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Multidimensional gravity in the nonrelativistic limit

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Exact solutions of the Poisson equation are found for multidimensional spaces with topology $M_{3+d} = \mathbb{R}^3 \times T^d$. These solutions describe smooth transitions from Newtonian behavior $1/r_3$ for distances bigger than the periods of the tori (the sizes of the extra dimensions) to multidimensional behavior $1/r_{3+d}^{1+d}$ in the small-distance limit. In the case of one extra dimension $d = 1$, a compact and elegant formula for the gravitational potential is found. It is shown that corrections to the gravitational constant in Cavendish-type experiments can be within the measurement accuracy of Newton's gravitational constant G_N. Models with test masses smeared over some (or all) extra dimensions are proposed. It is shown that in a 10-dimensional spacetime with three smeared extra dimensions the size of the remaining three extra dimensions can be enlarged up to submillimeter scales in case of a fundamental Planck scale $M_{\mathrm{Pl}(10)} \approx 1$ TeV. In models with all extra dimensions smeared, the gravitational potential coincides exactly with the Newtonian one. Nevertheless, the hierarchy problem can be solved in these models.

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I. INTRODUCTION

There are two well-known problems which are related to each other. These are the discrepancies in gravitational constant experimental data and the hierarchy problem. Discrepancies (see e.g. Fig. 2 in the "CODATA Recommended Values of the Fundamental Constants: 2006") are usually explained by extreme weakness of gravity. It is very difficult to measure Newton's gravitational constant G_N. Certainly, for this reason the geometry of an experimental setup can have an effect on the data. However, it may well be that the discrepancies can also be explained (at least partly) by the underlying fundamental theory. Formulas for the effective gravitational constant following from such a theory can be sensitive to the geometry of experiments. For example, if the correction to Newton's gravitational potential has the form of a Yukawa potential, then the force due to this potential at a given minimal separation per unit test-body mass takes a smallest value for two spheres and the greatest one for two planes (see e.g. [1]). The hierarchy problem—the huge gap between the electroweak scale $M_{\mathrm{EW}} \sim 10^3$ GeV and the Planck scale $M_{\mathrm{Pl}(4)} = 1.2 \times 10^{19}$ GeV—can be also reformulated in the following manner: why is gravity so weak? The smallness of G_N is the result of the relation $G_N = M_{\mathrm{Pl}(4)}^{-2}$. A natural explanation was proposed in [2,3]—actually gravity is strong: $G_{\mathcal{D}} = M_{\mathrm{Pl}(\mathcal{D})}^{-(2+d)} \sim M_{\mathrm{EW}}^{-(2+d)}$, but this happens in a ($\mathcal{D} = 4 + d$)-dimensional spacetime. It becomes weak when it is "smeared" over large extra dimensions: $G_N \sim G_{\mathcal{D}}/V_d$ where V_d is the volume of the internal space. To shed light on both of these problems from a new standpoint we intend to investigate multidimensional gravity in its nonrelativistic limit.

II. MULTIDIMENSIONAL GRAVITATIONAL POTENTIALS

It is of interest to generalize the well-known Newton's gravitational potential $\varphi(r_3) = -G_N m/r_3$ ($r_3 = |\mathbf{r}_3|$ is the length of a radius vector in three-dimensional space) to multidimensional spaces. Clearly, the result will depend on the topology of the models under investigation. We consider models where the ($D = 3 + d$)-dimensional spatial part has the factorizable geometry of a product manifold $M_D = \mathbb{R}^3 \times T^d$. $\mathbb{R}^3$ describes the three-dimensional flat external (our) space and T^d is a torus which corresponds to a d-dimensional internal space with volume V_d. Let $b \sim V_d^{1/d}$ be the characteristic size of the extra dimensions. Then, Gauss's flux theorem leads to the following asymptotic behavior of the gravitational potential (see e.g. [3]): $\varphi \sim 1/r_3$ for $r_3 \gg b$ and $\varphi \sim 1/r_{3+d}^{1+d}$ for $r_{3+d} \ll b$ with r_{3+d} as length of the radius vector in (3 + d)-dimensional space.

To get the exact expression for the D-dimensional gravitational potential, we start from the Poisson equation:

$$\triangle_D \varphi_D = S_D G_{\mathcal{D}} \rho_D(\mathbf{r}_D), \tag{1}$$

where $S_D = 2\pi^{D/2}/\Gamma(D/2)$ is the total solid angle (surface area of the ($D - 1$)-dimensional sphere of unit radius), $G_{\mathcal{D}}$ is the gravitational constant in the ($\mathcal{D} = D + 1$)-dimensional spacetime and $\rho_D(\mathbf{r}_D) = m\delta(x_1)\delta(x_2)\ldots\delta(x_D)$. In the case of an $\mathbb{R}^D$ topology, Eq. (1) has the following solution:

$$\varphi_D(\mathbf{r}_D) = -\frac{G_{\mathcal{D}} m}{(D-2) r_D^{D-2}}, \qquad D \geq 3. \tag{2}$$

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This is the unique solution to Eq. (1) which satisfies the boundary condition $\lim_{r_D\to+\infty}\varphi_D(\mathbf{r}_D)=0$. The gravitational constant $G_{\mathcal{D}}$ in (1) is normalized in such a way that the strength of gravitational field (acceleration of a test mass) takes the form: $-d\varphi_D/dr_D = -G_{\mathcal{D}}m/r_D^{D-1}$.

For $\mathbb{R}^3\times T^d$ topologies of the space we impose periodic boundary conditions in the directions of the extra dimensions: $\varphi_D(\mathbf{r}_3,\xi_1,\xi_2,\ldots,\xi_i,\ldots,\xi_d)=\varphi_D(\mathbf{r}_3,\xi_1,\xi_2,\ldots,\xi_i+a_i,\ldots,\xi_d)$, $i=1,\ldots,d$, where a_i denotes the period in the direction of the extra dimension ξ_i. Then the Poisson equation has the following solution (cf. also [3,4]):

$$\varphi_D(\mathbf{r}_3,\xi_1,\ldots,\xi_d) = -\frac{G_N m}{r_3}\sum_{k_1=-\infty}^{+\infty}\cdots\sum_{k_d=-\infty}^{+\infty}\times\exp\left[-2\pi\left(\sum_{i=1}^{d}\left(\frac{k_i}{a_i}\right)^2\right)^{1/2} r_3\right]\times\cos\left(\frac{2\pi k_1}{a_1}\xi_1\right)\ldots\cos\left(\frac{2\pi k_d}{a_d}\xi_d\right). \quad (3)$$

To get this result we, first, use the formula $\delta(\xi_i)=\frac{1}{a_i}\times\sum_{k=-\infty}^{+\infty}\cos(\frac{2\pi k}{a_i}\xi_i)$ and, second, impose the following relation between the gravitational constants in four- and $\mathcal{D}$-dimensional spacetimes:

$$\frac{S_D}{S_3}\cdot\frac{G_{\mathcal{D}}}{\prod_{i=1}^{d}a_i} = G_N. \quad (4)$$

The latter relation provides the correct limit when all $a_i\to 0$. In this limit the zero modes $k_i=0$ give the main contribution and we obtain $\varphi_D(\mathbf{r}_3,\xi_1,\ldots,\xi_d)\to -G_N m/r_3$. Equation (4) was widely used in large-extra-dimension approaches to solve the hierarchy problem [2,3]. It is convenient to rewrite (4) via fundamental Planck scales:

$$\frac{S_D}{S_3}\cdot M^2_{\mathrm{Pl}(4)} = M^{2+d}_{\mathrm{Pl}(\mathcal{D})}\prod_{i=1}^{d}a_i, \quad (5)$$

where $M_{\mathrm{Pl}(4)} = G_N^{-1/2} = 1.2\times10^{19}$ GeV and $M_{\mathrm{Pl}(\mathcal{D})}\equiv G_{\mathcal{D}}^{-1/(2+d)}$ are the fundamental Planck scales in four and $\mathcal{D}$ spacetime dimensions, respectively.

In the opposite limit when all $a_i\to+\infty$ the sums in Eq. (3) can be replaced by integrals. Using the standard integrals (e.g. from [5]) and relation (4), we can easily show that, for example, in the particular cases $d=1,2$ we get the desired result: $\varphi_D(\mathbf{r}_3,\xi_1,\ldots,\xi_d)\to -G_{\mathcal{D}}m/[(D-2)r_{3+d}^{1+d}]$.

III. ONE EXTRA DIMENSION

In the case of one extra dimension $d=1$ the series in Eq. (3) can be summed explicitly—either with the help of the Abel-Plana formula or we can simply use the tables of series [5]. As a result, we arrive at the compact and nice expression:

$$\varphi_4(\mathbf{r}_3,\xi) = -\frac{G_N m}{r_3}\frac{\sinh(\frac{2\pi r_3}{a})}{\cosh(\frac{2\pi r_3}{a})-\cos(\frac{2\pi\xi}{a})}, \quad (6)$$

where $r_3\in[0,+\infty)$ and $\xi\in[0,a]$. It is not difficult to verify that this formula has the correct asymptotic behavior for $r_3\gg a$ and $r_4\ll a$. Figure 1 demonstrates the shape of this potential. The dimensionless variables η_1 and η_2 are defined as $\eta_1\equiv r_3/a\in[0,+\infty)$ and $\eta_2\equiv\xi/a\in[0,1]$. With respect to the variable η_2, this potential has two minima at $\eta_2=0,1$ and one maximum at $\eta_2=1/2$. To show the form of the minimum at $\eta_2=0$ in more detail we continued the graph to negative values of $\eta_2\in[-1,1]$. The potential (6) is finite for any value of r_3 if $\xi\neq 0, a$ and tends to $-\infty$ as $-1/r_4^2$ if simultaneously $r_3\to 0$ and $\xi\to 0$, a (see Fig. 2). We would like to mention that in the particular case $\xi=0$ formula (6) was also found in [6].

With formulas (3) and (6) at hand, we can turn to the calculation of some elementary physical problems and compare the obtained results with the known Newtonian expressions. Some of these calculations can be found in our preprint [7]. As a good approximation it is usually sufficient to sum in (3) up to the first Kaluza-Klein modes $|k_i|=1(i=1,\ldots,d)$. Then, the terms with the largest periods a_i give the main contributions. If all test bodies are on the same brane ($\xi_i=0$) we obtain:

$$\varphi_D(\mathbf{r}_3,\xi_1=0,\ldots,\xi_d=0)\equiv\varphi_D(r_3) \approx -\frac{G_N m}{r_3}\left[1+\alpha\exp\left(-\frac{r_3}{\lambda}\right)\right], \quad (7)$$

where $\alpha=2s(1\le s\le d)$, $\lambda=a/(2\pi)$ and s is the number of extra dimensions with tori periods a_i equal (or approximately equal) to $a=\max a_i$. If $a_1=a_2=\ldots=a_d=a$, then $s=d$. Thus, the correction to Newton's potential has the form of a Yukawa potential. Test data of the gravitational inverse-square law (ISL) can now be used as observational bounds on possible gravity-related

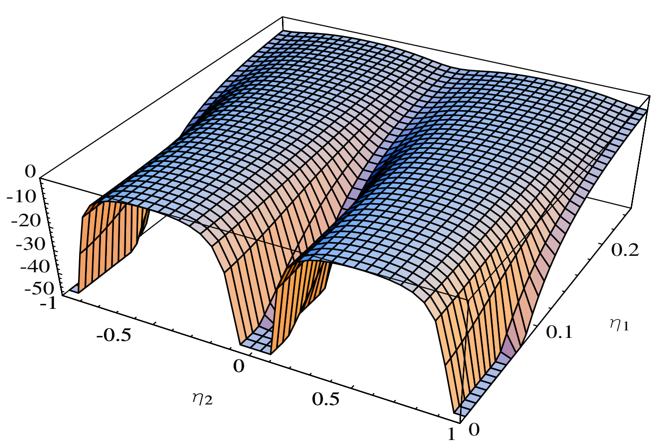

FIG. 1 (color online). Graph of function $\tilde{\varphi}(\eta_1,\eta_2)\equiv\varphi_4(\mathbf{r}_3,\xi)/(G_N m/a) = -\sinh(2\pi\eta_1)/[\eta_1(\cosh(2\pi\eta_1)-\cos(2\pi\eta_2))]$.

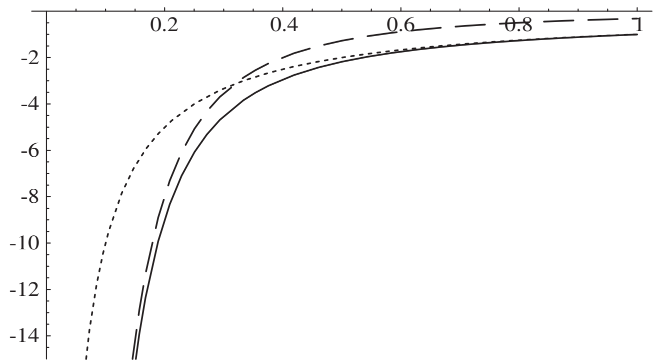

FIG. 2. Section $\xi = 0$ of potential (6). Solid line is $\tilde{\varphi}(\eta_1, 0) = -\sinh(2\pi\eta_1)/[\eta_1(\cosh(2\pi\eta_1) - 1)]$ which goes to $-1/\eta_1$ (dotted line) for $\eta_1 \to +\infty$ and to $-1/(\pi\eta_1^2)$ (dashed line) for $\eta_1 \to 0$.

Yukawa contributions. The overall diagram of the experimental bounds can be found in [1] (Fig. 5 therein) and we shall use these data for limitation a for given α.

In this approximation, the gravitational force between two spheres with masses m_1, m_2, radii R_1, R_2 and distance r_3 between the centers of the spheres reads:

$$F = -\frac{G_{N(\mathrm{eff})} m_1 m_2}{r_3^2}, \tag{8}$$

where

$$G_{N(\mathrm{eff})}(r_3) \approx G_N\left\{1 + \frac{9}{2} s\left(\frac{a}{2\pi R_1}\right)^2\left(\frac{a}{2\pi R_2}\right)^2 \frac{2\pi r_3}{a} \times \exp\left[-\frac{2\pi}{a}(r_3 - R_1 - R_2)\right]\right\} \equiv G_N(1 + \delta_G). \tag{9}$$

Here, we assumed $r_3 \geq R_1 + R_2$ and $R_1, R_2 \gg a/2\pi$.

IV. SMEARED EXTRA DIMENSIONS

In what follows, we consider models with asymmetric extra dimensions (cf. [8]) and topologies

$$M_D = \mathbb{R}^3 \times T^{d-p} \times T^p, \qquad p \leq d, \tag{10}$$

where we suppose that $(d - p)$ tori have the same "large" period a and p tori have "small" equal periods b. In this case, the fundamental Planck scale relation (5) reads

$$\frac{S_D}{S_3} \cdot M_{\mathrm{Pl}(4)}^2 = M_{\mathrm{Pl}(\mathcal{D})}^{2+d} a^{d-p} b^p. \tag{11}$$

Additionally, we assume that test bodies are uniformly smeared/spread over the small extra dimensions. This means that test bodies will have a finite thickness in small extra dimensions (thick brane approximation). For short, we shall call such small extra dimensions smeared extra dimensions. If $p = d$ then all extra dimensions are smeared.

It is not difficult to show that the gravitational potential does not feel smeared extra dimensions. We can prove this statement by three different methods. First, we can directly solve the D-dimensional Poisson Eq. (1) with periodic boundary conditions for the extra dimensions $\xi_{p+1}, \ldots, \xi_d$ and a mass density $\rho = (m/\prod_{i=1}^p a_i) \times \delta(\mathbf{r}_3)\delta(\xi_{p+1}) \ldots \delta(\xi_d)$. Second, we can average solutions (3) and (6) over dimensions $\xi_1, \ldots, \xi_p$ and take into account that $\int_0^a \cos(2\pi k\xi/a) d\xi = 0$. In the particular case of one extra dimension we can also show that

$$-\frac{G_N m}{a r_3} \sinh\left(\frac{2\pi r_3}{a}\right) \int_0^a \left[\cosh\left(\frac{2\pi r_3}{a}\right) - \cos\left(\frac{2\pi\xi}{a}\right)\right]^{-1} d\xi = -\frac{G_N m}{r_3}. \tag{12}$$

Finally, it is clear that in the case of test masses smeared over the extra dimensions, the gravitational field vector $\mathbf{E}_D = -\nabla_D \varphi_D$ does not have components with respect to these extra dimensions: $\mathbf{E}_D = E_D \mathbf{n}_{r_3}$. Thus, applying the Gauss's flux theorem to the Poisson equation, we obtain: $E_D(r_3) = -G_N m/r_3^2 \to \varphi_D(r_3) = -G_N m/r_3$. Therefore, all these three approaches show that in the case of p smeared extra dimensions the wave numbers $k_1, \ldots, k_p$ disappear from Eq. (3) and we should perform summation only with respect to $k_{p+1}, \ldots, k_d$.

V. EFFECTIVE GRAVITATIONAL CONSTANT

Coming back to the effective gravitational constant (9) in the case of topology (10) with p smeared extra dimensions, s in Eq. (9) is replaced by $(d - p)$. Now, we want to evaluate the corrections δ_G to the Newton's gravitational constant and to estimate their possible influence on experimental data. As it follows from Fig. 2 in the CODATA 2006, the most precise values of G_N were obtained in the University Washington and the University Zürich experiments [9,10]. They are $G_N/10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}} = 6.674215 \pm 0.000092$, and 6.674252 ± 0.000124, respectively. Let us consider two particular examples: the $(\mathcal{D} = 5)$-dimensional model with $d = 1$, $p = 0 \to \alpha = 2$ and the $(\mathcal{D} = 10)$-dimensional model with $d = 6$, $p = 3 \to \alpha = 6$. For these values of α, Fig. 5 in [1] gives the upper limits for $\lambda = a/(2\pi)$ correspondingly $\lambda \approx 2 \times 10^{-2}$ cm and $\lambda \approx 1.3 \times 10^{-2}$ cm. To calculate δ_G, we take parameters of the Moscow experiment [11]: $R_1 \approx 0.087$ cm for a platinum ball with mass $m_1 = 59.25 \times 10^{-3}$ g, $R_2 \approx 0.206$ cm for a tungsten ball with mass $m_2 = 706 \times 10^{-3}$ g and $r_3 = 0.3773$ cm. For both of these models we obtain $\delta_G \approx 0.0006247$ and $\delta_G \approx 0.0000532$, respectively. Both of these values are very close to the measurement accuracy of G_N in [9,10]. So, if the same accuracy can be achieved in the Moscow-type experiments, then, changing values of $R_{1,2}$ and r_3, we can reveal extra dimensions or obtain experimental limitations on the considered models.

In the Moscow experiment [11], measurements were carried out for two separations between the balls: $r_{3(1)} = 0.3773$ cm and $r_{3(2)} = 0.6473$ cm, and it was found that the experimental ratio of the gravitational forces between balls for these distances is

$$[F(r_{3(1)})/F(r_{3(2)})]_{\text{Mosc. exp.}} = 2.94 \pm 0.045. \quad (13)$$

The similar ratio for pure Newton's forces gives $[F(r_{3(1)})/F(r_{3(2)})]_{\text{Newton}} = r^2_{3(2)}/r^2_{3(1)} \approx 2.943321$ (this is an idealized value because we did not take into account uncertainties in measurements of the separation distances of the balls). In the case of our force (8) the ratio reads $[F(r_{3(1)})/F(r_{3(2)})]_{\text{our}} = [(1 + \delta_G(r_{3(1)}))/(1 + \delta_G(r_{3(2)}))] \times [r^2_{3(2)}/r^2_{3(1)}]$. If we apply this formula to our models with $d = 1$, $p = 0$ and $d = 6$, $p = 3$ we obtain for the ratio the values 2.945159 and 2.943477, respectively. Both of these values are within the accuracy of the measurements in (13). This accuracy is too rough to reveal the extra dimensions. To detect the extra dimensions, it is necessary to increase the accuracy.

VI. MODEL: $\mathcal{D} = 10$ WITH $d = 6$, $p = 3$

Let us consider in more detail the ($\mathcal{D} = 10$)-dimensional model with three smeared dimensions. Here, the structure of the spacial dimensions is very symmetric: three (our) external dimensions, three large extra dimensions with periods a and three small smeared extra dimensions with periods b. For b we put the limitation: $b \le b_{\max} = 10^{-17}$ cm which is usually taken for thick brane approximation. As we mentioned above, in the case of $\alpha = 6$, for a we should take a limitation $a \le a_{\max} = 8.2 \times 10^{-2}$ cm. To solve the hierarchy problem, the multidimensional Planck scale is usually considered from 1 TeV up to approximately 130 TeV (see e.g. [8,12]). To make some estimates, we take $M_{\min} = 1\,\text{TeV} \lesssim M_{\text{Pl}(10)} \lesssim M_{\max} = 50\,\text{TeV}$. Thus, as it follows from Eq. (11), the allowed values of a and b should satisfy inequalities:

$$\frac{S_9}{S_3}\frac{M^2_{\text{Pl}(4)}}{M^8_{\max}} \le a^3 b^3 \le \frac{S_9}{S_3}\frac{M^2_{\text{Pl}(4)}}{M^8_{\min}}. \quad (14)$$

Counting all limitations, we find the region allowed for a and b (shadow area in Fig. 3). In this trapezium, the upper horizontal and right vertical lines are the decimal logarithms of $a_{\max}$ and $b_{\max}$, respectively. The right and left inclined lines correspond to $M_{\text{Pl}(10)} = 1$ TeV and $M_{\text{Pl}(10)} = 50$ TeV, respectively. To illustrate this picture, we consider two points, A and B, on the line $M_{\text{Pl}(10)} = 1$ TeV. Here, we have $a = 0.82 \times 10^{-1}$ cm, $b = 10^{-21.5}$ cm for A and $a = 10^{-4}$ cm, $b = 10^{-18.6}$ cm for B. These values of large extra dimensions a are much larger than in the standard approach $a \sim 10^{(32/6)-17}$ cm $\approx 10^{-11.7}$ cm [2,3].

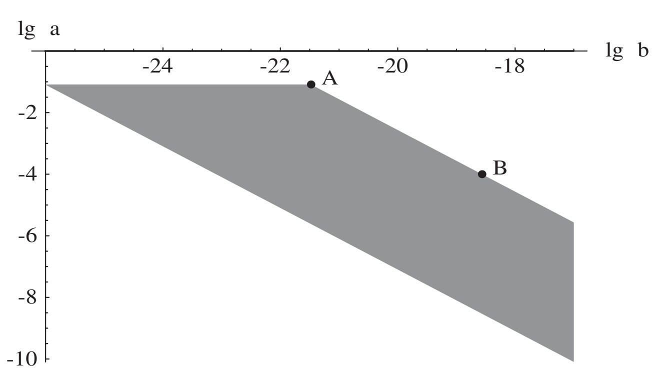

FIG. 3. Allowed region (shadow area) for periods of large (a) and smeared (b) dimensions in the model $\mathcal{D} = 10$ with $d = 6$, $p = 3$.

VII. MODEL: $\mathcal{D}$-ARBITRARY AND $d = p$

In this model, the test masses are smeared over all extra dimensions. Therefore, in the nonrelativistic limit, there is no deviation from Newton's law at all. It is worth noting that this result does not depend on the size of the smeared extra dimensions. The ISL experiments will not show any deviation from Newton's law with regard to the size b (see also Eq. (9) where $s = d - p = 0$). A similar reasoning is also applicable to Coulomb's law. It is necessary to suggest other experiments which can reveal the multidimensionality of our spacetime. Nevertheless, we can solve the hierarchy problem in this model because Eq. (11) (where $d = p$) still works. For example, in the case of bosonic string dimension $\mathcal{D} = 26$ we find $M_{\text{Pl}(26)} \approx 31$ TeV for $b = 10^{-17}$ cm. In the case $\mathcal{D} = 10$ we get $M_{\text{Pl}(10)} \approx 30$ TeV for $b = 5.59 \times 10^{-14}$ cm.

VIII. CONCLUSIONS

We have considered various generalizations of Newton's potential to the case of extra dimensions with multidimensional space topology $M_D = \mathbb{R}^3 \times T^d$. We have obtained exact solutions which describe a smooth transition from the Newtonian behavior $1/r_3$ for distances bigger than periods of tori (the extra dimension sizes) to multidimensional behavior $1/r_D^{D-2}$ in the opposite limit. In the case of one extra dimension, the gravitational potential is expressed via the compact and elegant formula (6).

In a Yukawa-potential approximation it was shown that the corrections to the gravitational constant in the Cavendish-type experiment can be within the measurement accuracy of G_N. It may provide signatures of the extra dimensions or experimental limitations on the parameters of multidimensional models.

Furthermore, we proposed models with test masses smeared over some or all of the extra dimensions. In this case, the gravitational potential does not feel the smeared dimensions. The number of smeared dimensions can be equal or less than the total number of the extra dimensions. Such an approach opens new remarkable possibilities.

For example in the case $\mathcal{D} = 10$ with three large and three smeared extra dimensions and $M_{\mathrm{Pl}(10)} = 1$ TeV, the large extra dimensions can be as big as the upper bound established by the ISL experiments for $\alpha = 6$, i.e. $a \approx 0.82 \times 10^{-1}$ cm. This value of a is many orders of magnitude larger than the rough estimate $a \approx 10^{-11.7}$ cm obtained from the fundamental Planck scale relation of the form of Eq. (5).

The limiting case where all extra dimensions are smeared is another interesting example. Here, there is no deviation from Newton's law at all. Nevertheless, the hierarchy problem can be solved in this model successfully.

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