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Quantum light-cone fluctuations in compactified spacetimes

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We treat the effects of compactified spatial dimensions on the propagation of light in the uncompactified directions in the context of linearized quantum gravity. We find that the flight times of pulses can fluctuate due to modification of the graviton vacuum by the compactification. In the case of a five-dimensional Kaluza-Klein theory, the mean variation in flight time can grow logarithmically with the flight distance. This effect is in principle observable, but too small to serve as a realistic probe of the existence of extra dimensions. This differs from the conclusion reached in an earlier work. We also examine the effect of the compactification on the widths of spectral lines, and find that there is a small line narrowing effect. This effect is also small for compactification well above the Planck scale, but might serve as a test of the existence of extra dimensions.

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I. INTRODUCTION

One of the key features expected in quantum theories of gravity is fluctuation of the classical light cone and possible effects on signal propagation. This possibility has been discussed in several contexts by numerous authors [1–11]. One approach is to study variations in the flight times of light pulses between a source and a detector [12–16]. This approach was used in Ref. [14] to study the effects of boundaries and periodic compactification of one space dimension. In particular, it was found that the fluctuation in flight time of a pulse propagating parallel to a plane boundary, or in a direction transverse to the compactified dimension, will tend to grow as the flight distance increases. This growth can be viewed as a cumulative effect of spacetime geometry fluctuations modified by the boundary or the compactification.

This effect is analogous to the local Casimir effect, whereby a boundary modifies the two-point function of the quantized electromagnetic field. This modification can produce observable effects, such as a force on an atom. This force was predicted theoretically by Casimir and Polder [17] in 1948, and measured experimentally by Sukenik *et al.* [18] in 1993. Another effect of modified electromagnetic vacuum fluctuations can be Brownian motion of charged test particles [19–22]. The presence of a boundary can alter the mean squared velocity or position of the particle. This modification is analogous to the effects of light-cone fluctuations when spacetime geometry fluctuations are modified.

In the present paper, we will be concerned with modification of light-cone fluctuations due to compact extra dimensions. Theories with extra dimensions were introduced into physics by Kaluza [23] and Klein [24], and have been the topic of many papers in recent years [25]. We wish to address the question of whether compact extra dimensions can give rise to observable effects on the propagation of light rays in the uncompactified directions. Such an effect could be a potential test for the existence of extra dimensions. A secondary purpose of this paper is to discuss quantization of linearized gravity in arbitrary numbers of flat spacetime dimensions and to give explicit expression for the graviton two-point functions in the transverse-tracefree (TT) gauge. In effect, we are searching for modifications of quantum effects in the uncompactified dimensions due to the presence of the extra dimensions. A somewhat different effect, whereby extra dimensions lead to changes in Casimir forces has recently been discussed by Cheng and others [26,27]. The present paper corrects earlier work by two of us, Refs. [15,28], in which a larger effect was claimed.

The outline of this paper is as follows: In Sec. II, we review how quantized linear metric perturbations may give rise to variations in flight times of pulses. In Sec. III, we discuss quantization of linearized metric perturbations and compute the graviton two-point function in the transverse-tracefree gauge for flat spacetime of arbitrary dimension. In particular, in Sec. III C we give a new derivation of the formula for Δt, the mean flight time variation. This result is used in Sec. IV to study light-cone fluctuations in a five-

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dimensional spacetime with one compact dimension. Some results for more than one compact dimension are also summarized. We also examine the issue of the correlation of successive pulses in this model in Sec. IV B. We turn to a different measure of light-cone fluctuations, the possible broadening of spectral lines, in Sec. V. Our results are discussed in Sec. VI. Appendix A provides a detailed treatment of the graviton two-point functions.

II. LIGHT-CONE FLUCTUATIONS AND FLIGHT TIME VARIATIONS

To begin, let us examine a $d = 4 + n$ dimensional flat spacetime with n extra dimensions. Consider a flat background spacetime with a linearized perturbation $h_{\mu\nu}$ propagating upon it, so the spacetime metric may be written as

$$ds^2 = (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu = -dt^2 + d\mathbf{x}^2 + h_{\mu\nu}dx^\mu dx^\nu, \tag{1}$$

where the indices μ, ν run through $0, 1, 2, 3, \ldots, 3 + n$. Let $\sigma(x, x')$ be one half of the squared geodesic distance between a pair of spacetime points x and x', and $\sigma_0(x, x')$ be the corresponding quantity in the flat background. In the presence of a linearized metric perturbation, $h_{\mu\nu}$, we may expand $\sigma = \sigma_0 + \sigma_1 + O(h^2_{\mu\nu})$. Here σ_1 is first order in $h_{\mu\nu}$. If we quantize $h_{\mu\nu}$, then quantum gravitational vacuum fluctuations will lead to fluctuations in the geodesic separation, and therefore induce light-cone fluctuations. In particular, we have $\langle\sigma_1^2\rangle \neq 0$, since σ_1 becomes a quantum operator when the metric perturbations are quantized. The quantum light-cone fluctuations give rise to fluctuations in the speed of light, which may produce a time delay or advance Δt in the arrival times of pulses.

We are concerned with how light-cone fluctuations characterized by $\langle\sigma_1^2\rangle$ are related to physical observable quantities. For this purpose, let us consider the propagation of light pulses between a source and a detector separated by a distance r on a flat background with quantized linear perturbations. In a perturbed spacetime, the pulse will travel on a path for which $\sigma = 0$, so that $\sigma_0 = -\sigma_1$ to leading order. For a pulse which is delayed by time Δt, which is much less than r, we have

$$\sigma_0 = \frac{1}{2}[-(r + \Delta t)^2 + r^2] \approx -r\Delta t, \tag{2}$$

in the coordinates of the unperturbed spacetime. This leads to

$$\Delta t = \frac{\sigma_1}{r}. \tag{3}$$

Square the above equation and take the average over a given quantum state of gravitons $|\phi\rangle$ (e.g. the vacuum states associated with compactification of spatial dimensions),

$$\Delta t_\phi^2 = \frac{\langle\phi|\sigma_1^2|\phi\rangle}{r^2}. \tag{4}$$

This result is, however, divergent due to the formal divergence of $\langle\phi|\sigma_1^2|\phi\rangle$. One can define an observable Δt by subtracting from Eq. (4) the corresponding quantity, Δt_0^2, for the vacuum state as follows:

$$\Delta t^2 = \Delta t_\phi^2 - \Delta t_0^2 = \frac{\langle\phi|\sigma_1^2|\phi\rangle - \langle 0|\sigma_1^2|0\rangle}{r^2} \equiv \frac{\langle\sigma_1^2\rangle_R}{r^2}. \tag{5}$$

In this case, we are dealing with the shift in the light-cone fluctuations due to a change in quantum state or spacetime topology. We do not attempt to treat the vacuum state of uncompactified Minkowski spacetime, but rather the dependence of the light-cone fluctuations on some parameter which can be varied. Therefore, the root-mean-squared deviation from the classical propagation time is given by

$$\Delta t = \frac{\sqrt{\langle\sigma_1^2\rangle_R}}{r}. \tag{6}$$

Note that Δt is the ensemble averaged deviation, not necessarily the expected variation in flight time, δt, of two pulses emitted close together in time. The latter is given by Δt only when the correlation time between successive pulses is less than the time separation of the pulses. This can be understood physically as due to the fact that the gravitational field may not fluctuate significantly in the interval between the two pulses. This point is discussed in detail in Ref. [13]. These stochastic fluctuations in the apparent velocity of light arising from quantum gravitational fluctuations are in principle observable, since they may lead to a spread in the arrival times of pulses from distant sources.

In order to find Δt in a particular situation, we need to calculate the quantum expectation value $\langle\sigma_1^2\rangle_R$ in any chosen quantum state $|\psi\rangle$, which can be shown to be given by [12,14]

$$\langle\sigma_1^2\rangle_R = \frac{1}{8}(\Delta r)^2 \int_{r_0}^{r_1} dr \int_{r_0}^{r_1} dr' n^\mu n^\nu n^\rho n^\sigma G^R_{\mu\nu\rho\sigma}(x, x'). \tag{7}$$

Although the previous derivations in Ref. [14] were given in 3 + 1 dimensions, the generalization to arbitrary dimensions is straightforward. Here $dr = |d\mathbf{x}|$, $\Delta r = r_1 - r_0$ and $n^\mu = dx^\mu/dr$. The integration is taken along the null geodesic connecting two points x and x', and

$$G^R_{\mu\nu\rho\sigma}(x, x') = \langle\psi|h_{\mu\nu}(x)h_{\rho\sigma}(x') + h_{\mu\nu}(x')h_{\rho\sigma}(x)|\psi\rangle \tag{8}$$

is the graviton Hadamard function, understood to be suitably renormalized. The gauge invariance of Δt, as given by Eq. (6), was analyzed in Ref. [14]. An alternative derivation which makes the gauge invariance more obvious is given in Sec. III C.

III. QUANTIZATION IN THE TRANSVERSE-TRACEFREE GAUGE

A. Minkowski spacetimes

We will use a quantization of the linearized gravitational perturbations $h_{\mu\nu}$ in flat spacetime with arbitrary dimension which retains only physical degrees of freedom. That is, we are going to work in a TT gauge defined by

$$h = h^{\mu}_{\mu} = 0, \qquad \partial^{\mu} h_{\mu\nu} = 0, \quad \text{and} \quad u^{\mu} h_{\mu\nu} = 0, \quad (9)$$

where u^{μ} is a timelike vector. In the frame of reference in which $u^{\mu} = (1, 0, 0, 0)$, the gravitational perturbations have only spatial components h_{ij}, satisfying the transverse, $\partial^i h_{ij} = 0$, and tracefree, $h^i_i = 0$ conditions. Here i, j run from 1 to $3 + n = d - 1$. These $2d$ conditions remove all of the gauge degrees of freedom and leave $\frac{1}{2}(d^2 - 3d)$ physical degrees of freedom. We write the quantized gravitational perturbation operator as

$$h_{ij} = \sum_{\mathbf{k},\lambda} [a_{\mathbf{k},\lambda} e_{ij}(\mathbf{k}, \lambda) f_{\mathbf{k}} + \text{H.c.}]. \qquad (10)$$

Here H.c. denotes the Hermitian conjugate, λ labels the $\frac{1}{2}(d^2 - 3d)$ independent polarization states, $f_{\mathbf{k}}$ is the mode function, and the $e_{\mu\nu}(\mathbf{k}, \lambda)$ are polarization tensors. The graviton creation and annihilation operators satisfy the usual commutation relation:

$$[a_{\mathbf{k},\lambda}, a^{\dagger}_{\mathbf{k}',\lambda'}] = \delta_{\mathbf{k},\mathbf{k}'} \delta_{\lambda,\lambda'}. \qquad (11)$$

This relation may be taken to be the fundamental quantization postulate. Units in which $32\pi G_d = 1$, where G_d is Newton's constant in d dimensions, and in which $\hbar = c = 1$ will be used in this paper, except as otherwise noted.

Let us now calculate the Hadamard function, $G_{\mu\nu\rho\sigma}(x, x')$, for gravitons in the Minkowski vacuum state in the transverse-tracefree gauge. (By Minkowski we mean flat spacetime with all dimensions uncompactified.) It follows that

$$G_{ijkl}(x, x') = \frac{2\text{Re}}{(2\pi)^{d-1}} \int \frac{d^{d-1}\mathbf{k}}{2\omega} \sum_{\lambda} e_{ij}(\mathbf{k}, \lambda) e_{kl}(\mathbf{k}, \lambda) \times e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} e^{-i\omega(t-t')}. \qquad (12)$$

The summation of polarization tensors in the transverse-tracefree gauge can be found using the tensorial argument in the appendix of Ref. [14].

$$\begin{aligned}\sum_{\lambda} e_{ij}(\mathbf{k}, \lambda) e_{kl}(\mathbf{k}, \lambda) = {} & \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{d-2}\delta_{ij}\delta_{kl} \\ & + \frac{2(d-3)}{d-2} \hat{k}_i \hat{k}_j \hat{k}_k \hat{k}_l \\ & + \frac{2}{d-2}(\hat{k}_i \hat{k}_j \delta_{kl} + \hat{k}_k \hat{k}_l \delta_{ij}) \\ & - \hat{k}_i \hat{k}_l \delta_{jk} - \hat{k}_i \hat{k}_k \delta_{jl} - \hat{k}_j \hat{k}_l \delta_{ik} \\ & - \hat{k}_j \hat{k}_k \delta_{il}, \end{aligned} \qquad (13)$$

where $\hat{k}_i = \frac{k_i}{k}$. We find that

$$\begin{aligned} G_{ijkl} = {} & \frac{2}{d-2}(2F_{ij}\delta_{kl} + 2F_{kl}\delta_{ij}) - 2F_{ik}\delta_{jl} - 2F_{il}\delta_{jk} \\ & - 2F_{jl}\delta_{ik} - 2F_{jk}\delta_{il} + \frac{4(d-3)}{d-2} H_{ijkl} \\ & + 2D(x, x')\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{d-2}\delta_{ij}\delta_{kl}\right). \end{aligned} \qquad (14)$$

Here $D(x, x')$, $F_{ij}(x, x')$ and $H_{ijkl}(x, x')$ are functions which are defined as follows:

$$D^n(x, x') = \frac{\text{Re}}{(2\pi)^{3+n}} \int \frac{d^{3+n}\mathbf{k}}{2\omega} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} e^{-i\omega(t-t')}, \qquad (15)$$

$$F^n_{ij}(x, x') = \frac{\text{Re}}{(2\pi)^{3+n}} \partial_i \partial'_j \int \frac{d^{3+n}\mathbf{k}}{2\omega^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} e^{-i\omega(t-t')}, \qquad (16)$$

and

$$H^n_{ijkl}(x, x') = \frac{\text{Re}}{(2\pi)^{3+n}} \partial_i \partial'_j \partial_k \partial'_l \int \frac{d^{3+n}\mathbf{k}}{2\omega^5} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} e^{-i\omega(t-t')}. \qquad (17)$$

These functions are calculated in Appendix A.

B. Flat spacetimes with periodic compactification

Let us now suppose that the extra n dimensions $z_1, \ldots, z_n$ are compactified with periodicity lengths $L_1, \ldots, L_n$, namely, spatial points z_i and $z_i + L_i$ are identified. For simplicity, we shall assume in this paper that $L_1 = \ldots = L_n = L$. The effect of imposition of the periodic boundary conditions on the extra dimensions is to restrict the field modes to a discrete set

$$f_{\mathbf{k}} = (2\omega(2\pi)^3 L^n)^{-(1/2)} e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}, \qquad (18)$$

with

$$k_i = \frac{2\pi m_i}{L}, \qquad i = 1, \ldots, n, \qquad m_i = 0, \pm 1, \pm 2, \pm 3, \ldots. \qquad (19)$$

Let us denote the associated vacuum state by $|0_L\rangle$. In order to calculate the gravitational vacuum fluctuations due to compactification of extra dimensions, we need the renormalized graviton Hadamard function with respect to the vacuum state $|0_L\rangle$, $G^R_{\mu\nu\rho\sigma}(x, x')$, which is given by a multiple image sum of the corresponding Hadamard function for the Minkowski vacuum, $G_{\mu\nu\rho\sigma}$:

$$G^R_{\mu\nu\rho\sigma}(t, z_i, t', z'_i) = \prod_{i=1}^{n} \sum_{m_i=-\infty}^{+\infty\prime} G_{\mu\nu\rho\sigma}(t, z_i, t', z'_i + m_i L). \tag{20}$$

Here the prime on the summation indicates that the $m_i = 0$ term is excluded and the notation

$$(t, \vec{x}, z_1, \ldots, z_n) \equiv (t, z_i) \tag{21}$$

has been adopted.

We are mainly concerned about how light-cone fluctuations arise in the usual uncompactified space as a result of compactification of extra dimensions. So we shall examine the case of a light ray propagating in one of the uncompactified dimensions. Take the direction to be along the x-axis in our four-dimensional world, then the relevant graviton two-point function is G_{xxxx}, which can be expressed as

$$\begin{aligned} G_{xxxx}(t, \vec{x}, z_i, t', \vec{x}', z'_i) = \frac{4(n+1)}{n+2}[&D(t, \vec{x}, z_i, t', \vec{x}', z'_i) \\ &- 2F_{xx}(t, \vec{x}, z_i, t', \vec{x}', z'_i) \\ &+ H_{xxxx}(t, \vec{x}, z_i, t', \vec{x}', z'_i)]. \end{aligned} \tag{22}$$

Assuming that the propagation goes from point $(a, 0, \ldots, 0)$ to point $(b, 0, \ldots, 0)$, we have

$$\begin{aligned} \langle\sigma_1^2\rangle &= \frac{1}{8}(b-a)^2 \int_a^b dx \int_a^b dx' G^R_{xxxx}(t, x, \mathbf{0}, t', x', \mathbf{0},) \\ &= \frac{1}{8}(b-a)^2 \int_a^b dx \int_a^b dx' \\ &\quad \times \prod_{i=1}^{n} \sum_{m_i=-\infty}^{+\infty\prime} G_{xxxx}(t, x, \mathbf{0}, t', x', 0, 0, m_1 L, \ldots, m_i L). \end{aligned} \tag{23}$$

With these results, we can in principle calculate light-cone fluctuations in spacetimes with an arbitrary number of flat extra dimensions. Recall that we are working in units in which the Newton's constant in d dimensions is $G_d = (32\pi)^{-1}$. In final results, it will be useful to convert to more familiar units using the relation that

$$\ell_P^2 = G_4 = G_d L^{4-d}, \tag{24}$$

where $\ell_P \approx 10^{-34}$ cm is the Planck length.

C. An alternative derivation of Δt

In this section, we wish to rederive Δt using the geodesic deviation equation. This derivation allows us to see the gauge invariance more clearly, and to discuss the issue of Lorentz invariance of light-cone fluctuations. Let us consider a pair of timelike geodesics with tangent vector u^μ, and n^μ as a unit spacelike vector pointing from one geodesic to the other (see Fig. 1). The geodesic deviation equation is given by

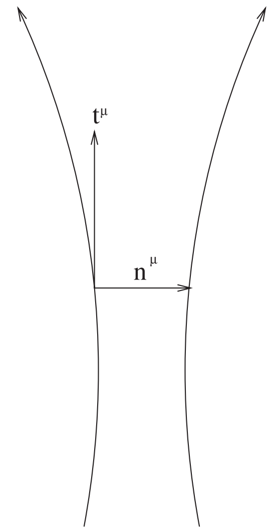

FIG. 1. A pair of nearby timelike geodesics. Here u^μ is a tangent vector along the geodesic, while n^μ is a unit spacelike vector pointing from one geodesic to the other.

$$\frac{D^2 n^\mu}{d\tau^2} = -R^\mu_{\alpha\nu\beta} u^\alpha n^\nu u^\beta, \tag{25}$$

where $R^\mu_{\alpha\nu\beta}$ is the Riemann tensor. The relative acceleration per unit proper length of particles on the neighboring geodesics is

$$\alpha \equiv n_\mu \frac{D^2 n^\mu}{d\tau^2} = -R_{\mu\alpha\nu\beta} n^\mu u^\alpha n^\nu u^\beta. \tag{26}$$

Thus if ds is the spatial distance between the two particles, then αds is their relative acceleration. It follows that the relative change in displacement of the two particles after a proper time T is

$$ds \int_0^T d\tau \int_0^\tau d\tau' \alpha(\tau', s), \tag{27}$$

Now consider the case of two observers (particles) separated by a finite initial distance s_0 as illustrated in Fig. 2.

We can find the relative change in displacement of these two observers by integrating on s:

$$\Delta s = \int_0^{s_0} ds \int_0^T d\tau \int_0^\tau d\tau' \alpha(\tau', s). \tag{28}$$

This is the relative displacement measured at the same moment of proper time for both observers.

Let us now consider a light signal sent from one observer to the other. If $\alpha = 0$, the distance traveled by the light ray is s_0. When $\alpha \neq 0$, this distance becomes $s_0 + \Delta s$, where now

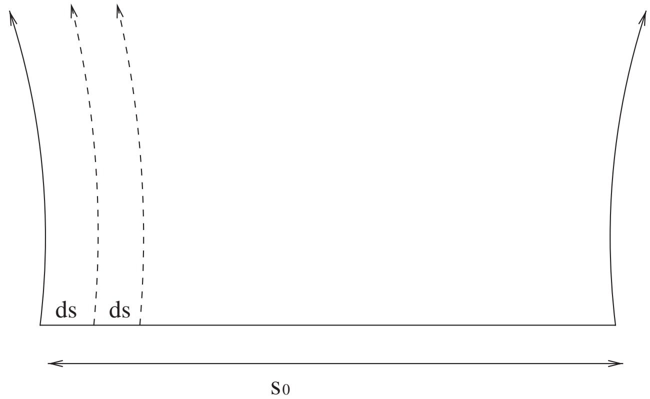

FIG. 2. Two timelike geodesics separated by a finite interval containing an infinite number of nearby geodesics.

$$\Delta s = \int_0^{s_0} ds \underbrace{\int_0^s d\tau \int_0^\tau d\tau' \alpha(\tau', s)}. \quad (29)$$

Here the under-braced integral is the displacement per unit s of a pair of observers at a distance s from the source. The domain of the final two integrations is illustrated in Fig. 3.

If gravity is quantized, the Riemann tensor will fluctuate around an average value of zero due to quantum gravitational vacuum fluctuations. This leads to $\langle\alpha\rangle = 0$, and hence $\langle\Delta s\rangle = 0$. Notice here that α becomes a quantum operator when metric perturbations are quantized. However, in general, $\langle(\Delta s)^2\rangle \neq 0$, and we have

$$\langle(\Delta s)^2\rangle = \int_0^{s_0} ds_1 \int_0^{s_0} ds_2 \int_0^{s_1} d\tau_1 \int_0^{\tau_1} d\tau_1' \int_0^{s_2} d\tau_2 \times \int_0^{\tau_2} d\tau_2' \langle\alpha(\tau_1', s_1)\alpha(\tau_2', s_2)\rangle. \quad (30)$$

Thus the root-mean-squared fluctuation in the flight path is $\sqrt{\langle(\Delta s)^2\rangle}$, which can also be understood as a fluctuation in the speed of light. It entails an intrinsic quantum uncertainty in the measurement of distance. Therefore, spacetime becomes fuzzy at a scale characterized by $\sqrt{\langle(\Delta s)^2\rangle}$. The integrand in Eq. (30) is obviously invariant under any coordinate transformation while the integral is gauge invariant within the linear approximation.

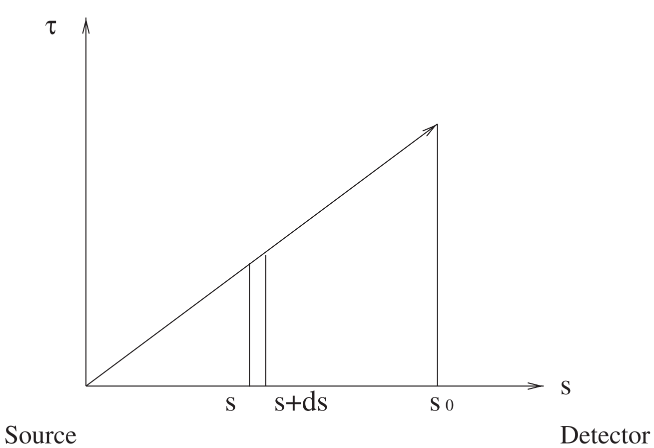

FIG. 3. The displacement, Δs, between a source and a detector is given by an integration within the triangular region.

We now wish to show that this gauge-invariant quantity is the same as Eq. (6) when calculated in the TT gauge. Choose a coordinate system where the source and the detector are both at rest, and suppose that the light ray propagates in the x-direction, then we have

$$u^\mu = (1, 0, 0, 0), \quad (31)$$

$$n^\mu = (0, 1, 0, 0), \quad (32)$$

and

$$\alpha = R_{xtxt} = -\frac{1}{2} h_{xx,tt}. \quad (33)$$

Substitution of the above results into Eq. (30) leads to

$$\begin{aligned}\langle(\Delta s)^2\rangle &= \int_0^r dx_1 \int_0^r dx_2 \int_0^{x_1} dt_1 \int_0^{t_1} dt_1' \int_0^{x_2} dt_2 \\ &\quad \times \int_0^{t_2} dt_2' \langle\alpha(t_1', s_1)\alpha(t_2', s_2)\rangle \\ &= \frac{1}{4}\int_0^r dx_1 \int_0^r dx_2 \langle h_{xx}(x_1, x_1) h_{xx}(x_2, x_2)\rangle \\ &= \frac{1}{r}\langle\sigma_1^2\rangle,\end{aligned} \quad (34)$$

where we have set $s_0 = r$ and used the fact that along the light ray $x = t$. Thus, one has

$$\Delta t = \frac{\sqrt{\langle\sigma_1^2\rangle}}{r} = \sqrt{\langle(\Delta s)^2\rangle} \quad (35)$$

which also demonstrates the gauge invariance of Δt.

Now we wish to discuss the rather subtle issue of the relation of light-cone fluctuations to Lorentz symmetry. It is sometimes argued that light-cone fluctuations are incompatible with Lorentz invariance. The most dramatic illustration of this arises when a time advance occurs, that is, when a pulse propagates outside of the classical light cone. In a Lorentz invariant theory, there will exist a frame of reference in which the causal order of emission and detection is inverted, so the pulse is seen to be detected before it was emitted. Thus the light-cone fluctuation phenomenon, if it is to exist at all, seems to be incompatible with strict Lorentz invariance.

Our view of the situation is the following: light-cone fluctuations respect Lorentz symmetry on the average, but not in individual measurements. The symmetry on the average insures that the mean light cone is that of classical Minkowski spacetime. The average metric is that of Minkowski spacetime provided that $\langle h_{\mu\nu}\rangle = 0$. However, a particular pulse effectively measures a spacetime geometry which is not Minkowskian and not Lorentz invariant. A simple model may help to illustrate this point. Consider a quantum geometry consisting of an ensemble of classical Schwarzschild spacetimes, but with both positive and negative values for the mass parameter M. (The fact that the $M < 0$ Schwarzschild spacetime has a naked singularity at $r = 0$ need not concern us. For the purpose of this model, we can confine our discussion to a region where $r \gg |M|$.) Suppose that this ensemble has $\langle M \rangle = 0$, but $\langle M^2 \rangle \neq 0$. It is well known that light propagation in a $M > 0$ Schwarzschild spacetime can exhibit a time delay relative to what would be expected in flat spacetime. This is the basis for the time delay tests of general relativity using radar signals sent near the limb of the sun. In the present model, however, the time difference is equally likely to be a time advance rather than a time delay. A measurement of the time difference amounts to a measurement of M. This model is Lorentz invariant on the average because $\langle M \rangle = 0$ and the average spacetime is Minkowskian. However, a specific measurement selects a particular member of the ensemble, which is generally not Lorentz invariant.

In addition to the fact that the mean metric is Minkowskian, there is another sense in which lightcone fluctuations due to compactification exhibit average Lorentz invariance. Note that Δs, and hence Δt, depends on the Riemann tensor correlation function $\langle R_{xtxt}(x_1) R_{xtxt}(x_2) \rangle$, which is invariant under Lorentz boosts along the x-axis. Thus if we were to repeat the above calculations of Δs in a second frame moving with respect to the first, the result will be the same. In both cases one is assuming that the detector is at rest relative to the source. This is a reflection of the Lorentz invariance of the spectrum of fluctuations, which is exhibited by the compactified flat spacetimes studied in this paper, but not by the Schwarzschild spacetime with a fluctuating mass.

IV. THE FIVE-DIMENSIONAL KALUZA-KLEIN MODEL

In this section, we will specialize to the case of one extra compactified dimension, so $d = 5$ and $n = 1$. This corresponds to the original Kaluza-Klein model [23,24].

A. Calculation of Δt

To begin, let us examine the influence of the compactification of the fifth (extra) dimension on the light propagation in our four-dimensional world, by considering a light ray traveling along the x-direction from point a to point b, which is perpendicular to the direction of compactification. Define

$$\rho = x - x', \qquad b - a = r \tag{36}$$

and note the fact that the integration in Eq. (23) is to be carried out along the classical null geodesic on which $t - t' = \rho$. To calculate Δt, we need the graviton two-point function component G_{xxxx}, which in this case is given by Eq. (14) as

$$G_{xxxx} = \frac{8}{3}(D - 2F_{xx} + H_{xxxx}). \tag{37}$$

The quantities in this expression may be computed from Eqs. (A2), (A11), (A14), (A16), and (A18), with the result

$$G_{xxxx}(t, x, 0, 0, 0, t', x', 0, 0, mL') = \frac{8}{3\pi^2}\frac{\rho^4 mL(5m^2L^2 - 3\rho^2)}{(\rho^2 + m^2L^2)^5}. \tag{38}$$

Thus, we have

$$\begin{aligned}\langle \sigma_1^2 \rangle_R &= \frac{1}{8} r^2 \int_a^b dx \int_a^b dx' G^R_{xxxx}(t, x, \mathbf{0}, t', x', \mathbf{0},) \\ &= \frac{1}{8} r^2 \int_a^b dx \int_a^b dx' \sum_{m=-\infty}^{+\infty\prime} G_{xxxx}(t, x, \mathbf{0}, t', x', 0, 0, mL) \\ &= \frac{2r^2}{9\pi^2 L} \sum_{m=1}^{\infty} \frac{\gamma^6}{m(m^2 + \gamma^2)^3},\end{aligned} \tag{39}$$

where we have introduced a dimensionless parameter $\gamma = r/L$. We are interested here in the case in which $\gamma \gg 1$. Thus the summation can be approximated by integration as follows:

$$\langle \sigma_1^2 \rangle_R = \frac{2r^2}{9\pi^2 L} \int_{1/\gamma}^{\infty} \frac{1}{x(x^2 + 1)^3} dx \tag{40}$$

which leads to

$$\langle \sigma_1^2 \rangle_R \approx \frac{2r^2}{9\pi^2 L} \ln \frac{r}{L}. \tag{41}$$

Note that this quantity increases as L decreases, for fixed r. This might come as a surprise, in light of the fact that the discrete momentum in the fifth dimension gives rise to a Kaluza-Klein tower of massive modes, each with a mass which is inversely proportional to L. Furthermore, large masses tend to give small contributions to radiative quantum effects. However, one can see that this line of reasoning is flawed by recalling the Casimir effect in a compact space. Consider, for example, the energy density of a massless scalar field in four spacetime dimensions, where one of the space dimensions is periodic with period L. The Casimir energy density, if it is nonzero, must be proportional to L^{-4} on dimensional grounds. An explicit calculation does yield a nonzero result of $-\pi^2/(90L^4)$. Similarly, Casimir energy density in higher dimensional compact spacetimes scale as inverse powers of the compactification scale. This is what should be expected for an expectation value of a local quantity in quantum theory, where systems confined in smaller spatial volumes undergo more violent quantum fluctuations. The same applies to integrals of such local quantities, such as appear in Eq. (41). Apparently, the effect of a sum over an infinite number of modes more than compensates for the large effective mass of each mode, at least for the purpose of calculating Casimir energy densities, or $\langle \sigma_1^2 \rangle_R$.

The mean deviation from the classical propagation time due to the light-cone fluctuations is

$$\Delta t \approx \sqrt{\frac{2}{9\pi^2 L}\ln\frac{r}{L}} = \sqrt{\frac{2}{9\pi^2 L}\ln\frac{r}{L}}\sqrt{32\pi G_5} = \sqrt{\frac{64}{9\pi}}\ell_P\sqrt{\ln\frac{r}{L}}, \tag{42}$$

where we have used Eq. (24). This result reveals that the mean deviation in the arrival time increases logarithmically with r, which contrasts with the square root growth in the four-dimensional case with one compactified spatial dimension [14]. It also grows as the size of the compactified dimension decreases. However, even if r is of cosmological size and L is near the Planck scale, Δt is never more than a couple of orders of magnitude larger than the Planck scale and hence unobservable in practice. Note that Eq. (42) corrects an erroneous result in Refs. [15,28], where a linear growth of Δt was found. This discrepancy can be traced to an error in the calculation of the five-dimensional graviton two-point function, now given by Eq. (38).

We have used the results in Appendix A to perform the analogous calculation for more than one compact dimension. For up to seven extra dimensions, the result for Δt is of the same form as Eq. (42), a logarithmic growth with r.

B. Correlation of pulses

The fluctuation in the flight time of pulses, Δt, can apply to successive pulses. However, Δt is the expected variation in the arrival times of two successive pulses only when they are uncorrelated [13]. To determine the correlation, we need to compare $|\langle\sigma_1^2\rangle|$ and $|\langle\sigma_1\sigma_1'\rangle|$. The latter quantity is defined by

$$\langle\sigma_1\sigma_1'\rangle = \frac{1}{8}(\Delta r)^2\int_0^r dr_1\int_0^r dr_2 n^\mu n^\nu n^\rho n^\sigma G^R_{\mu\nu\rho\sigma}(x_1, x_2), \tag{43}$$

where the r_1-integration is taken along the mean path of the first pulse, and the r_2-integration is taken along that of the second pulse. Here we will assume that $\Delta t \ll r$, so the slopes of the two mean paths are approximately unity. Let the time separation of the emission of the two pulses be T. Thus the two-point function in Eq. (43) will be assumed to be evaluated at $\rho = |\mathbf{x}_1 - \mathbf{x}_2| = |r_1 - r_2|$ and $\tau = |t_1 - t_2| = |r_1 - r_2 - T|$. If $|\langle\sigma_1\sigma_1'\rangle| \ll |\langle\sigma_1^2\rangle|$, two pulses are uncorrelated, and otherwise they are correlated.

In Appendix B, it is shown that

$$\langle\sigma_1\sigma_1'\rangle \approx \frac{r^2}{9\pi^2 L}\ln\frac{2r}{T}. \tag{44}$$

Compare this result with

$$\langle\sigma_1^2\rangle_R \approx \frac{2r^2}{9\pi^2 L}\ln\frac{r}{L}. \tag{45}$$

We can see that two successive pulses separated by T in time are only weakly correlated ($|\langle\sigma_1\sigma_1'\rangle| \ll |\langle\sigma_1^2\rangle_R|$) provided that

$$r \gg \frac{2L^2}{T}. \tag{46}$$

Equivalently,

$$T \gg \frac{2L^2}{r}. \tag{47}$$

However, if $r \ll L$, one can show, by series expansion, that $|\langle\sigma_1\sigma_1'\rangle| \ll |\langle\sigma_1^2\rangle_R|$, if $T \gg L$, and $|\langle\sigma_1\sigma_1'\rangle| \approx |\langle\sigma_1^2\rangle_R|$, when $T < L$.

A few comments are now in order about the physical picture behind our correlation results. It is natural to expect from the configuration that the dominant contributions to the light-cone fluctuation come from the graviton modes with wavelengths of the order of $\sim L$. In other words, the light cone fluctuates on a typical time scale of $\sim 1/L$. If the travel distance, r, is less than L, successive pulses are weakly correlated when their time separation is greater than the typical fluctuation time scale. Otherwise they are correlated because the quantum gravitational vacuum fluctuations are not significant enough in the interval between the pulses. However, if $r \gg 2L^2/T$, then successive pulses are in general weakly correlated. Thus the correlation time for large r is of order L^2/r, which is much smaller than the compactification scale L. We can understand this result as arising from the decrease in correlation as the pulses propagate over an increasing distance.

V. REDSHIFT FLUCTUATIONS

In this section, we will use a formalism based upon the Riemann tensor correlation function to calculate line broadening or narrowing due to spacetime geometry fluctuations [29]. Consider a source which emits signals at a mean frequency of ω_0 in its rest frame. The frequency detected by an observer is subject to Doppler and gravitational redshifts, and the geometry fluctuations will cause a fluctuation in the gravitational redshift. Let

$$\xi = \frac{\Delta\omega}{\omega_0} \tag{48}$$

be the fractional frequency shift. Now consider two successive signals sent from the source to the observer. The mean squared variation in ξ between these two signals due to geometry fluctuations is

$$\delta\xi^2 = \langle(\Delta\xi)^2\rangle - \langle\Delta\xi\rangle^2. \tag{49}$$

In Ref. [29], it is shown that this quantity may be expressed in terms of the Riemann tensor correlation function

$$C_{\alpha\beta\mu\nu\gamma\delta\rho\sigma}(x, x') = \langle R_{\alpha\beta\mu\nu}(x)R_{\gamma\delta\rho\sigma}(x')\rangle - \langle R_{\alpha\beta\mu\nu}(x)\rangle\langle R_{\gamma\delta\rho\sigma}(x')\rangle. \tag{50}$$

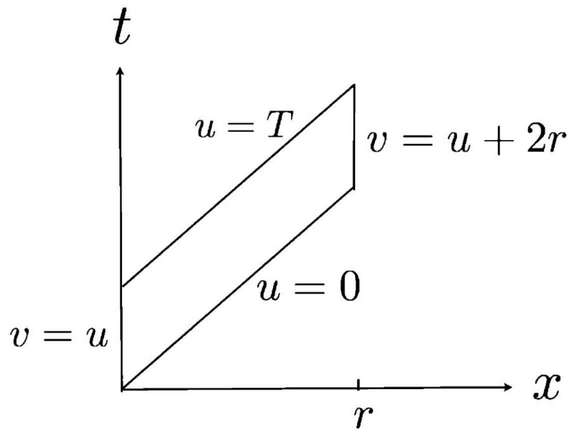

FIG. 4. The region of integration in the expressions for $\delta\xi^2$ is illustrated. The worldline of the source is $x = 0$, which is equivalent to $v = u$. Here $u = t - x$ and $v = t + x$. The worldline of the detector is $x = r$, or $v = u + 2r$. The first null ray is emitted at $t = 0$ and travels on the $u = 0$ line. The second ray is emitted at $t = T$ and travels on the $u = T$ line.

Specifically,

$$\delta\xi^2 = \int da \int da' C_{\alpha\beta\mu\nu\gamma\delta\rho\sigma}(x, x') t^\alpha k^\beta t^\mu k^\nu t^\gamma k^\delta t^\rho k^\sigma. \tag{51}$$

Here t^μ is the four-velocity of both the source and detector, which are assumed to be at rest with respect to one another, and k^ν is the tangent to the worldlines of the signals. The integrations in Eq. (51) are taken over the region bounded by the worldlines of the two signals and those of the source and detector, as illustrated in Fig. 4.

In our present problem, the average geometry is that of Minkowski spacetime, so that $\langle R_{\alpha\beta\mu\nu}(x)\rangle = 0$, and we have

$$\delta\xi^2 = \int da \int da' \langle R^\mu_{\ \alpha\nu\beta} R^\rho_{\ \gamma\sigma\lambda}\rangle t_\mu t_\rho t^\nu t^\sigma k^\alpha k^\beta k^\gamma k^\lambda. \tag{52}$$

Let $t^\mu = (1, 0, 0, 0)$ and $k^\mu = (1, 1, 0, 0)$, so that

$$\delta\xi^2 = \int da \int da' \langle R_{txtx}(x) R_{txtx}(x')\rangle. \tag{53}$$

The integrand in the above expression is given by Eqs. (22) and (33) to be

$$\langle R_{txtx} R_{txtx}\rangle = \frac{1}{4}\partial_t^4 G_{xxxx} = \frac{n+1}{n+2}\partial_t^4 (D - 2F_{xx} + H_{xxxx}) = \frac{n+1}{n+2}(\partial_t^2 - \partial_x^2) D, \tag{54}$$

where in the last step we used $\partial_t^4 H_{xxxx} = \partial_x^4 D$ and $\partial_t^4 F_{xx} = \partial_t^2 \partial_x^2 D$. Define $u = t - x$ and $v = t + x$ and write the above expression as

$$\langle R_{txtx} R_{txtx}\rangle = 8\frac{n+1}{n+2}\partial_u \partial_{u'} \partial_v \partial_{v'} D. \tag{55}$$

Then we find

$$\delta\xi^2 = 4\int da \int da' \partial_u \partial_{u'} \partial_v \partial_{v'} D = 8\frac{n+1}{n+2}\int_0^T du \int_0^T du' \int_u^{u+2r} dv \times \int_{u'}^{u'+2r} dv' \partial_v \partial_{v'} (\partial_u \partial_{u'} D). \tag{56}$$

Because D depends on v and v' only through $\Delta v = v - v'$, we find

$$\delta\xi^2 = 8\frac{n+1}{n+2}\int_0^T du \int_0^T du' [2(\partial_u \partial_{u'} D)|_{\Delta v = \Delta u} - (\partial_u \partial_{u'} D)|_{\Delta v = \Delta u + 2r} - (\partial_u \partial_{u'} D)|_{\Delta v = \Delta u - 2r}]. \tag{57}$$

A. Four-dimensional case

Here $n = 0$ and

$$D = \sum_{m=-\infty}^{\infty}{}' \frac{1}{4\pi^2(\Delta x^2 + m^2 L^2 - \Delta t^2)} = \frac{1}{2\pi^2}\sum_{m=1}^{\infty}\frac{1}{-\Delta v \Delta u + m^2 L^2}. \tag{58}$$

Use the relation

$$\int_0^T du \int_0^T du' = 2\int_0^T d\Delta u (T - \Delta u), \tag{59}$$

to write Eq. (57) as

$$\delta\xi^2 = T_1 + T_2 + T_3, \tag{60}$$

where

$$T_1 = 16\int_0^T d\Delta u (T - \Delta u)(\partial_u \partial_{u'} D)|_{\Delta v = \Delta u} = -\frac{16}{\pi^2}\sum_{m=1}^{\infty}\int_0^T d\Delta u (T - \Delta u)\frac{\Delta u^2}{(-\Delta u^2 + m^2 L^2)^3}, \tag{61}$$

$$T_2 = -8\int_0^T d\Delta u (T - \Delta u)(\partial_u \partial_{u'} D)|_{\Delta v = \Delta u + 2r} = -\frac{8}{\pi^2}\sum_{m=1}^{\infty}\int_0^T d\Delta u (T - \Delta u) \times \frac{(\Delta u + 2r)^2}{[-\Delta u(\Delta u + 2r) + m^2 L^2)]^3}, \tag{62}$$

and

$$T_3 = -8\int_0^T d\Delta u(T-\Delta u)(\partial_u\partial_{u'}D)|_{\Delta v=\Delta u-2r}$$
$$= -\frac{8}{\pi^2}\sum_{m=1}^{\infty}\int_0^T d\Delta u(T-\Delta u)$$
$$\times \frac{(\Delta u - 2r)^2}{[-\Delta u(\Delta u-2r)+m^2L^2)]^3}. \quad (63)$$

In the limits $r \gg T \gg L$, we obtain

$$T_1 \approx \frac{2}{3L^2}, \quad (64)$$

$$T_2 \approx \frac{Tr}{2\pi^2 L^4}\sum_{m=1}^{\infty}\frac{1}{m^4} - \frac{2}{3L^2}, \quad (65)$$

$$T_3 \approx -\frac{Tr}{2\pi^2 L^4}\sum_{m=1}^{\infty}\frac{1}{m^4} - \frac{2}{3L^2}. \quad (66)$$

Thus we find

$$\delta\xi^2 \approx -\frac{2\ell_P^2}{3L^2}. \quad (67)$$

The fact that $\delta\xi^2 < 0$ seems to imply a small narrowing of spectral lines. Unless L is very small, the natural line width of a spectral line is likely to be much larger in magnitude than this effect. This narrowing is analogous to the negative shifts in mean squared velocity of a charged or polarizable particle near a boundary found in Refs. [19,20]. Unlike Δt, this effect does not grow with increasing path length.

B. Five-dimensional case

Now we turn to the five-dimensional model with one compactified space dimension, which was studied in Sec. IV. Here

$$D = \frac{1}{4\pi^2}\,\mathrm{Re}\left[\sum_{m=1}^{\infty}\frac{1}{(-\Delta v\Delta u + m^2L^2)^{3/2}}\right]. \quad (68)$$

As in the four-dimensional case, we may write

$$\delta\xi^2 = \frac{4}{3}(T_1 + T_2 + T_3), \quad (69)$$

where

$$T_1 = 16\int_0^T d\Delta u(T-\Delta u)(\partial_u\partial_{u'}D)|_{\Delta v=\Delta u}, \quad (70)$$

$$T_2 = -8\int_0^T d\Delta u(T-\Delta u)(\partial_u\partial_{u'}D)|_{\Delta v=\Delta u+2r}, \quad (71)$$

and

$$T_3 = -8\int_0^T d\Delta u(T-\Delta u)(\partial_u\partial_{u'}D)|_{\Delta v=\Delta u-2r}. \quad (72)$$

In the limits $T \gg L$ and $r \gg T\&L$, we now obtain

$$T_1 \approx \frac{2}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^3L^3} = \frac{2\xi(3)}{\pi^2L^3}, \quad (73)$$

$$T_2 \approx \frac{2}{\pi^2}\sum_{m=1}^{\infty}\left[-\frac{3Tr}{m^5L^5} - \frac{1}{m^3L^3}\right], \quad (74)$$

$$T_3 \approx \frac{2}{\pi^2}\sum_{m=1}^{\infty}\left[\frac{3Tr}{m^5L^5} - \frac{1}{m^3L^3}\right]. \quad (75)$$

Thus we find in this case

$$\delta\xi^2 \approx -\frac{8\zeta(3)}{3\pi^2L^3}, \quad (76)$$

where $\zeta(3) \approx 1.20$ is a Riemann zeta function. We may use Eq. (24) to write this as

$$\delta\xi^2 \approx -\frac{256\zeta(3)\ell_P^2}{3\pi L^2}. \quad (77)$$

Again, we find a negative value for $\delta\xi^2$ which does not increase as the flight path length increases, and whose magnitude is determined by the ratio ℓ_P/L.

This negative value represents a small decrease in the bandwidth of the wave packet. Note that we are always working in an approximation of weak quantum gravity effects on a nearly fixed background geometry. This approximation requires that $L \gg \ell_P$, so that $|\delta\xi^2| \ll 1$. This guarantees that we would never have $|\Delta\omega| > \omega_0$. However, we also need to have that

$$\sqrt{|\delta\xi^2|} < \frac{\Delta_0\omega}{\omega_0}, \quad (78)$$

where $\Delta_0\omega$ is the original bandwidth of the wave packet. Otherwise the net squared bandwidth would become negative, which is not possible. Equation (78) follows provided that $\Delta_0\omega \gg 1/L$ and $\omega_0 \ll 1/\ell_P$. The latter condition is simply the requirement that the frequencies used to probe the spacetime geometry fluctuations be sub-Planckian. This seems reasonable to impose on the type of semiclassical analysis we are performing. The requirement that $\Delta_0\omega \gg 1/L$ follows if we require that our wave packets be localized on a spacetime scale less than L. This seems to be needed if we are to resolve geometry fluctuations whose natural length scale is L. With these two conditions, we obtain Eq. (78), which guarantees a positive net squared bandwidth.

Let us next comment on the relation between the result of Sec. IV that Δt grows with increasing path length, and the present result that $\delta\xi^2$ does not. If the crests of a plane wave could be treated as truly uncorrelated pulses, then we would expect the growth of Δt to lead to increasing line broadening with increasing r. When Eq. (47) is satisfied, the correlations between the pulses becomes weak, but is never completely absent. Apparently the remaining correlation is sufficient to prevent $\Delta\omega$ from increasing with increasing r. This might come as surprise, given that Δt can grow. The analogy of a charged particle coupled to electromagnetic vacuum fluctuations near a boundary, treated in Ref. [20] is useful here. In this latter case, the mean squared velocity of the particle can undergo a shift, but one which does not grow in time, although the mean squared displacement of the particle can grow in time. If the electric field fluctuation were not correlated, one would have obtained the linear growth characteristic of a random walk process. The lack of growth of the mean squared velocity can be traced to the fact that the time integral of the electric field correlation function vanishes in the limit of integration of an infinite time [30]. It is this property which enforces the strict anticorrelations that prevent growth. The same feature occurs in the present analysis. If the Riemann tensor correlation function were to be integrated over an infinite time, the result would be zero, due to the same type of anticorrelation as in the electromagnetic case.

VI. DISCUSSION AND CONCLUSIONS

We have treated the effects of compact extra flat spacetime dimensions on quantum light-cone fluctuations. One measure of light-cone fluctuations is variations in the flight times of pulses. We found in a five-dimensional model that this variation will grow as the logarithm of the flight distance. In principle this is an observable effect, but it is too small to be a realistic test of extra dimensions. We also examined the correlations of successive pulses. This correlation weakens as the pulse separation increases, but is always nonzero. This nonzero correlation is presumably responsible for the result that effect of geometry fluctuations on the width of spectral lines does not increase with increasing flight distance. In fact, we find a weak line narrowing effect in which the spacetime geometry fluctuations slightly reduce the natural line width. The fractional line narrowing effect is

$$\frac{\Delta\omega}{\omega} = \sqrt{|\langle\delta\xi^2\rangle|} = C\frac{\ell_P}{L}, \tag{79}$$

where C is a constant of order unity. The analysis in this paper assumes gravitons on a fixed background spacetime and is presumable only valid when the compactification scale is well above the Planck scale, $L \gg \ell_P$. Nonetheless, this suggests a possible observational signature of the existence of extra dimensions: a small, systematic narrowing of all spectral line from what would otherwise be expected.

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APPENDIX A: GRAVITON TWO-POINT FUNCTIONS

Here we evaluate the functions $D^n(x.x')$, $F^n_{ij}(x, x')$ and $H^n_{ijkl}(x, x')$ defined in Eqs. (15)–(17), respectively. Once these functions are given, the graviton two-point functions are easy to obtain. Define

$$R = |\mathbf{x} - \mathbf{x}'|, \qquad \Delta t = t - t', \qquad k = |\mathbf{k}| = \omega, \tag{A1}$$

and assume n extra dimensions, then

$$\begin{aligned} D^n(x, x') &= \frac{\mathrm{Re}}{(2\pi)^{3+n}}\int\frac{d^{(3+n)}\mathbf{k}}{2\omega}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}e^{-i\omega(t-t')} \\ &= \frac{\mathrm{Re}}{2(2\pi)^{3+n}}\int_0^\infty k^{n+1}e^{-ik\Delta t}dk\int_0^\pi d\theta_1\sin^{1+n}\theta_1 e^{ikR\cos\theta_1}\int_0^\pi d\theta_2\sin^n\theta_2\ldots.\int_0^\pi d\theta_{n+1}\sin\theta_{n+1}\int_0^{2\pi}d\theta_{n+2} \\ &= \frac{a_n\mathrm{Re}}{2(2\pi)^{3+n}}\int_0^\infty k^{n+1}e^{-ik\Delta t}dk\int_{-1}^1 e^{ikRx}(1-x^2)^{n/2}dx = \frac{a_n\mathrm{Re}}{(2\pi)^{3+n}}\int_0^\infty k^{n+1}e^{-ik\Delta t}dk\int_0^1(1-x^2)^{n/2}\cos(kRx)dx \\ &= \frac{a_n\sqrt{\pi}2^{n+1/2}\Gamma(\frac{n}{2}+1)\mathrm{Re}}{2(2\pi)^{3+n}}\frac{1}{R^{n+1/2}}\int_0^\infty k^{n+1/2}J_{n+1/2}(kR)e^{-ik\Delta t}dk \\ &= \frac{a_n\sqrt{\pi}2^{n+1/2}\Gamma(\frac{n}{2}+1)\mathrm{Re}}{2(2\pi)^{3+n}}\frac{1}{R^{n+1/2}}\lim_{\alpha\to 0^+ + i\Delta t}\int_0^\infty k^{n+1/2}J_{n+1/2}(kR)e^{-\alpha k}dk \\ &= \frac{a_n 2^n\Gamma(\frac{n}{2}+1)^2}{(2\pi)^{3+n}}\frac{1}{(R^2-\Delta t^2)^{n/2+1}} = \frac{\Gamma(\frac{n}{2}+1)}{4\pi^{n+4/2}}\frac{1}{(R^2-\Delta t^2)^{n/2+1}}. \end{aligned} \tag{A2}$$

Here we have defined

$$a_n = \int_0^\pi d\theta_2 \sin^n\theta_2 \ldots \int_0^\pi d\theta_{n+1}\sin\theta_{n+1}\int_0^{2\pi} d\theta_{n+2} = \frac{2\pi^{n/2+1}}{\Gamma(\frac{n}{2}+1)}, \tag{A3}$$

and used

$$\int_0^1 \cos(kRx)(1-x^2)^{n/2}dx = \frac{\sqrt{\pi}}{2}\Gamma\left(\frac{n}{2}+1\right)\left(\frac{2}{kR}\right)^{n+1/2} J_{n+1/2}(kR), \tag{A4}$$

and

$$\int_0^\infty e^{-\alpha x}J_\nu(\beta x)x^\nu dx = \frac{(2\beta)^\nu\Gamma(\nu+1/2)}{\sqrt{\pi}(\alpha^2+\beta^2)^{\nu+1/2}}, \qquad \mathrm{Re}\nu > -1/2. \tag{A5}$$

When n is odd, $D^n(x, x')$ should be taken to be zero when $R^2 < \Delta t^2$.

Let us now turn our attention to the calculation of F_{ij} and H_{jikl}. We find

$$\begin{aligned}
F^n_{ij}(x,x') &= \frac{\mathrm{Re}}{(2\pi)^{3+n}}\int d^{3+n}\mathbf{k}\,\frac{k_i k_j}{2\omega^3}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}e^{-i\omega(t-t')} \\
&= \frac{\mathrm{Re}}{2(2\pi)^{3+n}}\partial_i\partial'_j\int_0^\infty k^{n-1}e^{-ik\Delta t}dk\int_0^\pi d\theta_1 \sin^{1+n}\theta_1 e^{ikR\cos\theta_1}\int_0^\pi d\theta_2\sin^n\theta_2\ldots\int_0^\pi d\theta_{n+1}\sin\theta_{n+1}\int_0^{2\pi}d\theta_{n+2} \\
&= \frac{a_n\mathrm{Re}}{(2\pi)^{3+n}}\partial_i\partial'_j\int_0^\infty k^{n-1}e^{-ik\Delta t}dk\int_0^1(1-x^2)^{n/2}\cos(kRx)dx \\
&= \frac{a_n\sqrt{\pi}2^{n+1/2}\Gamma(\frac{n}{2}+1)\mathrm{Re}}{2(2\pi)^{3+n}}\partial_i\partial'_j\left(\frac{1}{R^{n+1/2}}\int_0^\infty k^{n-3/2}J_{n+1/2}(kR)e^{-ik\Delta t}dk\right) \\
&= \frac{a_n\sqrt{\pi}2^{n+1/2}\Gamma(\frac{n}{2}+1)}{2(2\pi)^{3+n}}\partial_i\partial'_j\left(\frac{\mathrm{Re}}{R^{n+1/2}}\int_0^\infty k^{n-3/2}J_{n+1/2}(kR)e^{-ik\Delta t}dk\right) \\
&= \frac{\mathrm{Re}}{2(2\pi)^{3+n/2}}\partial_i\partial'_j\left(\frac{1}{R^{n+1/2}}\int_0^\infty k^{n-3/2}J_{n+1/2}(kR)e^{-ik\Delta t}dk\right) \\
&= \frac{\mathrm{Re}}{2(2\pi)^{3+n/2}}\partial_i\partial'_j\left(\frac{n-1}{R^2}\frac{1}{R^{n-1/2}}\int_0^\infty k^{n-5/2}J_{n-1/2}(kR)e^{-ik\Delta t}dk - \frac{1}{R^{n+1/2}}\int_0^\infty k^{n-3/2}J_{n-3/2}(kR)e^{-ik\Delta t}dk\right),
\end{aligned} \tag{A6}$$

where we have utilized a recursive formula for Bessel functions

$$zJ_{\nu-1}(z) + zJ_{\nu+1}(z) = 2\nu J_\nu(z). \tag{A7}$$

Similarly, one finds that

$$\begin{aligned}
H^n_{ijkl}(x,x') &= \frac{\mathrm{Re}}{(2\pi)^{3+n}}\int d^{3+n}\mathbf{k}\,\frac{k_i k_j k_k k_l}{2\omega^5}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}e^{-i\omega(t-t')} \\
&= \frac{\mathrm{Re}}{2(2\pi)^{3+n/2}}\partial_i\partial'_j\partial_k\partial_l\left(\frac{n-1}{R^2}\frac{1}{R^{n-1/2}}\int_0^\infty k^{n-9/2}J_{n-1/2}(kR)e^{-ik\Delta t}dk\right. \\
&\quad \left. - \frac{1}{R^2}\frac{1}{R^{n+1/2}}\int_0^\infty k^{n-7/2}J_{n-3/2}(kR)e^{-ik\Delta t}dk\right).
\end{aligned} \tag{A8}$$

To proceed further with the calculation, we need to deal with the cases when n is odd or even separately.

1. The case of odd n

Assume $n = 2m + 1$ and define

$$S(m) = \frac{\mathrm{Re}}{R^{m+1}}\int_0^\infty k^{m-1}J_{m+1}(kR)e^{-ik\Delta t}dk, \qquad m \geq 0, \tag{A9}$$

$$T(m-1) = \frac{\mathrm{Re}}{R^{m+1}} \int_0^\infty k^{m-1} J_{m-1}(kR) e^{-ik\Delta t} dk$$
$$= \frac{\mathrm{Re}}{R^{m+1}} \lim_{\alpha\to 0^+ + i\Delta t} \int_0^\infty k^{m-1} J_{m-1}(kR) e^{-\alpha k} dk = \frac{2^{m-1}\Gamma(m-1/2)}{\sqrt{\pi}} \frac{\sqrt{R^2-\Delta t^2}}{R^2(R^2-\Delta t^2)^m},$$
$$= \frac{(2m-1)!!}{(2m-1)} \frac{\sqrt{R^2-\Delta t^2}}{R^2(R^2-\Delta t^2)^m}, \qquad m \geq 1, \tag{A10}$$

where we have appealed to integral (6.623.1) in Ref. [31]. The above result holds for $R^2 > \Delta t^2$, and $T(m-1)$ is zero when $R^2 < \Delta t^2$. Then it follows from Eq. (A6) that

$$F_{ij}^{2m+1} = \frac{1}{2(2\pi)^{m+2}} \partial_i \partial'_j (S(m)), \tag{A11}$$

and

$$S(m) = \frac{2m}{R^2} S(m-1) - T(m-1). \tag{A12}$$

Using the recursive relation Eq. (A12), we can show that

$$S(m) = \frac{(2m)!!}{R^{2m}} S(0) - \sum_{k=1}^{m} \frac{(2m)!!}{(2k)!!} \frac{T(k-1)}{R^{2m-2k}}$$
$$= \frac{(2m)!!}{R^{2m}} S(0) \left[1 - \sum_{k=1}^{m} \frac{(2k-1)!!}{(2k)!!(2k-1)} \frac{R^{2k}}{(R^2-\Delta t^2)^k} \right]$$
$$= -\frac{(2m)!!}{R^{2m}} S(0) \sum_{k=0}^{m} \frac{(2k+1)!!}{(2k)!!(2k+1)(2k-1)} \times \frac{R^{2k}}{(R^2-\Delta t^2)^k}. \tag{A13}$$

Here

$$S(0) = \frac{\mathrm{Re}}{R} \lim_{\alpha\to 0^+ + i\Delta t} \int_0^\infty \frac{J_1(kR)}{k} e^{-\alpha k} dk$$
$$= \frac{1}{R} \int_0^\infty \frac{J_1(kR)\cos(k\Delta t)}{k} dk$$
$$= \begin{cases} \frac{1}{R}\cos(\arcsin(\Delta t/R)) = \frac{\sqrt{R^2-\Delta t^2}}{R^2} & \text{for } R^2 > \Delta t^2, \\ 0 & \text{for } R^2 < \Delta t^2. \end{cases} \tag{A14}$$

If we define

$$Q(m) = \frac{\mathrm{Re}}{R^{m+1}} \int_0^\infty k^{m-3} J_{m+1}(kR) e^{-ik\Delta t} dk, \qquad m \geq 0, \tag{A15}$$

then it is easy to see that

$$H_{ijkl}^{2m+1} = \frac{1}{2(2\pi)^{m+2}} \partial_i \partial'_j \partial_k \partial'_l (Q(m)), \tag{A16}$$

and

$$Q(m) = \frac{2m}{R^2} Q(m-1) - \frac{1}{R^2} S(m-2). \tag{A17}$$

The above equation applies for $m \geq 2$. To use it to get a general expression, we need $Q(0)$, which can be calculated in the case of $R^2 < \Delta t^2$ as follows:

$$Q(0) = \frac{1}{R} \int_0^\infty \frac{1}{k^3} J_1(kR)\cos(k\Delta t) dk$$
$$= \lim_{\beta\to 0} \frac{1}{R} \int_0^\infty \frac{k^{-1}}{(k^2+\beta^2)} J_1(kR)\cos(k\Delta t) dk$$
$$= \lim_{\beta\to 0} \frac{1}{R} \frac{e^{-\beta\Delta t} I_1(\beta R)}{\beta^2} = \frac{1}{2\beta} - \frac{1}{2\Delta t}. \tag{A18}$$

This leads to a vanishing H_{ijkl}. However, in the case of $R^2 > \Delta t^2$, the calculation becomes a little complicated. First, let us write $Q(0)$ as

$$Q(0) = -\frac{1}{2R} \int_0^\infty \left(\frac{1}{k^2}\right)' J_1(kR)\cos(k\Delta t) dk$$
$$= \lim_{k\to 0} \left[-\frac{1}{4k}\cos(k\Delta t) \right] + P_1 + P_2 - \frac{1}{2} Q(0), \tag{A19}$$

where we have used the fact that

$$J'_1(x) = J_0(x) - \frac{1}{x} J_1(x), \qquad J_1(x) \sim \frac{x}{2}, \quad \text{as } \to 0, \tag{A20}$$

and defined

$$P_1 = -\frac{\Delta t}{2R} \int_0^\infty \frac{1}{k^2} J_1(kR)\sin(k\Delta t) dk, \tag{A21}$$

and

$$P_2 = \frac{1}{2} \int_0^\infty \frac{\cos(k\Delta t) J_0(kR)}{k^2} dk. \tag{A22}$$

Note that the first term in Eq. (A19) can be dropped although it is formally divergent. The reason is that it is only dependent on Δt and H_{ijkl} involves spatial differentiation. So, it follows that

$$Q(0) = \frac{2}{3}(P_1 + P_2). \tag{A23}$$

Our next task is then to evaluate P_1 and P_2, which can be done as follows:

$$\begin{aligned} P_1 &= \frac{\Delta t}{2R}\int_0^\infty \left(\frac{1}{k}\right)' J_1(kR)\sin(k\Delta t)dk \\ &= -\frac{\Delta t^2}{2R}\int_0^\infty J_1(kR)\cos(k\Delta t)\frac{dk}{k} \\ &\quad -\frac{\Delta t}{2}\int_0^\infty J_0(kR)\sin(k\Delta t)\frac{dk}{k} - P_1. \end{aligned} \tag{A24}$$

Substitution of integrals (6.693.1) and (6.693.2) in Ref. [31] into the above equation yields

$$P_1 = -\frac{\Delta t^2}{4R^2}\sqrt{R^2 - \Delta t^2} - \frac{\Delta t}{4}\arcsin(\Delta t/R). \tag{A25}$$

As for P_2, the calculation goes

$$\begin{aligned} P_2 &= -\frac{1}{2}\int_0^\infty \left(\frac{1}{k}\right)' \cos(k\Delta t)J_0(kR)dk \\ &= -\frac{\Delta t}{2}\int_0^\infty J_0(kR)\sin(k\Delta t)\frac{dk}{k} \\ &\quad -\frac{R}{2}\int_0^\infty J_1(kR)\cos(k\Delta t)\frac{dk}{k} \\ &= -\frac{\Delta t}{2}\arcsin(\Delta t/R) - \frac{\sqrt{R^2-\Delta t^2}}{2}, \end{aligned} \tag{A26}$$

where we have also discarded a formally divergent term dependent only on Δt since it does not contribute to H_{ijkl}. A combination of the above derived results finally leads to

$$Q(0) = -\frac{\sqrt{R^2-\Delta t^2}(\Delta t^2 + 2R^2)}{6R^2} - \frac{\Delta t}{2}\arcsin(\Delta t/R). \tag{A27}$$

We next need $Q(1)$, which can be calculated, using integral (6.693.5) in Ref. [31], as follows:

$$\begin{aligned} Q(1) &= \frac{1}{R^2}\int_0^\infty \frac{J_2(Rk)\cos(\Delta t k)}{k^2}dk = \begin{cases} \frac{1}{R^2}\left[\frac{R}{4}\cos(\arcsin(\Delta t/R)) + \frac{R}{12}\cos(3\arcsin(\Delta t/R))\right] & \text{for } R^2 > \Delta t^2, \\ 0 & \text{for } R^2 < \Delta t^2. \end{cases} \\ &= \begin{cases} \left(\frac{1}{3} - \frac{\Delta t^2}{3R^2}\right)\frac{\sqrt{R^2-\Delta t^2}}{R^2} = \left(\frac{1}{3} - \frac{\Delta t^2}{3R^2}\right)S(0) & \text{for } R^2 > \Delta t^2, \\ 0 & \text{for } R^2 < \Delta t^2. \end{cases} \end{aligned} \tag{A28}$$

In the above calculation, we have made use of the following trigonometric relations:

$$\cos(3x) = 4\cos^3(x) - 3\cos(x), \qquad \cos(\arcsin x) = \sqrt{1-x^2}. \tag{A29}$$

Therefore, for $m \geq 2$, one finds, using the recursive relation Eq. (A17),

$$\begin{aligned} Q(m) &= \frac{(2m)!!}{2R^{2m-2}}Q(1) - \frac{1}{R^2}\sum_{k=2}^{m}\frac{(2m)!!}{(2k)!!}\frac{S(k-2)}{R^{2m-2k}} \\ &= \frac{(2m)!!}{R^{2m-2}}\left[\frac{1}{2}Q(1) + \sum_{k=2}^{m}\sum_{j=0}^{k-2}\frac{(2j+1)!!}{2k(2k-2)(2j)!!(2j+1)(2j-1)}\frac{R^{2j}}{(R^2-\Delta t^2)^j}S(0)\right]. \end{aligned} \tag{A30}$$

This expression can be simplified if we note that

$$\sum_{k=j+2}^{m}\frac{1}{k(k-1)} = \sum_{k=2}^{m}\frac{1}{k(k-1)} - \sum_{k=2}^{j+1}\frac{1}{k(k-1)} = \frac{m-j-1}{m(j+1)}, \tag{A31}$$

and

$$\begin{aligned} &\sum_{k=2}^{m}\sum_{j=0}^{k-2}\frac{(2j+1)!!}{2k(2k-2)(2j)!!(2j+1)(2j-1)}\frac{R^{2j}}{(R^2-\Delta t^2)^j}S(0) \\ &= \sum_{j=0}^{m-2}\sum_{k=j+2}^{m}\frac{(2j+1)!!}{2k(2k-2)(2j)!!(2j+1)(2j-1)}\frac{R^{2j}}{(R^2-\Delta t^2)^j}S(0) \\ &= \sum_{j=0}^{m-2}\frac{(m-j-1)(2j+1)!!}{4m(j+1)(2j)!!(2j+1)(2j-1)}\frac{R^{2j}}{(R^2-\Delta t^2)^j}S(0). \end{aligned} \tag{A32}$$

So, we have in this case

$$D^{2m+1} = \begin{cases} \frac{(2m+1)!!}{2(2\pi)^{m+2}} \frac{1}{(R^2-\Delta t^2)^{m+3/2}}, & \text{for } R^2 > \Delta t^2, \\ 0 & \text{for } R^2 < \Delta t^2, \end{cases} \tag{A33}$$

$$F^{2m+1}_{ij} = -\frac{1}{2(2\pi)^{m+2}} \partial_i \partial'_j \bigg(\frac{(2m)!!}{R^{2m}} S(0) \sum_{k=0}^{m} \frac{(2k+1)!!}{(2k)!!(2k+1)(2k-1)} \frac{R^{2k}}{(R^2-\Delta t^2)^k} \bigg), \tag{A34}$$

and

$$H^{2m+1}_{ijkl} = \frac{1}{2(2\pi)^{m+2}} \partial_i \partial'_j \partial_k \partial'_l \bigg\{ \frac{(2m)!!}{R^{2m-2}} \bigg[\frac{1}{2} Q(1) + \sum_{j=0}^{m-2} \frac{(m-j-1)(2j+1)!!}{4m(j+1)(2j)!!(2j+1)(2j-1)} \frac{R^{2j}}{(R^2-\Delta t^2)^j} S(0) \bigg] \bigg\}, \tag{A35}$$

for $m \geq 2$, while for $m = 0$ and $m = 1$, H_{ijkl} can be found by using Eq. (A16), (A27), and (A28).

2. The case of even n

Let $n = 2m$ with $m = 1, 2, 3 \ldots$. The graviton two-point functions for $m = 0$ corresponding to the usual four-dimensional spacetime have been given previously [14]. The analog of Eq. (A11) for this case is

$$F^{2m}_{ij} = \frac{1}{2(2\pi)^{m+3/2}} \partial_i \partial'_j \bigg(S\bigg(m - \frac{1}{2} \bigg) \bigg). \tag{A36}$$

Here

$$S(m-1/2) = \frac{2m-1}{R^2} S\bigg(m - \frac{3}{2} \bigg) - T\bigg(m - \frac{3}{2} \bigg). \tag{A37}$$

Using this recursive relation, we can express $S(m-1/2)$ in terms of $S(1/2)$ which is calculated, by employing

$$J_{n+1/2}(z) = (-1)^n z^{n+1/2} \sqrt{\frac{2}{\pi}} \frac{d^n}{(zdz)^n} \bigg(\frac{\sin z}{z} \bigg), \tag{A38}$$

to be

$$\begin{aligned} S(1/2) &= \frac{1}{R^{3/2}} \int_0^\infty k^{-1/2} J_{3/2}(Rk) \cos(\Delta t k) dk \\ &= -\sqrt{\frac{2}{\pi}} \frac{1}{R^3} \int_0^\infty \frac{d}{dk} \bigg(\frac{\sin(Rk)}{k} \bigg) \cos(\Delta t k) dk \\ &= -\sqrt{\frac{2}{\pi}} \bigg(\frac{1}{R^3} \frac{\sin(Rk)\cos(\Delta t k)}{k} \bigg|_0^\infty \\ &\quad + \frac{\Delta t}{R^3} \int_0^\infty \frac{\sin(Rk)\sin(\Delta t k)}{k} dk \bigg) \\ &= \sqrt{\frac{2}{\pi}} \bigg(\frac{1}{R^2} - \frac{\Delta t}{4R^3} \ln\bigg(\frac{R+\Delta t}{R-\Delta t} \bigg)^2 \bigg). \end{aligned} \tag{A39}$$

It then follows that

$$\begin{aligned} S(m-1/2) &= \frac{(2m-1)!!}{R^{2m-2}} S(1/2) - \sum_{k=2}^{m} \frac{(2m-1)!!}{(2k-1)!!} \\ &\quad \times \frac{T(k-3/2)}{R^{2m-2k}} \\ &= \frac{(2m-1)!!}{R^{2m}} \sqrt{\frac{2}{\pi}} \bigg[1 - \frac{\Delta t}{4R} \ln\bigg(\frac{R-\Delta t}{R-\Delta t} \bigg)^2 \\ &\quad - \frac{1}{R^2} \sum_{k=2}^{m} \frac{2^{k-2}\Gamma(k-1)}{(2k-1)!!} \frac{R^{2k}}{(R^2-\Delta t^2)^{k-1}} \bigg]. \end{aligned} \tag{A40}$$

Similarly, one has for H^{2m}_{ijkl}

$$H^{2m}_{ijkl} = \frac{1}{2(2\pi)^{m+3/2}} \partial_i \partial'_j \partial_k \partial'_l \bigg(Q\bigg(m - \frac{1}{2} \bigg) \bigg), \tag{A41}$$

and

$$Q(m-1/2) = \frac{2m-1}{R^2} Q\bigg(m - \frac{3}{2} \bigg) - \frac{1}{R^2} S\bigg(m - \frac{5}{2} \bigg). \tag{A42}$$

Now the calculation becomes a little tricky. First, let us note that H^0_{ijkl} has already been given [14] and the recursive relation Eq. (A42) can only be applied when $m \geq 3$. So, we need both H^2_{ijkl} and H^4_{ijkl} or $Q(1/2)$ and $Q(3/2)$ as our basis to use the recursive relation for a general expression. Because there is an infrared divergence in the $Q(1/2)$ integral, so, as we did in the four-dimensional case, we will introduce a regulator β in the denominator of the integrand and then let β approach 0 after the integration is performed. Noting that

$$J_{3/2}(z) = \sqrt{\frac{2}{\pi z}} \bigg(\frac{\sin z}{z} - \cos z \bigg), \tag{A43}$$

we obtain

$$Q(1/2) = \frac{1}{R^{3/2}} \int_0^\infty k^{-5/2} J_{3/2}(kR) \cos(kt) dk$$
$$= \sqrt{\frac{2}{\pi}} \left(\frac{1}{R^3} \int_0^\infty \frac{dk}{k^4} \sin kR \cos k\Delta t - \frac{1}{R^2} \int_0^\infty \frac{dk}{k^3} \cos kR \cos k\Delta t \right)$$
$$= \sqrt{\frac{2}{\pi}} \lim_{\beta\to 0} \left(-\frac{1}{R^3} \frac{1}{2\beta} \frac{\partial}{\partial\beta} \int_0^\infty \frac{\sin kR \cos k\Delta t}{k^2+\beta^2} dk + \frac{1}{R^2} \frac{1}{2\beta} \frac{\partial}{\partial\beta} \int_0^\infty \frac{k \cos kR \cos k\Delta t}{k^2+\beta^2} dk \right). \quad \text{(A44)}$$

We next use

$$\int_0^\infty \frac{\sin(ax)\cos(bx)}{\beta^2+x^2} dx = \frac{1}{4\beta} e^{-a\beta} \{ e^{b\beta} \mathrm{Ei}[\beta(a-b)] + e^{-b\beta} \mathrm{Ei}[\beta(a+b)] \} - \frac{1}{4\beta} e^{a\beta} \{ e^{b\beta} \mathrm{Ei}[-\beta(a+b)] + e^{-b\beta} \mathrm{Ei}[-\beta(a-b)] \}, \quad \text{(A45)}$$

$$\int_0^\infty \frac{x\cos(ax)\cos(bx)}{\beta^2+x^2} dx = -\frac{1}{4} e^{-a\beta} \{ e^{b\beta} \mathrm{Ei}[\beta(a-b)] + e^{-b\beta} \mathrm{Ei}[\beta(a+b)] \} - \frac{1}{4} e^{a\beta} \{ e^{b\beta} \mathrm{Ei}[-\beta(a+b)] + e^{-b\beta} \mathrm{Ei}[-\beta(a-b)] \}, \quad \text{(A46)}$$

where $\mathrm{Ei}(x)$ is the exponential-integral function, and the fact that, when x is small,

$$\mathrm{Ei}(x) \approx \gamma + \ln|x| + x + \frac{1}{4}x^2 + \frac{1}{18}x^3 + O(x^4), \quad \text{(A47)}$$

where γ is the Euler constant. After expanding $Q(1/2)$ around $\beta = 0$ to the order of β^2, one finds

$$Q(1/2) = \lim_{\beta\to 0} \sqrt{\frac{2}{\pi}} \left(\frac{5}{18} - \frac{1}{3}\gamma - \frac{1}{3}\ln(\beta) - \frac{1}{6}\ln(R^2-\Delta t^2) - \frac{\Delta t^2}{6R^2} + \frac{\Delta t}{8R}\left(\frac{\Delta t^2}{3R^2} - 1\right) \ln\left(\frac{R+\Delta t}{R-\Delta t}\right)^2 \right). \quad \text{(A48)}$$

Note, however, that what we need is H_{ijkl} which involves differentiation of $Q(1/2)$, therefore we can discard the constant and divergent terms in $Q(1/2)$ as far as H_{ijkl} is concerned. To calculate $Q(3/2)$, let us recall that

$$Q(3/2) = \frac{3}{R^2} Q(1/2) - \frac{1}{R^2} S(-1/2) \quad \text{(A49)}$$

and note that $S(-1/2)$ is given by $\sqrt{2/\pi}$ times Eq. (A19) in Ref. [14] Thus, we have

$$Q(3/2) = \sqrt{\frac{2}{\pi}} \left(-\frac{1}{6R^2} - \frac{\Delta t^2}{2R^4} + \frac{\Delta t}{8R^3}\left(\frac{\Delta t^2}{R^2} - 1\right) \times \ln\left(\frac{R+\Delta t}{R-\Delta t}\right)^2 \right). \quad \text{(A50)}$$

With $Q(3/2)$ at hand, it is easy to show that for an arbitrary $m \geq 3$

$$Q(m-1/2) = \frac{(2m-1)!!}{3R^{2m-4}} Q(3/2) - \frac{1}{R^2} \sum_{k=3}^{m} \frac{(2m-1)!!}{(2k-1)!!} \frac{S(k-5/2)}{R^{2m-2k}}$$
$$= \frac{(2m-1)!!}{R^{2m-4}} \left(\frac{1}{3} Q(3/2) - \sum_{k=3}^{m} \frac{1}{(2k-1)(2k-3)} S(1/2) + \frac{1}{R^4} \sqrt{\frac{2}{\pi}} \sum_{k=3}^{m} \sum_{j=2}^{k-2} \frac{2^{j-2}\Gamma(j-1)}{(2j-1)!!(2k-1)(2k-3)} \frac{R^{2j}}{(R^2-\Delta t^2)^{j-1}} \right)$$
$$= \frac{(2m-1)!!}{R^{2m-4}} \left(\frac{1}{3} Q(3/2) - \sum_{k=3}^{m} \frac{1}{(2k-1)(2k-3)} S(1/2) + \frac{1}{R^4} \sqrt{\frac{2}{\pi}} \sum_{j=2}^{m-2} \frac{(m-j-1)2^{j-2}\Gamma(j-1)}{(2m-1)(2j+1)!!} \frac{R^{2j}}{(R^2-\Delta t^2)^{j-1}} \right). \quad \text{(A51)}$$

Here in the last step, we have made use of the following results

$$\sum_{k=j+2}^{m} \frac{1}{(2k-1)(2k-3)} = \sum_{k=2}^{m} \frac{1}{(2k-1)(2k-3)} - \sum_{k=2}^{j+1} \frac{1}{(2k-1)(2k-3)} = \frac{m-j-1}{(2m-1)(2j+1)}, \quad \text{(A52)}$$

and

$$\sum_{k=3}^{m}\sum_{j=2}^{k-2} f(j)g(k) = \sum_{k=4}^{m}\sum_{j=2}^{k-2} f(j)g(k) = \sum_{j=2}^{m-2} f(j)\sum_{k=2+j}^{m} g(k). \tag{A53}$$

Consequently, we obtain

$$D^{2m} = \frac{2^m m!}{(2\pi)^{m+2}}\frac{1}{(R^2-\Delta t^2)^{m+1}}, \tag{A54}$$

$$F_{ij}^{2m} = \frac{1}{(2\pi)^{m+2}}\partial_i\partial_j'\left\{\frac{(2m-1)!!}{R^{2m}}\left[1-\frac{\Delta t}{4R}\ln\left(\frac{R-\Delta t}{R-\Delta t}\right)^2 - \frac{1}{R^2}\sum_{k=2}^{m}\frac{2^{k-2}\Gamma(k-1)}{(2k-1)!!}\frac{R^{2k}}{(R^2-\Delta t^2)^{k-1}}\right]\right\}, \tag{A55}$$

and

$$H_{ijkl}^{2m} = \frac{1}{(2\pi)^{m+2}}\partial_i\partial_j'\partial_k\partial_l'\left\{\frac{(2m-1)!!}{R^{2m-4}}\left[\frac{\Delta t}{24R^3}\left(\frac{\Delta t^2}{R^2}-1\right)\ln\left(\frac{R+\Delta t}{R-\Delta t}\right)^2 - \frac{1}{18R^2} - \frac{\Delta t^2}{6R^4}\right.\right.$$
$$\left.\left. - \sum_{k=3}^{m}\frac{1}{(2k-1)(2k-3)}\left(\frac{1}{R^2}-\frac{\Delta t}{4R^3}\ln\left(\frac{R+\Delta t}{R-\Delta t}\right)^2\right) + \frac{1}{R^4}\sum_{j=2}^{m-2}\frac{(m-j-1)2^{j-2}\Gamma(j-1)}{(2m-1)(2j+1)!!}\frac{R^{2j}}{(R^2-\Delta t^2)^{j-1}}\right)\right. \tag{A56}$$

APPENDIX B: DERIVATION OF EQ. (44)

Here we wish to give the details of the derivation of Eq. (44), the expression for $\langle\sigma_1\sigma_1'\rangle$ in the limit of large r. The relevant graviton two-point function can be expressed as

$$\begin{aligned} &G_{xxxx}(t,x,0,0,0,t',x',0,0,nL)|_{t-t'=\rho-T} \\ &= \frac{1}{3\pi^2}\frac{n^4L^4 - 10n^2L^2\beta(\rho,T)^2 - 15\beta(\rho,T)^4}{\beta(\rho,T)^3(\rho^2+n^2L^2)^2} \\ &+ \frac{4}{3\pi^2}\frac{3n^4L^4+20n^2L^2\beta(\rho,T)^2+5\beta(\rho,T)^4}{\beta(\rho,T)(\rho^2+n^2L^2)^3} \\ &- \frac{4}{3\pi^2}\frac{18n^4L^4\beta(\rho,T)+20n^2L^2\beta(\rho,T)^3}{(\rho^2+n^2L^2)^4} \\ &+ \frac{64}{3\pi^2}\frac{n^4L^4\beta(\rho,T)^3}{(\rho^2+n^2L^2)^5}, \end{aligned} \tag{B1}$$

where

$$\beta(\rho,T) \equiv \sqrt{n^2L^2+2\rho T - T^2}. \tag{B2}$$

Utilizing the following integration relation

$$\int_a^b dx\int_a^b dx' f(x-x') = \int_0^r (r-\rho)[f(\rho)+f(-\rho)]d\rho, \tag{B3}$$

one finds that

$$\langle\sigma_1\sigma_1'\rangle = -\frac{r^2}{36\pi^2}\sum_{n=1}^{\infty}\left[-\frac{2(4n^2L^2-T^2)\sqrt{n^2L^2-T^2}}{n^4L^4} + \frac{h(r,T)+h(-r,T)}{(n^2L^2+r^2)^3}\right], \tag{B4}$$

where

$$h(r,T) \equiv \beta(r,T)[4n^4L^4 + n^2L^2(6r-T)(2r+T) + 3r^2(2r+T)^2]. \tag{B5}$$

A few things are to be noticed here: (1) We need to drop the terms when the square root is imaginary. (2) It can be shown that the above expression for $\langle\sigma_1\sigma_1'\rangle$ reduces to $\langle\sigma_1^2\rangle$ when $T=0$, as it should. (3) The asymptotic behavior of the summand when $n\to\infty$, is $\sim\frac{1}{n^3}$ hence the summation converges.

To proceed, let us now assume that $r \gg T, L$, then

$$\begin{aligned}\langle\sigma_1\sigma_1'\rangle \approx -\frac{r^2}{36\pi^2}\sum_{n=1}^{\infty}\Bigg[&-\frac{2(4n^2L^2-T^2)\sqrt{n^2L^2-T^2}}{n^4L^4} \\ &+ \frac{\sqrt{n^2L^2-2rT}}{(n^2L^2+r^2)^3}[4n^4L^4+12r^2n^2L^2+12r^4] \\ &+ \frac{\sqrt{n^2L^2+2rT}}{(n^2L^2+r^2)^3}[4n^4L^4+12r^2n^2L^2+12r^4]\Bigg],\end{aligned} \tag{B6}$$

and

$$p \equiv \sqrt{\frac{2rT}{L^2}} \gg 1 \tag{B7}$$

is a huge number. Thus for the second term in Eq. (B6), the sum should only start from $n=p$. We can now split the summation into two parts, i.e. terms with $n\le p$ and those with $n>p$. Using the asymptotic form of the summand for the part with $n>p$ and defining $m=[T/L]$, where [ ] denotes the integer part, one has

$$\langle\sigma_1\sigma_1'\rangle \approx \frac{r^2}{36\pi^2 L}\left(\sum_{n=m}^{p}\frac{2(4n^2-m^2)\sqrt{n^2-m^2}}{n^4} - \sum_{n=1}^{p}\frac{4\sqrt{n^2+p^2}}{(n^2+r^2/L^2)^3}[n^4+3n^2r^2/L^2+3r^4/L^4] + \sum_{n=p}^{\infty}\frac{6T^2}{n^3L^3}\right). \tag{B8}$$

Hence, it follows that

$$\langle\sigma_1\sigma_1'\rangle < \frac{r^2}{36\pi^2 L}\left(\int_m^p dn\frac{2(4n^2-m^2)\sqrt{n^2-m^2}}{n^3} + \sum_{n=1}^{p}\frac{4\sqrt{n^2+p^2}}{(n^2+r^2/L^2)^3}[n^4+3n^2r^2/L^2+3r^4/L^4] + \sum_{n=p}^{\infty}\frac{6T^2}{n^3L^3}\right). \tag{B9}$$

Let us now evaluate the above expression term by term. One has, keeping in mind that $p \gg 1$, that

$$\int_m^p dn\frac{2(4n^2-m^2)\sqrt{n^2-m^2}}{n^3} \approx 8\ln\frac{2p}{m} \approx 4\ln\frac{2r}{T}, \tag{B10}$$

$$\sum_{n=1}^{p}\frac{4\sqrt{n^2+p^2}}{(n^2+r^2/L^2)^3}[n^4+3n^2r^2/L^2+3r^4/L^4] \approx \int_{1/p}^{1}dx\frac{\sqrt{1+x^2}}{(x^2+r/2T)^3}(4x^4+6x^2r/T+3r^2/T^2) \approx 12[\sqrt{2}+\mathrm{Coth}^{-1}(\sqrt{2})]\frac{T}{r}, \tag{B11}$$

and

$$\sum_{n=p}^{\infty}\frac{1}{n^3} = -\frac{1}{2}\Psi(2,p) \sim \frac{1}{2}\frac{1}{p^2} = \frac{1}{4}\frac{L}{r}\frac{L}{T}. \tag{B12}$$

Here we have used Eq. (B7) and the asymptotic expansion for $\Psi(2,x)$

$$\Psi(2,x) \approx -\frac{1}{x^2} - \frac{1}{x^3} - \frac{1}{2x^4} + O(1/x^6), \tag{B13}$$

where function $\Psi(n,x)$ is defined as

$$\Psi(n,x) = \frac{d^n\psi(x)}{dx^n}, \qquad \psi(x) = \frac{d}{dx}\ln\Gamma. \tag{B14}$$

Keeping only the dominant terms, we obtain Eq. (44).

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