asymptotically decreasing P_r can only occur for $A \geq 0$. We then get $P_r \sim r^{-2}$ for $A = 0$ and $P_r \sim r^{-7/2}$ for $A > 0$. From these observations, it follows that the inertial and gravitating mass given by (1.17) and (1.18), do not converge in the case $A = 0$ ("radiation ball"). Constant density solutions ($A = -1$) do not exist as well.

Expanding Eq. (1.16) around the origin, we further observe the following relation:

$$\left(1 - \frac{3}{2}A\right)\frac{P_r''(0)}{P_0} + 2B''(0) = 0, \tag{2.4}$$

suggesting that the value $A = 2/3$ should play a role in the solutions. Integrating the equations we obtained finite mass solutions for $0 < A < 2/3$. Figure 1(a) contains graphic representations of three solutions in this range. Actually, we solved a dimensionless version of Eqs. (1.15), (1.16), and (2.1) for B, P_r/P_0, and ρ/P_0 in terms of $x = r(\alpha P_0)^{1/4}$. It is clear from the plots that the gravitational

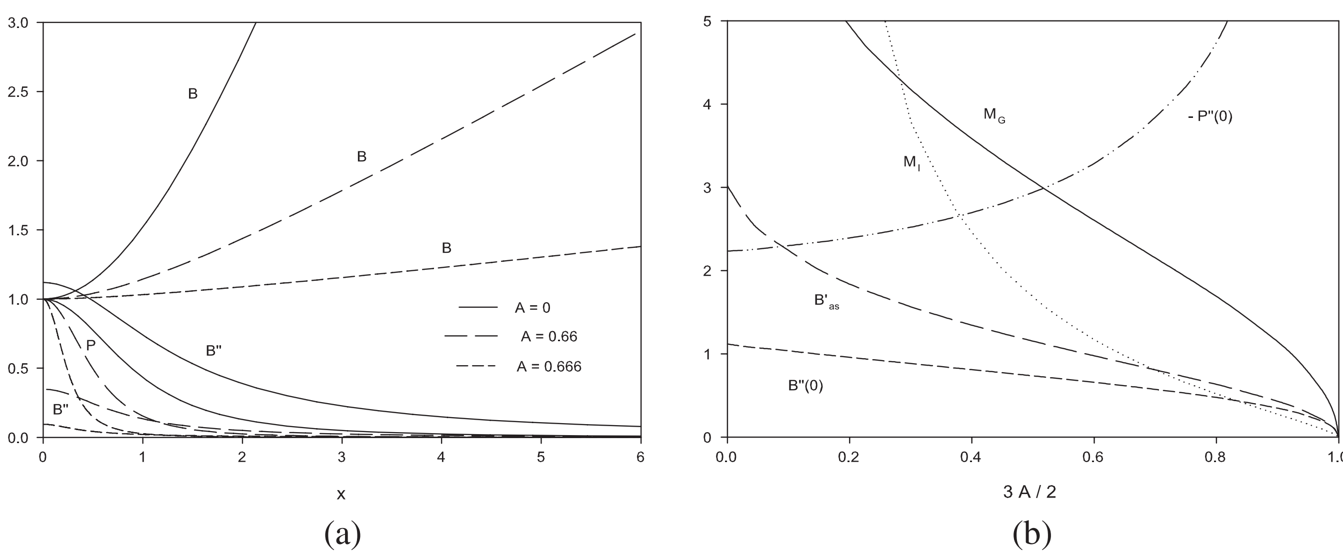


FIG. 1. Perfect fluid solutions with $\kappa = 0$: (a) the profiles of three solutions; (b) plots of several characteristics of the solutions as a function of the parameter A.

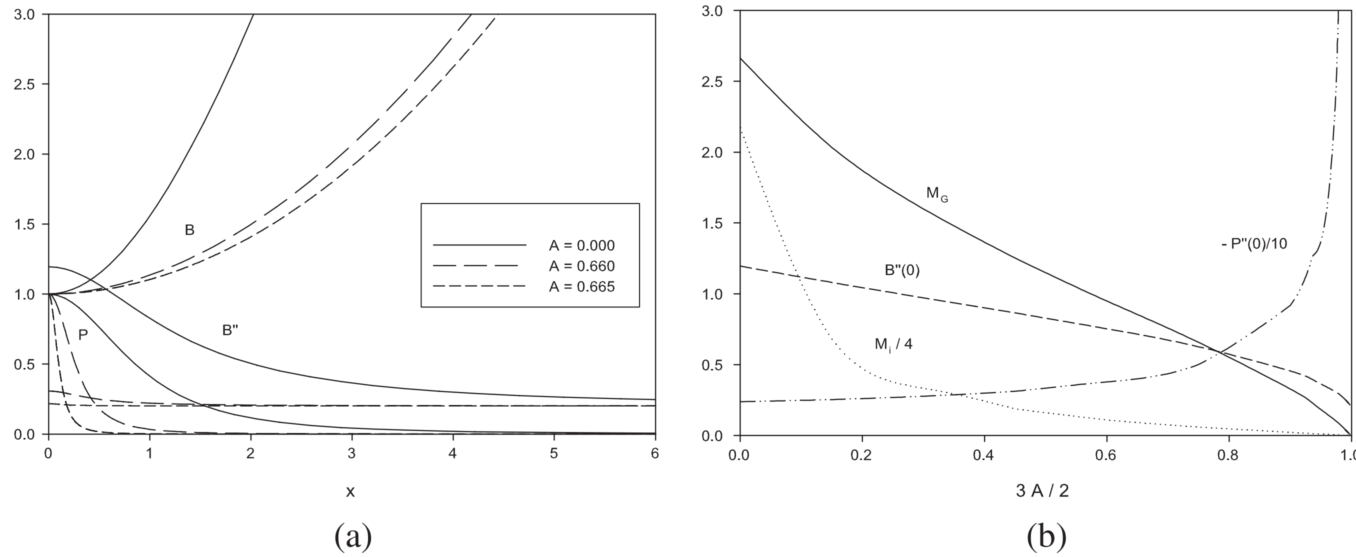


FIG. 2. Perfect fluid solutions with $\kappa = 0.1$: (a) the profiles of three solutions; (b) plots of several characteristics of the solutions as a function of the parameter A. Note that unlike the $\kappa = 0$ case, here $B''(0)$ does not vanish as $A \to 2/3$.

potential is asymptotically linear, which is the required form in order to explain the galactic rotation curves within this context [1,2]. However, closer inspection shows that the $1/r$ component, which is necessary for the recovery of the Newtonian (Schwarzschild) behavior in smaller scales, is missing. This is reflected by the fact that the coefficient of the $1/r$ term, c_2 [see (1.19)] diverges.

Several parameters characterizing the solutions [namely, the masses, the values $B''(0)$, $P_r''(0)$, and $B'(\infty)$] are depicted on Fig. 1(b); it shows, in particular, that the solution is well defined in the limit $A = 0$. In fact, in this case, we have $P_r = P_0/B^2$ but the masses are infinite. The limit $A \to 2/3$ is more subtle. It seems indeed that in this limit the function $B(r)$ approaches $B = 1$ on the full space, while the function P_r becomes more and more concentrated around the origin and $|P_r''(0)| \to \infty$. We checked that the relation (2.4) is obeyed. At the same time the masses approach zero.

It is expected to also consider the equations for $A > 2/3$. We were able to obtain solutions in this case. Our numerical results strongly suggest, however that no globally regular solutions exist there. The solutions have $B''(0) < 0$ and the function $B(r)$ approaches zero at some finite value $r = r_0$. At the same time, the pressure becomes singular for $r \to r_0$, suggesting that the solution is singular. Of course, the numerical construction of such solutions cannot be achieved directly for $r \in [0, \infty]$; in fact, we proceed on a small interval $r \in [0, r_{\max}]$ and gradually increase $r_{\max}$.

B. Solutions with $\kappa > 0$

The pattern of the solutions is very similar to the case $\kappa = 0$. In particular, regular solutions are also limited to $A < 2/3$ and singularities appear for $A > 2/3$. It is worth noticing that the conservation equation, Eq. (1.16) implies now $P_r \sim r^{-4}$ (for $A \geq 0$), so that the masses are finite in the limit $A \to 0$. Of course the values of the masses depend on the value adopted for $B''(0)$. Another difference with respect to the case $\kappa = 0$, is that now the second mass parameter [Eq. (1.19)] converges and a $1/r$ term appears in the gravitational potential, but with a wrong sign. Moreover, the convergence is related to the nonvanishing cosmological term κ, so this potential is of a universal nature rather than of local one.

In the case $\kappa < 0$, the field $B(r)$ has a node at a finite r say, $r = r_0$, which leads to a singularity of the matter function. This is just the de Sitter horizon, which is related to the fact that de Sitter space does not admit a globally static coordinate system. We will not consider the possibility of asymptotically de Sitter space further in this work apart from a few instances.

III. BOSON STARS

Among all the higher-order gravitational theories [8,9], CG is unique in the sense that it is based on an additional symmetry principle. The conformal symmetry imposes severe limitations on the allowed matter sources. When matter is described in terms of an energy-momentum tensor it should be traceless as mentioned above already. Similarly, the matter Lagrangian is very much constrained, but the Abelian Higgs model is essentially still consistent with the conformal symmetry provided the scalar field "mass term" is replaced with the appropriate "conformal coupling" term, which introduces a coupling to the Ricci scalar R. The matter Lagrangian, which we will use here, is therefore

$$\mathcal{L}_m = \frac{1}{2}(D_\mu \Phi)^*(D^\mu \Phi) - \frac{1}{12} R|\Phi|^2 - \frac{\lambda}{4}|\Phi|^4 - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}, \tag{3.1}$$

and the resulting field equations are

$$D_\mu D^\mu \Phi + \lambda|\Phi|^2\Phi + \frac{R}{6}\Phi = 0, \tag{3.2}$$

$$\nabla_\mu F^{\mu\nu} = -\frac{ie}{2}[\Phi^*(D^\nu\Phi) - \Phi(D^\nu\Phi)^*] = J^\nu. \tag{3.3}$$

The gravitational field equations are (1.3) with

$$T_{\mu\nu} = T^{(\text{minimal})}_{\mu\nu} + \frac{1}{6}(g_{\mu\nu}\nabla^\lambda\nabla_\lambda|\Phi|^2 - \nabla_\mu\nabla_\nu|\Phi|^2 - G_{\mu\nu}|\Phi|^2) \tag{3.4}$$

$T^{(\text{minimal})}_{\mu\nu}$ being the ordinary ("minimal") energy-momentum tensor of the Abelian Higgs model and $G_{\mu\nu}$ is the Einstein tensor.

The simplest spherically symmetric localized solution of this system is the boson star [10–12], which requires a global U(1) symmetry only—that is, $A_\mu = 0$ and $\Phi = f(r)e^{i\omega t}$. This yields a global conserved charge that is responsible for its existence.

The components of the energy-momentum tensor are [after use of the Φ equation, Eq. (3.2)]

$$T^0_0 = \frac{5}{6}\frac{\omega^2 f^2}{B} + \frac{B}{6}f'^2 - \frac{\lambda}{12}f^4 + \frac{B'}{12}(f^2)' + \frac{1}{18}\left(B'' + \frac{B'}{r} + \frac{1-B}{r^2}\right)f^2, \tag{3.5}$$

$$T^r_r = -\frac{1}{2}\frac{\omega^2 f^2}{B} - \frac{B}{2}f'^2 + \frac{\lambda}{4}f^4 - \frac{1}{12}\left(B' + \frac{4B}{r}\right)(f^2)' - \frac{1}{6}\left(\frac{B'}{r} - \frac{1-B}{r^2}\right)f^2, \tag{3.6}$$

$$T^\theta_\theta = T^\varphi_\varphi = -\frac{1}{6}\frac{\omega^2 f^2}{B} + \frac{B}{6}f'^2 - \frac{\lambda}{12}f^4 + \frac{B}{6r}(f^2)' - \frac{1}{18}\left(\frac{1}{2}B'' - \frac{B'}{r} + \frac{2(1-B)}{r^2}\right)f^2. \tag{3.7}$$

Since there is only one independent metric component, it is obvious that not all the field Eqs. (1.3) are independent. Actually, there is only one independent equation, and we may use the third-order one

$$W^r_r - \frac{\alpha}{2}T^r_r = 0. \tag{3.8}$$

However, a much simpler form is again obtained by using (1.10) giving therefore the following fourth-order equation for the metric component $B(r)$:

$$\frac{(rB)''''}{r} = -\frac{\alpha}{B}\left[\frac{2\omega^2 f^2}{B} + Bf'^2 - \frac{\lambda}{2}f^4 + \frac{1}{4}\left(B' + \frac{2B}{r}\right)(f^2)' - \frac{R}{12}f^2\right]. \tag{3.9}$$

For the scalar field we have the second-order equation

$$\frac{(r^2Bf')'}{r^2} + \left(\frac{\omega^2}{B} - \frac{R}{6}\right)f - \lambda f^3 = 0, \tag{3.10}$$

where one should also write explicitly $R = 2(1-B)/r^2 - 4B'/r - B''$ by (1.6).

The inertial mass and gravitational mass of these boson stars are given by equations like (1.17) and (1.18) with the necessary adaptations

$$M_I = 4\pi\int_0^\infty drr^2 T^0_0(r), \tag{3.11}$$

$$M_G = 12\pi\int_0^\infty drr^2(T^0_0(r) - T^r_r(r))/B(r), \tag{3.12}$$

where T^0_0 and T^r_r are given by (3.5) and (3.6). The second mass parameter is defined in analogy with (1.19):

$$c_2 = \frac{\alpha}{4}\int_0^\infty drr^4(T^0_0(r) - T^r_r(r))/B(r). \tag{3.13}$$

The boson star has also a global charge (particle number), which is given by

$$Q = 4\pi\omega\int_0^\infty drr^2 f^2(r)/B(r). \tag{3.14}$$

It is interesting to note that the field equations form an autonomous system as a result of the transformation

$$B(r) = V(r)r^2; \qquad f(r) = \varphi(r)/r; \qquad r = 1/u, \tag{3.15}$$

and they also simplify considerably

$$\begin{aligned} &\frac{1}{\alpha}V'''' + (\varphi')^2 - \frac{1}{2}\varphi\varphi'' + \frac{3\omega^2\varphi^2}{2V^2} = 0 \\ &(V\varphi')' + \frac{\omega^2}{V}\varphi + \frac{V''-2}{6}\varphi - \lambda\varphi^3 = 0. \end{aligned} \tag{3.16}$$

Here, of course $' = d/du$. These field equations may be obtained from the following "reduced Lagrangian":

$$L_{\text{red}} = \frac{1}{6\alpha}(V'')^2 + \frac{V}{2}(\varphi')^2 - \frac{\omega^2\varphi^2}{2V} + \frac{1}{6}\left(1 - \frac{V''}{2}\right)\varphi^2 + \frac{\lambda}{4}\varphi^4. \tag{3.17}$$

There is also a "conserved energy" K (such that $K' = 0$):

$$K = \frac{1}{6\alpha}((V'')^2 - 2V'V''') + \frac{V}{2}(\varphi')^2 + \frac{\omega^2\varphi^2}{2V} + \frac{V'}{6}\varphi\varphi' - \frac{1}{6}\varphi^2 - \frac{\lambda}{4}\varphi^4, \tag{3.18}$$

whose value is not free but fixed to be $K = 2/3\alpha$ since Eq. (3.18) is equivalent to (3.8). Moreover, if we define a third degree of freedom W, such that $W = V''$, the equations of motion can be derived from the following "ordinary" second-order Lagrangian

$$L_2 = \frac{V}{2}(\varphi')^2 - \frac{\omega^2\varphi^2}{2V} + \frac{1}{6}\varphi^2 + \frac{\lambda}{4}\varphi^4 + \frac{1}{6}V'\varphi\varphi' - \frac{1}{6\alpha}(W^2 + 2V'W'). \tag{3.19}$$

IV. BOSON STARS: NUMERICAL RESULTS

In the absence of explicit solutions (not even to the simple autonomous system), we approached the system of Eqs. (3.9) and (3.10) numerically. Using an appropriate rescaling $r \to Cr$ and $f \to Ff$, the coupling constants λ, α scale with a factor C^2F^2, while ω scales by C. Using these rescalings, we can set $\omega = \alpha = 1$ in the equations and study the solutions for several values of the coupling constant λ. If we denote by $\tilde{f}(x)$ and $\tilde{B}(x)$ the solution with $\omega = \alpha = 1$ and a given λ, the solutions with general values of ω and α and self-coupling $\alpha\lambda$ are

$$f(r) = \frac{\omega}{\sqrt{\alpha}}\tilde{f}(\omega r); \qquad B(r) = \tilde{B}(\omega r). \tag{4.1}$$

It is also easy to see that the charge Q is independent of the parameter ω and the mass scales like ω/α.

Since we chose to solve the fourth-order Eq. (3.9), Eq. (3.8) serves as a constraint. Taking the derivatives of the left-hand side of (3.8) with respect to r and eliminating the maximal derivatives B'''' and f'' by using (3.9) and (3.10), leads to an expression that vanishes identically. This implies that the two equations we chose to solve guarantee that the combination $W_r^r - \frac{\alpha}{2}T_r^r$ is constant so the constraint will be automatically fulfilled (for any r) by a consistent choice of the boundary conditions (such that the constant is 0).

We first discuss the solutions in the case where the function $B(r)$ is asymptotically linear; that is to say $B''(\infty) = 0$, or $\kappa = 0$ in (1.11). The relevant set of boundary conditions for solutions of this type is

$$\begin{aligned} &B(0) = 1, \qquad B'(0) = 0, \qquad B'''(0) = 0,\\ &f'(0) = 0, \qquad B''(\infty) = 0, \qquad f(\infty) = 0. \end{aligned} \tag{4.2}$$

For a better understanding of the numerical results, it is instructive to analyze the asymptotic possible behavior of the solutions. The asymptotic form of the B field, i.e. $B(r) \sim B_1 r$, enforces the function $f(r)$ to obey asymptotically a hypergeometric equation whose solutions are of the form

$$f(r) = \frac{F_0}{r}\sin\left(\frac{\log(\omega r)}{B_1} + \varphi\right), \qquad r \to \infty, \tag{4.3}$$

where F_0, φ are constants. As a consequence, the function $f(r)$ oscillates asymptotically and necessarily develops nodes, rendering the numerical integration technically difficult. We manage however to construct the solution by replacing the condition $f(\infty) = 0$ by $f(r_0) = 0$ imposing by hand the first zero r_0 of the function $f(r)$. Proceeding this way, we obtained strong numerical evidences that a continuum family of solutions exist, labeled by r_0. In particular, the values $B''(0)$, $f(0)$, B_1 are fixed by r_0. Unfortunately, the integrated energy density and particle number densities of these solutions behave according to

$$\int dr\frac{1}{r}(\sin(\log(\omega r)/B_1 + \varphi))^2 \sim \int dy \sin^2(y + \varphi), \qquad y = \log(\omega r). \tag{4.4}$$

The corresponding mass and particle number are then

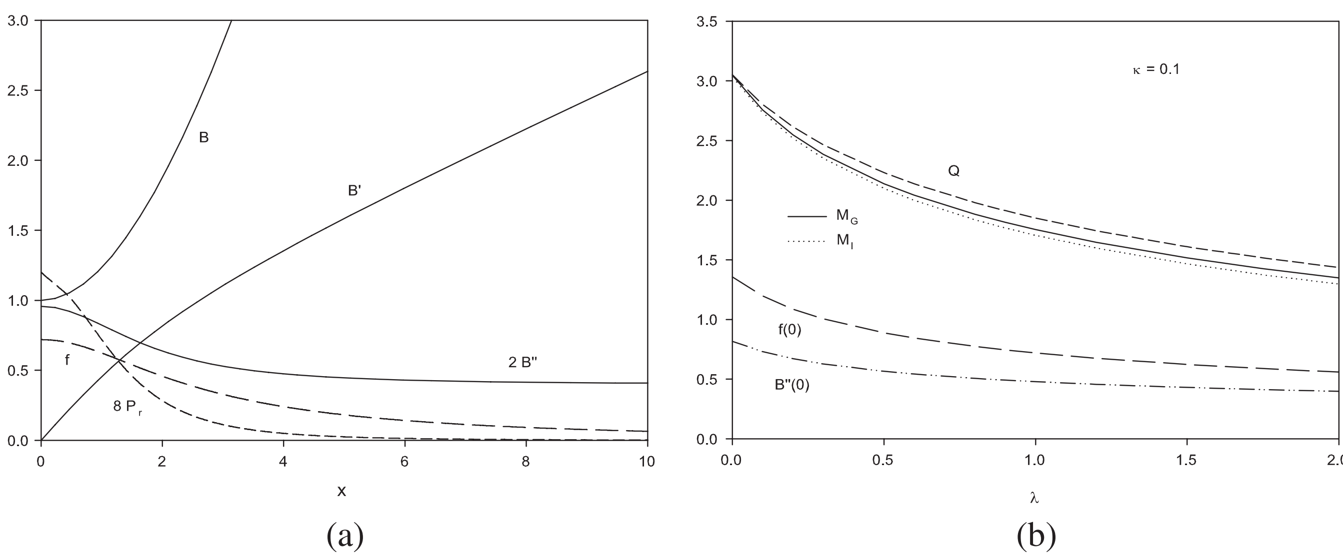


FIG. 3. Boson stars with $\kappa = 0.1$. (a) the profiles of a typical conformal boson star solution with $\lambda = 1$; (b) plots of several characteristics of the solutions as a function of λ: The inertial mass M_I and particle number Q are plotted in units 8π, while the gravitational mass M_G is given in units 4π.

infinite. In other words, in this theory boson star solutions with a linear gravitational potential do not exist .

On the other hand, boson stars exist with a quadratic gravitational potential which corresponds to an asymptotically AdS space (a negative cosmological constant). Setting $B''(\infty) = 2\kappa$, we obtain the asymptotic form

$$B(r) = \kappa r^2 + B_1 r + B_0 + \ldots, \qquad f = f_1/r + f_2/r^2 + \ldots \tag{4.5}$$

Finite mass solution needs to impose the stronger decay at infinity such that $f_1 = 0$. We obtained strong numerical evidences that such solutions exist, that is we solved numerically the field equations for a wide range of the self-coupling parameter λ. A typical profile is presented in Fig. 3(a) with $\lambda = 1$ and $\kappa = 0.1$. Several physical characteristics of the solutions are plotted for $\lambda \in [0, 2]$ in Fig. 3(b). Surprisingly, the solutions seems to persist in the absence of self-interaction ($\lambda = 0$).

Because of the asymptotic behavior mentioned above, the second mass parameter (3.13) is also finite, namely, the gravitational potential contains in this case too a wrong sign $1/r$ term.

V. SCALAR-TENSOR CONFORMAL GRAVITY

CG has been criticized from several aspects both phenomenological and formal. Several authors claim that predictions in the weak field limit disagree with solar system observations [13], yield wrong light deflection [14] (see however suggestions [15,16] for circumventing the difficulties) or more generally, the exterior solution (1.11) with $\kappa = 0$, $c_1 > 0$, and $c_2 < 0$ (which yields the desired behavior) cannot be matched to any source with a "reasonable" mass distribution [17]. In this respect we have found in the previous sections that boson stars indeed cannot produce such a behavior. On the other hand, the "anisotropic" polytropes [Eq. (2.1)] may present a linear potential, but the $1/r$ component is missing.

Other authors find evidence for tachyons or ghosts [18] or raise the fact that only null geodesics are physically meaningful in this theory since the "standard" point particle Lagrangian is not conformally invariant [19].

This last point (and possibly some of the former) can be easily corrected and can serve as a starting point for a consistent conformal theory by adding a real scalar field and turning the theory into a scalar-tensor theory. The conformally invariant point particle Lagrangian will be

$$L_{pp} = -S\sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}, \tag{5.1}$$

where S is a real scalar field with the usual conformal transformation laws. The gravitational Lagrangian (1.2) will be modified to

$$\mathcal{L}_g = \frac{1}{\alpha}\left(-\frac{1}{2}C_{\kappa\lambda\mu\nu}C^{\kappa\lambda\mu\nu} + \frac{1}{2}\nabla_\lambda S \nabla^\lambda S - \frac{1}{12}RS^2 - \frac{\nu}{4}S^4\right), \tag{5.2}$$

where ν is a possible self-coupling parameter.

The field equations will be modified accordingly. First of all, there will be an additional scalar field equation

$$\nabla_\mu \nabla^\mu S + \nu S^3 + \frac{R}{6}S = 0. \tag{5.3}$$

Second, we turn to the tensorial Eqs. (1.3). Technically, the modification is just an additional energy-momentum tensor $S_{\mu\nu}$ in the right-hand side of (1.3), namely,

$$W_{\mu\nu} = \frac{\alpha}{2}T_{\mu\nu} + \frac{1}{2}S_{\mu\nu}, \tag{5.4}$$

where

$$S_{\mu\nu} = \partial_\mu S \partial_\nu S - g_{\mu\nu}\left(\frac{1}{2}\nabla_\lambda S \nabla^\lambda S - \frac{\nu}{4}S^4\right) + \frac{1}{6}(g_{\mu\nu}\nabla^\lambda\nabla_\lambda S^2 - \nabla_\mu\nabla_\nu S^2 - G_{\mu\nu}S^2). \tag{5.5}$$

But in principle the scalar field should be considered as a gravitational degree of freedom, which is stressed by the absence of the coupling constant α in front of $S_{\mu\nu}$.

The simplest case to be studied is static spherically symmetric vacuum solutions, which within this framework are obtained by solving the following simplified version of Eqs. (3.9) and (3.10) for $B(r)$ and $S(r)$:

$$\frac{(rB)''''}{r} + \frac{1}{B}\left[BS'^2 - \frac{\nu}{2}S^4 + \frac{1}{4}\left(B' + \frac{2B}{r}\right)(S^2)' - \frac{R}{12}S^2\right] = 0$$
$$\frac{(r^2BS')'}{r^2} - \frac{R}{6}S - \nu S^3 = 0, \tag{5.6}$$

where as before, R should be expressed in terms of $B(r)$ using (1.6). Since the equation

$$W_r^r = (1/2)S_r^r \tag{5.7}$$

is of third order, it will be necessary to assure its validity, and this will be done as before by using consistent boundary conditions.

The difference with respect to the boson stars discussed above, is that now we may allow singular solutions in analogy with the Schwarzschild solution of standard GR. The no-hair theorem, which precludes black holes with scalar hair, is evidently not applicable in the present context.

Actually, one may prefer to study the system in a different gauge where by conformal transformation the scalar field is a constant, $S(x^\mu) = S_0$. This simplifies considerably the general field Eqs. (5.3), (5.4), and (5.5) and gives immediately the result $R = -6\nu S_0^2$. However, after transforming to a constant $S(x^\mu)$, one cannot use the

"Mannheim gauge" [Eq. (1.5)] anymore. The metric tensor will have two independent components, and the relatively simple expressions for the Bach tensor W^{μ}_{ν} will become quite cumbersome.

We therefore chose to stick to the "Mannheim gauge" and to use S as a second degree of freedom. On the other hand, the "effective metric" that a point particle experiences is $\bar{g}_{\mu\nu} = S^2 g_{\mu\nu}$—see (5.1). Consequently, the interpretation of the solutions is now quite different: it is now $\bar{g}_{\mu\nu}$ that has the physical significance, and the question of the gravitational potential should be answered by analyzing $\bar{g}_{00} = S^2(r)B(r)$ rather than $B(r)$.

As for the purely tensorial case with a scalar field, the vacuum scalar-tensor theory yields an autonomous system as well. We repeat the transformation (3.15) now with $\Sigma(u) = S(1/u)/u$ and get the equations of motion

$$\begin{aligned} V'''' + (\Sigma')^2 - \frac{1}{2}\Sigma\Sigma'' = 0 \\ (V\Sigma')' + \frac{V'' - 2}{6}\Sigma - \nu\Sigma^3 = 0, \end{aligned} \tag{5.8}$$

the "reduced Lagrangian"

$$L_{\rm ST} = \frac{1}{6}(V'')^2 + \frac{V}{2}(\Sigma')^2 + \frac{1}{6}\left(1 - \frac{V''}{2}\right)\Sigma^2 + \frac{\nu}{4}\Sigma^4, \tag{5.9}$$

and the "conserved energy" $K_{\rm ST}$ (now $K_{\rm ST} = 2/3$):

$$\begin{aligned} K_{\rm ST} = {}& \frac{1}{6}((V'')^2 - 2V'V''') + \frac{V}{2}(\Sigma')^2 + \frac{V'}{6}\Sigma\Sigma' \\ & - \frac{1}{6}\Sigma^2 - \frac{\nu}{4}\Sigma^4. \end{aligned} \tag{5.10}$$

The second-order Lagrangian is in this case just (3.19) without the ω term:

$$\begin{aligned} L_{2\rm ST} = {}& \frac{V}{2}(\Sigma')^2 + \frac{1}{6}\Sigma^2 + \frac{\nu}{4}\Sigma^4 + \frac{1}{6}V'\Sigma\Sigma' \\ & - \frac{1}{6}(W^2 + 2V'W'). \end{aligned} \tag{5.11}$$

VI. VACUUM SOLUTIONS

A. Schwarzschild-like solutions

Equations (5.6) possess a three-parameter family of explicit solutions given by

$$\begin{aligned} B(r) = {}& (1 + r/a)^2 - \frac{r_h}{r}\frac{(1 + r/a)^3}{(1 + r_h/a)} \\ & + \frac{\nu S_0^2 r_h^2}{2}\left(\frac{r^2}{r_h^2} - \frac{r_h}{r}\frac{(1 + r/a)^3}{(1 + r_h/a)^3}\right), \\ S(r) = {}& \frac{S_0}{1 + r/a}, \end{aligned} \tag{6.1}$$

where r_h, a, S_0 are free parameters; a and S_0 fix, respectively, the scale of the radial coordinate r and of the scalar function $S(r)$. This solution describes a family of black hole space-times with a regular horizon at $r = r_h$ satisfying $B(r_h) = 0$.

Assuming for simplicity $\nu = 0$, these solutions have the following behavior:

$$\begin{aligned} B'(r_h) = \frac{1}{a} + \frac{1}{r_h}, \qquad B(r \to 0) = -\frac{r_h}{(a + r_h)r}, \\ B(r \to \infty) = \frac{r^2}{a(a + r_h)} + O(r). \end{aligned} \tag{6.2}$$

The metric $g_{\mu\nu}$ is therefore asymptotically anti-de Sitter (or de Sitter) space with a cosmological constant $\Lambda = -3/(a(a + r_h))$. These formulas can be generalized for $\nu \neq 0$ but they become more involved; the case $\nu = 0$ is sufficient to illustrate our results. Fixing the parameters r_h and a (or Λ), the solutions (6.1) can be of three different forms according to the value of a:

(i) $a > 0$: a single horizon at $r = r_h$.

(ii) $-r_h < a < 0$: a regular horizon at $r = r_h$ hidden by a doubly degenerate horizon at $r = \tilde{r}_h > r_h$ with $\tilde{r}_h = r_h + \sqrt{r_h^2 + 16\kappa a^4}$. $\Lambda < 0$ in this case.

(iii) $a < -r_h$: a regular horizon at $r = r_h$ and, inside, a doubly degenerate horizon at $r = \tilde{r}_h < r_h$ with $\tilde{r}_h = r_h - \sqrt{r_h^2 + 16\kappa a^4}$. $\Lambda > 0$ in this case.

Note however that although $B(r)$ is quite similar to the solutions of the purely tensorial CG, the actual gravitational field "felt" by a point particle is very different, since the components of the relevant metric tensor are those of $\bar{g}_{\mu\nu} = S^2 g_{\mu\nu}$, namely, $S^2(r)B(r)$, $S^2(r)/B(r)$, and $r^2S^2(r)$. Already here we can notice that $S^2(r)B(r)$ increases with r much less steeply, and actually goes asymptotically to a constant. Moreover, the circumferential radius $rS(r)$ is also bounded. These solutions are therefore closed. On the other hand, the limit $a \to \infty$ gives rise to yet another kind of solution with constant $S(r)$ and a "purely Schwarzschild" $B(r)$. It is just a special case of a whole family of open solutions that will be discussed below in Sec. VI B 2.

B. General black hole solutions

The family of solutions discussed in the previous section are entirely determined by the scale of the scalar field and by the value of the horizon r_h and the parameter a. In particular, the values of the horizon, of the derivative $B'(r_h)$ and of the cosmological constant are not independent. However, since the equation determining the metric field is of the fourth order, more general solutions are expected. In absence of a generalization of the explicit form (6.1), we investigated the equations by numerical methods. The first step in this direction consists of establishing the most general set of appropriate boundary conditions. Prior to this step, the following scale invariance of Eqs. (5.6) has to be fixed:

$$r \to Cr, \qquad S \to \frac{S}{C}, \qquad B \to B, \tag{6.3}$$

where C is a constant. We will fix this arbitrary scale by imposing a particular value for $S_h \equiv S(r_h)$. So we define a dimensionless scalar field[2] S/S_h and a radial variable $x = r/|a|$. For the vacuum solution (6.1), this scale fixing yields the relation $S_0 = S_h(1 + x_h)$.

Solutions presenting a regular horizon at $x = x_h$ require the following conditions:

$$B(x_h) = 0, \qquad B'(x_h) = b, \qquad \mathcal{G}|_{x=x_h} = 0,$$
$$\mathcal{H}|_{x=x_h} = 0, \qquad S(x_h) = 1, \qquad B''(\infty) = B_2 \equiv 2\kappa a^2 \tag{6.4}$$

where the symbols $\mathcal{G}$, $\mathcal{H}$ represent (respectively) the conditions of regularity of Eq. (5.6) at the horizon and the constraint (5.7):

$$\mathcal{G} = 6B'S' - S\left(\frac{2}{x_h^2} - \frac{4B'}{x_h} - B''\right) - 6\nu B^3, \tag{6.5}$$

$$\mathcal{H} = 2x_h^2(4(B')^2 - S^2) + 2x_h^3(B'S^2 - 4B'B'') + x_h^4(2(B'')^2 - 4B'B''' + 2B'SS' - 3\nu S^4) - 8. \tag{6.6}$$

The normalization chosen for the field $S(x)$ in (6.4) fixes the rescaling (6.3). The constants b, B_2 are *a priori* independent. They encode the deviation with respect to the vacuum solution (6.1) where the relation between them is fixed to give

$$B_2 = \frac{2(bx_h - 1)^2}{bx_h^3}, \tag{6.7}$$

as found by eliminating the parameter a from $B'(r_h)$ and $B''(\infty)$ of (6.2). This demonstrates, in particular, that, to any positive value of B_2 (i.e. negative Λ), two solutions of the form (6.1) are available. One of these solutions presents a doubly degenerate horizon at $\tilde{x}_h = x_h + \sqrt{x_h^2 + 8/B_2}$, corresponding to $a < 0$ in (6.1).

Our numerical results show strong evidence that the analytic solutions can be deformed for generic values of b, B_2 or, put differently, are just special cases of a much wider family of vacuum solutions of the scalar-tensor conformal theory. These new solutions can be characterized by their expansion around the horizon,

$$B(x) = b(x - x_h) + \frac{b_2}{2}(x - x_h)^2 + \frac{b_3}{6}(x - x_h)^3 + \ldots,$$
$$S(x) = 1 + s(x - x_h) + O((x - x_h)^2), \tag{6.8}$$

as well as by their asymptotic behavior

$$B(x) = \frac{B_2}{2}x^2 + B_1 x + B_0 + \frac{B_-}{x} + O(x^{-2}),$$
$$S = \frac{S_1}{x} + O(x^{-2}), \tag{6.9}$$

where the parameters B_2, b have to be set in the boundary conditions, while the parameters B_1, B_0, B_-, and S_1 of the scalar field can be determined from the numerical solutions.

Because of the decay $S \sim 1/x$, the combination $BS^2(x)$ approaches asymptotically the constant $B_2 S_1^2/2$. This is encouraging since it yields a nondegenerate point particle Lagrangian [see Eq. (5.1)] in the asymptotic region.

For $B_2 = 0$, the asymptotic expansion involves "log" terms, in particular $S(x) \sim (S_1 + S_2 \log(x))/x + O(1/x^2)$ and the expansion of $B(x)$ is more involved. The black hole solution approaching a de Sitter space-time asymptotically (i.e. with negative B_2 or positive Λ) can also be constructed, presenting a cosmological horizon at some radius $x = x_c$ with $x_c > x_h$.

We now discuss the new solutions for $B_2 > 0$.

1. Case $\nu = 0$

We first discuss solutions in the case $\nu = 0$. Two such solutions are presented in Fig. 4 for $x_h = 0.5$, $B_2 = 2/(1 + x_h) = 4/3$. Here, the analytic solution corresponding to $b = 3$ is compared to the numerical solution corresponding to $b = 1$.

A natural question consists of determining the domain of existence of the solutions with a fixed x_h in the plane b, B_2. Our numerical investigations reveal that, for fixed x_h and

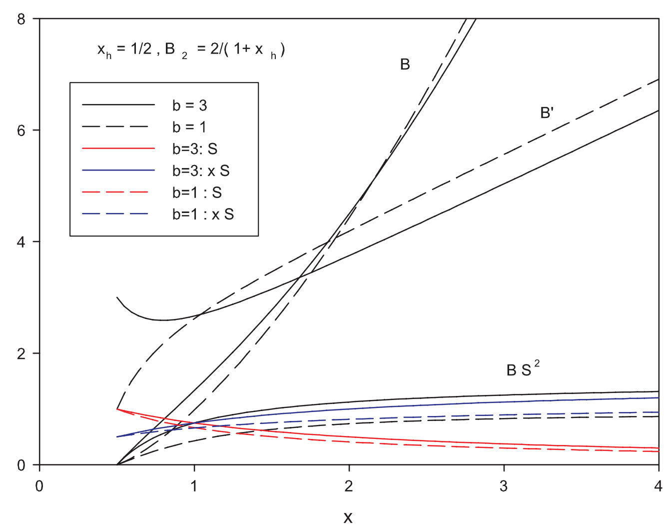


FIG. 4 (color online). First branch vacuum solutions for the case $\nu = 0$: Two profiles for $B_2 = 4/3$ with $b = 1$ and $b = 3$ (analytic solution). Color online distinguishes between the curves. In the black and white version notice that $S(x)$ curves are the two decreasing ones, while those of $B(x)S^2(x)$ start on the x axis.

[2]We will still use S for it too.

B_2, black holes exist for $0 \leq b \leq b_{\max}$ where the maximal value $b_{\max}$ depends on x_h, B_2. In the limit $b \to 0$, the horizon becomes extremal. For $x_h = 0.5$, we find $b_{\max} \approx 3.7, 3.06, 2.3$, respectively, for $B_2 = 7/3, 4/3, 0$. The values of b corresponding to the analytic solutions are

$$\begin{aligned} &b \approx \{1.17, 3.41\} \quad \text{for } B_2 = 7/3, \\ &b = \left\{\frac{4}{3}, 3\right\} \quad \text{for } B_2 = 4/3, \\ &b = 2 \quad \text{for } B_2 = 0. \end{aligned} \tag{6.10}$$

The numerical solutions therefore exist for larger values of the parameter b than the analytic ones. Table I summarizes these results.

The parameters b_2, b_3, B_1 are plotted as functions of b in Fig. 5 for $B_2 = 4/3$ (branches labeled "1"). The evolution of the parameters b_2, b_3 clearly determines the critical phenomenon stopping the solution at $b = b_{\max}$. The property that solutions do not exist for $b > b_{\max}$ suggests that a new branch of solutions should occur for $b < b_{\max}$, joining the first branch in the limit $b \to b_{\max}$. This was confirmed by the numerics: we indeed managed to construct a second family of solutions presenting this property. The corresponding data is presented in Fig. 5 by the lines labeled with a symbol "2." Decreasing the parameter b along the second branch, we observe very peculiar properties. In particular, the functions $S(x)$ and $B(r)$ stop to be monotonically decreasing, but present, respectively, a local minimum and a local maximum at two different radii, which are rather close to the horizon. For $b \to 0$, the position of the local extrema slowly move to the horizon and result in large variation of the derivatives $S'(x)$, $B''(x)$, $B'''(x)$ in the region of the horizon. The numerical results suggest strongly that the solutions tend to a configuration where $S(x)$ presents a singularity at the horizon. This appears on Fig. 5 where the parameters b_2, b_3, s are plotted as functions of b.

Profiles of three solutions of the second branch are shown in Fig. 6. Note that the effective metric exhibits a similar behavior as above and is very different from purely tensorial CG: the gravitational potential, which is encoded in $S^2(r)B(r)$, increases much less steeply and tends asymptotically to a constant, the space-time seems to have only a bounded extension since the circumferential radial distance $rS(r)$ has a finite limit as $r \to \infty$ as well as the proper radial distance $\int dr S(r)/\sqrt{B(r)}$.

TABLE I. Summary of results for the parameter b for three values of B_2.

	$B_2 = 7/3$	$B_2 = 4/3$	$B_2 = 0$
$b_{\max}$	3.7	3.06	2.3
$b_{\text{analytic1}}$	3.41	3	2
$b_{\text{analytic2}}$	1.17	4/3	2

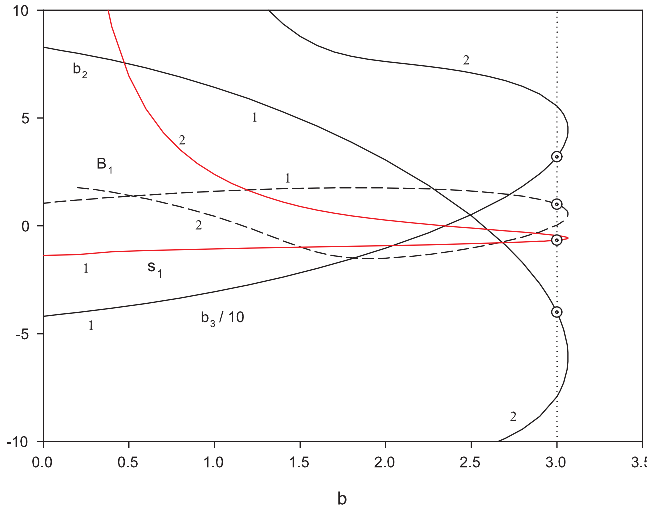


FIG. 5 (color online). The two branch structure of the $\nu = 0$ solutions. The bullets indicate the corresponding analytic solutions. For the meaning of the various parameters, see the text.

2. Case $\nu \neq 0$

For $\nu \neq 0$, the analytic solutions with fixed x_h, B_2 are real as long as the condition

$$B_2^2 x_h^2 + 6B_2 \nu x_h^2 + 8B_2 - 3\nu^2 x_h^2 - 8\nu \geq 0 \tag{6.11}$$

holds. This defines bounds of the parameter ν. We have also tried to deform the numerical solutions available for $\nu = 0$ to the case $\nu \neq 0$. The features of the solutions are basically similar (see Fig. 7). Keeping the parameters b, B_2 fixed and increasing ν, it turns out that the coefficient S_1 of the scalar field [defined in Eq. (6.9)] increases rapidly and diverges when the coupling constant ν approaches a critical value. For example, setting $b = 1$, $B_2 = 4/3$, we find that the main branch exists for $0 \leq \nu \leq 1.65$, while the second branch exists for $0 \leq \nu \leq 0.075$.

Contrary to our expectation, the two solutions available for $\nu = 0$ do not converge to a common solution while increasing ν gradually.

All of the above-mentioned solutions are closed ones. However, other kinds of black hole solutions exist, which are open and thus of course much more relevant in order to deal with the astrophysical and cosmological issues discussed in the introduction. These solutions are characterized (in addition to the horizon) by the nonvanishing value of $S(\infty)$, which fixes the cosmological constant parameter by $\kappa = \nu S^2(\infty)/2$. They have the asymptotic behavior

$$\begin{aligned} &B(r) \sim \kappa r^2 + (B_2 \log(r) + B_1)r + B_0, \\ &S(r) \sim \sqrt{\frac{2\kappa}{\nu}} + S_1 \frac{\log(r)}{r}, \end{aligned} \tag{6.12}$$

where B_0, B_1, B_2, S_1 are constants. Figure 8 contains a graphic representation of typical solutions.

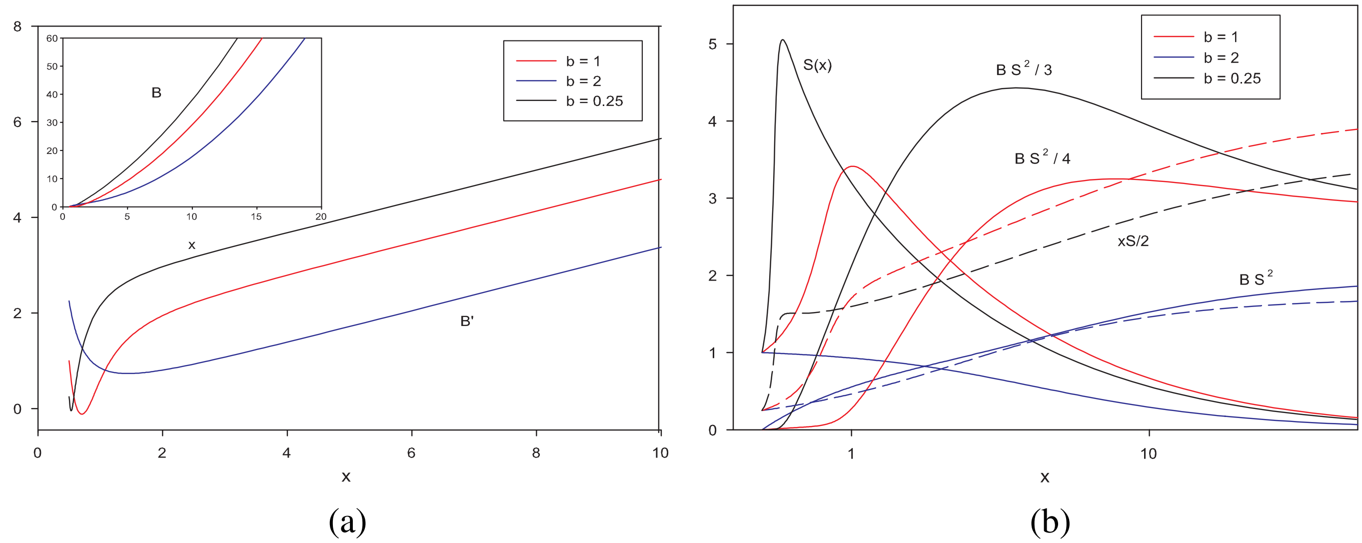


FIG. 6 (color online). Second branch vacuum solutions for the case $\nu = 0$: Three profiles for $B_2 = 1/3$ with $b = 2$, $b = 1$, and $b = 0.25$. (a) curves for $B(x)$ and $B'(x)$; (b) curves for $S(x)$, $xS(x)$, and $S^2(x)B(x)$.

C. Regular solutions

Apart from the black hole solutions discussed in the previous section, the system (5.6) also admits regular solutions in $r \in [0, \infty]$, which may be viewed as gravitational solitons in this scalar-tensor theory. Taking advantage of the symmetry (6.3), the boundary conditions can all be fixed to be

$$B(0) = 1, \qquad B'(0) = 0, \qquad B'''(0) = 0, \qquad S'(0) = 0, \qquad B''(\infty) = 2\kappa \tag{6.13}$$

with $S(0)$ as an additional input, which we will allow to vary in a certain range. It turns out that a family of solutions exists, labeled by $S(0)$. The solutions behave asymptotically according to (6.9), and several profiles of such solitons are presented in Fig. 9.

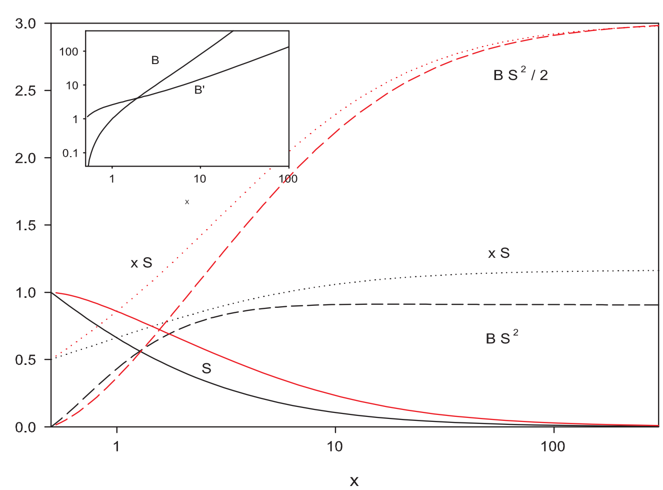


FIG. 7 (color online). A typical solution with $\nu > 0$ (online red) compared with a $\nu = 0$ (online black) solution.

Actually, this kind of solution must also exist for the boson star system of Sec. III as a completely static self-gravitating solution—a novelty in the CG with no analogue in standard GR. We have not found them at this time because we used ω in order to rescale the dimensional variables. Note however that the interpretation of these static solutions is very different in both cases: Here, it is a gravitational soliton in a scalar-tensor theory, while the same solution within the purely tensorial CG describes a self-gravitating scalar field, i.e. boson star, with the peculiar property that it is purely static. The dotted line in Fig. 13 below shows the (inertial) mass of such a static boson star as a function of the central value of the scalar field $S(0)$. The oscillatory curve is very similar to the one found for the usual boson stars in GR [10–12].

The striking feature about these regular solutions is that the field B deviates only a little from the form $B(r) = 1 + \kappa r^2$. The numerical results indicates that the difference $1 - (B''(\infty)/B''(0))$ is positive and of the order of a few percents (we checked that this is not a numerical artefact). As a consequence, these regular solutions are essentially characterized by their cosmological constant.

If we examine the family of black holes with a fixed Λ and decreasing r_h, it turns out that the maximal value $b_{\max}$ of the parameter b increases. The numerical results then strongly suggest that the profile of the regular solution is approached on the interval $r \in]0, \infty[$ by the black holes corresponding to the second branch. The convergence cannot be extended toward the point at the origin because of the different condition of the metric field: $B(r_h) = 0$ for black holes, $B(0) = 1$ for the soliton.

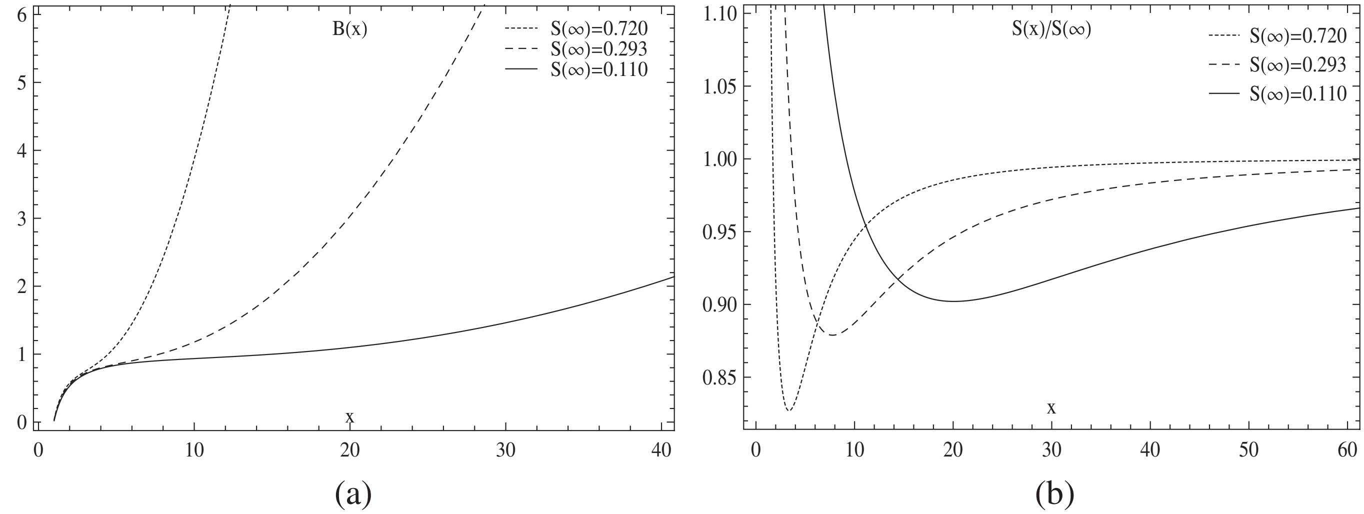


FIG. 8. Open solutions of the vacuum scalar-tensor system with $\nu = 0.2$ for three asymptotic values of the scalar field.

VII. PERFECT FLUID SOLUTIONS IN SCALAR-TENSOR CONFORMAL GRAVITY

Having investigated the scalar-tensor vacuum solutions and especially obtaining open solutions, we now proceed to couple matter sources to this system. The first is the perfect fluid with a polytropic equation of state. In this case we are confronted with the following set of equations:

$$\frac{(rB)''''}{r} + \frac{1}{B}\left[BS'^2 - \frac{\nu}{2}S^4 + \frac{1}{4}\left(B' + \frac{2B}{r}\right)(S^2)' - \frac{R}{12}S^2\right] = -\frac{3\alpha}{2B}(\rho + P_r), \tag{7.1}$$

$$\frac{(r^2BS')'}{r^2} - \frac{R}{6}S - \nu S^3 = 0, \tag{7.2}$$

which are supplemented by the conservation law (1.16).

We have found two types of regular solutions to the equations above distinguished by $S(\infty)$ being either zero or nonzero. This corresponds to closed or open space-time geometries, respectively. For the two cases, the field $B(r)$ satisfies the same boundary conditions as in the purely tensorial conformal gravity, namely,

$$B(0) = 1, \qquad B'(0) = 0, \qquad B'''(0) = 0, \qquad B''(\infty) = 2\kappa. \tag{7.3}$$

The boundary conditions on the function $S(r)$ are different

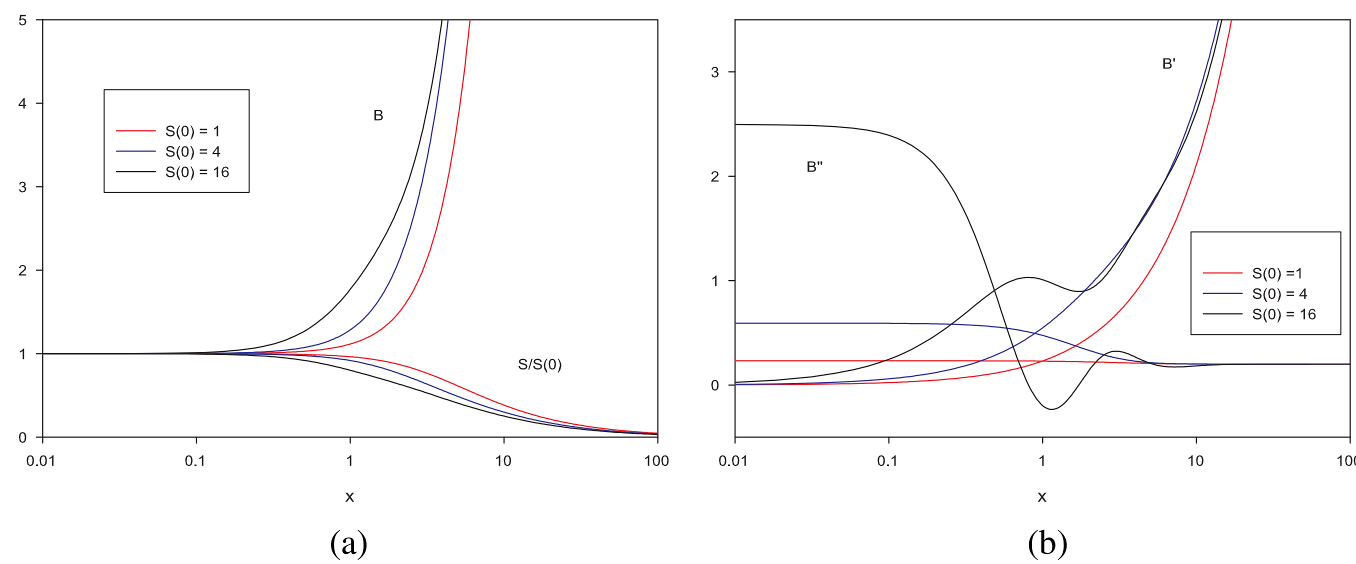


FIG. 9 (color online). Several profiles of the regular solutions of the scalar-tensor system with $\nu = 0$ for different central value of the scalar field: $S(0) = 1$, $S(0) = 4$, $S(0) = 16$. Notice the oscillations for large $S(0)$.

for the two solutions. Setting $\nu = 0$, we find solutions with the asymptotic behavior

$$B(r) \sim \kappa r^2 + B_1 r + B_0, \qquad S(r) \sim \frac{S_1}{r}, \qquad P_r(r) \sim \frac{P_1}{r^4}. \tag{7.4}$$

The space-time associated with these solutions is closed since the function $rS(r)$ varies on a finite range. Similarly, the proper radial distance is bounded from above. The solutions of this type can be deformed for $\nu > 0$; however, they do not exist for large values of ν. The function $S(r)$ indeed develops a singularity at a finite value of r when ν approaches a critical value $\nu = \nu_c$.

The solutions of the second type that we constructed are open and exist for generic nonzero values of the coupling constant ν; they are characterized by $S(\infty) > 0$ and obey asymptotically

$$\begin{aligned} &B(r) \sim \kappa r^2 + (B_2 \log(r) + B_1) r + B_0, \\ &S(r) \sim \sqrt{\frac{2\kappa}{\nu}} + S_1 \frac{\log(r)}{r}, \qquad P_r(r) \sim \frac{P_1}{r^4}, \end{aligned} \tag{7.5}$$

where B_0, B_1, B_2, S_1 are constants. This form was checked both analytically and numerically. The corresponding space-time is open since $rS(r)$ is unbounded from above. Let us point out two features of these solutions: (i) They do not possess a regular limit for $\nu \to 0$, as seen, e.g. from $S(\infty) = \sqrt{2\kappa/\nu}$. (ii) Non-analytical terms (log terms) appear in the asymptotic expansion of the fields B and S. These terms seem to be related to the fact that the field S does not go to zero for $r \to \infty$.

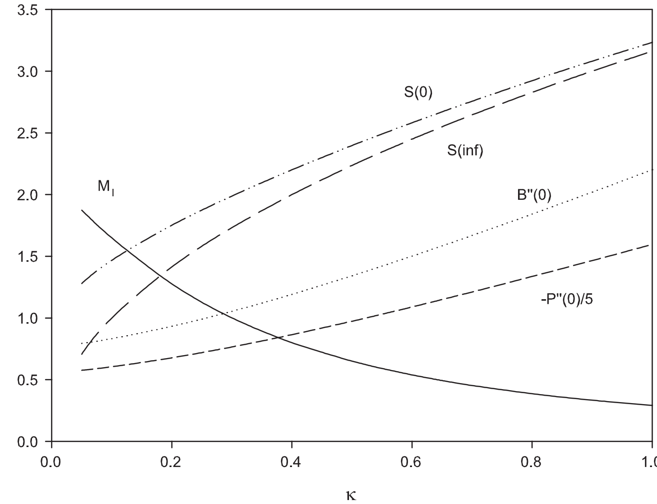


FIG. 11. Perfect fluid solutions in the scalar-tensor theory: dependence on the parameter κ. The other parameters are $\nu = 0.2$, $A = 1/3$.

Figure 10(a) shows the profiles of two solutions with $\nu = 1$. The functions $B(r)$ and $P_r(r)$ are quite similar to those in the "pure tensor" theory. The main difference is that point particles are now consistently coupled to the gravitational field through the new field $S(r)$. The coupling is described now by the combination $S^2(r)B(r)$, and it is obvious that besides the cosmological r^2 behavior, we recover the linear potential with logarithmic modifications. Further study is required in order to check the relation of this new kind of solution to observational data. Figure 10(b) presents the dependence on the polytropic index A of the main properties of the solutions. These properties as a function of the cosmological constant parameter κ are shown in Fig. 11.

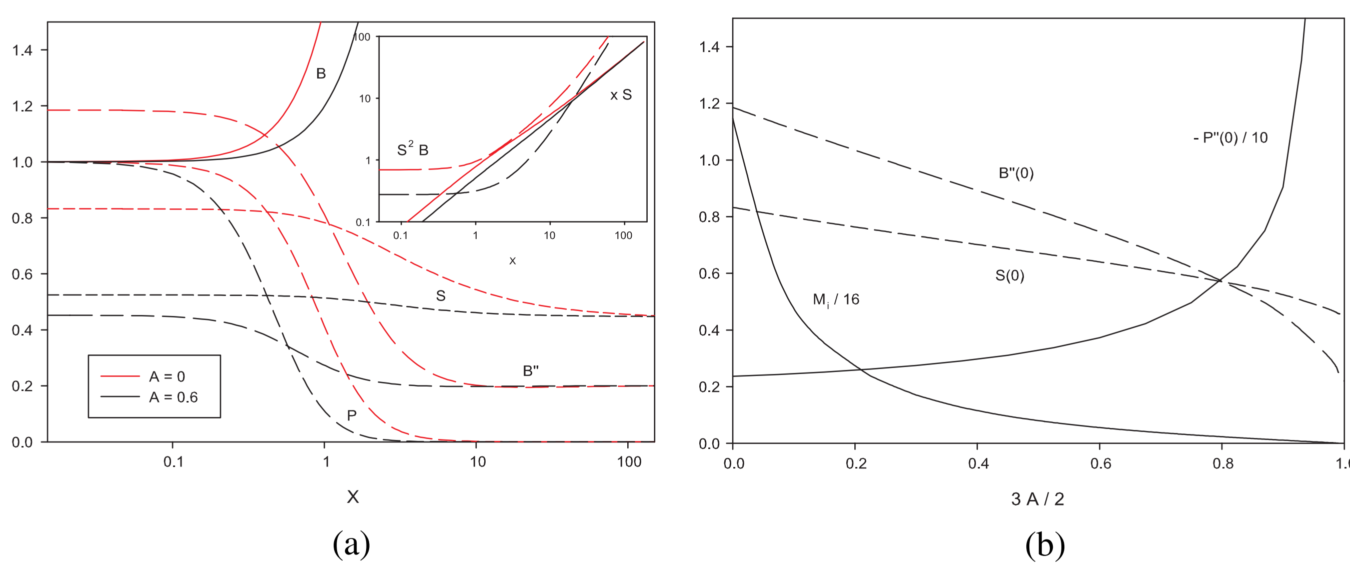


FIG. 10 (color online). Perfect fluid open solutions in the scalar-tensor theory with $\nu = 1$. $\kappa = 0.1$: (a) the profiles of two solutions; (b) plots of several characteristics of the solutions as a function of the parameter A.

VIII. BOSON STARS IN SCALAR-TENSOR CONFORMAL GRAVITY

Next we move to the complex scalar field, i.e. boson stars. In this case, the Lagrangian density is the sum of all the previous terms with a possible additional coupling between the "gravitational scalar field" $S(x)$ and the other scalar Φ, namely, $-\mu S^2|\Phi|^2/2$ with μ dimensionless (real) parameter. The field equations will contain two second-order equations for the two scalar fields:

$$\frac{(r^2Bf')'}{r^2} + \left(\frac{\omega^2}{B} - \frac{R}{6} - \mu S^2\right)f - \lambda f^3 = 0, \tag{8.1}$$

$$\frac{(r^2BS')'}{r^2} - \left(\frac{R}{6} + \alpha\mu f^2\right)S - \nu S^3 = 0, \tag{8.2}$$

and a fourth-order equation for $B(r)$

$$\begin{aligned}\frac{(rB)''''}{r} &+ \frac{1}{B}\left[BS'^2 - \frac{\nu}{2}S^4 + \frac{1}{4}\left(B' + \frac{2B}{r}\right)(S^2)' - \frac{R}{12}S^2\right] \\ &= -\frac{\alpha}{B}\left[\frac{2\omega^2f^2}{B} + Bf'^2 - \frac{\lambda}{2}f^4 - \mu S^2f^2 \right.\\ &\quad \left. + \frac{1}{4}\left(B' + \frac{2B}{r}\right)(f^2)' - \frac{R}{12}f^2\right].\end{aligned} \tag{8.3}$$

The space of solutions is quite large and defined by two types of parameters: those which appear in the field equations, namely, ν, λ and μ and parameters (integration constants) which specify the solutions like $S(\infty)$, κ etc.

A systematic survey of all possible solutions is beyond the scope of this work. Here, we present the main properties of several families of solutions in limited but typical regions of parameter space. Here, too we find closed as well as open solutions.

We addressed the system (8.1), (8.2), and (8.3) numerically. We started by fixing the different coupling constants according to $\nu = \mu = 0$, the constant α can then be set to $\alpha = 1$ by a rescaling of f. A rescaling of the radial variable and of the field $S(x)$ allows one to fix $\omega = 1$. In the reduced system fixed this way, we further assumed $\lambda = 1$. Regular solutions to the equations can then be constructed with the following boundary conditions:

$$B(0) = 1, \qquad B'(0) = 0, \qquad B'''(0) = 0, \qquad B''(\infty) = 2\kappa \tag{8.4}$$

for the metric function, and

$$S(0) = S_0, \qquad S'(0) = 0, \qquad f'(0) = 0, \qquad f(r \to \infty) = \frac{f_2}{r^2} \tag{8.5}$$

for the two scalar functions. Here, S_0 is an arbitrary constant. In the numerical analysis, we set $\kappa = 0.1$. In the limit $S_0 \to 0$, we have $S(x) = 0$ and the boson star solutions of Sec. IV are recovered. We have studied how the boson star solution available in CG is deformed by the additional scalar field and discovered a rather unexpected pattern, which we now discuss.

A. Closed solutions

Increasing the parameter S_0 gradually, we observe that the boson star gets continuously deformed by the new scalar field. It turns out that the scalar field of the boson star tends uniformly to the null function for some critical value $S_0 = S_c$; with our values of the coupling constants,

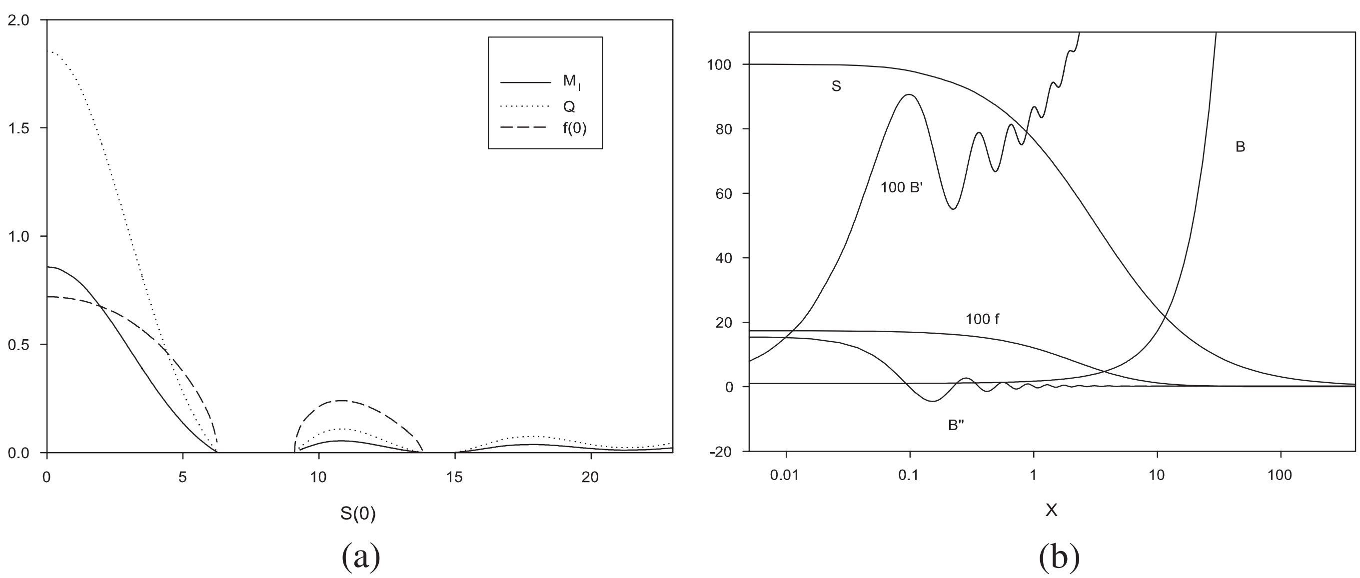


FIG. 12. Boson stars in the scalar-tensor theory corresponding to $\lambda = 1$, $\nu = \mu = 0$, $\kappa = 0.1$. (a) mass, particle number Q and the value $f(0)$ as a function of $S(0)$; (b) details of the profile for $S(0) = 100$. The high value of $S(0)$ was chosen to get noticeable oscillations—see the text.

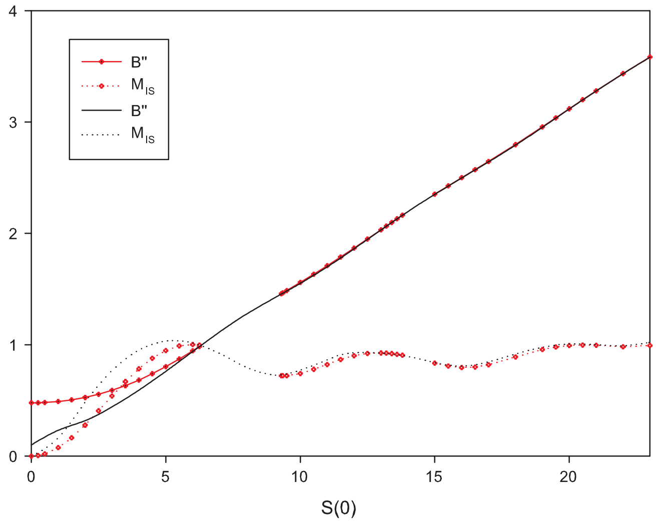


FIG. 13 (color online). The value $B''(0)$ and the inertial mass of the field S for STBS (online red) and STR soliton (online black) as functions of $S(0)$. Notice the gaps in the STBS (red) curves where no solutions exist. This can be used to distinguish between the cases in a black and white plot.

we find $S_c \approx 6.26$. Accordingly, the mass corresponding to the boson star (i.e. supported by the field $f(r)$) tends to zero in this limit, along with the particle number Q. These features are illustrated by Fig. 12(a). So, for $0 \leq S_0 \leq S_c$ two regular solutions coexist: the scalar-tensor boson star (STBS) and the scalar-tensor regular solution (STR—the gravitational soliton) with $f(r) = 0$. The values of $B''(0)$ and the inertial mass M_{IS} of the scalar field S (only) are represented in Fig. 13.

However, this is not the end of the story. Indeed, while we continue to increase S_0, it turns out that STBS solutions reappear for $S_0 > 9.1$ and that again, the two non-trivial solutions exist for $9.1 < S_0 < 13.8$. After still another gap of the nonexistence of STBS solutions, they again reappear for $S_0 > 15$ and seem to then coexist with the STR (regular) solution as suggested by Figs. 12(a) and 13.

It is challenging to find a full analytical explanation of the features just discussed above, and so far we have not fully succeeded. Possibly, an explanation is to be found in the fact that for large values of $S(0)$ the function $B(r)$ develops some oscillations near the origin. These oscillations, which appear more clearly when looking at $B'(r)$ and $B''(r)$ [see Fig. 12(b)], are due to a term

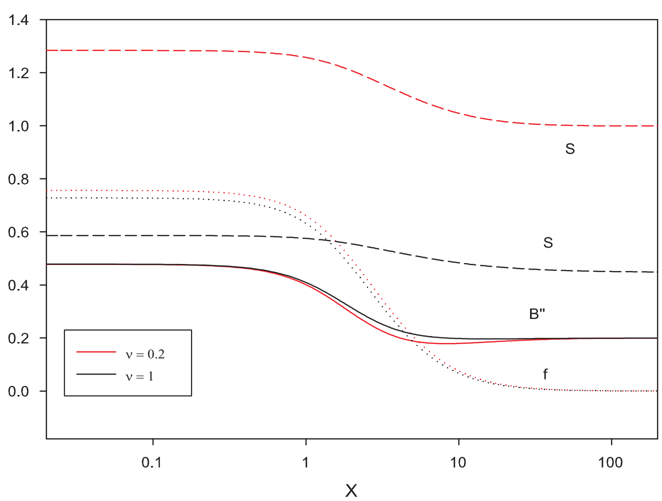
$$Y'' + \frac{S^2}{B} Y + \text{subdominant terms} = 0, \qquad Y \equiv B'' \tag{8.6}$$

contained in the equations, which leads to visible oscillations when S_0 is sufficiently large. A detailed numerical study of the "gap-structured" phenomenon of the STBS solution shows that (i) the function $B''(r)$ is monotonically decreasing for $S_0 < 6.26$; (ii) a local minimum of B'' occur somewhere for $S_0 \in [9.1, 13.8]$. This strongly suggests a connection between the oscillations of the function B'' and the pattern of STBS solutions.

In fact, the occurrence of oscillations appears for the STR solution already, i.e. in the absence of a boson star. This property is illustrated in Fig. 9, where the profiles of B, S, B', B'' are superposed for three values of $S(0)$.

B. Open solutions

Finally, we turn to very different kind of solutions, namely, the open ones. Along with the case of polytropes discussed in Sec. VII, the open solutions are characterized



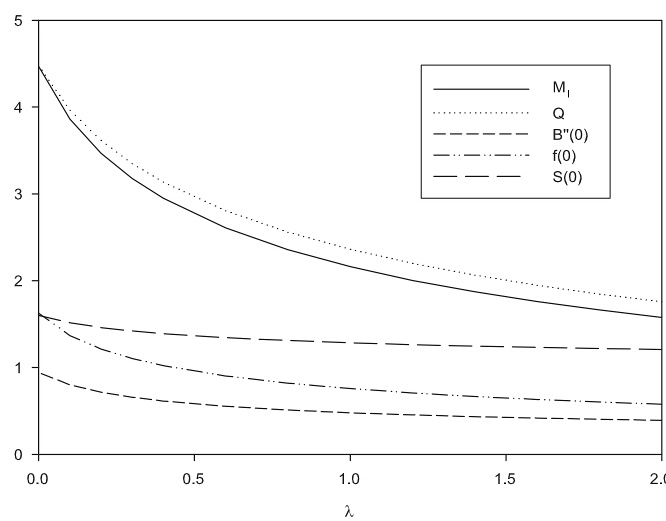


FIG. 14 (color online). Open boson star solutions in the scalar-tensor theory: (a) two profiles for $\nu = 0.2$ and $\nu = 1$ with $\lambda = 1$, $\mu = 0$, $\kappa = 0.1$; (b) plots of several characteristics of the solutions as a function of the parameter λ with $\nu = 0.2$, $\mu = 0$, $\kappa = 0.1$.

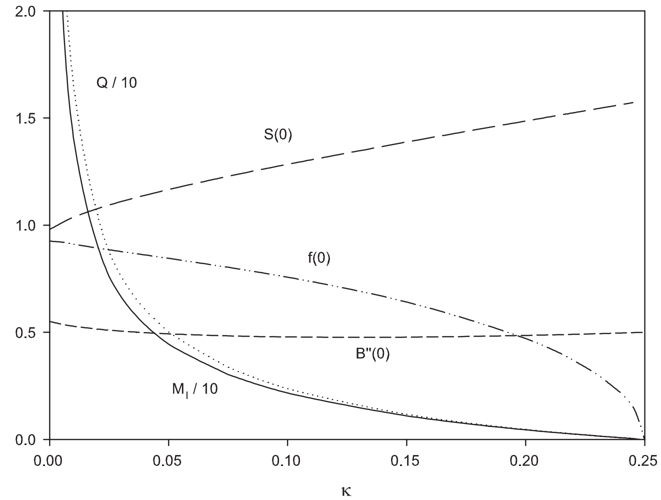


FIG. 15 (color online). Open boson star solutions in the scalar-tensor theory: dependence on the parameter κ. The other parameters are $\lambda = 1$, $\nu = 0.2$, $\mu = 0$.

by $S(\infty) = \sqrt{2\kappa/\nu}$, and the corresponding asymptotic expansion presents log terms. The pattern of solutions of these nonlinear equations is rich and presents several bifurcations in the space of coupling constants. Fixing the different coupling constants, the solutions are even not unique since they are characterized by parameters at the boundary (or integration constant like κ or $S(0)$), generating continuous families of solutions.

Figure 14(a) shows two typical open boson star profiles for two values of ν. The quite weak dependence on ν makes it sufficient to present in Fig. 14(b) the dependence of the mass, charge, and other quantities on the self-coupling λ. On the other hand, the physical quantities are quite sensitive to the parameter κ as is clearly apparent from Fig. 15. We see that boson stars exist only up to a maximal value of κ and that the STBS system bifurcate into a regular STR.

IX. CONCLUSION

We have analyzed several types of spherically symmetric solutions in the minimal CG and in a scalar-tensor extension. The polytrope solutions in the minimal case have an asymptotically linear gravitational potential and contain a ‘‘wrong sign’’ Newtonian component of a $1/r$ term only in asymptotically anti-de Sitter space, which is generated by a ‘‘cosmological integration constant’’ (κ). These are in line with previous results showing that the generic exterior gravitational fields in this theory have a behavior that is very difficult to settle with observations.

The linear potential may be considered an advantage in the sense that applying it in a galactic scale may provide an explanation for the rotation curves without invoking dark matter. However, since there is no *a priori* reason to refrain from trying CG in a solar system scale, the absence of the (attractive) Newtonian potential seems to be a drawback. Of course, it is always possible to claim that ‘‘conformal polytropes’’ made of matter, which satisfies $T^\mu_\mu = 0$, is a rather special kind of material that may be found in galactic and intergalactic scale, but this line of argument effectively brings dark matter through the ‘‘back door.’’

From this point of view, a scalar field can be viewed as a more conventional matter source, as scalar fields are ubiquitous in theoretical physics. Since the conformal coupling to gravity adds to the energy-momentum tensor terms, which render the energy density non positive definite, it might be expected that the gravitational fields of boson stars turn out to have different behavior. However, we found that boson stars exist only in (asymptotically) anti-de Sitter space, otherwise the mass and charge of the solutions do not converge. Similarly, the gravitational potential of these boson stars is linear with a $+1/r$ additional contribution.

The motivation for the scalar-tensor extension was to search for possible different behaviors. Moreover, from a formal point of view the scalar-tensor CG is the simplest theory of its kind, which has physically meaningful timelike geodesics, or in other words, couples consistently to point particles.

The pattern of the solutions we have found in this case turns out to be very rich and presents unexpected features, especially with respect to the purely tensorial CG. First, in the vacuum sector we found Schwarzschild-like (or more accurately, Schwarzschild–AdS-like) solutions as well as closed solutions with finite radial extension. In addition, there exist regular (‘‘solitonlike’’) solutions, which have no analogue in ordinary GR.

When matter sources are added, the resulting solutions are classified similarly for both perfect fluid polytropes and boson stars: There are closed solutions that although interesting on their own right cannot be considered as relevant in a four-dimensional astrophysical or cosmological context. A second type is open solutions with a gravitational potential that contains the standard (by now) linear term modified with logarithmic corrections whose observational relevance needs further study.

The kinds of equations we have solved (fourth order) are unconventional but could be treated with a good accuracy by our numerical methods, which appear in this case to be indispensable.

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