

# Teukolsky-Starobinsky identities: A novel derivation and generalizations

Plamen P. Fiziev\*

*Department of Theoretical Physics, University of Sofia, Boulevard 5 James Bourchier, Sofia 1164, Bulgaria  
and BLTF, JINR, Dubna, 141980 Moscow Region, Russia*

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We present a novel derivation of the Teukolsky-Starobinsky identities, based on properties of the confluent Heun functions. These functions define analytically all exact solutions to the Teukolsky master equation, as well as to the Regge-Wheeler and Zerilli ones. The class of solutions, subject to Teukolsky-Starobinsky type of identities is studied. Our generalization of the Teukolsky-Starobinsky identities is valid for the already studied linear perturbations to the Kerr and Schwarzschild metrics, as well as for large new classes of such perturbations which are explicitly described in the present article. Symmetry of parameters of confluent Heun's functions is shown to stay behind the behavior of the known solutions under the change of the sign of their spin weights. A new efficient recurrent method for calculation of Starobinsky's constant is described.

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## I. INTRODUCTION

Today the Teukolsky-Starobinsky identities (TSI) are essential ingredients of the theory of perturbations to a gravitational field of Kerr black holes. For spin one and two the TSI were discovered together with their first applications in the study of perturbations of rotating relativistic objects pioneered by Teukolsky [1] and Starobinsky [2]. An independent derivation and detailed study of TSI was given by Chandrasekhar in [3,4] and by other authors in [5]. Later on a proper generalization of the TSI for all physically interesting spins, as well as the TSI in the presence of a nonzero cosmological constant was found and justified in [6]. Some more recent applications can be found in [7].

All known derivations of TSI are based on a direct use of the famous Teukolsky master equation (TME), or its generalizations, and on some very special properties of the solutions in use. For example, one utilizes the regularity and integrability of the used angular solutions, parameters with real values, etc.

The TME describes the perturbations  $\Psi(t, r, \theta, \varphi)$  of fields of all interesting spin weights  $s = 0, \pm 1/2, \pm 1, \pm 3/2, \pm 2$  in Kerr background in terms of Newman-Penrose scalars. Various significant results and references may be found in [1–8].

The key feature is that in Boyer-Lindquist coordinates one can separate the variables using the ansatz  $\Psi(t, r, \theta, \varphi) = \exp(-i\omega t + im\varphi)R(r)S(\theta)$ . Thus, a pair of two connected differential equations arises—the Teukolsky angular equation (TAE):

$$\frac{1}{\sin\theta} \frac{d}{d\theta} \left( \sin\theta \frac{d}{d\theta} S(\theta) \right) + (W(\theta) + E)S(\theta) = 0, \quad (1.1)$$

$$W(\theta) = a^2 \omega^2 \cos^2 \theta - 2sa\omega \cos \theta - (m^2 + s^2 + 2ms \cos \theta)/$$

$\sin^2 \theta$ , and the Teukolsky radial equation (TRE):

$$\Delta^{-s} \frac{d}{dr} \left( \Delta^{s+1} \frac{d}{dr} R(r) \right) + (V(r) - E)R(r) = 0, \quad (1.2)$$

$V(r) = 4is\omega r + 2ma\omega - a^2\omega^2 + s(s+1) + \frac{K^2 - 2is(r-M)K}{\Delta}$ . The real parameter  $a = J/M \geq 0$  is related with angular momentum  $J$  of the Kerr metric ( $a < M$  for black holes, or  $a > M$  for naked singularities),  $M$  is the Keplerian mass of the Kerr solution,  $K = \omega(r^2 + a^2) - ma$ ,  $\Delta = (r - r_+) \times (r - r_-) = r^2 - 2Mr + a^2$ . The two *complex* separation constants  $\omega$  and  $E$  are to be determined using the boundary conditions of the problem.

At present the usage of the Kerr metric for the description of astrophysical black holes is widely accepted. The astrophysical application of the naked singularities seems to be more problematic because of their instability [9]. However, such instability may be useful for construction of models of astrophysical explosions like gamma ray bursts [10]. Our method for the derivation of the generalized TSI is applicable to both cases, despite of the fact that some quantities (like  $r_{\pm}$ ) are complex for the naked singularities.

The TSI for angular Teukolsky's function  ${}_s S(\theta)$  of arbitrary spin weight  $s$  can be written in the form

$$\begin{aligned} {}_{1-s}\mathcal{L}_{2-s}\mathcal{L} \dots {}_{s-1}\mathcal{L}_s \mathcal{L}({}_+ S(\theta)) &= {}_s\mathfrak{G}({}_- S(\theta)), \\ {}_{1-s}\mathcal{L}^\dagger_{2-s}\mathcal{L}^\dagger \dots {}_{s-1}\mathcal{L}^\dagger_s \mathcal{L}^\dagger({}_- S(\theta)) &= {}_s\mathfrak{G}({}_+ S(\theta)); \end{aligned} \quad (1.3)$$

where

$$\begin{aligned} {}_n\mathcal{L} &= \partial_\theta + \left( \frac{m}{\sin\theta} - a\omega \sin\theta \right) + n \cot\theta, \\ {}_n\mathcal{L}^\dagger &= \partial_\theta - \left( \frac{m}{\sin\theta} - a\omega \sin\theta \right) + n \cot\theta. \end{aligned} \quad (1.4)$$

\*fiziev@phys.uni-sofia.bg

Using the operators

$${}_n\mathcal{D} = \partial_r + i\frac{K}{\Delta} + \frac{n}{\Delta}\partial_r\Delta, \quad {}_n\mathcal{D}^\dagger = \partial_r - i\frac{K}{\Delta} + \frac{n}{\Delta}\partial_r\Delta, \quad (1.5)$$

one obtains the TSI for the radial Teukolsky function  ${}_sR(r)$  of arbitrary spin weight  $s$  in the form [11]:

$$\begin{aligned} \Delta^s({}_0\mathcal{D})^{2s}(\Delta^s{}_sR(r)) &= {}_s\mathfrak{D}(-{}_sR(r)), \\ \Delta^s({}_0\mathcal{D}^\dagger)^{2s}(-{}_sR(r)) &= {}_s\mathfrak{D}^*(\Delta^s{}_sR(r)). \end{aligned} \quad (1.6)$$

In Eqs. (1.3) and (1.6)  ${}_s\mathfrak{C}$  and  ${}_s\mathfrak{D}$  are the corresponding Starobinsky constants [12]. The standard calculation of these constants for arbitrary  $s$  is quite complicated [3,4].

In the widespread derivation of TSI, proposed for the first time in [3], one supposes to work with *real* frequencies  $\omega$ . This is explicitly stressed in [3], as well as in some of the articles in [5,6]. As a result, the operators  $\mathcal{D}$  and  $\mathcal{D}^\dagger$  can be considered conjugated; this property is systematically used for derivation of the TSI [3–6]. However, the well-known boundary problems, related to black holes and naked singularities, yield complex frequencies  $\omega$  with certainly nonzero imaginary part  $\Im(\omega) \neq 0$ . (See, for example, Refs. [4,8] and the recent articles [13], where a new boundary problem was studied and some very preliminary results of our attempts to apply singular solutions of TAE to the description of collimated relativistic jets are reported.) The physical reason is clear: The systems under consideration are open physical systems and their energy is not conserved. Hence, for them  $\Im(\omega) \neq 0$ . Our derivation of TSI does not imply complex conjugation of the corresponding quantities. It is valid for complex frequencies  $\omega$ , as well as for complex values of all other parameters. Despite the fact that our derivation is quite different in form, it is closely related to the original one, outlined in [1,2]. This original method is also valid for complex frequencies.

It has been well known for a long time [14] that for  $a \neq M$  [15] the TAE (1.1) and TRE (1.2) can be reduced to the confluent Heun ordinary differential equation [16]. Recently in [17] all classes of the exact solutions to TAE (1.1) and TRE (1.2) were described using the confluent Heun function  $\text{HeunC}(\alpha, \beta, \gamma, \delta, \eta, z)$ —a unique particular solution of the confluent Heun equation, written here in the simplest uniform shape:

$$H'' + \left(\alpha + \frac{\beta + 1}{z} + \frac{\gamma + 1}{z - 1}\right)H' + \left(\frac{\mu}{z} + \frac{\nu}{z - 1}\right)H = 0. \quad (1.7)$$

The function  $\text{HeunC}(\alpha, \beta, \gamma, \delta, \eta, z)$  is defined as a solution of Eq. (1.7) which is regular in vicinity of the singular point  $z = 0$  and subject to the normalization condition  $\text{HeunC}(\alpha, \beta, \gamma, \delta, \eta, 0) = 1$  [16]. The parameters  $\alpha, \beta, \gamma, \delta, \eta$ , introduced in [16] and used in the widespread computer package MAPLE, are related with  $\mu$  and  $\nu$ , ac-

ording to the equations  $\mu = \frac{1}{2}(\alpha - \beta - \gamma + \alpha\beta - \beta\gamma) - \eta$  and  $\nu = \frac{1}{2}(\alpha + \beta + \gamma + \alpha\gamma + \beta\gamma) + \delta + \eta$ .

The other particular solutions to Eq. (1.7) are *not* termed “confluent Heun’s functions,” according to the accepted modern terminology [16]. The reason is that in the general case other solutions can be represented in a nontrivial way in terms of solutions  $\text{HeunC}(\alpha, \beta, \gamma, \delta, \eta, z)$ . An example of the application of this specific property of the Eq. (1.7) is the basic formula (2.1) in the next section. Therefore from a computational point of view it is sufficient to study the Taylor series of this standard local solution and its analytical continuation in the complex plane  $\mathbb{C}_z$ . Thus, the instrumental use of the confluent Heun function  $\text{HeunC}(\alpha, \beta, \gamma, \delta, \eta, z)$  is much more advantageous than the simple fact, recognized already in [14], that the TRE and TAE can be reduced to the Eq. (1.7).

Essential novel properties of the confluent Heun functions  $\text{HeunC}(\alpha, \beta, \gamma, \delta, \eta, z)$  were studied for the first time in the recent article [18]. These properties are prerequisites for our derivation of the TSI in the next section. The subclass of  $\delta_N$ -confluent-Heun’s-functions,

$$\text{HeunC}_N(\alpha, \beta, \gamma, \eta, z) = \text{HeunC}(\alpha, \beta, \gamma, \delta_N, \eta, z), \quad (1.8)$$

was introduced assuming the fulfilment of the “ $\delta_N$ -condition,”

$$\delta_N = -\alpha(N + 1 + (\beta + \gamma)/2), \quad (1.9)$$

for some fixed nonnegative integer  $N = 0, 1, 2, \dots$

For functions (1.8) and their associate functions

$$\text{HeunC}_N^*(\alpha, \beta, \gamma, \eta, z) = \text{HeunC}(\alpha_N^*, \beta_N^*, \gamma_N^*, \delta_N^*, \eta_N^*, z), \quad (1.10)$$

$$\begin{aligned} \alpha_N^* &= \alpha, & \beta_N^* &= \beta + N + 1, & \gamma_N^* &= \gamma + N + 1, \\ \frac{\delta_N^*}{\alpha_N^*} &= \frac{\delta}{\alpha} + N + 1, & \eta_N^* &= \eta + \frac{(N + 1)(N + 1 - \alpha)}{2}, \end{aligned} \quad (1.11)$$

$\alpha = \alpha - \beta - \gamma$ ; the following simple basic relation is available:

$$\begin{aligned} \frac{d^{N+1}}{dz^{N+1}} \text{HeunC}_N(\alpha, \beta, \gamma, \eta, z) \\ = \mathfrak{F}_N \text{HeunC}_N^*(\alpha, \beta, \gamma, \eta, z). \end{aligned} \quad (1.12)$$

$\mathfrak{F}_N = (N + 1)!v_{N+1}(\alpha, \beta, \gamma, -\alpha(N + 1 + (\beta + \gamma)/2), \eta)$  is a constant related to the coefficients in the Taylor series

$$\text{HeunC}(\alpha, \beta, \gamma, \delta, \eta, z) = \sum_{n=0}^{\infty} v_n(\alpha, \beta, \gamma, \delta, \eta)z^n. \quad (1.13)$$

These new results are the mathematical basis of the present article. We derive TSI using the above properties of the confluent Heun function  $\text{HeunC}(\alpha, \beta, \gamma, \delta, \eta, z)$  and show that the Starobinsky constant is related to its Taylor

series coefficients. This yields a new effective method for the calculation of Starobinsky's constant for any spin.

Our derivation of the TSI, being simpler, universal, and straightforward, supplements and justifies the usual approach. It is uniform and valid for both the TAE and TRE. It extends the TSI to new classes of solutions to the TME, described below. Thus the known fascinating properties of the standard solutions of the TME acquire a natural mathematical framework and proper extension.

Moreover, the same derivation yields Teukolsky-Starobinsky like identities for the Regge-Wheeler equation (RWE) and Zerilli equation (ZE). To our knowledge, until now the TSI were not known for the RWE and ZE.

Our approach is in accord with the Chandrasekhar expectation that the Newman-Penrose formalism for perturbations of the Kerr metric may "...enable us to discover new classes of identities among the special functions of mathematical physics when they occur as solutions of Einstein's equations" and his suggestion to "...include Teukolsky's functions among "special" functions of mathematical physics..." [3]. Today it is clear that for realization of this general idea we need to use the Heun functions and some of their generalizations [19].

## II. THE EXACT SOLUTIONS TO THE TAE, TRE, AND RWE IN TERMS OF THE CONFLUENT HEUN FUNCTION AND THE GENERALIZED TSI

The exact local solutions around the singular points  $z_{\pm} = 0$  ( $\Leftrightarrow z_{\mp} = 1$ ) of the three differential equations listed in the introduction—the TRE, TAE, and RWE—can be written in the following common and universal form [17]:

$${}_s X_{\omega, E, m, \{\sigma\}}^{\pm}(z_{\pm}) = \varrho(z_{\pm}) e^{\alpha z_{\pm}/2} z_{\pm}^{\beta/2} z_{\mp}^{\gamma/2} \times \text{HeunC}(\alpha, \beta, \gamma, \delta, \eta, z_{\pm}). \quad (2.1)$$

The TSI turn out to be a representation of the basic relation (1.12) in terms of the corresponding  $\delta_N$  solutions of type (2.1). Indeed, let us use the short notation  $X_N(z_{\pm})$ ,  $X_N^*(z_{\pm})$  for the  $\delta_N$  solutions (2.1) which correspond to the functions (1.8) and (1.10) of argument  $z_{\pm}$  in the place of the factor  $\text{HeunC}(\alpha, \beta, \gamma, \delta, \eta, z_{\pm})$  in Eq. (2.1). Then, using relations (1.11) and  $\varrho^* = \varrho$ , we obtain from (1.12) the following universal form of the generalized Teukolsky-Starobinsky identities:

$$(z_+ z_-)^{(N+1)/2} \hat{D}_{\pm}^{N+1} X_N(z_{\pm}) = \mathfrak{F}_N X_N^*(z_{\pm}), \quad (2.2)$$

where

$$\hat{D}_{\pm} = \frac{d}{dz_{\pm}} - \frac{\alpha}{2} \mp \frac{\beta}{2} \frac{1}{z_{\pm}} \pm \frac{\gamma}{2} \frac{1}{z_{\mp}} - \frac{1}{\varrho} \frac{d\varrho}{dz_{\pm}}. \quad (2.3)$$

For compactness, in the right-hand side of Eqs. (2.1) and (2.3) we are using the following abbreviated notation:  $\alpha = \sigma_{\alpha} \alpha_{\pm}$ ,  $\beta = \sigma_{\beta} \beta_{\pm}$ ,  $\gamma = \sigma_{\gamma} \gamma_{\pm}$ ;  $\{\sigma\} = \{\sigma_{\alpha}, \sigma_{\beta}, \sigma_{\gamma}\}$ ,  $\sigma_{\alpha, \beta, \gamma} = \pm 1$ . Thus, we obtain 16 local solutions to any

of the TRE, TAE, and RWE, making use of the confluent Heun's function  $\text{HeunC}(\alpha, \beta, \gamma, \delta, \eta, z)$  of different argument  $z_+$  or  $z_-$  in the right-hand side of Eq. (2.1) and changing the signs of the parameters  $\alpha, \beta, \gamma$ .

The difference among the solutions to the above three problems (the TRE, TAE, and RWE) is in the values of the parameters and arguments in their common form (2.1):

(1) According to [17], for the TAE (1.1)  $\varrho(z_{\pm}) = 1$  and

$$\begin{aligned} \alpha_{\pm} &= \pm 4a\omega, & \beta_{\pm} &= s \mp m, \\ \gamma_{\pm} &= s \pm m, & \delta_{\pm} &= \pm 4sa\omega, \\ \eta_{\pm} &= \frac{m^2 + s^2}{2} \mp 2sa\omega - a^2\omega^2 - E; & (2.4) \\ z_+ &= z_+(\theta) = (\cos(\theta/2))^2, \\ z_- &= z_-(\theta) = (\sin(\theta/2))^2. \end{aligned}$$

(2) For the TRE (1.2)  $\varrho(z_{\pm}) = \Delta^{-s/2}$ . Using the quantities  $p = \sqrt{r_+/r_-} - \sqrt{r_-/r_+}$ ,  $\Omega_{\pm} = \sqrt{r_{\mp}/r_{\pm}}/(r_+ + r_-)$ ,  $\Omega_a = (\Omega_+ + \Omega_-)/2 = 1/(2a)$ ,  $\Omega_g = \sqrt{\Omega_+ \Omega_-} = 1/(2M)$  we have

$$\alpha_{\pm} = \pm 2i\omega(r_+ - r_-) = \pm ip\omega/\Omega_a, \quad (2.5a)$$

$$\beta_{\pm} = s \pm 2i(m - \omega/\Omega_{\mp})/p, \quad (2.5b)$$

$$\gamma_{\pm} = s \mp 2i(m - \omega/\Omega_{\pm})/p, \quad (2.5c)$$

$$\delta_{\pm} = \alpha_{\pm}(s - i\omega/\Omega_g), \quad (2.5d)$$

$$\begin{aligned} \eta_{\pm} &= -E + s^2 + m^2 + \frac{2m^2\Omega_a^2 - \omega^2}{p^2\Omega_a^2} \\ &\quad - \frac{(2m\Omega_a - \omega)^2}{p^2\Omega_g^2} - \frac{1}{2} \left( s - i \frac{\omega\Omega_{\pm}}{\Omega_a\Omega_g} \right)^2; \end{aligned} \quad (2.5e)$$

$$z_+ = \frac{r - r_-}{r_+ - r_-}, \quad z_- = \frac{r_+ - r}{r_+ - r_-}. \quad (2.5f)$$

(3) For the Schwarzschild metric the discussed results (in terms of the Weyl scalars) can be obtained from the case of the Kerr metric in the limit  $a \rightarrow 0$ ; see [17]. An independent treatment is possible, making use of the RWE. Then in units  $2M = 1$  we have [17,20]

$$\varrho = r, \quad E = l(l+1), \quad l = |s|, |s| + 1, \dots; \quad (2.6a)$$

$$\alpha_{\pm} = \pm 2i\omega, \quad \beta_{\pm} = \left\{ \begin{array}{c} 2s \\ 2i\omega \end{array} \right\}, \quad \gamma_{\pm} = \left\{ \begin{array}{c} 2i\omega \\ 2s \end{array} \right\}, \quad (2.6b)$$

$$\delta_{\pm} = \pm 2\omega^2, \quad \eta_{\pm} = \left\{ \begin{array}{c} -E + s^2 \\ -E + s^2 + 2\omega^2 \end{array} \right\}; \quad (2.6c)$$

$$z_+ = r, \quad z_- = 1 - r. \quad (2.6d)$$

The knowledge of the 16 local solutions (2.1) to the each of the TRE, TAE, and RWE makes it possible to write down their local general solutions as linear combinations of different pairs of linearly independent particular solutions. Then we can formulate different boundary problems for the TRE, TAE, and RWE [17]. For example, one can impose the standard black hole boundary conditions on the solutions of the TRE and combine them with the standard regularity requirement of the solutions to the TAE on the poles  $\theta = 0, \pi$ . This way we obtain the standard quasi-normal modes of the Kerr black holes in terms of confluent Heun's functions [17]. Considering polynomial solutions of the TRE and TAE we are able to describe collimated one-way running waves of perturbations to the Kerr metric in terms of confluent Heun's polynomials [17]. One may hope that the proper usage of the singular solutions to the TAE may help us to describe the creation of relativistic jets by such collimated running waves; see [13,17]. These important physical problems need a separate careful investigation.

In the present article we are concentrating our efforts on the mathematical properties of the solutions related to the TSI. Knowledge of these properties is necessary for further progress in the field. As we shall see in the next sections, it is natural to formulate these common general properties in terms of confluent Heun's functions.

### III. $\delta_N$ -CONFLUENT HEUN'S FUNCTIONS AND SOLUTIONS TO THE TAE, TRE, AND RWE

The fulfilment of the  $\delta_N$  condition for solutions to the TAE, TRE, and RWE is tightly related to the specific values of the parameters (2.4), (2.5), and (2.6). One can find a detailed general consideration of this problem in [17].

For the TAE the  $\delta_N$  condition yields two different cases:

$$\begin{aligned} {}_s N + 1 = 2|s| \geq 1, \quad |s| \geq 1/2, \\ \sigma_\alpha = \sigma_\beta = \sigma_\gamma = -\sigma. \end{aligned} \quad (3.1)$$

$$\begin{aligned} {}_s N_m + 1 = |s| \pm m\sigma \geq 1, \\ \sigma_\alpha = -\sigma_\beta = \sigma_\gamma = -\sigma. \end{aligned} \quad (3.2)$$

For the TRE the  $\delta_N$  condition yields four different cases. In the first one relations (3.1) are also valid. Note that, there,  $\sigma = \text{sign}(s)$ . In the other three cases instead of the integer  $N$ , the frequencies  ${}_s \omega_{N,m,\sigma_\alpha,\sigma_\beta,\sigma_\gamma}^\pm$  are fixed:

$${}_s \omega_{N,m,\mp,\pm,\pm}^+ = {}_s \omega_{N,m,\mp,\pm,\pm}^- = \pm i \frac{N+1}{4M}, \quad (3.3)$$

$$\begin{aligned} {}_s \omega_{N,m,\mp,\mp,\pm}^+ &= {}_s \omega_{N,m,\mp,\pm,\mp}^- \\ &= m\Omega_+ \pm \frac{i}{4M} \left(1 - \frac{r_-}{r_+}\right) (N+1 \mp s), \end{aligned} \quad (3.4a)$$

$$\begin{aligned} {}_s \omega_{N,m,\pm,\mp,\pm}^+ &= {}_s \omega_{N,m,\pm,\pm,\mp}^- \\ &= m\Omega_- \pm \frac{i}{4M} \left(\frac{r_+}{r_-} - 1\right) (N+1 \pm s). \end{aligned} \quad (3.4b)$$

The  $\delta_N$  condition yields the following cases for the RWE:

$$\begin{aligned} {}_s N + 1 = |s| \geq 1, \quad \sigma_\alpha = \sigma_\gamma, \\ \sigma_\beta = -\sigma, \quad \text{or} \quad \sigma_\alpha = \sigma_\beta, \quad \sigma_\gamma = -\sigma, \end{aligned} \quad (3.5)$$

$${}_s \omega_{N\sigma_\beta\sigma_\gamma}^+ = \frac{i\sigma_\gamma}{2} (N+1 + \sigma_\beta s), \quad \sigma_\alpha = -\sigma_\beta; \quad (3.6a)$$

$${}_s \omega_{N\sigma_\beta\sigma_\gamma}^- = \frac{i\sigma_\beta}{2} (N+1 + \sigma_\gamma s), \quad \sigma_\alpha = -\sigma_\gamma. \quad (3.6b)$$

Until recently only solutions of a very special kind were studied and used in the existing literature. For these special solutions the relations (3.1) take place and the  $\delta_N$  condition is fulfilled automatically and simultaneously for both the TRE and TAE.

As we have seen, there exist a lot of other interesting solutions to the TRE, TAE, and RWE, listed above, for which the  $\delta_N$  condition yields important extra restrictions on the free parameters of the problem. We call “ $\delta_N$ -solution” any solution (2.1) to the TAE, TRE, or RWE, subject to the  $\delta_N$  condition (1.9). In Sec. IV we will use the term “ $\delta_N$ -solution” for the solutions (4.6b) to the ZE, having in mind the fulfillment of the  $\delta_N$  condition (1.9), too.

Up to now only  $\delta_N$ -solutions to the TRE and TAE which obey the relations (3.1) were studied. The  $\delta_N$  solutions, which obey the other relations—(3.2), for the TAE, (3.3) and (3.4), for the TRE, and (3.6), for the RWE—are new. Their physical meaning and applications still have to be recovered. In the next section we derive the generalized TSI, valid for all  $\delta_N$  solutions.

### IV. A NOVEL DERIVATION OF THE TEUKOLSKY-STAROBINSKY IDENTITIES

For all eight classes of  $\delta_N$  solutions to the TRE, TAE, and RWE described in Sec. III, one can derive in a uniform way identities of the Teukolsky-Starobinsky type, starting from the basic form (2.2). Relations (2.3), (2.4), and (2.5) yield the following explicit representations of the generalized TSI [21]:

(1) In the case of the TAE,



$$(\sin\theta)^{N+1} \left( \frac{\hat{d}_{\theta,\pm}}{\sin\theta} \right)^{N+1} S_N^\pm(\theta) = \mathfrak{C}_N S_N^{\pm\star}(\theta), \quad (4.1a)$$

$$\hat{d}_{\theta,\pm} = \frac{d}{d\theta} + \sigma_\alpha a \omega \sin\theta + \frac{(\sigma_\beta - \sigma_\gamma)s \mp (\sigma_\beta + \sigma_\gamma)m}{2 \sin\theta} \mp \frac{(\sigma_\beta + \sigma_\gamma)s \mp (\sigma_\beta - \sigma_\gamma)m}{2} \cot\theta, \quad (4.1b)$$

$$S_N^\pm(\theta) = X_N(z_\pm), \quad (4.1c)$$

$$\mathfrak{C}_N = (\pm 1)^{N+1} \mathfrak{R}_N. \quad (4.1d)$$

Disentangling the differential operators in Eq. (4.1a), we can represent it in a form similar to Eq. (1.3):

$$\overleftarrow{\prod}_{k=0}^{N+1} (\hat{d}_{\theta,\pm} - k \cot\theta) S_N^\pm(\theta) = \mathfrak{C}_N S_N^{\pm\star}(\theta). \quad (4.2)$$

Here an ordered operator product is used. The arrow indicates the operator ordering and points in the direction of the increase of the integer  $k$ .

(2) In the case of the TRE,

$$\Delta^{(N+1)/2} (\hat{d}_{r,\pm})^{N+1} R_N^\pm(r) = \mathfrak{D}_N R_N^{\pm\star}(r), \quad (4.3a)$$

$$\begin{aligned} \hat{d}_{r,\pm} &= \frac{d}{dr} - i\sigma_\alpha \omega \\ &\pm i \frac{\sigma_\gamma(m - \omega/\Omega_\pm)}{p(r - r_+)} \\ &\mp i \frac{\sigma_\beta(m - \omega/\Omega_\mp)}{p(r - r_-)}, \end{aligned} \quad (4.3b)$$

$$\begin{aligned} R_N^\pm(r) &= (r - r_+)^{s(1-\sigma_\gamma)/2} \\ &\times (r - r_-)^{s(1-\sigma_\beta)/2} \\ &\times X_N(z_\pm), \end{aligned} \quad (4.3c)$$

$$\mathfrak{D}_N = (\pm 1)^{N+1} (-1)^{(N+1)/2} \mathfrak{R}_N. \quad (4.3d)$$

It is easy to check that in the case of Eqs. (3.1) relations (4.1) and (4.3) produce the standard TSI (1.3) and (1.6) with Starobinsky's constants  $\mathfrak{C}_N$  and  $\mathfrak{D}_N$ .

To reach this result one has to apply the new non-trivial hidden symmetry of the parameters of  $\delta_N$ -confluent Heun's functions, considered as functions of the spin weight  $s$  in the special case  $N + 1 = 2|s|$ . Then

$$\begin{aligned} \{\alpha_\pm^\star(s), \beta_\pm^\star(s), \gamma_\pm^\star(s), \delta_\pm^\star(s), \eta_\pm^\star(s)\} \\ = \{\alpha_\pm(-s), \beta_\pm(-s), \gamma_\pm(-s), \delta_\pm(-s), \eta_\pm(-s)\}. \end{aligned} \quad (4.4)$$

Equation (4.4) follows from (1.11), (2.4), and (2.5), if  $N + 1 = 2|s|$ . As a result, in the case (3.1)  $X_N^\star(s, z_\pm) = X_N(-s, z_\pm)$  is a solution to *the same*

TRE or TAE but with spin weight  $(-s)$ . The last fact was discovered for the first time in [1], as a special property of the particular solutions to the TRE or TAE considered there. As seen, we obtain a natural explanation of this phenomenon using the symmetry properties of  $\delta_N$ -confluent Heun's functions.

Now it remains only to point out that

(A) From Eqs. (4.1) for  $\sigma_\alpha = \sigma_\beta = \sigma_\gamma = -\sigma$  and  $n = s - k$  one obtains

$$\begin{aligned} \hat{d}_{\theta,+} - k \cot\theta &= {}_n\mathcal{L}, & S_N^+(\theta) &= {}_{+s}S(\theta), \\ S_N^{+\star}(\theta) &= {}_{-s}S(\theta) \end{aligned}$$

when  $\sigma = +1$ ; or

$$\begin{aligned} \hat{d}_{\theta,+} - k \cot\theta &= {}_n\mathcal{L}^\dagger, & S_N^+(\theta) &= {}_{-s}S(\theta), \\ S_N^{+\star}(\theta) &= {}_{+s}S(\theta) \end{aligned}$$

when  $\sigma = -1$ . Hence, for these particular values of the parameters Eq. (4.2) coincides with the standard angular TSI (1.3).

(B) From Eqs. (4.3) for  $\sigma_\alpha = \sigma_\beta = \sigma_\gamma = -\sigma$  one obtains

$$\begin{aligned} \hat{d}_{r,+} &= {}_0\mathcal{D}, & R_N^+(r) &= \Delta^s {}_{+s}R(r), \\ R_N^{+\star}(r) &= {}_{-s}R(r), \end{aligned}$$

and  $\mathfrak{D}_N = {}_s\mathfrak{D}$  when  $\sigma = +1$ ; or

$$\begin{aligned} \hat{d}_{r,+} &= {}_0\mathcal{D}^\dagger, & R_N^+(r) &= {}_{-s}R(r), \\ R_N^{+\star}(r) &= \Delta^s {}_{+s}R(r), \end{aligned}$$

and  $\mathfrak{D}_N = {}_s\mathfrak{D}^*$  when  $\sigma = -1$ . Hence, for these particular values of the parameters Eq. (4.3a) coincides with the standard radial TSI (1.6).

To our knowledge, the TSI for the other seven cases—Eqs.(3.2), (3.3), (3.4a), (3.4b), (3.5), (3.6a), and (3.6b)—have not been studied up to now. In particular, relation (4.3c) extends to all cases (3.3) and (3.4) the transformation invented for the case (III.1) in [1,4].

(3) For RWE (in units  $2M = 1 \Rightarrow \Delta_0 = r(r-1)$ ),

$$\Delta_0^{(N+1)/2} (\hat{d}_{r,0,\pm})^{N+1} R_{N,0}^{\pm}(r) = \mathfrak{D}_{N,0} R_{N,0}^{\pm}(r), \quad (4.5a)$$

$$\begin{aligned} \hat{d}_{r,0,\pm} &= \frac{d}{dr} - i\sigma_\alpha \omega \\ &\quad - \frac{\sigma_\beta - \sigma_\gamma}{2} (s + i\omega) \\ &\quad \mp \frac{\sigma_\beta + \sigma_\gamma}{2} (s - i\omega), \end{aligned} \quad (4.5b)$$

$$R_{N,0}^{\pm}(r) = r^{-1} X_N(z_{\pm}), \quad (4.5c)$$

$$\mathfrak{D}_{N,0} = (\pm 1)^{N+1} \mathfrak{R}_N. \quad (4.5d)$$

For TSI (4.5) we have  $N + 1 = |s|$ .

(4) The case of ZE is slightly different. In it we may proceed in the following way. The solutions  $Z_0^{\pm}(r)$  to the Zerilli equation are known to be simply related with the solutions  $R_0^{\pm}(r)$  to the RWE [4]. We write down the corresponding relations (in units  $2M = 1$ ) in the following form (see the appendix):

$$i\bar{\omega} R_0^{\pm}(r) = \left(1 - \frac{1}{r}\right) e^{\psi(r)} \frac{d}{dr} (e^{-\psi(r)} Z_0^{\pm}(r)), \quad (4.6a)$$

$$i\bar{\omega} Z_0^{\pm}(r) = \left(1 - \frac{1}{r}\right) e^{-\psi(r)} \frac{d}{dr} (e^{\psi(r)} R_0^{\pm}(r)). \quad (4.6b)$$

Let us introduce the notation  $Z_{N,0}^{\pm}(r)$  for  $\delta_N$  solutions to the ZE, and  $Z_{N,0}^{\pm}(r)$  for corresponding associate solutions, and the new operator  $z\hat{d}_{r,0,\pm} = \hat{d}_{r,0,\pm} + \Delta_0^{-1}$ . Then after substitution of (4.6a) into Eq. (4.5a) we obtain the following non-standard Teukolsky-Starobinsky-like identities for the solutions to the ZE:

$$\begin{aligned} \Delta_0^{(N+1)/2} (z\hat{d}_{r,0,\pm})^{N+1} \left( e^{\psi(r)} \frac{d}{dr} (e^{-\psi(r)} Z_{N,0}^{\pm}(r)) \right) \\ = \mathfrak{D}_{N,0} \left( e^{\psi(r)} \frac{d}{dr} (e^{-\psi(r)} Z_{N,0}^{\pm}(r)) \right). \end{aligned} \quad (4.7)$$

## V. A NEW METHOD FOR CALCULATION OF STAROBINSKY'S CONSTANTS $\mathfrak{C}_N$ AND $\mathfrak{D}_N$

Equations (4.3d), (4.1d), and (4.5d) show that in all cases the corresponding Starobinsky constants coincide (up to known numerical factors) with the coefficients  $v_n(\alpha, \beta, \gamma, \delta, \eta)$  in the Taylor series expansion (1.13) of  $\delta_N$ -confluent Heun's function evaluated for proper values of the parameters. These coefficients can be determined by the three-terms recurrence relation [16,18]:

$$A_n v_n = B_n v_{n-1} + C_n v_{n-2}, \quad n = 1, \dots, \infty, \quad (5.1)$$

with the initial condition  $v_{-1} = 0$ ,  $v_0 = 1$ . For  $\delta_N$ -confluent Heun's functions in (5.1) we have coefficients

$$\begin{aligned} A_n &= 1 + \frac{\beta}{n}, & B_n &= 1 - \frac{\kappa + 1}{n} + \frac{\kappa - \mu}{n^2}, \\ C_n &= \frac{\alpha}{n^2} (n - N - 2). \end{aligned} \quad (5.2)$$

Relations (5.1) and (5.2) give a very efficient new method for calculation of Starobinsky's constant for all values of the spin  $|s|$ , confirming the known results and easily solving the problem for all cases (3.1), (3.2), (3.3), (3.4), (3.5), and (3.6) of validity of the  $\delta_N$  condition.

## VI. CONCLUSION

In the present article, we showed that TSI exist for a large class of physical problems when the background vacuum metric is of Petrov type D. We constructed explicitly the TSI for different perturbations of any spin to the Kerr metric, as well as to the Schwarzschild one. Our derivation of TSI is valid for complex values of the separation constants  $\omega$  and  $E$ , as well as for complex values of all other parameters. The obtained results can be applied to both the black holes and naked singularities.

The exact solutions to the TRE, TAE, RWE, and ZE were briefly described in terms of confluent Heun's functions. One can find more details in the articles [17]. Having in hands these explicit solutions we are able

- (1) To write down the corresponding general solutions.
- (2) To formulate different boundary problems.
- (3) To derive all properties of the solutions using explicitly the properties of Heun's functions.

In particular this article accomplished the following aims:

- (i) We found that the TSI and all their possible generalizations have a common origin—the recently unveiled property (1.12) of the confluent Heun function, related with the  $\delta_N$  condition (1.9); see [13]. As a result of this condition, the multiple derivative of order  $(N + 1)$  of a given  $\delta_N$ -confluent Heun's function becomes proportional to another (associated) confluent Heun's function. The coefficient is precisely  $(N + 1)!$  times the  $(N + 1)$ -th coefficient in the Taylor series expansion of the first function. This algebraic property is universal and not related to the usually considered specific boundary problems, nor with other properties of their solutions like regularity, integrability, etc. Using the relation of the exact solutions at hand with the confluent Heun's function, one can express this property directly in terms of the very solutions. Thus one obtains the TSI for the TRE, TAE, RWE, and ZE in a uniform and universal way.

- (ii) We discussed all classes of  $\delta_N$  solutions to the above problems. For the specific class of  $\delta_N$  solutions, defined by relation (3.1), the novel symmetry (4.4) of the parameters of  $\delta_N$ -confluent Heun's functions (1.8) ensures that its multiple derivative of order  $(N + 1)$  is a solution to the *same* Heun equation. Thus, this symmetry yields a second solution to the same TME equation. As a result, we obtain the known pair of solutions to the TME with the opposite signs of their spin weights  $(\pm s)$  [1].
- (iii) We extended the familiar results for TSI to all possible cases on a firm and general mathematical basis. Thus, using the  $\delta_N$  condition we found the possible generalizations of the TSI.
- (iv) We made transparent the origin of some of the properties of the solutions to the TRE, TAE, RWE, and ZE, relating them with the specific form of the parameters in confluent Heun's functions, which solves these equations. For example, such unexpected property is the symmetry (4.4), which is specific for large class of solutions to TRE and TAE. This symmetry does not take place for all confluent  $\delta_N$ -Heun's functions.
- (v) Using the recurrence relation for the Taylor series expansion of the confluent Heun's function we proposed a new efficient and universal method for the calculation of Starobinsky's constants.

Thus, the confluent Heun function was proven to be an adequate tool for solving the above problems. The usage of this function gives a natural treatment and deeper understanding of the Teukolsky-Starobinsky identities, as well as of the solutions to the Teukolsky equations. The results, presented here, are a basis for future physical applications.

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### APPENDIX: ON THE RELATION BETWEEN THE SOLUTIONS TO THE RWE AND ZE

The relation between the solutions to the RWE and ZE were discovered by Chandrasekhar; see, for example, [4] and the references therein. We slightly modify the form of these relations using the notation  $Z_0^\pm(r)$  for solutions to the Zerilli equation and  $R_0^\pm(r)$  for solutions to the RWE. These functions have additional constant phases in comparison with the functions  $Z^{(\pm)}(r)$ , used in [4]. (Note that the meaning of the signs  $\pm$  in our notation is completely different; see [17].) Our choice of these additional phases ensures the appearance of the *common* factor  $i\bar{\omega}$  simultaneously in the two of the Eqs. (4.6).

As usual, here we consider only the ZE for gravitational waves with spin  $|s| = 2 \Rightarrow N + 1 = 2$ . Then in units  $2M = 1$  we obtain

$$e^{\psi(r)} = |r - 1|^{\omega_l} e^{\omega_l r} \left( 1 + \frac{3}{(l-1)(l+2)r} \right)^{-1},$$

$$\omega_l = \frac{1}{6}(l-1)l(l+1)(l+2), \quad l = 2, 3, \dots, \quad (\text{A1})$$

$$\bar{\omega} = \sqrt{\omega^2 + \omega_l^2} \neq 0 \quad \text{for } \omega \neq \pm i\omega_l.$$

The Eqs. (4.6) do not relate the solutions  $Z_0^\pm(r)$  and  $R_0^\pm(r)$  when  $\omega = \pm i\omega_l$ . Instead, in this case one obtains from the Eqs. (4.6) the well known algebraically special solutions in the following explicit quasipolynomial form:

$$R_0^\pm(r) = \text{const} \times e^{+\psi(r)}, \quad (\text{A2a})$$

$$Z_0^\pm(r) = \text{const} \times e^{-\psi(r)}. \quad (\text{A2b})$$

Thus we see that the case of the algebraically special solutions (A2) is a degenerate one.

The system of the ordinary differential equations (4.6) can be easily split to the decoupled RWE and ZE, written in the form

$$R'' + \bar{\omega}^2 R = (-\psi'' + \psi'^2)R, \quad (\text{A3a})$$

$$Z'' + \bar{\omega}^2 Z = (+\psi'' + \psi'^2)Z. \quad (\text{A3b})$$

Here the prime denotes differentiation with respect to the tortoise coordinate  $r_* = r + \ln|r - 1|$ . This splitting reflects the physical independence of the axial and polar perturbations to the Schwarzschild metric.

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