Microscopic entropy of the three-dimensional rotating black hole of Bergshoeff-Hohm-Townsend massive gravity

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Asymptotically anti–de Sitter rotating black holes for the Bergshoeff-Hohm-Townsend massive gravity theory in three dimensions are considered. In the special case when the theory admits a unique maximally symmetric solution, apart from the mass and the angular momentum, the black hole is described by an independent ''gravitational hair'' parameter, which provides a negative lower bound for the mass. This bound is saturated at the extremal case, and since the temperature and the semiclassical entropy vanish, it is naturally regarded as the ground state. The absence of a global charge associated with the gravitational hair parameter reflects itself through the first law of thermodynamics in the fact that the variation of this parameter can be consistently reabsorbed by a shift of the global charges, giving further support to consider the extremal case as the ground state. The rotating black hole fits within relaxed asymptotic conditions as compared with the ones of Brown and Henneaux, such that they are invariant under the standard asymptotic symmetries spanned by two copies of the Virasoro generators, and the algebra of the conserved charges acquires a central extension. Then it is shown that Strominger's holographic computation for general relativity can also be extended to the Bergshoeff-Hohm-Townsend theory; i.e., assuming that the quantum theory could be consistently described by a dual conformal field theory at the boundary, the black hole entropy can be microscopically computed from the asymptotic growth of the number of states according to Cardy's formula, in exact agreement with the semiclassical result.

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I. INTRODUCTION

The new theory of massive gravity in three dimensions, recently proposed by Bergshoeff, Hohm, and Townsend (BHT) [1], has naturally earned a great deal of attention since it enjoys many remarkable properties. The theory is described by the parity-invariant action

$$I_{\rm BHT} = \frac{1}{16\pi G}\int d^3x\sqrt{-g}\left[R - 2\lambda - \frac{1}{m^2}K\right], \quad (1)$$

where

$$K = R_{\mu\nu}R^{\mu\nu} - \tfrac{3}{8}R^2, \quad (2)$$

which yields fourth order field equations for the metric. Noteworthy, since at the linearized level they are equivalent to the Fierz-Pauli equations for a massive spin-2 field, ghosts are ''exorcized'' from the theory [1–3]. As a consequence, the BHT theory appears to be unitary [4] and renormalizable [5]. A variety of exact solutions have been found [3,6–10], its locally supersymmetric extension is known [11], and further aspects have been developed in [12].

In the special case $m^2 = \lambda$, the theory possesses a unique maximally symmetric solution and it acquires additional interesting features, as it is the enhancement of gauge invariance for the linearized theory, such that the graviton is described by a single degree of freedom [3] being ''partially massless'' [13–16]. For the nonlinear theory this is reflected in the fact that the anti–de Sitter (AdS) waves propagate a single scalar degree of freedom whose mass saturates the Breitenlohner-Freedman bound [7]. It is also known that in this case, the Brown-Henneaux boundary conditions can be consistently relaxed, which enlarges the space of admissible solutions so as to include rotating black holes, gravitational solitons, kinks, and wormholes [8].

In what follows we will focus on the asymptotically AdS rotating black hole found in [8]. The solution is described in terms of two global charges, being the mass and the angular momentum, as well as by an additional ''gravitational hair'' parameter, which provides a negative lower bound for the mass. This bound is saturated at the extremal case, and since the temperature and the semiclassical entropy vanish, it is naturally regarded as the ground state. As revisited in the next section, this sort of extremality is due to the gravitational hair and it turns out to be stronger than extremality due to rotation. In Sec. III it is shown that the absence of a global charge associated with the gravitational hair parameter is reflected in the first law of thermodynamics through the fact that the variation of this parameter can

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be consistently reabsorbed by a shift of the global charges, giving a remarkably strong support to consider the extremal case as the ground state. Since the rotating black hole fits within relaxed asymptotic conditions as compared with the ones of Brown and Henneaux [17], such that they are invariant under the standard asymptotic symmetries spanned by two copies of the Virasoro generators, and the algebra of the conserved charges acquires a central extension, Sec. IV is devoted to showing that Strominger's holographic result for general relativity [18] can also be extended to the BHT theory; i.e., assuming that the quantum theory could be consistently described by a dual conformal field theory at the boundary, the black hole entropy can be microscopically computed from the asymptotic growth of the number of states according to Cardy's formula, in exact agreement with the semiclassical result. Ending remarks are made in Sec. V.

II. ROTATING BLACK HOLE

The BHT theory (1) for the special case $m^2 = \lambda = -\frac{1}{2l^2}$ admits the following rotating black hole solution [8]:

$$ds^2 = -NFdt^2 + \frac{dr^2}{F} + r^2(d\phi + N^\phi dt)^2, \tag{3}$$

where N, N^ϕ, and F are functions of the radial coordinate r, given by

$$\begin{aligned} N &= \left[1 + \frac{bl^2}{4H}(1 - \Xi^{1/2})\right]^2, \\ N^\phi &= -\frac{a}{2r^2}(4GM - bH), \\ F &= \frac{H^2}{r^2}\left[\frac{H^2}{l^2} + \frac{b}{2}(1 + \Xi^{1/2})H + \frac{b^2l^2}{16}(1 - \Xi^{1/2})^2 \right. \\ &\quad \left. - 4GM\Xi^{1/2}\right], \end{aligned} \tag{4}$$

and

$$H = \left[r^2 - 2GMl^2(1 - \Xi^{1/2}) - \frac{b^2l^4}{16}(1 - \Xi^{1/2})^2\right]^{1/2}. \tag{5}$$

Here $\Xi := 1 - a^2/l^2$, and the rotation parameter a is bounded in terms of the AdS radius according to $-l \leq a \leq l$. The solution is then described by two global charges, where M is the mass and $J = Ma$ is the angular momentum, as well as by an additional [19] "gravitational hair" parameter b.

The rotating black hole is a conformally flat asymptotically AdS spacetime, and depending on the range of the parameters M, a, and b, the solution possesses an ergosphere and a singularity that can be surrounded by event and inner horizons. In the case of $b = 0$, the solution reduces to the Banados-Teitelboim-Zanelli black hole [20,21], while when the gravitational hair parameter is switched on ($b \neq 0$), the spacetime is no longer of constant curvature and the solutions splits in two branches according to the sign of b. The event horizon radius, the temperature, and the entropy are given by $r_+ = \gamma \bar{r}_+$, $T = \gamma^{-1}\bar{T}$, and $S = \gamma\bar{S}$, respectively, where $\gamma^2 = \frac{1}{2}(1 + \Xi^{-1/2})$, and $\bar{r}_+$, $\bar{T}$, $\bar{S}$ correspond to the radius of the event horizon, the temperature, and the entropy for the static case. Thus, the angular velocity of the horizon turns out to be

$$\Omega_+ = \frac{1}{a}(\Xi^{1/2} - 1), \tag{6}$$

and the Hawking temperature and the entropy can be explicitly expressed as

$$T = \frac{1}{\pi l}\Xi^{1/2}\sqrt{2G\Delta M(1 + \Xi^{1/2})^{-1}}, \tag{7}$$

$$S = \pi l\sqrt{\frac{2}{G}\Delta M(1 + \Xi^{1/2})}, \tag{8}$$

where

$$\Delta M := M - M_0 = M + \frac{b^2l^2}{16G}. \tag{9}$$

Note that the rotating Banados-Teitelboim-Zanelli black hole ($b = 0$) possesses twice the entropy obtained from general relativity, i.e., $S = \frac{A_+}{2G}$.

The black hole described by (3) fulfills

$$M^2 \geq \frac{J^2}{l^2}. \tag{10}$$

This bound is saturated when the rotation parameter is given by $a^2 = l^2$, so that the angular velocity of the horizon is $\Omega_+^2 = \frac{1}{l^2}$ and the temperature (7) vanishes. This is an extremal case since the event and inner horizons coincide, and for $b \neq 0$ they are on top of the singularity which becomes null and is located at

$$r_+^2 = r_-^2 = r_s^2 = 2Gl^2\Delta M. \tag{11}$$

Note that for $a^2 = l^2$ the entropy (8) reduces to $S = \pi l\sqrt{\frac{2}{G}\Delta M}$.

The case $b < 0$ is particularly interesting since the black hole mass is allowed to be negative up to certain extent, and it is bounded in terms of the gravitational hair parameter according to

$$M \geq M_0, \tag{12}$$

with

$$M_0 = -\frac{b^2l^2}{16G}. \tag{13}$$

This opens the possibility of having a different kind of stronger extremality. Indeed, the bound (12) is saturated in the case of $M = M_0$, so that the metric describes an extremal black hole for which the event and the inner horizons coincide,

$$r_+^2 = r_-^2 = \frac{b^2 l^4}{8} \Xi^{1/2}(1 + \Xi^{1/2}),$$

always enclosing a timelike singularity located at

$$r_s^2 = \frac{b^2 l^4}{8} \Xi^{1/2}(\Xi^{1/2} - 1). \tag{14}$$

Remarkably, for $M = M_0$, not only the temperature but also the entropy vanishes, as is shown by Eqs. (7) and (8). Thus, it is natural to regard the case of $M = M_0$ as the ground state, not only because it is the lower bound for the mass allowed by cosmic censorship, but also because, since the entropy vanishes, it would correspond to a single nondegenerate microscopic state.

Note that this kind extremality is due to the existence of the gravitational hair parameter and it can be attained for any value of the rotation parameter a within its allowed range, so that the angular momentum is $J_0 = M_0 a$, and the extremal horizon has an angular velocity given by (6).

As explained in Sec. IV, at the extremal case $M = M_0$, not only the entropy but also both left and right temperatures vanish, while for the extremal case $J^2 = M^2 l^2$, only one of these temperatures vanishes, let us say the left, while neither the right temperature nor the entropy do. Thus, also in this sense, extremality due to gravitational hair is stronger than extremality due to rotation.

III. GRAVITATIONAL HAIR, FIRST LAW OF THERMODYNAMICS, AND THE GROUND STATE

Following the Deser-Tekin approach [22], it was shown in [8] that the rotating black hole (3) possesses only two global charges, the mass M and the angular momentum $J = Ma$, where the massless Banados-Teitelboim-Zanelli black hole was chosen as the reference background. Thus, because of the absence of a global charge associated to b, it was dubbed as the gravitational hair parameter.

The absence of a global charge associated with b is then reflected in the first law of thermodynamics through the fact that no chemical potential can be associated to it, and hence the variation of this parameter has to be consistently reabsorbed by a shift of the global charges.

This can be explicitly seen as follows: According to Eqs. (7) and (8), the product of the temperature and the variation of the entropy are given by

$$TdS = \Xi^{1/2} dM + \frac{bl^2}{8G}\Xi^{1/2} db - \frac{1}{a}(1 - \Xi^{1/2})\left(M + \frac{b^2 l^2}{16G}\right) da,$$

and taking into account Eqs. (6) and (13), this equation reduces to

$$d(M - M_0) = TdS - \Omega_+ d(J - J_0), \tag{15}$$

where M_0 and $J_0 = M_0 a$ correspond to the mass and the angular momentum of the extremal case, respectively.

As expected, the dependence on the gravitational hair parameter is entirely absorbed by a shift of the global charges. Remarkably, Eq. (15) means that the shift is precisely such that the first law is fulfilled provided the global charges (the mass and the angular momentum) are measured with respect to the ones of the extremal case that saturates the bound (12).

This provides further strong support to consider the extremal case as the ground state. Using this fact, in the next section it is shown that the entropy of the rotating black hole (8) can be microscopically computed from the asymptotic growth of the number of states of the dual theory.

IV. MICROSCOPIC ENTROPY OF THE ROTATING BLACK HOLE

As shown in [8], the rotating black hole (3) fits within a relaxed set of asymptotic conditions as compared with the one of Brown and Henneaux [17], being such that they are invariant under the standard asymptotic symmetries spanned by two copies of the Virasoro generators. The algebra of the conserved charges also acquires a central extension being twice the value found for general relativity, i.e.,

$$c_\pm = c = \frac{3l}{G}. \tag{16}$$

Choosing the extremal case as the reference background, the only nonvanishing surface integrals for the rotating black hole are then the ones associated with the left and right Virasoro generators $L_0^\pm$, given by

$$\Delta_\pm = \tfrac{1}{2}(l\Delta M \pm \Delta J) = \tfrac{1}{2}\Delta M(l \pm a). \tag{17}$$

where $\Delta M = M - M_0$, and $\Delta J = \Delta M a$, are mass and the angular momentum.

Regarding this as the starting point, one can see that Strominger's result for general relativity [18] works also for the BHT theory described by (1). Strominger holographic computation relies on an observation pushed forward more than 20 years ago by Brown and Henneaux [17], who suggested that since the asymptotic symmetry group of general relativity with a negative cosmological constant in three dimensions is generated by two copies of the Virasoro algebra, a consistent quantum theory of gravity should be described in terms of a two-dimensional conformal field theory. This is currently interpreted in terms of the AdS/CFT correspondence [23].

Hence, assuming that the quantum theory for BHT massive gravity exists and it is consistently described by a dual conformal field theory at the boundary, the physical states must form a representation of the algebra with a central charge given by (16), and if the CFT fulfills some physically sensible properties, the asymptotic growth of the number of states is given by Cardy's formula.

Therefore, as explained above, since the black hole (3) can be regarded as excitations of the ground state, which corresponds to the extremal case $M = M_0$, the entropy can be computed in the microcanonical ensemble as the logarithm of the density of states, given by

$$S = 2\pi\sqrt{\frac{c_+}{6}\Delta_+} + 2\pi\sqrt{\frac{c_-}{6}\Delta_-}, \quad (18)$$

where $c_\pm$ is given by (16), and $\Delta_\pm$ in Eq. (17) correspond to the eigenvalues of $L_0^\pm$. Thus, Eq. (18) reduces to

$$S = \pi l\sqrt{\frac{\Delta M}{G}}\left(\sqrt{1+\frac{a}{l}} + \sqrt{1-\frac{a}{l}}\right), \quad (19)$$

$$= \pi l\sqrt{\frac{2}{G}(1+\Xi^{1/2})\Delta M}, \quad (20)$$

which exactly agrees with the semiclassical result in Eq. (8).

Note that since left and right movers are decoupled, they can be at equilibrium at different temperatures $T_\pm$. In the canonical ensemble, the entropy acquires the form

$$S = \frac{\pi^2 l}{3}(c_+T_+ + c_-T_-), \quad (21)$$

and since the free energy is given by

$$F = (\beta_+\Delta_+ + \beta_-\Delta_-)l^{-1} - S = \beta\Delta M + \beta\Omega_+\Delta J - S, \quad (22)$$

the left and right temperatures turn out to be

$$T_\pm = \frac{T}{1 \pm l\Omega_+} = \frac{1}{2\pi l}(1 \mp l\Omega_+)\sqrt{2G\Delta M(1+\Xi^{1/2})}. \quad (23)$$

Then, by virtue of Eqs. (16) and (23) it is simple to verify that formula (21) exactly reproduces the black hole entropy (8) as well.

Note that for the extremal black holes case due to rotation, $J^2 = Ml^2$, for which $\Omega_+^2 = \frac{1}{l^2}$, only one of these temperatures vanishes, let us say the left, while the right temperature is given by $T_+ = \frac{1}{\pi l}\sqrt{2G\Delta M(1+\Xi^{1/2})}$ and they have a nonvanishing entropy $S = \pi l\sqrt{\frac{2}{G}\Delta M}$. It is reassuring then to verify that for the extremal case due to gravitational hair, $M = M_0$, not only the entropy but also both left and right temperatures vanish, as it has to be for a suitable ground state.

V. DISCUSSION AND COMMENTS

It was shown that the semiclassical entropy of the rotating black hole (3) can be microscopically reproduced from Cardy's formula (18), where the ground state turns out to be given by the extremal case $M = M_0$. It is worth pointing out that the computations can be extended perfectly well even for a case that they were not intended for, $b > 0$. The subtlety is related to the fact that for $b > 0$, the black hole configuration with $M = M_0$ suffers certain pathologies. Nevertheless, as was shown in [8], in this case the theory also admits a gravitational soliton for $M = M_0$. Thus, since the spacetime is regular everywhere, the soliton provides a suitable nondegenerate state that can be naturally regarded as the ground state. This point is left for future detailed discussion.

Since the rotating black hole (3) is conformally flat, it solves the BHT field equations for the special case $m^2 = \lambda$ even in the presence of the topological mass term, and it is simple to verify that our results extend to this case. The vanishing of the Cotton tensor should also imply that the rotating black hole is conformally related to the matching of different solutions of constant curvature by means of an improper conformal transformation, as it occurs for the static case [24].

It would also be interesting to explore whether the bounds (10) or (12) could be derived from the supergravity theory proposed in [11], or from some extension of it.

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