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Spontaneous compactification and nonassociativity

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We consider the Freund-Rubin-Englert mechanism of compactification of $N = 1$ supergravity in 11 dimensions. We systematically investigate both well-known and some new solutions of the classical equations of motion in 11 dimensions. In particular, we show that any threeform potential in 11 dimension is given locally by the structure constants of a geodesic loop in an affinely connected space.

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I. INTRODUCTION

The Kaluza-Klein mechanism of spontaneous compactification works as follows [1–4]. The equations describing gravity and matter fields in d dimensions are considered. A vacuum solution of the equations in which d-dimensional spacetime M is of the form $M_4 \times K$ is searched. Here M_4 is a maximally symmetric four-dimensional space (de Sitter space, anti–de Sitter space or Minkowski space) and K is a compact manifold (as a rule this is an Einstein space). The representation of M in the form of a direct product induces the block diagonal form of the vacuum metric

$$g_{MN} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & g_{mn} \end{pmatrix}, \tag{1}$$

where $g_{\mu\nu}$ and g_{mn} are components of g_{MN} defined on M_4 and K, respectively. In one's turn, the block representation of g_{MN} is compatible with the Einstein equations

$$R_{MN} - \tfrac{1}{2}g_{MN}R = T_{MN} - \Lambda g_{MN}, \tag{2}$$

if components of the energy-momentum tensor of matter fields are

$$T_{\mu\nu} = k_1 g_{\mu\nu}, \qquad T_{mn} = k_2 g_{mn}. \tag{3}$$

If we consider the interaction of gravity with matter fields without the potential term $g_{MN}V(\varphi)$, then $T_{00} > 0$ and the constant k_1 is negative. In addition, if the cosmological constant $\Lambda = 0$, then it follows from (3) that M_4 is an anti–de Sitter space. On the contrary, if T_{MN} contains the potential term, then the solution $\varphi = \text{const}$ is equivalent to the introduction of the Λ term and the space M_4 may be flat [5,6].

Now we briefly consider the Freund-Rubin-Englert mechanism [7,8] of spontaneous compactification of $d = 11$ supergravity. In the Bose sector of this theory the equations of motion (Einstein equations and equations for the antisymmetric gauge field strength) have the form [9]

$$R_{MN} - \tfrac{1}{2}g_{MN}R = 12(8F_{MPQR}F_N{}^{PQR} - g_{MN}F_{SPQR}F^{SPQR}), \tag{4}$$

$$F^{MNPQ}{}_{;M} = -\frac{\sqrt{2}}{24}\varepsilon^{NPQM_1\ldots M_8}F_{M_1M_2M_3M_4}F_{M_5M_6M_7M_8}, \tag{5}$$

where $\varepsilon^{M_1\ldots M_r}$ is a fully antisymmetric covariant constant tensor such that $\varepsilon_{1\ldots r} = |g|^{1/2}$. The Freund and Rubin solution [7] is

$$F_{\mu\nu\sigma\lambda} = \rho\varepsilon_{\mu\nu\sigma\lambda}, \tag{6}$$

where ρ is a real constant, and all other components of F_{MNPQ} are zero. A more composite solution is obtained if we assume that $F_{MNPQ} \neq 0$ not only for the spacetime components, assuming the Freund-Rubin form, but also for the internal space components. Such solutions (Englert solution [8]) were first constructed on the sphere $\mathbb{S}^7$ with torsion. The Englert solution is to set

$$F_{\mu\nu\sigma\lambda} = \rho\varepsilon_{\mu\nu\sigma\lambda}, \tag{7}$$

$$F_{mnpq} = \lambda\partial_{[q}S_{mnp]}, \tag{8}$$

where $S_{mnp} = S_{[mnp]}$ is a suitable totally antisymmetric torsion tensor.

Note that the connection between an antisymmetric gauge field strength and a torsion defined by (8) has an universal character in the 11-dimensional supergravity. Bars and McDowell [10] have shown that the g_{MN}/A_{MNP} gravity-matter system may be reinterpreted, in first-order formalism, as a pure gravity theory with torsion S_{MNP} such that

$$A_{MNP} = \lambda S_{[MNP]}, \tag{9}$$

$$F_{MNPS} = \partial_{[S}A_{MNP]}. \tag{10}$$

In this sense, even the Abelian gauge invariance of A_{MNP} may be regarded as a spacetime symmetry of pure gravity. Besides, the deformation,

$$\mathring{\Gamma}_{MNP} \to \mathring{\Gamma}_{MNP} + S_{[MNP]}, \tag{11}$$

of the Riemann connection converts M into an affinely

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connected space with torsion. Note that such an interpretation is possible only for three-index fields since the torsion tensor is rank 3.

II. PRELIMINARIES

Out next goal is to think about algebraic properties of the threeform potential and its field strength. The requisite concepts are geodesic loops in an affinely connected space and its tangent algebras. In [11] Kikkawa introduced the idea of a geodesic loop in an affinely connected space. In that article it is proved that in a neighborhood of each point of an affinely connected space we can define, in a natural way, the operation of multiplication, relative to which this neighborhood becomes a local loop. Let us recall Kikkawa's construction.

Let M be an affinely connected space and e an arbitrary point in it. In a neighborhood of this point, we introduce a binary operation as follows. Let x and y be two points belonging to this neighborhood. We connect them to the point e by geodesic lines $\widehat{ex}$ and $\widehat{ey}$. Then we translate the arc $\widehat{ex}$ in a parallel way into the position $\widehat{yz}$. The point z is then by definition the product of the points x and y, i.e. $z = xy$. We denote the obtained binary system by the symbol G_e. It is easy to prove that e is the unity element of G_e, the equations $ax = b$ and $ya = b$ are uniquely solvable for all $a, b \in G_e$, and G_e satisfies the identity

$$xx^2 = x^2x. \tag{12}$$

Thus, G_e is a local monoassociative loop. The loop G_e is said to be a geodesic loop of the affinely connected space M.

To study geodesic loops in affinely connected spaces we may use a method which is usually employed to study the local structure of Lie groups. Despite the lack of associativity in geodesic loops, this method enables us to uniquely define binary $[x, y]$ and ternary (x, y, z) operations in their tangent spaces T_e and to construct the local algebras [12,13]. These operations are expressed in terms of the coordinates of the vectors $x, y, z \in T_e$ as follows:

$$[x, y]^i = 2\alpha^i_{jk}x^j y^k, \tag{13}$$

$$(x, y, z)^i = 2\beta^i_{jkl}x^j y^k z^l. \tag{14}$$

If $\alpha^i_{jk} = 0$, then the corresponding geodesic loop G_e is Abelian, if $\beta^i_{jkl} = 0$, then the loop G_e is associative, i.e. it is a local Lie group. The tensors α^i_{jk} and β^i_{jkl} are called the fundamental tensors of the geodesic loop G_e [14]. They satisfy the relation

$$\beta^i_{[jkl]} = \alpha^m_{[jk}\alpha^i_{l]m}, \tag{15}$$

which follows from (12). The right-hand side of the relation is obtained from the Jacobiator of the vectors x, y, and z. This is the reason that relation (15) is called the generalized Jacobi identity.

Now suppose M is an affinely connected space with torsion and Γ^i_{jk} is a metric-compatible affine connection, the difference with the Levi-Civita connection is given by the contortion tensor. It is easy to prove that $S_{ijk} = S_{[ijk]}$. Therefore

$$\Gamma_{ijk} = \overset{\circ}{\Gamma}_{ijk} + S_{ijk} \tag{16}$$

is a metric-compatible affine connection with skew-symmetric torsion. We choose the torsion and curvature tensors of M in the form

$$S^i_{jk} = \Gamma^i_{[jk]}, \tag{17}$$

$$R^i_{jkl} = \partial_k\Gamma^i_{jl} - \partial_l\Gamma^i_{jk} + \Gamma^m_{jl}\Gamma^i_{mk} - \Gamma^m_{jk}\Gamma^i_{ml}. \tag{18}$$

It is known [14] that for any geodesic loop constructed in a neighborhood of $e \in M$, the fundamental tensors can be expressed using values of the torsion and curvature tensors in e by the formulas

$$\alpha^i_{jk} = -S^i_{jk}, \tag{19}$$

$$4\beta^i_{jkl} = -2\nabla_l S^i_{jk} - R^i_{jkl}. \tag{20}$$

Thus, noncommutativity and nonassociativity of the local geodesic loops are intimately related to torsion and curvature of the space M. Note that the formulas (19) and (20), full antisymmetry of the torsion tensor, and the Bianchi identities

$$R^i_{[jkl]} + 2\nabla_{[j}S^i_{kl]} + 4S^m_{[jk}S^i_{l]m} = 0, \tag{21}$$

$$\nabla_{[k}R^{ij}{}_{lm]} - 2R^{ij}{}_{n[k}S^n_{lm]} = 0, \tag{22}$$

will play an important part in following constructions. Note also that the first Bianchi identity (21) is obtained if we substitute (19) and (20) in the identity (15).

III. THE FREUND-RUBIN TYPE SOLUTIONS

In this section, we begin an analysis of the $M_4 \times K$ compactification of $d = 11$ supergravity. We consider the Bose sector of this theory as a pure gravity theory with torsion. We show that the torsion is given locally by structure constants of a geodesic loop. Then we apply this result to an analysis of the Freund-Rubin solution.

A. Geodesic groups

We consider the $M = M_4 \times K$ compactification of $d = 11$ supergravity. Obviously, M is a Riemann space with a metric of the block diagonal form (1). We deform the Riemannian connection by the rule (16) and convert M into an affinely connected space with a fully antisymmetric torsion tensor S_{ijk}. Since the projections $M_4 \times K \to M_4$

and $M_4 \times K \to K$ are Riemannian submersions, we can construct objects in M by pulling back the objects from M_4 and K along these projections. In particular, we consider geodesic loops in M_4 and in K separately in order to later pull them back to M in this way.

Suppose e is an arbitrary point in M_4 or K. In a neighborhood of this point, we define a geodesic loop G_e. Then using the full skew-symmetry of S_{ijk}, we rewrite the tensors (19) and (20) in the form

$$\alpha_{ijk} = -S_{ijk}, \tag{23}$$

$$4\beta_{ijkl} = -2S_{ijk;l} + 6S^m_{[ij}S_{kl]m} - R_{ijkl}. \tag{24}$$

Now suppose the loop G_e is associative. Then we have the identities

$$\beta_{ijkl} = 0, \tag{25}$$

$$\alpha^m_{[jk}\alpha^i_{l]m} = 0, \tag{26}$$

instead of (15). It follows from here that the tangent algebra A_G of the local loop G_e in the point e is a Lie algebra. We prove that this algebra is compact if $e \in K$. Indeed, we choose the basic $\{e_i\}$ in A_G such that

$$[e_i, e_j] = \alpha_{ij}{}^k e_k, \tag{27}$$

and define the bilinear form

$$(e_i, e_j) = g_{ij}. \tag{28}$$

Using the full skew-symmetry of α_{ijk}, we prove the equality

$$([e_i, e_j], e_k) = (e_i, [e_j, e_k]). \tag{29}$$

If the point $e \in K$, then the inner product (28) is Euclidean and hence it is positive definite. Therefore the Lie algebra A_G is compact. It is well known (see e.g. [15]) that any compact Lie algebra is reductive, and hence isomorphic to the direct sum of a semisimple Lie subalgebra and the center, which is Abelian. Therefore if the algebra A_G is non-Abelian, then we have one of the following isomorphisms:

$$A_G \simeq su(2) \oplus su(2) \oplus u(1), \qquad A_G \simeq su(2) \oplus u(1) \oplus u(1) \oplus u(1) \oplus u(1). \tag{30}$$

If the point $e \in M_4$, then the inner product (28) is Lorentzian. The classification of Lorentzian metric Lie algebras is known [16]. Any such algebra is isomorphic to one of the following metric Lie algebras:

(1) Abelian with Lorentzian inner product,
(2) $su(2) \oplus u(1)$ with a timelike inner product on $u(1)$,
(3) $su(1, 1) \oplus u(1)$ with Lorentzian inner product on $su(1, 1)$ and Euclidean on $u(1)$,
(4) the Nappi-Witten solvable Lie algebra [17].

We find an explicit form of the Lie brackets (27) for the above Lie algebras. Obviously, the algebra A_G is Abelian if and only if all $\alpha_{ijk} = 0$. Suppose 1, 2, and 3 are spatial indexes. Then

$$A_G \simeq su(2) \oplus u(1) \quad \text{if only } \alpha_{123} \neq 0, \qquad A_G \simeq su(1, 1) \oplus u(1) \quad \text{if only } \alpha_{124} \neq 0. \tag{31}$$

In order that to find the Lie brackets for the Nappi-Witten Lie algebra, we note that the algebra has the following explicit description:

$$[J, P_i] = \varepsilon_{ij}P_j, \qquad [P_i, P_j] = \varepsilon_{ij}T, \qquad [T, J] = [T, P_i] = 0. \tag{32}$$

This algebra is a central extension of the $d = 2$ Poincaré algebra to which it reduces if one sets $T = 0$. We suppose

$$e_i = P_i, \qquad e_3 = J - T, \qquad e_4 = J. \tag{33}$$

Then it is easily shown that

$$\alpha_{123} = \alpha_{124} = 1, \qquad \alpha_{134} = \alpha_{234} = 0. \tag{34}$$

Thus, if the group G_e is non-Abelian, then any nonzero torsion defined in the point $e \in K$ or M_4 is given by structure constants of the algebras (30) or (31) and (32) respectively. It follows from (9) that it is true for any threeform potential with components that take nonzero expectation values in M_4 or K.

Now, we consider the identities (23) and (24). Since the curvature tensor R_{ijkl} is skew symmetric in the last two indexes and the torsion tensor is fully antisymmetric, it follows from (24) that

$$-R_{ijkl} = 2S_{ijk;l} = 2\partial_{[l}S_{ijk]}. \tag{35}$$

Hence the tensor R_{ijkl} also is fully antisymmetric. We sum the Bianchi identity on the indexes i, m and use the antisymmetry of S_{ijk} and R_{ijkl}. Then we get

$$R^i_{jkl;i} = S^i_{n[j}R^n_{kl]i}. \tag{36}$$

It follows from the identities (26) and (35) that

$$R^i_{jkl;i} = -2S^i_{n[j}S^n_{kl];i} = 0. \tag{37}$$

Since the threeform potential and its gauge field strength are connected with the torsion by the relations (9) and (10), we get

$$F_{ijkl}{}^{;i} = 0. \tag{38}$$

Thus, if we consider the Bose sector of $d = 11$ supergravity as a pure gravity theory with torsion, then the gauge field strength must satisfy Eq. (38).

Further, suppose e is an arbitrary point in M_4 and G_e is a geodesic group defined in a neighborhood of e. Using (16), we represent the curvature tensor (18) in the form

$$R^i_{jkl} = \mathring{R}^{\,i}_{\ jkl} + (S^i_{jl;k} - S^i_{jk;l} + S^m_{jl}S^i_{mk} - S^m_{jk}S^i_{ml}). \quad (39)$$

Since the tensor S_{ijk} is fully antisymmetric, it follows from (35) that the Ricci tensor

$$\mathring{R}_{ij} = S^m_{ik}S^k_{mj}. \quad (40)$$

Suppose the tangent Lie algebra A_G is non-Abelian. Then it is isomorphic to one of the algebras (31) and (32). Therefore in a suitable local coordinate system the components of α_{ijk} satisfy to one of the conditions (31) and (34). Then it follows from (40) that components of the Ricci tensor in the point e have the form

$$\mathring{R}_{i4} = 0, \qquad \mathring{R}_{i3} = 0, \quad \text{or} \quad \mathring{R}_{i1} = \mathring{R}_{i2} = 0. \quad (41)$$

Since the metric g_{ij} is nondegenerate, it follows from here that $\mathring{R}_{ij} \neq \lambda g_{ij}$ if λ is nonzero, i.e. the spacetime M_4 is non-Einstein. Now suppose G_e is a non-Abelian geodesic group defined in a neighborhood of $e \in K$. Using the isomorphisms (30) and arguing as above, we easy prove that the internal space K is non-Einstein. Thus, if $M = M_4 \times K$ is an Einstein space, then the geodesic loop defined in a neighborhood of any point of M is either nonassociative or it is an Abelian group. It follows from here that there is no Freund-Rubin background $M_4 \times K$ of 11-dimensional supergravity where M_4 and K are Lie groups with bi-invariant metric. This, of course, is a known result.

B. The Freund-Rubin Ansatz

Now we start to analyze the Freund-Rubin solution. Suppose M_4 is a four-dimensional Riemann spacetime of signature $(+++-)$. We deform the Riemannian connection by the rule (16) and convert M_4 into an affinely connected space with a fully antisymmetric torsion tensor S_{ijk}. Let e be an arbitrary point in M_4. In a neighborhood of this point, we define a geodesic loop G_e and consider the right-hand side of the identity (15). Since the tensor $\alpha_{ijk} = g_{is}\alpha^s_{jk}$ is fully antisymmetric and its indices take only four different values, we have

$$\alpha^m_{[jk}\alpha^i_{l]m} = 0, \quad (42)$$

$$\beta^i_{[jkl]} = 0, \quad (43)$$

instead of (15). It follows from (42) that with respect to the operation of commutation the tangent algebra A_G is a Lie algebra. Suppose the algebra A_G is non-Abelian. Then arguing as above, we see that it has the form (31) or (32). Thus, any nonzero torsion in $e \in M_4$ is given by the structure constants of these algebras. It follows from (9) that it is true for any threeform potential with the components taking nonzero expectation values in M_4. Note that the similar assertion for S^7 torsion and the Cayley structure constants was proved in [18], where the Englert's solution [8] was analyzed.

Antisymmetrizing the curvature tensor (39) on i, j, and k, we get

$$R_{[ijkl]} = -2S_{ijk;l} = -2\partial_{[l}S_{ijk]}. \quad (44)$$

Hence the gauge field strength

$$F_{ijkl} = \lambda' R_{[ijkl]}. \quad (45)$$

Substituting F_{ijkl} in the Bianchi identity (22) and summing on i and m, we obtain

$$F^i_{jkl;i} = S^i_{n[j}F^n_{kl]i}. \quad (46)$$

Since the tensor indices in (46) take only four different values, the covariant derivative

$$F_{ijkl;m} = 0. \quad (47)$$

Hence, the gauge field strength F_{ijkl} is proportional to the fully antisymmetric covariant constant 4 tensor, i.e. it must have the form (6). Note that this assertion is true for any four-dimensional Riemann spacetime of Lorentzian signature. Indeed, using the full skew symmetry of S_{ijk}, the Jacobi identity (42), and the equality (39), we may represent the fundamental tensor (20) in the form

$$4\beta_{ijkl} = S^m_{ij}S_{klm} - \mathring{R}_{ijkl}. \quad (48)$$

Since the Riemannian curvature tensor is satisfied, the Bianchi identity

$$\mathring{R}^{\,i}_{\ [jkl]} = 0, \quad (49)$$

and the conditions (42) and (43) do not impose any restrictions on the spacetime M_4.

IV. THE ENGLERT'S TYPE SOLUTIONS

In this section, we continue the investigation of the $M = M_4 \times K$ compactification of $d = 11$ supergravity. We again consider the Bose sector of this theory as a pure gravity theory with torsion. However now we suppose that the matter fields have nonvanishing components in the internal space K. We analyze the Englert's type solutions and prove that the torsion is given locally by the Cayley structure constants.

A. Geodesic Moufang Loops

We recall (see i.e. [19]) that the algebra $\mathbb{O}$ of octonions (Cayley algebra) is a real linear algebra with the canonical basis $1, e_1, \ldots, e_7$ such that

$$e_ie_j = -\delta_{ij} + c_{ijk}e_k, \quad (50)$$

where the structure constants c_{ijk} are completely antisymmetric and nonzero and equal to unity for the seven combinations (or cycles)

$$(ijk) = (123), (145), (167), (246), (275), (374), (365).$$

The algebra of octonions is not associative but alternative, i.e. the associator,

$$(x, y, z) = (xy)z - x(yz), \tag{51}$$

is totally antisymmetric in x, y, z. The algebra $\mathbb{O}$ permits the involution (antiautomorphism of period two) $x \to \bar{x}$ such that the elements

$$t(x) = x + \bar{x} \quad \text{and} \quad n(x) = \bar{x}x \tag{52}$$

are in $\mathbb{R}$. It is easy to prove that the quadratic form $n(x)$ is positive definite and permits the composition

$$n(xy) = n(x)n(y). \tag{53}$$

It follows from here that the set

$$\mathbb{S} = \{x \in \mathbb{O} \mid n(x) = 1\} \tag{54}$$

is closed relative to the multiplication in $\mathbb{O}$ and hence it is an analytic loop. The loop $\mathbb{S}$ is the unique, up to isomorphism, analytic compact simple nonassociative Moufang loop. The tangent algebra of $\mathbb{S}$ is isomorphic to the seven-dimensional commutator subalgebra

$$\mathbb{M} = \{x \in \mathbb{O}^{(-)} \mid t(x) = 0\}. \tag{55}$$

The algebra $\mathbb{M}$ is the unique compact simple non-Lie Malcev algebra and it satisfies the identity

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 6(x, y, z). \tag{56}$$

The algebra $\mathbb{M}$ has the canonical basis $e_1, \ldots, e_7$. Using (50) we can find the commutators and associators of the basis elements

$$[e_i, e_j] = 2c_{ijk}e_k, \tag{57}$$

$$(e_i, e_j, e_k) = 2c_{ijkl}e_l, \tag{58}$$

where c_{ijkl} is a completely antisymmetric nonzero tensor equal to unity for the seven combinations

$$(ijkl) = (4567), (2367), (2345), (1357), (1364), (1265), (1274).$$

Now let K be a compact seven-dimension Riemann space and G_e be a geodesic Moufang loop defined in a neighborhood of $e \in K$. If the loop G_e is associative, then its tangent algebra is either Abelian or a compact Lie algebra of the form (30). Suppose G_e is a nonassociative Moufang loop. Then arguing as above, we prove that its tangent algebra is a compact seven-dimension non-Lie Malcev algebra. Since any such algebra is isomorphic to the algebra (55), it follows that G_e is locally isomorphic to the loop (54). Therefore we have the identity

$$\beta^i_{jkl} = \alpha^m_{[jk}\alpha^i_{l]m}, \tag{59}$$

instead of (15). It can easily be checked that this identity is equivalent to the Malcev identity (56). On the other hand, it follows from (39) that

$$\tfrac{1}{2}R_{[ijk]l} = S^m_{[ij}S_{k]lm} - S_{ijk;l}. \tag{60}$$

Hence the tensors β_{ijkl} and $S_{ijk;l}$ are fully antisymmetric. Then it follows from (24) that the curvature tensor R_{ijkl} is also fully antisymmetric.

Conversely, let G_e be a geodesic loop and R_{ijkl} be a fully antisymmetric tensor. Then

$$\tfrac{1}{2}R_{ijkl} = S^m_{[ij}S_{kl]m} - S_{ijk;l}, \tag{61}$$

and hence the tensor $S_{ijk;l}$ is fully antisymmetric. Again, using the identity (24), we prove the fully antisymmetry of β_{ijkl}. Therefore we have the Malcev identity (59) instead of (15) and hence G_e is a Moufang loop. Thus, the geodesic loop G_e is Moufang if and only if the curvature tensor R_{ijkl} is fully antisymmetric. In addition, any non-Abelian geodesic Moufang loop in K is locally isomorphic to either the Lie group (30) or the nonassociative Moufang loop (54).

B. The Englert's Ansatz

We consider the Bose sector of $d = 11$ supergravity as a pure gravity theory with torsion and suppose that the matter fields have nonvanishing components in the internal space K. We suppose that K is an Einstein space and G_e is a geodesic Moufang loop defined in a neighborhood of $e \in K$. As we proved above, G_e is nonassociative and hence its tangent algebra is isomorphic to the algebra $\mathbb{M}$. We select the basis $\tilde{e}_1, \ldots, \tilde{e}_7$ in $\mathbb{M}$ such that

$$[\tilde{e}_i, \tilde{e}_j] = 2kc_{ijk}\tilde{e}_k, \tag{62}$$

$$(\tilde{e}_i, \tilde{e}_j, \tilde{e}_k) = 2k^2c_{ijkl}\tilde{e}_l, \tag{63}$$

where k is a real constant. By comparing these equalities with (13) and (14), we get the following relations:

$$\alpha_{ijk} = kc_{ijk}, \tag{64}$$

$$\beta_{ijkl} = k^2c_{ijkl}. \tag{65}$$

It is known [20] that the tensors c_{ijk} and c_{ijkl} are connected by self-duality relations. For the fundamental tensors of G_e these relations are

$$\varepsilon^{npqlijk}k\alpha_{ijk} = 6\beta^{npql}, \tag{66}$$

$$\varepsilon^{npqlijk}\beta_{ijkl} = 24k\alpha^{npq}. \tag{67}$$

In addition, the following identities are true:

$$\alpha_{ijm}\alpha^{ijn} = 6k^2\delta^n_m, \tag{68}$$

$$\beta_{mijk}\beta^{nijk} = 24k^4\delta^n_m, \tag{69}$$

$$\alpha_{im}{}^{j}\alpha_{jn}{}^{k}\alpha_{kp}{}^{i} = 3k^2\alpha_{mnp}. \tag{70}$$

Now we substitute the Freund-Rubin ansatz (6) in the equations of motion (5). We obtain

$$F^{mnpq}{}_{;m} = \sqrt{2}\rho\varepsilon^{npqijkl}F_{ijkl}, \tag{71}$$

where $\varepsilon^{npqijkl}$ is the fully antisymmetric covariant constant 7 tensor. Further, it follows from the Bianchi identity (36) and the identity (61) that

$$\begin{aligned}\tfrac{1}{2}R_{mnpq}{}^{;m} &= S_{tm[n}S^{t;m}{}_{pq]} - S_{mnp;q}{}^{;m} \\ &= S^{l}{}_{m[n}S^{mt}{}_{p}S_{q]tl} - S^{l}{}_{m[n}S^{m}{}_{pq];l}.\end{aligned} \tag{72}$$

Using (68) and (70), we get

$$S_{npq;m}{}^{;m} + 4k^2S_{npq} = 0. \tag{73}$$

Moreover, it follows from (61) that

$$S_{npq;m} = \partial_{[m}S_{npq]}. \tag{74}$$

Note that these formulas are true for the geodesic loop of any point in the Einstein space K. Note also that the identity (73) generalizes the Englert's identity that was found in the work [8].

By taking into account the obtained identities, we rewrite Eqs. (71) as

$$4k^2S^{npq} + \sqrt{2}\rho\varepsilon^{npqijkl}S_{ijk;l} = 0. \tag{75}$$

We will find solutions of these equations in the form

$$S_{mnp;q} = hS_{t[mn}S^{t}{}_{pq]}. \tag{76}$$

Such ansatz converts (75) into the (anti)self-duality equations. In order to obtain a value of h, we find

$$S_{mnp;q}S^{rnp;q} = h^2\beta_{mnpq}\beta^{rmnpq} = 24h^2k^4\delta^r_m. \tag{77}$$

On the other hand, it follows from (73) that

$$S_{mnp;q}S^{rnp;q} = -S_{mnp}S^{rnp;q}{}_{;q} = 24k^4\delta^r_m. \tag{78}$$

By comparing (77) and (78), we get $h = \pm 1$. Substituting the ansatz (76) in (75), we obtain the values

$$k = \pm 6\sqrt{2}\rho. \tag{79}$$

Obviously, the obtained solution is self-dual as $h = 1$ and anti–self-dual as $h = -1$. Besides, it follows from (61) that the curvature tensor

$$R_{ijkl} = 0 \quad \text{if } h = 1, \tag{80}$$

$$R_{ijkl} \neq 0 \quad \text{if } h = -1. \tag{81}$$

Note that the equality (80) is a necessary and sufficient condition of parallelizibility of K. Precisely this condition was used in [8] for a construction of solution of $d = 11$ supergravity on the sphere $\mathbb{S}^7$. It follows from (81) that this condition is not obligatory. Thus, if we choose

$$F_{mnpq} = \pm\lambda S_{t[mn}S^{t}{}_{pq]} \tag{82}$$

and take into account the conditions (79), we get self-dual and anti–self-dual solutions of Eqs. (71).

Now we find restrictions that must lay on the spaces K and M_4. To this end, using (61) and (76), we rewrite Eqs. (39) in the form

$$\overset{\circ}{R}{}^{i}{}_{jkl} = S^{i}_{jt}S^{t}_{kl} - hS^{i}_{jk;l}, \tag{83}$$

where $h = \pm 1$. By summing on the indices i and k, we obtained the Ricci tensor

$$\overset{\circ}{R}{}_{mn} = 6k^2g_{mn}. \tag{84}$$

It follows from here that the K is really an Einstein space. Substituting (84) and

$$F_{\mu\sigma\rho\lambda}F_{\nu}{}^{\sigma\rho\lambda} = -6\rho^2g_{\mu\nu}, \tag{85}$$

$$F_{mrpq}F_{n}{}^{rpq} = 24k^4\lambda^2g_{mn}, \tag{86}$$

in the Einstein equation (4), we get

$$\overset{\circ}{R}{}_{\mu\nu} = -10k^2g_{\mu\nu}, \qquad 2\lambda^2 = (12k)^{-2}. \tag{87}$$

It follows from here that the four-dimensional spacetime M_4 is the anti–de Sitter space. Note that all constants in the solutions are defined by the condition $\tilde{e}_i = ke_i$, i.e. its values depend only on a selection of basis in the Malcev algebra $\mathbb{M}$.

V. CONCLUSION

In this paper, we have studied the Freund-Rubin-Englert mechanism of the $M_4 \times K$ compactification of $d = 11$ supergravity. We have shown that any threeform potential in 11 dimensions is given locally by the structure constants of a geodesic loop in an affinely connected space. In particular, we have shown that any threeform potential with components that take nonzero expectation values in M_4 is given by structure constants of the Lie algebras (31) and (32). We have found the Englert's type solution of $d = 11$ supergravity on the Einstein space K and shown that the corresponding threeform potentials are given locally by the Cayley structure constants. The solution is such that the affine curvature tensor of K is nonzero. It follows from here that the space K may be not parallelizable. Since everything considered in the paper is of a local character, the global geometry of K is not clear though.

Note that the geodesic loops method that was used in the paper may be applied to the analysis of M theory compactifications on singular manifolds with G_2 holonomy [21,22]. The point is that in addition to the compact Moufang loop $\mathbb{S}$ there exists a noncompact nonassociative Moufang loop that is analytically isomorphic to the space $\mathbb{S}^3 \times \mathbb{R}^4$. This is exactly the asymptotically conical manifolds with G_2 holonomy.

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