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Global-local duality in eternal inflation

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We prove that the light-cone time cutoff on the multiverse defines the same probabilities as a causal patch with initial conditions in the longest-lived metastable vacuum. This establishes the equivalence of two measures of eternal inflation which naively appear very different (though both are motivated by holography). The duality can be traced to an underlying geometric relation which we identify.

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I. INTRODUCTION

In an eternally inflating spacetime, anything that is not completely forbidden will happen infinitely many times. To define relative probabilities, various regularization procedures, or "measures," have been explored, including [1–22]. Some measures are formulated as geometric cutoffs: The relative probability of events of type I and J is defined in terms of the ratio of the number of occurrences of each type of event, N_I and N_J, in some finite portion of the spacetime.

Geometric cutoffs proposed so far can be classified as "global" or "local." Global cutoffs define a time slicing in the multiverse and compute relative probabilities as a late-time limit:

$$\frac{p_I}{p_J} = \lim_{t\to\infty} \frac{N_I(t)}{N_J(t)}, \tag{1}$$

where $N_I(t)$ is the number of occurrences prior to the time t. The result depends strongly on the choice of time foliation, so there are many inequivalent ways to define probabilities by a global cutoff.

Local cutoffs consider the number of events in a finite neighborhood of a single inextendible timelike geodesic in the multiverse. Relative probabilities are defined by the number of occurrences in this finite neighborhood, averaged over initial conditions and possible decoherent histories:

$$\frac{p_I}{p_J} = \frac{\langle N_I(t)\rangle}{\langle N_J(t)\rangle}. \tag{2}$$

The result depends on how the neighborhood is defined, and on the initial conditions used, so that there are many inequivalent measures that can be obtained from local cutoffs. Interestingly, however, both local prescriptions studied so far [9,15] have a global "dual."

The first global-local duality was described in Ref. [15]: The (global) scale factor time cutoff [1–5,11,14,15] is dual to the (local) "fat geodesic" cutoff, in which the neighborhood of the geodesic is chosen to have fixed physical volume, and one averages over geodesics starting in a particular vacuum: that which occupies the greatest proper volume fraction at late scale factor time. This duality is somewhat limited, because the definition of scale factor time is ambiguous in collapsed regions such as galaxies [14,15]. The scale factor and fat geodesic duality holds only in universes without collapsed regions, where the global cutoff is unambiguous.[1]

In this paper, we will prove another global-local duality: The (global) light-cone time cutoff [22] is dual to the (local) "causal patch" cutoff [9], in which the relevant neighborhood of the geodesic g is defined as the causal past $C(g)$ of the entire geodesic. The duality holds if one averages over causal patches generated by geodesics starting in a particular vacuum: that which occupies most horizon volumes at late light-cone time.

Our proof generalizes a much less powerful argument given in Ref. [22], which proceeded by showing that the difference between relative probabilities computed from two different global cutoffs (light-cone time and scale factor time) is the same as the difference between relative probabilities computed from two local cutoffs (causal patch and fat geodesic). The known scale factor and fat geodesic duality [15] then implied the claimed light-cone and causal patch duality. Of course, that argument could only be as general as the scale factor and fat geodesic duality it relied on, so it applied only in everywhere-expanding universes. Additional assumptions rendered the argument still less general: it applied only to multiverse regions that are homogeneous, isotropic, and spatially flat on the horizon scale.

Our present proof eliminates all of the above restrictions. We will establish the light-cone and causal patch duality directly, without interposing another, less general global-local duality. We will assume only that the universe is eternally inflating. At the center of our proof is a simple

[1]Fat geodesics are always well defined, so it is natural to ask if the duality can be made more general. However, it is not clear whether there exists a global foliation that reduces to scale factor time in expanding regions but reproduces the probabilities computed from fat geodesics even in collapsed regions.

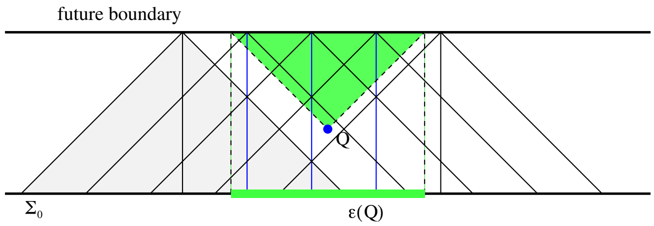

FIG. 1 (color online). Geodesics (thin vertical lines) emanating from an initial surface Σ_0 define an ensemble of causal patches (the leftmost is shaded grey [light]) with a particular mix of initial conditions. The causal patch measure assigns to the event Q a weight proportional to the number of patches that contain Q. Notice that Q is contained precisely in those causal patches whose generating geodesics (blue) enter the causal future of Q, $I^+(Q)$ (shaded green [dark]). In the continuum limit, the weight of Q is therefore proportional to the volume, $\epsilon(Q)$, of the projection of $I^+(Q)$ onto Σ_0. This observation is crucial to our proof of equivalence to the light-cone time cutoff. The light-cone time of Q is defined as $t(Q) \equiv -\frac{1}{3}\log\epsilon(Q)$.

geometric relation. Let Q be some event in the multiverse, and let g be a timelike geodesic (which need not contain the event Q). *The causal patch* $C(g)$ *will contain the event* Q, *if and only if the geodesic* g *enters the causal future of* Q. This is shown in Fig. 1.

Our argument proceeds by using the same family of geodesics that define light-cone time, to also define an ensemble of causal patches. The light-cone time of the event Q is defined as (minus the log of) the fraction of geodesics that enter the causal future of Q, which by the above relation is the same as the ensemble-fraction of causal patches that will contain Q. This implies that the local and the global cutoff will yield the same relative probability for different types of events, as long as all events occur at the same light-cone time. However, the ensemble-fraction depends on light-cone time, decreasing exponentially as the geodesics are diluted by the cosmological expansion. Thus, the causal patch ensemble will weight later events exponentially less than the light-cone cutoff.

The two measures will nevertheless agree, if this discrepancy affects all types of events equally, i.e., if the ratio of the rates at which events of different types occur is independent of time. But this is precisely what happens in the late-time attractor regime of the light-cone slicing, when $N_I(t)$ grows exponentially with time, with an I-independent coefficient. Therefore, if we use the attractor regime to define the initial conditions for the ensemble of causal patches, both measures will agree.[2] In a generic landscape, the attractor regime is completely dominated by the longest-lived de Sitter vacuum, so this amounts to starting all but a negligible fraction of causal patches in this dominant vacuum.

[2]It is not necessary to use the attractor regime as an initial condition in the global measure, since the relative probabilities in Eq. (1) are dominated by events occurring at late times in any case.

A. Outline

In Sec. II, we show that a spacelike hypersurface Σ_0, together with a family of geodesics puncturing it, defines an ensemble of causal patches with specific initial conditions. The weight of a particular event Q, according to the causal patch measure, has a geometric representation as the volume occupied on Σ_0 by those geodesics that end up in the causal future of Q. The causal patch probability for an event of type I is the sum of the volumes associated with all events of type I occurring in the spacetime. Sec. III contains the proof of the light-cone and causal patch duality. The proof uses aspects of the universal late-time behavior of the light-cone slicing, which are derived in Sec. IV.

B. Discussion

We can prove only that the light-cone time and causal patch cutoffs yield the same measure, not that they yield the correct measure. To identify which, if any, of the extant proposals is correct, one can proceed in two ways: either phenomenologically (mostly, by falsification), or by derivation from a fundamental theory for which there exists independent evidence.

The phenomenological approach has been quite fruitful [14,15,23–41]. Measures make predictions, some of which are robust independently of the details of the landscape of vacua. A number of global cutoffs are ruled out because of predictions that conflict dramatically with observation [10,27,30,42–49]. It is interesting that both the scale factor time cutoff and the light-cone time cutoff, which have so far[3] evaded such problems, have a local dual. For example, no natural local dual is known for the proper time cutoff [2–5,48], a measure that is ruled out observationally by the youngness paradox [30,42–46,48,49].

[3]A potential phenomenological problem for both measures is the so-called staggering problem. In the Bousso-Polchinski model [50] of the string landscape (and perhaps more generally), the dominant vacuum can only decay to vacua with smaller cosmological constant if the resulting cosmological constant is negative. Thus, the dominant vacuum can populate the landscape efficiently only by first transitioning to vacua with higher cosmological constant. Such upward jumps are exponentially suppressed at least by the difference in horizon entropy of the two de Sitter vacua. As pointed out in Ref. [26] (in the context of a different measure in which the same issue arises), this can lead to a staggered probability distribution: a few vacua are strongly favored over all others. This would eliminate most of the landscape, and thus its ability [50,51] to solve the cosmological constant problem. As shown by Schwartz-Perlov and collaborators [33,35,52], this problem is absent for certain ranges of reasonable model parameters. It remains to be seen whether the string theory landscape falls into this range. Similarly, both measures may be dominated by Boltzmann brains, but only if the string landscape contains sufficiently long-lived vacua [15,27,36].

The second approach—the derivation of a measure from a unified fundamental theory, say, string theory—is less well developed. However, there may be general principles that must govern such a theory, and which we may already discern, and we can apply such principles to the measure problem.

We are not aware of any principle supporting the scale factor cutoff or the fat geodesic cutoff. Meanwhile, both sides of the duality we establish here—the light-cone time cutoff and the causal patch cutoff—are, in different ways, motivated by the holographic principle. The necessity of restricting the description of space-time to a single causal-patch was first discovered by studying the holographic properties of black holes [53]. The light-cone time slicing [22] was constructed in response to the proposal [16] that the holographic UV-IR connection of the AdS/CFT correspondence should have a multiverse analogue. Both cutoffs are defined in terms of null hypersurfaces (the event horizon of a geodesic defines the causal patch; the future light-cone of a point defines its light-cone time); and indeed, null hypersurfaces are essential to a general formulation of the holographic principle [54–56].

The AdS/CFT analogy is most compelling in eternally inflating vacua (or more precisely, in eternal domains [22]). This suggests that the light-cone time cutoff (and thus, the causal patch) may not apply to regions with vanishing or negative cosmological constant. The analogy also suggests that the global cutoff may not be sharp,[4] but should be smeared on time scales of order $|\Lambda_i|^{-1/2}$, where Λ_i is the cosmological constant of vacuum i. It is intriguing that uncertainties of this magnitude appear to provide just enough room for resolving two phenomenological problems: The cutoffs appear to give too much weight to vacua with negative cosmological constant [40]; moreover, they give rise to divergences in supersymmetric vacua with vanishing cosmological constant [37,57], where the horizon scale diverges. A refinement of the light-cone time and causal patch cutoff may be needed for these regions.

These limitations illustrate that one can only get so far by extrapolation and analogy, or by formulating and falsifying purely geometric proposals. Nevertheless, we are encouraged by the recent confluence of phenomenological and first-principle support for the light-cone time and causal patch cutoff (or some closely related prescription). If this proves to be the right direction, we will have discovered more than a measure: we will know that in the multiverse, both the causal patch and the future boundary have special significance. We may be approaching a milestone, at which the phenomenological study of the measure problem begins to yield constraints on the fundamental description of the landscape and the multiverse.

[4]We are grateful to B. Freivogel, A. Guth, and A. Vilenkin for stressing this point to us.

II. THE CAUSAL PATCH CUTOFF

The causal patch measure assigns to events of type I and J the relative probability

$$\frac{\hat{P}_I}{\hat{P}_J} = \frac{\langle \hat{N}_I \rangle}{\langle \hat{N}_J \rangle}, \tag{3}$$

where $\langle \hat{N}_I \rangle$ is the expectation value of the number of such events in a particular space-time region: the *causal patch*, defined as the past of an inextendible geodesic g orthogonal to some initial spatial hypersurface Σ_0:

$$C(\Sigma_0, g) \equiv I^-(g) \cap I^+(\Sigma_0). \tag{4}$$

That is, the causal patch consists of those points to the future of Σ_0 from which some point on g can be reached by a timelike curve (Fig. 2). The boundary ∂C of the causal patch in the spacetime M consists of a null and a spacelike portion. The null portion is the *event horizon*

$$E(\Sigma_0, g) \equiv \partial C(\Sigma_0, g) \cap I^+(\Sigma_0). \tag{5}$$

The spacelike portion is the subset of Σ_0 contained within the event horizon,

$$\sigma_0(\Sigma_0, g) \equiv \partial C(\Sigma_0, g) \cap \Sigma_0, \tag{6}$$

which we shall call the *initial patch*.

A. Ensemble of histories and initial conditions

The appearance of an expectation value, $\langle \hat{N}_I \rangle$, in the above definition indicates that we are considering an ensemble of causal patches. Let us take Z identical copies of Σ_0 and pick the same starting point for geodesics. Because of decoherent quantum effects, the resulting Z causal patches will not be identical. For example, the initial vacuum α may decay at different times or into different vacua, etc. [9]. Given initial conditions, the probabilities for different decoherent histories can be computed as usual from local dynamical laws. The expectation value is defined as

$$\langle \hat{N}_I \rangle = \lim_{Z \to \infty} Z^{-1} \sum_{\nu=1}^{Z} N_I(\nu), \tag{7}$$

where $N_I(\nu)$ is the number of times the outcome I occurs in

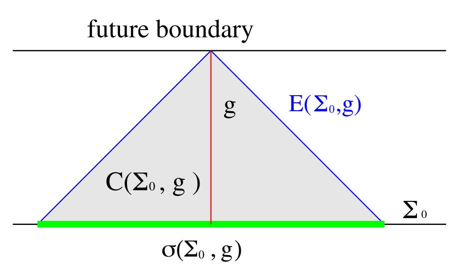

FIG. 2 (color online). A geodesic g starting from an initial surface Σ_0 defines a causal patch C (shaded region), event horizon, E, and initial patch σ.

the νth causal patch. We assume that any observations of interest involve large enough observers or apparatuses that $N_I(\nu)$ (as well as the spacetime geometry) is definite in each decoherent history. This is certainly true for all observations we make.

In general, the ensemble average, $\langle \hat{N}_I \rangle$, will depend on the choice of initial conditions. A theory of initial conditions might instruct us to start in one particular initial state and no other, as was implicitly assumed above. In general, however, it may define an ensemble of initial conditions. For example, it may tell us to start in the empty metastable de Sitter vacuum α with probability $p_\alpha^{(0)}$, with $\sum_\alpha p_\alpha^{(0)} = 1$.[5] In this case, we should enlarge the ensemble of Eq. (7) and include a weighted average over of initial conditions. Equation (7) still holds, but instead of constructing all Z causal patches from the same initial surface Σ_0, we construct $Zp_\alpha^{(0)}$ patches from an initial surface Σ_0^α which is in vacuum α. More generally, the initial patch could be in a terminal vacuum, or it may contain matter and radiation or more than one vacuum; in this case the sum would run over a larger class of possible initial regions. Such refinements will not play an important quantitative role in this paper, assuming only that the initial conditions have nonzero support in at least one long-lived metastable vacuum.

If the initial vacuum is a long-lived metastable de Sitter vacuum α, then the size of the initial patch $\sigma_0(\Sigma_0^\alpha, g_\alpha)$ is essentially independent of the future evolution (Fig. 2). Its boundary is given by the event horizon of the de Sitter space α, a sphere of radius $H_\alpha^{-1} = (\Lambda_\alpha/3)^{-1/2}$. This holds true even if the geodesic later enters a vacuum with very small cosmological constant, like ours. The area of the event horizon will become large, but only after the decay. If the decay happens h Hubble times after Σ_0, it will change the horizon size on Σ_0^α by an amount of order $\exp(-h)$ relative to the event horizon of an eternal de Sitter space with cosmological constant Λ_α. For generic metastable vacua, h is typically exponentially large, so the horizon area on Σ_0^α has radius H_α^{-1} to superexponential accuracy, independently of future decays.

Because of this property, we may choose Σ_0^α in the above ensemble to be as small as a single horizon volume, or *patch of type* α, which we denote as $\mathring{\alpha}$. Geometrically, it is defined as a three-dimensional ball of radius H_α^{-1} with Euclidean metric, i.e., as the interior of the event horizon on a spatially flat slice of de Sitter space with cosmological constant Λ_α. Its proper spatial volume is

$$v_\alpha = \frac{4\pi}{3} H_\alpha^{-3}. \tag{8}$$

(The flat three-geometry is chosen for later convenience: The interior of most horizon regions of metastable vacua on surfaces of constant light-cone time is indeed flat to great accuracy.)

B. Global representation of the ensemble

We have defined probabilities in terms of an ensemble of causal patches, averaging both over initial conditions and over decoherent histories. It is easy to see that one can represent the ensemble of Z distinct causal patches in a single large geometry, by enlarging the initial surface Σ_0 to include Z nonoverlapping horizon volumes, of which a fraction $p_\alpha^{(0)}$ is in vacuum α. Let us write this schematically as

$$\Sigma_0 \supset \sum_\alpha (Zp_\alpha^{(0)})\mathring{\alpha}. \tag{9}$$

By constructing one causal patch from each initial patch $\mathring{\alpha}$ (Fig. 3), one recovers the ensemble that appears in Eq. (7). In this representation, events $N_I(\nu)$ can be thought of as occurring in the same universe for different ν (though they will not all be accessible to the same observer).

Conversely, we can regard any large initial hypersurface Σ_0, along with a set of timelike geodesics originating from Σ_0, as defining an ensemble of initial conditions for the causal patch measure. For example, let Σ_0 be spatially flat, containing a volume $\bar{Z}p_\alpha^{(0)} v_\alpha$ of each de Sitter vacuum α, where $\bar{Z}$ is very large. The region occupied by vacuum α need not be connected, but we will assume that each portion has volume much greater than v_α, so that boundary effects[6] can be neglected. (This assumption will be satisfied on the surfaces of constant light-cone time that we will consider as initial surfaces below.) In addition to Σ_0, we must specify the Z points at which orthogonal geodesics should be erected, defining Z causal patches. If we choose these points to form, say, a rectangular grid, with spacing $2H_\alpha^{-1}$ in regions of vacuum α, then we will have defined an ensemble consisting of $Zp_\alpha^{(0)}$ nonoverlapping causal patches starting with vacuum α, where $Z/\bar{Z}$ is a number of order unity that depends on the grid shape and does not affect relative probabilities.

Since we have already assumed that boundary effects are not important, we can be sure that the statistical properties of the ensemble will not change if we increase the density of geodesics, for example, by including another geodesic

[5]We will aim to use lower case variables (e.g., p) and indices (such as $i, j, \ldots$) when referring to vacua. Greek indices $\alpha, \beta, \ldots$ refer specifically to metastable de Sitter vacua; the longest-lived metastable vacuum is called $*$. Indices $m, n, \ldots$ refer to terminal vacua (vacua with $\Lambda \le 0$). We will use capitalized variables ($N, P, \ldots$) and indices ($I, J, \ldots$) to refer to events.

[6]Regions of different vacua are separated by two-dimensional boundaries. Near the boundaries, general relativity imposes nontrivial constraints on the geometry and extrinsic curvature of Σ_0. Physically, a boundary will typically consist of a domain wall that typically expands into the region of higher cosmological constant.

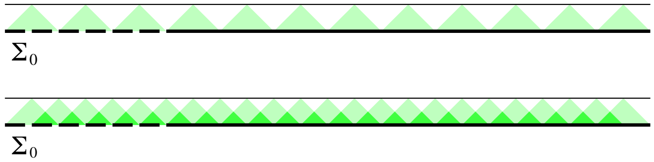

FIG. 3 (color online). An ensemble of causal patches (shaded triangles) can be represented in a single large geometry (upper panel). Suppose that initial conditions require starting in one of two particular de Sitter vacua, with probability $p_1^{(0)} = 0.25$ and $p_2^{(0)} = 0.75$. Let Σ_0 be a spacelike hypersurface containing a very large number of both types of de Sitter horizon regions, so that we can choose large numbers $Zp_1^{(0)}$ (dashed lines) and $Zp_2^{(0)}$ (solid lines) of nonoverlapping initial patches. Then relative probabilities for events of type I and J are given directly by the ratio N_I/N_J of the numbers of such events in the causal patch regions. Conversely, any Σ_0 and set of geodesics emanating from it defines an ensemble of causal diamonds. Increasing the density of geodesics enlarges the ensemble (bottom panel); an event occurring, say, in two different patches counts twice. If each vacuum region contains many horizon volumes, this will not change the statistical properties of the ensemble.

midway between any pair of neighboring starting points on Σ_0 (Fig. 3). The patches will now overlap, and the same event may be counted by more than one patch. But each event will be overcounted by the same factor, so this will not affect relative probabilities. More generally, relative probabilities will be unchanged as long as the density of geodesics in regions of vacuum α is given by

$$\rho_\gamma(0) = z/v_\alpha, \tag{10}$$

for any $z \geq 1$. It is convenient to renormalize Eq. (7):

$$\langle \hat{N}_I \rangle = z^{-1} \lim_{Z\to\infty} Z^{-1} \sum_{\nu=1}^{Zz} N_I(\nu). \tag{11}$$

This allows us to take the limit $z \to \infty$ without changing relative probabilities or encountering divergences.

C. Probability as initial volume

The geometric picture we have developed for the ensemble average allows us to represent the probability for a certain type of event in terms of volumes on Σ_0. Consider a particular event Q of type I, as shown in Fig. 1. This event will be included in any causal patch whose generating geodesic g enters the chronological future of Q, $I^+(Q)$. Therefore, its total probability is proportional to the number of geodesics entering $I^+(Q)$. If we had chosen to place geodesics at a fixed density per proper volume on Σ_0, the probability of Q would thus be proportional to the volume, $\epsilon(Q)$, on Σ_0, of those geodesics that enter $I^+(Q)$. Since we have instead chosen to consider a fixed number of geodesics per horizon patch $\mathring{\alpha}$, the probability of Q is equal to the *patch number* $\pi(Q)$:

$$\hat{P}(Q) = (Zz)^{-1}\pi(Q), \tag{12}$$

where the patch number is defined as the fraction of a patch, on Σ_0, taken up by the starting points of the geodesics that enter $I^+(Q)$:

$$\pi(Q) \equiv \frac{\epsilon(Q)}{v_\alpha}. \tag{13}$$

In other words, π is the volume of the starting points measured in units of the horizon volume given in Eq. (8).

Because any two nonoverlapping horizon patches on Σ_0 are likely to remain causally disconnected, their causal patches cannot both contain Q. Therefore we have $\pi(Q) \lesssim 1$. If Q occurs after many Hubble times of de Sitter expansion, then $\pi(Q)$ will be exponentially small. Therefore, we can neglect the probability that the starting points cover more than one vacuum; indeed, Eq. (13) assumes that all geodesics that enter the future of Q started in the same vacuum.

Since $\pi(Q)$ is independent of Z, any individual event Q will have vanishing probability in the large Z limit. But we are interested in the probability for events of type I, not just in one particular instance of such an event. In the global picture of eternal inflation, events of any type will occur infinitely many times in the future of Σ_0. The probability for an event of type I, according to the global representation of the causal patch measure we have developed, is the sum of the patch number of each instance (Fig. 1):

$$\hat{P}_I \propto \sum_{Q\in I} \hat{P}(Q), \tag{14}$$

where the sum is over all events of type I and $\hat{P}(Q)$ is defined in Eq. (12). The notation "$\propto$" indicates that an I-independent normalization factor has been dropped. Thus, Eq. (14) defines relative probabilities for events of type I and J.

III. EQUIVALENCE TO THE LIGHT-CONE TIME CUTOFF

Light-cone time is defined as follows [22]: Let $\gamma(\Sigma_0')$ be the congruence of geodesics orthogonal to the hypersurface Σ_0', and let Q be an event in the future of Σ_0'. The light-cone time t at Q is defined in terms of the patch number[7] $\pi(Q)$, on Σ_0', of the starting points of those geodesics that enter the future of Q, $I^+(Q)$:

$$t(Q) = -\tfrac{1}{2}\log\pi(Q). \tag{15}$$

[7]In Ref. [22], the light-cone time was defined in terms of the proper volume of starting points on Σ_0'. This distinction can be absorbed into a deformation of the initial hypersurface. Because relative probabilities are independent of the choice of Σ_0', they are, in particular, unaffected by this modification. The present choice will serve us better for formal reasons.

In the light-cone cutoff measure, the relative probability of events of type I and type J is defined as the limit

$$\frac{\check{P}_I}{\check{P}_J} = \lim_{t\to\infty} \frac{N_I(t)}{N_J(t)}, \tag{16}$$

where $N_I(t)$ is the number of events Q_I of type I whose light-cone time is less than t. We will now show that this measure is equivalent to the causal patch measure defined in the previous section, with a suitable choice of initial hypersurface Σ_0.

The main ingredient of this proof is the following assumption: At late times, the number of events of any type I grows at the same universal exponential rate,

$$\langle N_I \rangle = \check{N}_I e^{\gamma t} + O(e^{\varphi t}), \tag{17}$$

with $0 < \gamma < 3$, up to subdominant effects, $\varphi < \gamma$, whose relative contribution can be neglected at late times. Moreover, the number of horizon patches of metastable vacua grows at the same universal rate:

$$\langle n_\alpha \rangle = \check{n}_\alpha e^{\gamma t} + O(e^{\varphi t}). \tag{18}$$

We will later justify this assumption rigorously and derive the values of $\check{N}_I$ and γ from parameters of the landscape. For now, we may take universal exponential growth to be a defining characteristic of eternal inflation.

By Eqs. (16) and (17), the light-cone measure gives probabilities

$$\check{P}_I \propto \check{N}_I. \tag{19}$$

Let us compare this to the causal patch measure with initial conditions defined in the manner described in Sec. II B. Specifically, we consider the ensemble of patches generated by the geodesics in γ, starting from a large hypersurface Σ_0 which we take to be a surface of constant light-cone time $t_0 > 0$.

Consider the geodesics that enter the future light-cone of a point $Q \in \Sigma_0$. By definition, they occupy a patch number $\pi = e^{-3t_0}$ on Σ_0'. Moreover, if Q lies in a vacuum de Sitter region,[8] then the same geodesics occupy exactly 1 horizon patch on Σ_0, and are orthogonal to Σ_0. (This follows, for example, from the arguments given in Ref. [22], which apply in the vacuum limit.) Therefore, if we started with z' geodesics per horizon volume on Σ_0', there will be

$$z = z'\pi(t_0) = z' e^{-3t_0} \tag{20}$$

geodesics per horizon volume on Σ_0. In particular, the number of geodesics per horizon volume is constant on Σ_0, so the construction summarized in Eq. (11) can be applied.

[8] We shall find in the following section that this is the case for all but a superexponentially small fraction of the volume of Σ_0, which can be neglected at this stage. This does not mean that we will be neglecting regions containing matter when we count events.

Let us choose t_0 so large that the correction term in Eq. (18) can be neglected. Then the surface Σ_0 will satisfy Eq. (9) with $p_\alpha^{(0)} \propto \check{n}_\alpha$. By Eq. (18), increasing t_0 any further is equivalent to increasing Z in Eq. (9), so it leaves relative probabilities untouched.

By Eq. (14), the causal patch measure defines relative probabilities

$$\hat{P}_I \propto \int_{t_0}^\infty dt \frac{d\langle N_I(t)\rangle}{dt} Z^{-1}\pi(t). \tag{21}$$

Note that π depends only on t: Because light-cone time is defined in terms of patch number, the patch number of an event Q depends only on the light-cone time at which it takes place. Substituting Eq. (17), the integral is trivial,

$$\hat{P}_I \propto \int_{t_0}^\infty dt \gamma \check{N}_I e^{(\gamma-3)t} \propto \check{N}_I \propto \check{P}_I, \tag{22}$$

and we find that normalized probabilities are the same as in the light-cone cutoff measure. (We remind the reader that "$\propto$" signifies equality up to I-independent factors, which do not affect relative probabilities.)

IV. PROPERTIES OF THE LIGHT-CONE CUTOFF

In this section we will establish a number of key properties of light-cone time, including the results used in the previous section for the proof of equivalence to the causal patch measure, Eqs. (17) and (18). We will begin with two simple examples and then consider the general case. Since it is clear from the context which quantities should be thought of as expectation values, we will omit the brackets $\langle\rangle$ in the interest of readability.

A. Pure de Sitter

Let us first consider a completely stable vacuum with positive cosmological constant $3H_*^2$, which we call $*$. Strictly, this case is outside the scope of this paper, since there are no terminal vacua, but it provides a useful starting point. The metric of the corresponding de Sitter geometry, in flat coordinates, is

$$ds^2 = -dT^2 + H_*^{-2} e^{2H_* T}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{23}$$

Let us choose Σ_0' to be a finite volume of the hypersurface $T = 0$, with radius $r_0 \gg 1$. The orthogonal congruence γ consists of the comoving worldlines at fixed (r, θ, ϕ). It follows trivially from the symmetries of this choice that surfaces of constant T must also be surfaces of constant light-cone time, but it will be instructive to derive the relation $t(T)$. Consider a point Q at time T; by homogeneity, we can assume $r = 0$ without loss of generality. The future light-cone of Q has comoving radius $e^{-H_* T}$ at future infinity. The proper volume, on Σ_0', of the geodesics entering this light-cone is $\epsilon(Q) = \frac{4\pi}{3H_*^3}\exp(-3H_* T)$. Since the volume of a single horizon patch is $v = \frac{4\pi}{3H_*^3}$, the patch number is $\pi(Q) = \exp(-3H_*T)$, and the light-cone time is $t(Q) = -\frac{1}{3}\log\pi(Q) = H_*T$. In terms of light-cone time, the metric is

$$ds^2 = H_*^{-2}(-dt^2 + e^{2t}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]). \tag{24}$$

It follows that the number of horizon patches is given by

$$n_*(t) = \check{n}_* \exp(3t), \tag{25}$$

with $\check{n}_* = r_0^3$. Pure de Sitter space is in a thermal state, and events occur at a Boltzmann-suppressed rate per Hubble volume and Hubble time. Let κ_{I*} be the rate at which events of type I (e.g., the formation of a Boltzmann brain) occur. Then

$$N_I = \kappa_{I*}n_*(t). \tag{26}$$

Therefore, Eqs. (17) and (18) are satisfied with $\gamma = 3$ and $\check{N}_I = \kappa_{I*}\check{n}_*$.

B. Single metastable vacuum

We have claimed that $\gamma < 3$; this holds in any landscape that has terminal vacua, or sinks. To see this, let us now consider the case of a single metastable de Sitter vacuum, which we call $*$. It can decay into terminal vacua by the nucleation of bubbles, at small dimensionless rates κ_{m*} per Hubble volume and Hubble time. Let us choose the same initial surface as in the previous example of a stable de Sitter vacuum. Wherever the vacuum has not decayed, the metric is described by Eq. (23), and the relation $t = H_*T$ will hold.

Let us find the correction to Eq. (25) due to decays. For small total decay rate $\kappa_* \equiv \sum_m \kappa_{m*} \ll 1$, we can treat decays as a small perturbation of the global geometry; that is, we will work at leading order in κ_*. The expected number of nucleation events dN between the time t and $t + dt$ is given by $\frac{4\pi}{3}\kappa_* H_*^4$ times the enclosed physical four-volume:

$$\frac{dN}{dt} = \kappa_* n_*(t). \tag{27}$$

Note that we are not distinguishing between decays into different terminal vacua at this stage.

Let us assume model parameters such that all initial bubble radii are much smaller than the de Sitter horizon H_*^{-1}. Then the evolution of a bubble can be approximated by the future light-cone of the nucleation event. Again, by homogeneity, we can consider a decay at $r = 0$, at time t_n. At the time t, the bubble will have comoving radius $r_b(t, t_n) = e^{-t_n} - e^{-t}$. It will have destroyed a physical volume $\frac{4\pi}{3H_*^3} r_b^3 \exp(3t)$ of the vacuum $*$, corresponding to

$$\frac{d}{dN}\delta n_* = -(e^{t-t_n} - 1)^3 \tag{28}$$

lost horizon patches per bubble. (Here we have neglected collisions between bubbles, which is legitimate at leading order in κ_*.) This can be written as $\frac{d}{dN}\delta n_* = e^{3(t-t_n)}[1 - O(e^{-(t-t_n)})]$.

It follows that at late times, $t - t_n \gg 1$, the bubble occupies precisely the volume that a single horizon patch at t_n would have expanded to by the time t, up to exponentially small corrections [6]. Thus, we will make a negligible error by assuming that the bubble forms immediately at its asymptotic comoving size, and treating the bubble wall as comoving. This simplifies the derivation of the evolution equation for $n_*(t)$. During a time dt, the de Sitter expansion produces $3n_*(t)dt$ new horizon volumes, and $\kappa_* n_*(t)dt$ horizon patches are lost to decay. Thus,

$$\frac{dn_*}{dt} = (3 - \kappa_*)n_*(t), \tag{29}$$

and it follows that

$$n_*(t) = \check{n}_* e^{(3-\kappa_*)t}. \tag{30}$$

Therefore, Eq. (18) is satisfied with $\gamma = 3 - \kappa_*$ and $\check{n}_* = r_0^3$.

The number of terminal bubbles of type m produced between t and $t + dt$ is

$$\frac{dN_m}{dt} = \kappa_{m*} n_*(t). \tag{31}$$

At late times, all bubbles of type m are statistically equivalent, because their production is a local effect in an empty de Sitter region. Therefore, the expected number of events of type I per bubble, dN_I/dN_m, will depend only on the type of bubble, and on the time since bubble nucleation, $\tau \equiv t - t_n$.

To find the total number of events of type I at late times, we integrate over all types of bubbles and all nucleation times:

$$N_I(t) = \kappa_{I*}n_*(t) + \sum_m \int_0^t \left(\frac{dN_I}{dN_m}\right)_{t-t_n} \left(\frac{dN_m}{dt}\right)_{t_n} dt_n. \tag{32}$$

(The first term is analogous to Eq. (26) and takes into account events that occur in the de Sitter vacuum.) Combining the above equations we find

$$N_I(t) = \left(\kappa_{I*} + \sum_m N_{Im}\kappa_{m*}\right) n_*(t), \tag{33}$$

where

$$N_{Im} \equiv \int_0^\infty d\tau e^{-\gamma\tau} \left(\frac{dN_I}{dN_m}\right)_\tau \tag{34}$$

is independent of time. Therefore, Eq. (17) is satisfied with

$$\check{N}_I = \left(\kappa_{I*} + \sum_m N_{Im}\kappa_{m*}\right)\check{n}_*. \tag{35}$$

The upper limit of integration in Eq. (34) should strictly be t, so this result is valid only at late times, but this is the only regime relevant for computing relative probabilities. For the measure to be well-defined, the indefinite integral must converge. This will be the case if dN_I/dN_m diverges nowhere and grows less rapidly than $e^{\gamma\tau}$ at large τ. If the terminal vacuum m has negative cosmological constant, then these conditions are satisfied. Although events can arise with fixed density on infinite spatially open hypersurfaces inside the bubble, at any finite τ only a finite portion of every open slice is included, so the integral is finite for finite t. At late times, the size of this portion will grow no faster than $\exp(2\tau)$. For small κ_*, this is slower than $\exp(\gamma\tau)$, so the integral remains finite as $t \to \infty$. This explains why the "edges" of the bubble do not contribute a divergence. Near the center, the same conclusion follows from the fact that vacua with negative cosmological constant crunch after a finite proper time. (Light-cone time is formally infinite at the singularity, but it will be finite one Planck time before the big crunch, where the semiclassical description breaks down.)

However, if the vacuum m has vanishing cosmological constant, and if it contains events of type I, then N_{Im} can diverge. In this case, the light-cone cutoff does not succeed in regularizing the spacetime. Possible resolutions are discussed in Ref. [22]. The potential divergences in $\Lambda = 0$ vacua do not affect our claim of equivalence to the causal patch cutoff, since the latter would encounter the same divergence [37,57]. For the purposes of this paper, we will exclude the interiors of $\Lambda = 0$ bubbles (defined more rigorously as "hat domains" in Ref. [22]). This means we will be computing relative probabilities for events not occurring in such regions.

C. General landscape

Consider a theory such as the string landscape, which contains metastable de Sitter vacua $\alpha, \beta, \ldots$ and terminal vacua $m, n, \ldots$. We will assume that the metastable vacua are long-lived, $\kappa_\alpha \ll 1$; states that do not satisfy this condition can be treated as excited states in the vacua they decay into. In this limit, and for the purpose of computing the abundances of horizon patches of each metastable vacuum, n_α, we may neglect transitory effects such as bubble expansion and the initial presence of matter and radiation, which affect the size and growth of de Sitter regions only in an exponentially small fraction of their lifetime and volume. The analysis preceding Eq. (29) now yields the rate equation

$$\frac{dn_\alpha}{dt} = (3 - \kappa_\alpha)n_\alpha + \sum_\beta \kappa_{\alpha\beta} n_\beta. \tag{36}$$

The first term corresponds to the de Sitter expansion and to the loss of horizon patches due to the decay of the vacuum α. The final sum, which did not appear in the previous subsection, describes the production of α patches by other metastable vacua β.

This matrix equation takes exactly the same form[9] as Eq. (37) in Ref. [6], and it has the same mathematical solution, which takes the form given in Eq. (18):

$$n_\alpha(t) = \check{n}_\alpha e^{\gamma t} + O(e^{\varphi t}). \tag{37}$$

Here γ is the largest eigenvalue of the matrix $M_{\alpha\beta}$, and $\check{n}_\alpha$ is the corresponding eigenvector; φ is the second-largest eigenvalue. Arguments given in the appendices of Ref. [6] generalize straightforwardly to show that $\varphi < \gamma < 3$.

Since the decay of metastable vacua is an exponentially suppressed tunneling process, the decay rates will vary enormously, and there is generically one vacuum with much longer life time than all others. We will call this the dominant vacuum, $*$. A straightforward generalization of arguments presented in Ref. [26] shows that the above eigenvector is dominated by the $*$ vacuum, and the associated eigenvalue is related to its total decay rate, κ_*,

$$\check{n}_\alpha \approx \delta_{\alpha *}, \qquad \gamma \approx 3 - \kappa_*, \tag{38}$$

to exponentially good approximation.

We conclude that at late times, the number of patches of every vacuum grows at a universal rate, governed by the decay rate of the longest-lived metastable vacuum. Since the growth is exponential, this asymptotic regime will completely dominate over all earlier transitory regimes, and we can compute probabilities from it alone. Therefore, we may as well assume that the initial surface Σ_0' is already in the asymptotic regime, allowing us to drop terms of order $e^{\varphi t}$ and smaller.

To obtain an expression for the number of events of type I and derive Eq. (17), we can now proceed in close analogy with Eqs. (31)–(35). At the time t, bubbles of type i are produced at the rate

$$\frac{dN_i}{dt} = \sum_\alpha \kappa_{i\alpha} n_\alpha(t). \tag{39}$$

The total number of events of type I is

$$N_I(t) = \kappa_{I*} n_*(t) + \sum_{i \neq *}^{i} \int_0^t \left(\frac{dN_I}{dN_i}\right)_{t-t_n} \left(\frac{dN_i}{dt}\right)_{t_n} dt_n \tag{40}$$

[9] However, the equation is for a different physical variable: In Ref. [6], it is for the volume occupied by the vacuum α; here it is for the number of horizon patches of vacuum α. Consequently, the dominant vacuum we find below is exactly the same as the vacuum dominating the scale factor cutoff. But because of the difference in measures, it dominates in a different sense: In the light-cone cutoff, it dominates the number of horizon patches, whereas in the scale factor measure it dominates the proper volume. (There is another distinction, which is trivial: The term $3n_\alpha$ on the right-hand side is absent in Ref. [6], because the volume fractions rather than total volume are described.)

$$= \left(\kappa_{I*}\check{n}_* + \sum_i \sum_\alpha N_{Ii}\kappa_{i\alpha}\check{n}_\alpha\right)e^{\gamma t}, \quad (41)$$

where

$$N_{Ii} \equiv {}^i\!\int_0^\infty d\tau e^{-\gamma\tau}\left(\frac{dN_I}{dN_i}\right)_\tau. \quad (42)$$

To avoid overcounting, the integral should run only over a single bubble of vacuum i, excluding regions of other vacua nucleated inside the i bubble; this restriction is denoted by index i appearing on the upper left of the integration symbol. We conclude that Eq. (17) is satisfied with $\check{N}_I = (\kappa_{I*}\check{n}_* + \sum_i \sum_\alpha N_{Ii}\kappa_{i\alpha}\check{n}_\alpha)$.

The integral N_{Im} will be finite, and the measure well defined, under the condition identified in the previous subsection: the absence of $\Lambda = 0$ vacua or at least of observations therein. In particular, there is no divergence associated with the thermal productions of events at late times in metastable vacua α, since the number of such events in a single bubble grows like the number of horizon patches, which is by definition slower than the growth rate $e^{\gamma\tau}$ of the dominant vacuum.

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