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Cosmological matching conditions for gravitational waves at second order

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We compute the second-order matching conditions for tensor metric perturbations at an abrupt change in the equation of state. For adiabatic perturbations on large scales the matching hypersurface coincides with a uniform-density hypersurface. We show that in the uniform-density gauge both the tensor perturbation and its time-derivative are continuous in this case. For nonadiabatic perturbations, the matching hypersurface need not coincide with a uniform-density hypersurface, and the tensor perturbation in the uniform-density gauge may be discontinuous. However, we show that in the Poisson gauge both the tensor perturbation and its time derivative are continuous for adiabatic or nonadiabatic perturbations. As an application we solve the evolution equation for second-order tensor perturbations on large scales for a constant equation of state, and we use the matching conditions to evolve the solutions through the transition from an inflationary era to a radiation era. We show that in the radiation era the resulting free part of the large-scale tensor perturbation (constant mode) is slow-roll suppressed in both the uniform-density and Poisson gauges. Thus, we conclude that second-order gravitational waves from slow-roll inflation are suppressed.

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I. INTRODUCTION

Recent precise measurements of the cosmic microwave background anisotropies have revealed the nature of the primordial perturbations [1], and future experiments such as Planck [2] will improve these measurements significantly. However, cosmic microwave background anisotropies and large-scale structure probe density perturbations only on large scales, and there is no direct way to observe density perturbations on smaller scales except for the case where large density perturbations form primordial black holes [3,4]. Recently, it has been recognized that gravitational waves can be used to probe small scales physics like preheating [5–7] and density perturbations on small scales [8,9]. At first order, tensor perturbations decouple from density perturbations. However, at second order, density perturbations can generate tensor perturbations [10]. The second-order tensor perturbations have been calculated during inflation [11], in the radiation era [8] and in the matter era [10,12–17]. While the amplitude of the second-order tensor perturbations is generally very small (determined by the square of the amplitude of density perturbations), there are several interesting situations where the second-order tensor perturbations give interesting observational consequences. For example, if density perturbations are large enough to form primordial black holes, then these density perturbations can generate sizable second-order gravitational waves [9]. In the matter era, the second-order tensor perturbation remains constant, which enhances the power spectrum of the tensor perturbations [14]. This happens either in the late-time matter era or in an early-time matter era due to the oscillations of the inflaton field after inflation [17]. The first-order tensor perturbations generated during inflation are model dependent, and their amplitude can be very small. However, the amplitude of the second-order tensor perturbations is determined completely by that of the density perturbations. Thus, this gives a lower bound for the tensor to scalar ratio [18].

There are still several important issues in the second-order generation of gravitational waves. At second order, the tensor perturbations are not gauge invariant under the first-order gauge transformations [19]. Thus, they depend on the choice of the gauge for first-order perturbations. It is however possible to construct gauge-invariant tensor perturbations by eliminating the gauge degrees of freedom [20–24]. A commonly used gauge is the Poisson gauge [12,19,25] and almost all calculations of second-order gravitational waves have been done in the Poisson gauge. During inflation, another commonly used gauge is the uniform-density gauge. This gauge is often used to calculate higher-order correlation functions for the curvature perturbation [26], and these calculations involve the second-order tensor perturbations because they are coupled to the curvature perturbations at higher order [27]. In fact, the amplitudes of tensor perturbations in these two gauges can be very different. In the Poisson gauge, the amplitude of the second-order tensor perturbations during inflation is suppressed by slow-roll parameters but it is not in the uniform-density gauge.

The question is then whether the amplitude of gravitational waves *after* inflation is slow-roll suppressed, or in-

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deed whether large tensor perturbations can be generated at second order by the transition to a radiation dominated universe. In order to address this problem, we should track the evolution of the tensor perturbations during the transition from inflation to the radiation era.

At first order, matching conditions for cosmological perturbations have been developed in cases where there is an abrupt change in the expansion rate at the matching surface [28–31], such as reheating at the end of inflation. In this paper, we extend this matching condition to second-order perturbations and derive matching conditions for second-order tensor perturbations on large scales. Matching conditions will be derived in two cases. For adiabatic perturbations on large scales a sudden transition in the equation of state must occur at a specific density. Thus, we apply the matching conditions across a uniform-density hypersurface. We will also consider the case where the transition surface is given by a fixed value of some field χ, which need no longer coincide with a uniform-density hypersurface in the presence of nonadiabatic perturbations. In this case the matching conditions are applied across a uniform-χ hypersurface. These matching conditions are applied to the second-order tensor perturbations generated during inflation, and we calculate the second-order tensor perturbations in the radiation era after reheating in both the Poisson gauge and the uniform-density gauge. We will show that the constant mode of the second-order gravitational waves on large scales, corresponding to the free part of the tensor perturbations, is the same in the two gauges in the radiation era, and it is suppressed by slow-roll parameters.

This paper is organized as follows: In Sec. II, we summarize the gauge transformations at the first and second order and derive solutions for the metric perturbations on large scales in the Poisson gauge and in the uniform-density gauge. In Sec. III, the evolution equation for the second-order tensor perturbations is derived. General solutions for the second-order tensor perturbations are derived on large scales with a constant equation of state, and we check the consistency of solutions in two different gauges under the gauge transformation. In Sec. IV, matching conditions for perturbations are developed at the first and second order in the cases where there is an abrupt change in the expansion rate at the matching surface. We consider both adiabatic and nonadiabatic matchings. In Sec. V, we apply the matching conditions to the second-order tensor perturbations generated during slow-roll inflation and calculate the evolution of the second-order tensor perturbations on large scales in the radiation era. Section V is devoted to conclusions.

II. GAUGE-INVARIANT PERTURBATIONS

In this section, after introducing the necessary notation and the perturbed metric, we shall review the first and second-order general gauge transformations for the perturbations. After that, using some simplifying assumptions, we will obtain the gauge transformation rules for second-order tensor perturbations. We will also derive the solutions for first-order scalar metric perturbations on large scales.

A. Perturbed metric

Throughout this work Greek indices $\mu, \nu, \ldots$ take values from 0 to 3, while Latin indices $i, j, \ldots$ denote spatial indices and can take values from 1 to 3. The components of a perturbed spatially flat Friedman-Robertson-Walker metric can be written as

$$g_{00} = -a(\eta)^2\bigg(1 + 2\sum_{r=1}^{+\infty}\frac{1}{r!}A^{(r)}\bigg), \tag{1}$$

$$g_{0i} = a(\eta)^2\sum_{r=1}^{+\infty}\frac{1}{r!}B_i^{(r)}, \tag{2}$$

$$g_{ij} = a(\eta)^2\bigg[\bigg(1 - 2\sum_{r=1}^{+\infty}\frac{1}{r!}C^{(r)}\bigg)\delta_{ij} + \sum_{r=1}^{+\infty}\frac{1}{r!}C_{ij}^{(r)}\bigg], \tag{3}$$

where η denotes conformal time and $C_{ij}^{(r)}$ is traceless, i.e. $C_i^{i(r)} = 0$, where the index was raised with δ^{ij}. The superscript (r) indicates the order of the perturbation. The perturbations depend on conformal time and position $\mathbf{x}$. The inverse metric up to second order can be found in the Appendix. We decompose the perturbations in scalar, vector and tensor parts as

$$B_i^{(r)} = \partial_i B^{(r)} + S_i^{(r)}, \tag{4}$$

where $\partial^i S_i^{(r)} = 0$.

$$C_{ij}^{(r)} = 2D_{ij}E^{(r)} + \partial_i F_j^{(r)} + \partial_j F_i^{(r)} + h_{ij}^{(r)}, \tag{5}$$

where $\partial^i F_i^{(r)} = 0$ and $h_i^{i(r)} = \partial^i h_{ij}^{(r)} = 0$. The operator D_{ij} is defined as $D_{ij} = \partial_i\partial_j - \frac{1}{3}\delta_{ij}\partial^2$. The energy density ρ and the four velocity of matter are perturbed like

$$\rho = \rho_{(0)} + \sum_{r=1}^{+\infty}\frac{1}{r!}\delta\rho^{(r)}, \tag{6}$$

$$u^\mu = \frac{1}{a}\bigg(\delta_0^\mu + \sum_{r=1}^{+\infty}\frac{1}{r!}v^{\mu(r)}\bigg). \tag{7}$$

u^μ is normalized as $u^\mu u_\mu = -1$, and this implies that $v^{0(r)}$ is related to the lapse perturbation $A^{(r)}$. The velocity perturbation is decomposed as

$$v^{i(r)} = \partial^i v^{\|(r)} + v^{\perp i(r)}. \tag{8}$$

B. Gauge transformations

Bruni *et al.* [19] have shown us that given a tensor T, its perturbations in two different gauges X, and Y are related at first order as

$$\delta T^{Y^{(1)}} - \delta T^{X^{(1)}} = \mathcal{L}_\xi^{(1)} T^{(0)}, \tag{9}$$

and at second order we have

$$\delta T^{Y^{(2)}} - \delta T^{X^{(2)}} = (\mathcal{L}_{\xi^{(2)}} + \mathcal{L}^2_{\xi^{(1)}})T^{(0)} + 2\mathcal{L}_{\xi^{(1)}}\delta T^{X^{(1)}}, \tag{10}$$

where $\xi^{(1)}$ and $\xi^{(2)}$ are the gauge shifts at first and second order respectively. The Lie derivative along a vector field ξ of a rank two covariant tensor T is given by $\mathcal{L}_\xi T_{\mu\nu} = T_{\mu\nu,\lambda}\xi^\lambda + \xi^\lambda_{,\mu}T_{\lambda\nu} + \xi^\lambda_{,\nu}T_{\mu\lambda}$. We also decompose $\xi^{\mu(r)}$ is scalar and vector parts as

$$\xi^{0(r)} = \alpha^{(r)}, \qquad \xi^{i(r)} = \partial^i\beta^{(r)} + d^{i(r)}, \tag{11}$$

where $\partial_i d^{i(r)} = 0$.

1. First order

At first order, the gauge transformations, Eq. (9), can be written as

$$\tilde{A}^{(1)} = A^{(1)} + \alpha^{(1)\prime} + \mathcal{H}\alpha^{(1)}, \tag{12}$$

$$\tilde{C}^{(1)} = C^{(1)} - \tfrac{1}{3}\partial^2\beta^{(1)} - \mathcal{H}\alpha^{(1)}, \tag{13}$$

$$\tilde{B}_i^{(1)} = B_i^{(1)} - \alpha^{(1)}_{,i} + \beta^{(1)\prime}_{,i} + d_i^{(1)\prime}, \tag{14}$$

$$\tilde{C}_{ij}^{(1)} = C_{ij}^{(1)} + 2D_{ij}\beta^{(1)} + d^{(1)}_{i,j} + d^{(1)}_{j,i}, \tag{15}$$

where prime denotes the derivative with respect to conformal time and the conformal Hubble rate is $\mathcal{H} = \frac{a'}{a}$. The last two equations imply

$$\tilde{B}^{(1)} = B^{(1)} - \alpha^{(1)} + \beta^{(1)\prime}, \tag{16}$$

$$\tilde{S}_i^{(1)} = S_i^{(1)} + d_i^{(1)\prime}, \tag{17}$$

$$\tilde{E}^{(1)} = E^{(1)} + \beta^{(1)}, \tag{18}$$

$$\tilde{F}_i^{(1)} = F_i^{(1)} + d_i^{(1)}, \tag{19}$$

$$\tilde{h}_{ij}^{(1)} = h_{ij}^{(1)}. \tag{20}$$

Four-dimensional scalars, like the energy density or the pressure, have the transformation rule

$$\delta\tilde{\rho} = \delta\rho + \rho^{(0)\prime}\alpha^{(1)}, \qquad \delta\tilde{P} = \delta P + P^{(0)\prime}\alpha^{(1)}. \tag{21}$$

The spatial component of the four velocity transform as

$$\tilde{v}^{i(1)} = v^{i(1)} - \partial^i\beta^{(1)\prime} - d^{i(1)\prime}, \tag{22}$$

or equivalently

$$\tilde{v}^{\parallel(1)} = v^{\parallel(1)} - \beta^{(1)\prime}, \qquad \tilde{v}^{\perp i(1)} = v^{\perp i(1)} - d^{i(1)\prime}, \tag{23}$$

while the temporal component transforms as $A^{(1)}$ because of the normalization constraint $v^{0(1)} = -A^{(1)}$.

Following Bardeen [32], we can define gauge-invariant quantities by specifying completely the choice of coordinates. The longitudinal or Poisson gauge is defined at first order by requiring $B_P^{(1)} = E_P^{(1)} = 0$, which from Eqs. (16) and (18) fixes $\beta_P^{(1)} = -E^{(1)}$ and $\alpha_P^{(1)} = B^{(1)} - E^{(1)\prime}$, where the subscript P denotes the Poisson gauge quantity. The remaining scalar metric perturbations then have the gauge-invariant definitions

$$\Phi \equiv A_P^{(1)} = A^{(1)} + \mathcal{H}(B^{(1)} - E^{(1)\prime}) + (B^{(1)} - E^{(1)\prime})', \tag{24}$$

$$\Psi \equiv C_P^{(1)} = C^{(1)} - \mathcal{H}(B^{(1)} - E^{(1)\prime}) + \tfrac{1}{3}\partial^2 E^{(1)}, \tag{25}$$

The gauge-invariant density perturbation corresponding to density perturbations in this specific gauge is, from Eq. (21),

$$\delta\rho_P^{(1)} = \delta\rho + \rho^{(0)\prime}(B^{(1)} - E^{(1)\prime}). \tag{26}$$

An alternative set of gauge-invariant variables can be constructed by working on uniform-density hypersurfaces for which $\delta\rho_{\rm UD} = 0$, which from Eq. (21) fixes $\alpha_{\rm UD}^{(1)} = -\delta\rho/\rho^{(0)\prime}$. The curvature perturbation in the uniform-density gauge is thus the gauge-invariant variable

$$-\zeta \equiv C_{\rm UD}^{(1)} + \frac{1}{3}\partial^2 E_{\rm UD}^{(1)} = C^{(1)} + \frac{1}{3}\partial^2 E^{(1)} + \frac{\mathcal{H}}{\rho^{(0)\prime}}\delta\rho^{(1)}. \tag{27}$$

Here and in the rest of this work we have choose the threading of the time slices, $\beta_{\rm UD}^{(1)} = -E^{(1)}$, such that $E_{\rm UD}^{(1)} = 0$. The gauge-invariant definition of the scalar part of the shift vector (16) in the uniform-density gauge is given as

$$B_{\rm UD}^{(1)} = B^{(1)} - E^{(1)\prime} + \frac{\delta\rho^{(1)}}{\rho^{(0)\prime}} = \frac{\delta\rho_P^{(1)}}{\rho^{(0)\prime}} = -\left(\frac{\zeta + \Psi}{\mathcal{H}}\right). \tag{28}$$

In the absence of anisotropic stress, the Einstein equations in the Poisson gauge require $\Phi = \Psi$ and the energy constraint equation gives a relation between the curvature perturbation in the uniform-density and Poisson gauges

$$\zeta = -\frac{2\mathcal{H}^2 - \mathcal{H}'}{\mathcal{H}^2 - \mathcal{H}'}\Psi - \frac{\mathcal{H}}{\mathcal{H}^2 - \mathcal{H}'}\Psi' + \frac{1}{3(\mathcal{H}^2 - \mathcal{H}')}\partial^2\Psi. \tag{29}$$

In the next section, when writing the equations of motion for second-order tensor perturbation, we will neglect terms with more than two spatial gradients. We also ignore non-adiabatic perturbations. This implies that we can take $\zeta' = 0$ at zeroth order in the gradient expansion. The scalar part of the Einstein equations gives

$$a^{-2}(6\mathcal{H}(C^{(1)\prime} + \mathcal{H}A^{(1)}) - 2\partial^2 C^{(1)}) = -\kappa_G^2 \delta\rho^{(1)}, \quad (30)$$

where κ_G is the gravitational constant. In the uniform-density gauge $\delta\rho^{(1)} = 0$, thus on large scales $A_{\rm UD} = 0$. For a perfect fluid with a constant equation of state w we have $\zeta = V =$ constant on large scales, and Eq. (29) gives the solution

$$\Psi = -\frac{3+3w}{5+3w}V + W\eta^{-((5+3w)/(1+3w))}, \quad (31)$$

where $W = W(\mathbf{x})$ is a constant of integration to be determined by the first-order junction conditions. If $w = 1/3$, the previous equation simplifies to give

$$\Psi = -\tfrac{2}{3}V + W\eta^{-3}. \quad (32)$$

2. Second order

The second-order gauge transformation can be found in Bruni *et al.* For the trace-free spatial metric perturbation it reads

$$\begin{aligned}\tilde{C}_{ij}^{(2)} &= C_{ij}^{(2)} + 2(C_{ij}^{(1)\prime} + 2\mathcal{H}C_{ij}^{(1)})\alpha^{(1)} + 2C_{ij,k}^{(1)}\xi^{k(1)} + 2(-4C^{(1)} + \alpha^{(1)}\partial_0 + \xi^{k(1)}\partial_k + 4\mathcal{H}\alpha^{(1)})(d_{(i,j)}^{(1)} + D_{ij}\beta^{(1)}) \\ &\quad + 2\Big[(2B_{(i}^{(1)} - \alpha_{,(i}^{(1)} + \xi_{(i}^{(1)\prime})\alpha_{,j)}^{(1)} - \frac{1}{3}\delta_{ij}(2B^{k(1)} - \alpha^{,k(1)} + \xi^{k(1)\prime})\alpha_{,k}^{(1)}\Big] \\ &\quad + 2\Big[(2C_{(i|k|}^{(1)} + \xi_{k,(i}^{(1)} + \xi_{(i,|k|}^{(1)})\xi_{,j)}^{(1)k} - \frac{1}{3}\delta_{ij}(2C_{lk}^{(1)} + \xi_{k,l}^{(1)} + \xi_{l,k}^{(1)})\xi^{k,l(1)}\Big] + 2(d_{(i,j)}^{(2)} + D_{ij}\beta^{(2)}). \end{aligned} \quad (33)$$

We can define gauge-invariant second-order tensor perturbations in the Poisson gauge as

$$\begin{aligned}\tilde{h}_{ij\ P}^{(2)} &= h_{ij}^{(2)} + 2\Big[(C_{ij}^{(1)\prime} + 2\mathcal{H}C_{ij}^{(1)})\alpha^{(1)} + C_{ij,k}^{(1)}\xi^{k(1)} + (-4C^{(1)} + \alpha^{(1)}\partial_0 + \xi^{k(1)}\partial_k + 4\mathcal{H}\alpha^{(1)})(d_{(i,j)}^{(1)} + D_{ij}\beta^{(1)}) \\ &\quad + (2B_{(i}^{(1)} - \alpha_{,(i}^{(1)} + \xi_{(i}^{(1)\prime})\alpha_{,j)}^{(1)} + (2C_{(i|k|}^{(1)} + \xi_{k,(i}^{(1)} + \xi_{(i,|k|}^{(1)})\xi_{,j)}^{(1)k}\Big]^{TT}, \end{aligned} \quad (34)$$

where all the quantities in the right-hand side of the previous equation are in an arbitrary gauge, and the parameters of the gauge transformations are explicitly given as $\alpha^{(1)} = B^{(1)} - E^{(1)\prime}$, $\beta^{(1)} = -E^{(1)}$, $d_i^{(1)} = -F_i^{(1)}$ and $\xi^{i(1)} = \partial^i\beta^{(1)} + d^{i(1)}$. In a similar way, a gauge-invariant definition of the uniform-density second-order tensor perturbation is

$$\begin{aligned}\tilde{h}_{ij\ \rm UD}^{(2)} &= h_{ij}^{(2)} + 2[(C_{ij}^{(1)\prime} + 2\mathcal{H}C_{ij}^{(1)})\alpha^{(1)} + C_{ij,k}^{(1)}\xi^{k(1)} + (-4C^{(1)} + \alpha^{(1)}\partial_0 + \xi^{k(1)}\partial_k + 4\mathcal{H}\alpha^{(1)})(d_{(i,j)}^{(1)} + D_{ij}\beta^{(1)}) \\ &\quad + (2B_{(i}^{(1)} - \alpha_{,(i}^{(1)} + \xi_{(i}^{(1)\prime})\alpha_{,j)}^{(1)} + (2C_{(i|k|}^{(1)} + \xi_{k,(i}^{(1)} + \xi_{(i,|k|}^{(1)})\xi_{,j)}^{(1)k}]^{TT}, \end{aligned} \quad (35)$$

where $\alpha^{(1)} = -\delta\rho^{(1)}/\rho^{(0)\prime}$, $\beta^{(1)} = -E^{(1)}$, $d_i^{(1)} = -F_i^{(1)}$ and $\xi^{i(1)} = \partial^i\beta^{(1)} + d^{i(1)}$.

In this work, we will neglect the effect of first-order vector and tensor perturbations in order to simplify the second-order equations as much as possible. In many models of the very early Universe, such as slow-roll inflation, which we consider later, vector and tensor perturbations are suppressed with respect to scalar perturbations.

TABLE I. Time gauge shift $\alpha^{(1)}$ at first order between our different gauges.

$\alpha^{(1)}$	Poisson	Uniform density
Poisson	0	$-\delta\rho^{(1)}/\rho^{(0)\prime}$
Uniform density	$B^{(1)}$	0

First-order primordial vector and tensor perturbations, if they do exist, would provide an additional, independent source of second-order gravitational waves.

We choose to set $E^{(r)}$ to zero in both Poisson and uniform-density gauges. This implies that in the gauge transformations between these two gauges the parameter $\beta^{(1)}$ is zero (and $d^{i(1)} = 0$). The first-order temporal gauge shift, $\alpha^{(1)}$, is given in Table I. For example, if we start in the Poisson gauge and if we do a gauge change with $\alpha^{(1)} = -\delta\rho^{(1)}/\rho^{(0)\prime}$, then we obtain the quantity in the uniform-density gauge (tilde gauge), as given in Eqs. (12)–(15). The Poisson gauge is defined by $B^{(1)} = E^{(1)} = F_i^{(1)} = 0$ and the uniform-density gauge is $\delta\rho^{(1)} = E^{(1)} = F_i^{(1)} = 0$. With the previous assumptions and with the choice $E^{(1)} = 0$, the gauge transformation Eq. (33) for the traceless perturbation $C_{ij}^{(2)}$ simplifies considerably to

$$\tilde{C}_{ij}^{(2)} = C_{ij}^{(2)} + 2\bigg[(2B_{(i}^{(1)} - \alpha_{,(i}^{(1)})\alpha_{,j)}^{(1)} - \frac{1}{3}\delta_{ij}(2B^{k(1)} - \alpha^{,k(1)})\alpha_{,k}^{(1)}\bigg] + 2(d_{(i,j)}^{(2)} + D_{ij}\beta^{(2)}), \tag{36}$$

which gives the following rule for transforming second-order tensors

$$\tilde{h}_{ij}^{(2)} = h_{ij}^{(2)} + 2[(2B_{(i}^{(1)} - \alpha_{,(i}^{(1)})\alpha_{,j)}^{(1)}]^{TT}, \tag{37}$$

where TT means the transverse and traceless part of the terms in between square brackets. In the case of transforming from the Poisson gauge to the uniform-density gauge the previous equation further simplifies to give

$$\tilde{h}_{ij\ \mathrm{UD}}^{(2)} - h_{ij\ P}^{(2)} = \frac{2}{\mathcal{H}^2}[(\zeta + \Psi)(\zeta + \Psi)_{,ij}]^{TT}. \tag{38}$$

The right-hand side of this equation leads to different solutions for second-order tensor perturbation in uniform-density or Poisson gauges. Nonetheless we have constructed gauge-invariant expressions, (34) and (35), for the second-order tensor perturbation in the uniform-density and Poisson gauges, and the difference between them is also manifestly gauge invariant.

III. TENSOR PERTURBATION EQUATIONS

In this section, we will solve the first and second-order large-scale equations of motion for tensor perturbations in a fluid with a constant equation of state, in both the Poisson and the uniform-density gauges. At the end of the section, we shall present the second-order gauge transformations and the relations between the integration constants that appear in the two gauges.

A. First order

The first-order tensor perturbation is gauge invariant, and its equation of motion on large scales is

$$h_{ij}^{(1)''} + 2\mathcal{H}h_{ij}^{(1)'} = 0, \tag{39}$$

which can be easily solved

$$h_{ij}^{(1)} = Y_{ij} + Z_{ij}\eta^{-((3-3w)/(1+3w))}, \tag{40}$$

where Y_{ij} and Z_{ij} are transverse and traceless integration constants to be determined by the initial conditions. During matter or radiation eras, Z_{ij} is a decaying mode and the growing mode is constant in time, as expected for first-order gauge-invariant tensor perturbations on superhorizon scales.

B. Second order

In this subsection we will present the equation of motion for second-order tensor perturbations in the uniform-density gauge and in the Poisson gauge. Note that we neglect first-order vector and tensor perturbations, and we choose $E^{(r)} = 0$. We also ignore first and second-order anisotropic stress.

Using the second-order Einstein equations from Bartolo *et al.* review [33] we get

$$G_j^{i(2)^{TT}} = \kappa_G^2 T_j^{i(2)^{TT}}, \tag{41}$$

where

$$T_j^{i(2)^{TT}} = (\rho^{(0)} + P^{(0)})[\partial^i v^{\|(1)}(\partial_j v^{\|(1)} + \partial_j B^{(1)})]^{TT}, \tag{42}$$

$$G_j^{i(2)^{TT}} = a^{-2}\left(\frac{1}{4}h_j^{i(2)''} + \frac{1}{2}\mathcal{H}h_j^{i(2)'} - \frac{1}{4}\partial^2 h_j^{i(2)}\right) + a^{-2}[\partial^i A\partial_j A + 2A\partial^i\partial_j A - 2C\partial^i\partial_j A - \partial_j A\partial^i C - \partial^i A\partial_j C + 3\partial^i C\partial_j C + 4C\partial^i\partial_j C + 2\mathcal{H}\partial^i B\partial_j A + 4\mathcal{H}A\partial^i\partial_j B + A'\partial^i\partial_j B + 2A\partial^i\partial_j B' + \partial^2 B\partial^i\partial_j B - \partial_j\partial^k B\partial^i\partial_k B - 2\mathcal{H}\partial^i C\partial_j B - 2\mathcal{H}\partial^i B\partial_j C - \partial^i C'\partial_j B + \partial_j C'\partial^i B - \partial^i C\partial_j B' - \partial_j C\partial^i B' - 2C\partial^i\partial_j B' + C'\partial^i\partial_j B - 4\mathcal{H}C\partial_j\partial^i B]^{TT}, \tag{43}$$

where in the previous equation we omitted the superscript (1) to indicate the order of the perturbations, and we corrected the typos of equation (A.43) of Ref. [33].

1. Poisson gauge

The equation of motion for second-order tensor perturbations in the Poisson gauge is known and can be found in Baumann *et al.* [14] (see also [8,25,34]). Under our assumptions (for constant w) it simplifies to

$$h_{ij\ P}^{(2)''} + 2\mathcal{H}h_{ij\ P}^{(2)'} - \partial^2 h_{ij\ P}^{(2)} = 8\bigg[\partial_i\Psi\partial_j\Psi + \frac{2}{3(1+w)\mathcal{H}^2}\partial_i(\Psi' + \mathcal{H}\Psi) \times \partial_j(\Psi' + \mathcal{H}\Psi)\bigg]^{TT}. \tag{44}$$

Using the solution for Ψ as in Eq. (31), the equation of motion can be written in a simple form as

$$h_{ij}^{(2)''}{}_P + 2\mathcal{H}h_{ij}^{(2)'}{}_P = 8\frac{3+3w}{5+3w}[\partial_i V\partial_j V]^{TT} + 4(5+3w) \times [\partial_i W\partial_j W]^{TT}\eta^{-2((5+3w)/(1+3w))}, \tag{45}$$

where we drop the gradient terms on the left-hand side of (44). This is the equation of motion for second-order tensor perturbations, at leading order in a gradient expansion. It can be solved to give

$$h_{ij}^{(2)}{}_P = Y_{ijP} + Z_{ijP}\eta^{-((3-3w)/(1+3w))} + 4\frac{(1+3w)(3+3w)}{(5+3w)^2}[\partial_i V\partial_j V]^{TT}\eta^2 + \frac{(1+3w)^2}{2}[\partial_i W\partial_j W]^{TT}\eta^{-(8/(1+3w))}, \tag{46}$$

where Y_{ijP} and Z_{ijP} are integration constants.

2. Uniform-density gauge

Using Einstein Eqs. (41), we can derive the equation of motion for second-order tensor perturbations in the uniform-density gauge as

$$h_j^{i(2)''}{}_{\rm UD} + 2\mathcal{H}h_j^{i(2)'}{}_{\rm UD} - \partial^2 h_j^{i(2)}{}_{\rm UD} = -4S_{ij}^{TT}, \tag{47}$$

where the source is given by

$$S_{ij} = \partial^i A\partial_j A + 2A\partial^i\partial_j A + 2\zeta\partial^i\partial_j A + \partial_j A\partial^i\zeta + \partial^i A\partial_j\zeta + 3\partial^i\zeta\partial_j\zeta + 4\zeta\partial^i\partial_j\zeta + 2\mathcal{H}\partial^i B\partial_j A + 4\mathcal{H}A\partial^i\partial_j B + A'\partial^i\partial_j B + 2A\partial^i\partial_j B' + \partial^2 B\partial^i\partial_j B - \partial_j\partial^k B\partial^i\partial_k B + 2\mathcal{H}\partial^i\zeta\partial_j B + 2\mathcal{H}\partial^i B\partial_j\zeta + \partial^i\zeta'\partial_j B - \partial_j\zeta'\partial^i B + \partial^i\zeta\partial_j B' + \partial_j\zeta\partial^i B' + 2\zeta\partial^i\partial_j B' - \zeta'\partial^i\partial_j B + 4\mathcal{H}\zeta\partial_j\partial^i B - \frac{3}{2}\partial_j(-\zeta' + \mathcal{H}A)\partial^{-2}\partial^i\left(\zeta' + \frac{\mathcal{H}}{\rho^{(0)} + P^{(0)}}\delta P\right), \tag{48}$$

and we have used the first-order Einstein equations to simplify the result. Using Eqs. (28) and (31), $A_{\rm UD} = 0$ on large scales, ignoring higher derivatives terms and the gradient term on the left-hand side of (47), we can write the equation of motion (47) for second-order tensor perturbations in the uniform-density gauge at leading order in a gradient expansion as

$$h_{ij}^{(2)''}{}_{\rm UD} + 2\mathcal{H}h_{ij}^{(2)'}{}_{\rm UD} = 4[\partial_i V\partial_j V]^{TT}. \tag{49}$$

This can be easily solved

$$h_{ij}^{(2)}{}_{\rm UD} = Y_{ij\rm UD} + Z_{ij\rm UD}\eta^{-((3-3w)/(1+3w))} + 2\frac{1+3w}{5+3w}[\partial_i V\partial_j V]^{TT}\eta^2, \tag{50}$$

where $Y_{ij\rm UD}$ and $Z_{ij\rm UD}$ are integration constants to be determined by the initial conditions.

The second-order gauge transformation (38) between these two gauges is

$$h_{ij}^{(2)}{}_{\rm UD} = h_{ij}^{(2)}{}_P - 2\left(\frac{1+3w}{5+3w}\right)^2[\partial_i V\partial_j V]^{TT}\eta^2 - \frac{(1+3w)^2}{2} \times [\partial_i W\partial_j W]^{TT}\eta^{-(8/(1+3w))} - \frac{(1+3w)^2}{5+3w} \times [\partial_i W\partial_j V + \partial_j W\partial_i V]^{TT}\eta^{((-3+3w)/(1+3w))}. \tag{51}$$

We have checked that Eqs. (49) and (45) are related by (51) [ignoring higher gradients]. This provides a good consistency check on our calculation. In a similar way, one can confirm that the solutions of Eqs. (49) and (45) are related by (51), and we get two relations between the integration constants in the two gauges as

$$Y_{ijP} = Y_{ij\rm UD}, \quad Z_{ijP} = Z_{ij\rm UD} + \frac{(1+3w)^2}{5+3w}[\partial_i W\partial_j V + \partial_j W\partial_i V]^{TT}. \tag{52}$$

We note that the gauge dependence of the second-order tensor perturbation, (51), affects only the amplitude of the time-dependent parts of the perturbation in Poisson or uniform-density gauges, and leaves the growing mode, Y_{ij}, unaffected on large scales.

IV. MATCHING CONDITIONS

In this section, we shall derive the matching conditions for the metric and the extrinsic curvature up to second order. These matching conditions are applicable on large scales and when there is an abrupt change in the expansion rate at the matching surface. We will first consider the case where this transition happens at a specific energy density, as it does for adiabatic perturbations. We will then discuss the more general situation, where the transition is not determined by the density but by some other field χ. We call this case the nonadiabatic matching. (See section 9 of Ref.[24] for the definition of nonadiabatic perturbations of multiple fluids or fields.) In the final subsection, we will apply these matching conditions to the second-order tensor perturbations coming from first-order scalar-perturbations.

Matching the induced metric and extrinsic curvature of a uniform-density or uniform-χ hypersurface specifies a physical model for the matching, but does not specify the gauge. We will choose to work in a gauge in which the matching surface coincides with a constant-η hypersurface. This is not a physical restriction; it is simply a choice of gauge, reflecting our freedom to choose the time slicing.

The metric of the constant η hypersurfaces (in the perturbed spacetime) is

$$q_{ij} = g_{ij} = a(\eta)^2\bigg[\bigg(1 - 2\sum_{r=1}^{+\infty}\frac{1}{r!}C^{(r)}\bigg)\delta_{ij} + \sum_{r=1}^{+\infty}\frac{1}{r!}C^{(r)}_{ij}\bigg]. \tag{53}$$

The unit timelike vector field orthogonal to these surfaces is $N^\mu = (N^0, N^i)$ with the normalization $N^\mu N_\mu = -1$. q_{ij} is given by $q_{\mu\nu} = g_{\mu\nu} + N_\mu N_\nu$, which together with Eq. (53) implies $N_\mu = (N_0, \mathbf{0})$. The inverse induced metric is $q^{\mu\nu} = g^{\mu\nu} + N^\mu N^\nu$, and it gives

$$\begin{aligned} q^{ij(0)} &= g^{ij(0)}, \qquad q^{ij(1)} = g^{ij(1)},\\ q^{ij(2)} &= g^{ij(2)} + a^{-2}B^{i(1)}B^{j(1)}. \end{aligned} \tag{54}$$

The extrinsic curvature of the constant η hypersurfaces is

$$K_{\mu\nu} = \tfrac{1}{2}\mathcal{L}_{N^\lambda}q_{\mu\nu} = \tfrac{1}{2}(q_{\mu\nu,\sigma}N^\sigma + N^\sigma_{,\mu}q_{\sigma\nu} + N^\sigma_{,\nu}q_{\mu\sigma}). \tag{55}$$

Defining the trace and the traceless parts of the extrinsic curvature like

$$K = q^{ij}K_{ij}, \qquad \tilde{K}_{ij} = K_{ij} - \tfrac{1}{3}q_{ij}K, \tag{56}$$

the junction conditions are

$$[q_{ij}]^{+}_{-} = 0, \qquad [K_{ij}]^{+}_{-} = 0, \tag{57}$$

which are equivalent to

$$[q_{ij}]^{+}_{-} = 0, \qquad [\tilde{K}_{ij}]^{+}_{-} = 0, \qquad [K]^{+}_{-} = 0. \tag{58}$$

A. Background

In the background,

$$N^{0(0)} = \mp a^{-1}, \qquad N^{i(0)} = 0, \qquad N^{(0)}_0 = \pm a, \tag{59}$$

and

$$K^{(0)}_{ij} = \mp a'\delta_{ij}. \tag{60}$$

Therefore,

$$K^{(0)} = \mp 3\frac{\mathcal{H}}{a}, \qquad \tilde{K}^{(0)}_{ij} = 0. \tag{61}$$

Continuity of the induced metric and its extrinsic curvature, Eq. (57), at zeroth-order thus requires that a and a' are continuous, i.e.,

$$[a]^{+}_{-} = 0, \qquad [\mathcal{H}]^{+}_{-} = 0. \tag{62}$$

These matching conditions can be used to give the initial conditions for the scale factor and its derivative after a sudden transition in the equation of state from a w_- era to a w_+ era (such as from inflation to a radiation domination era). The solution of the background Einstein equations for the conformal Hubble rate is

$$\mathcal{H} = \bigg(\frac{1 + 3w}{2}\eta + C_1\bigg)^{-1}, \tag{63}$$

where C_1 is an integration constant. From the continuity of the scale factor and its first derivative we can obtain

$$\begin{aligned} \mathcal{H} &= \bigg(\frac{1 + 3w_-}{2}\eta + C_{1-}\bigg)^{-1} \quad \text{for } \eta < \eta_*,\\ \mathcal{H} &= \bigg(\frac{1 + 3w_+}{2}\eta + C_{1+}\bigg)^{-1} \quad \text{for } \eta_* < \eta, \end{aligned} \tag{64}$$

where the constant C_{1+} is

$$C_{1+} = C_{1-} + \tfrac{3}{2}\eta_*(w_- - w_+). \tag{65}$$

We can write the previous equations as

$$\begin{aligned} \mathcal{H} &= \bigg(\frac{1 + 3w_-}{2}\eta_-\bigg)^{-1} \quad \text{for } \eta_- < \eta_{-*},\\ \mathcal{H} &= \bigg(\frac{1 + 3w_+}{2}\eta_+\bigg)^{-1} \quad \text{for } \eta_{+*} < \eta_+, \end{aligned} \tag{66}$$

where the new time variables $\eta_\pm$ are

$$\eta_\pm = \eta + \frac{2}{1 + 3w_\pm}C_{1\pm}, \tag{67}$$

and the transition times in the new variables are

$$\begin{aligned} \eta_{-*} &= \eta_* + \frac{2}{1 + 3w_-}C_{1-},\\ \eta_{+*} &= \frac{1 + 3w_-}{1 + 3w_+}\eta_* + \frac{2}{1 + 3w_+}C_{1-}. \end{aligned} \tag{68}$$

The subscripts $+$ ($-$) denote that the quantity should be evaluated at a time after (before) the transition time η_*.

B. At first order

The first-order matching conditions have been calculated previously in Refs. [29–31].

Continuity of the induced metric at first-order leads to the conservation of the spatial metric perturbations $C^{(1)}$ and $C^{(1)}_{ij}$, or equivalently

$$\begin{aligned} [C^{(1)}]^{+}_{-} &= 0, \qquad [E^{(1)}]^{+}_{-} = 0,\\ [F^{(1)}_i]^{+}_{-} &= 0, \qquad [h^{(1)}_{ij}]^{+}_{-} = 0. \end{aligned} \tag{69}$$

The first-order correction to the ortho-normal vector field is

$$N^{0(1)} = \pm a^{-1} A^{(1)}, \qquad N^{i(1)} = \pm a^{-1} B^{i(1)}, \qquad N_0^{(1)} = \pm a A^{(1)}, \tag{70}$$

and the extrinsic curvature is then

$$K_{ij}^{(1)} = \pm \frac{a}{2}[2\delta_{ij}(2\mathcal{H} C^{(1)} + \mathcal{H} A^{(1)} + C^{(1)\prime}) + B_{j,i}^{(1)} + B_{i,j}^{(1)} - 2\mathcal{H} C_{ij}^{(1)} - C_{ij}^{(1)\prime}], \tag{71}$$

or equivalently

$$K^{(1)} = \pm a^{-1}[3(\mathcal{H} A^{(1)} + C^{(1)\prime}) + \partial^2 B^{(1)}], \qquad \tilde{K}_{ij}^{(1)} = \pm a\left(B_{(i,j)}^{(1)} - \frac{1}{3}\delta_{ij}\partial^2 B^{(1)} - \frac{1}{2} C_{ij}^{(1)\prime}\right). \tag{72}$$

The junction condition (57) for the extrinsic curvature gives

$$[3(\mathcal{H} A^{(1)} + C^{(1)\prime}) + \partial^2 B^{(1)}]_-^+ = 0, \tag{73}$$

$$[B^{(1)} - E^{(1)\prime}]_-^+ = 0, \tag{74}$$

$$[S_j^{(1)} - F_j^{(1)\prime}]_-^+ = 0, \tag{75}$$

$$[h_{ij}^{(1)\prime}]_-^+ = 0. \tag{76}$$

Note that Eq. (73) combined with Eqs. (69). and the Einstein equations, Eq. (30), enforces continuity of energy density across the hypersurface

$$[\delta\rho^{(1)}]_-^+ = 0. \tag{77}$$

1. Adiabatic matching

For adiabatic perturbations on large scales a sudden transition in the equation of state must occur at a specific density. Thus, we apply the matching conditions across uniform-density hypersurface [29]. We can use the freedom in the choice of spatial coordinates on the matching hypersurface to set $E^{(1)} = 0$. In this case the matching conditions (69) and (74) across a uniform-density hypersurface reduce to

$$[\zeta]_-^+ = 0, \qquad [\Psi]_-^+ = 0. \tag{78}$$

Note that the remaining junction condition for scalar perturbations, Eq. (73), reduces to $[\zeta']_-^+ = 0$ on large scales, which is automatically satisfied for adiabatic perturbations since $\zeta' = 0$. Taking the growing mode for adiabatic perturbations on large scales before the transition

$$\Psi_- = -\frac{3 + 3w_-}{5 + 3w_-} V_- = \text{const}, \tag{79}$$

then the junction conditions (78), together with the general solution for constant w, Eq. (31), can be used to determine the solution of Ψ after a sudden transition as

$$\Psi_+ = -\frac{3 + 3w_+}{5 + 3w_+} V_+ + W_+ \eta_+^{-((5+3w_+)/(1+3w_+))}, \tag{80}$$

where $V_+ = V_- = \zeta_-$ and the amplitude of the decaying mode, W_+, is given by

$$W_+ = \frac{3 + 3w_+}{5 + 3w_+} \eta_{+*}^{((5+3w_+)/(1+3w_+))} \times \left(1 - \frac{3 + 3w_-}{5 + 3w_-}\frac{5 + 3w_+}{3 + 3w_+}\right) V_-. \tag{81}$$

2. Nonadiabatic matching

To consider a more general matching condition we will consider the case where the transition surface is given by a fixed value of some field, χ. The scalar metric perturbations on a uniform-χ hypersurface can be given in terms of the metric perturbations and χ-field perturbations in an arbitrary gauge as

$$-\zeta_\chi = C_\chi^{(1)} + \frac{1}{3}\partial^2 E_\chi^{(1)} = C^{(1)} + \frac{1}{3}\partial^2 E^{(1)} + \mathcal{H}\frac{\delta\chi^{(1)}}{\chi^{(0)\prime}}, \tag{82}$$

$$B_\chi^{(1)} - E_\chi^{(1)\prime} = B^{(1)} - E^{(1)\prime} + \frac{\delta\chi^{(1)}}{\chi^{(0)\prime}}. \tag{83}$$

Departures from adiabaticity are characterized by perturbations of the χ field with respect to the total density

$$\mathcal{S}_\chi \equiv \mathcal{H}\left(\frac{\delta\rho^{(1)}}{\rho^{(0)\prime}} - \frac{\delta\chi^{(1)}}{\chi^{(0)\prime}}\right). \tag{84}$$

We will recover the adiabatic matching conditions when $\mathcal{S}_\chi = 0$ and χ is unperturbed on uniform-density hypersurfaces, so that the uniform-χ hypersurface coincides with a uniform-density hypersurface.

The usual gauge-invariant curvature perturbations can then be given as

$$\zeta = \zeta_\chi - \mathcal{S}_\chi, \tag{85}$$

$$\Psi = -\zeta_\chi - \mathcal{H}(B_\chi^{(1)} - E_\chi^{(1)\prime}). \tag{86}$$

The junction conditions for scalar metric perturbations (69) and (74) require

$$[\zeta_\chi]_-^+ = 0, \qquad [B_\chi^{(1)} - E_\chi^{(1)\prime}]_-^+ = 0, \tag{87}$$

while Eq. (73) enforces energy conservation across the hypersurface, Eq. (77) and thus

$$[\mathcal{S}_\chi]_-^+ = \left(\frac{1 + w_-}{1 + w_+} - 1\right)\mathcal{S}_{\chi-}. \tag{88}$$

Expressing Eqs. (87) in terms of the usual gauge-invariant curvature perturbations we then have

$$[\zeta]_-^+ = \left(1 - \frac{1+w_-}{1+w_+}\right)\mathcal{S}_{\chi-}, \qquad [\Psi^{(1)}]_-^+ = 0. \tag{89}$$

Analogously to the previous subsection, these junction conditions can be used to evolve the solution of Ψ across the transition. The solution is given by Eqs. (79) and (80), where now the integration constants after the transition are given by

$$V_+ = \zeta_{\chi-} + \frac{1+w_-}{1+w_+}(V_- - \zeta_{\chi-}),$$
$$W_+ = \left(\frac{3+3w_+}{5+3w_+}V_+ - \frac{3+3w_-}{5+3w_-}V_-\right)\eta_*^{((5+3w_+)/(1+3w_+))}. \tag{90}$$

C. At second order

Continuity of the induced metric at second order leads to the obvious extension of the first-order junction conditions (69) for the spatial metric perturbations $C^{(2)}$ and $C_{ij}^{(2)}$, or equivalently

$$[C^{(2)}]_-^+ = 0, \qquad [E^{(2)}]_-^+ = 0,$$
$$[F_i^{(2)}]_-^+ = 0, \qquad [h_{ij}^{(2)}]_-^+ = 0. \tag{91}$$

At second order, in the ortho-normal vector field we have

$$N^{0(2)} = \pm\frac{a^{-1}}{2}[B_i^{(1)}B^{i(1)} - 3(A^{(1)})^2 + A^{(2)}],$$
$$N_0^{(2)} = \pm\frac{a}{2}[B_i^{(1)}B^{i(1)} - (A^{(1)})^2 + A^{(2)}], \tag{92}$$

$$N^{i(2)} = \pm a^{-1}\left(-A^{(1)}B^{i(1)} + 2C^{(1)}B^{i(1)} - B_j^{(1)}C^{ij(1)} + \frac{B^{i(2)}}{2}\right). \tag{93}$$

The extrinsic curvature tensor is

$$K_{ij}^{(2)} = \pm a\delta_{ij}\left[\mathcal{H}C^{(2)} + \frac{1}{2}C^{(2)\prime} + \frac{1}{2}\mathcal{H}A^{(2)} - 2\mathcal{H}A^{(1)}C^{(1)} - A^{(1)}C^{(1)\prime} + \frac{1}{2}\mathcal{H}(B_k^{(1)}B^{k(1)} - 3(A^{(1)})^2) - B^{k(1)}C_{,k}^{(1)}\right]$$
$$\pm a\left[-\frac{\mathcal{H}}{2}C_{ij}^{(2)} - \frac{1}{4}C_{ij}^{(2)\prime} + \mathcal{H}A^{(1)}C_{ij}^{(1)} + \frac{1}{2}A^{(1)}C_{ij}^{(1)\prime} + \frac{1}{2}B^{k(1)}C_{ij,k}^{(1)} - A^{(1)}B_{(j,i)}^{(1)} + 2C_{,(i}^{(1)}B_{j)}^{(1)} - B_k^{(1)}C_{(j,i)}^{k(1)} + \frac{B_{(j,i)}^{(2)}}{2}\right]. \tag{94}$$

The trace and the traceless part read

$$K^{(2)} = \pm\frac{1}{2a}[3(\mathcal{H}A^{(2)} + C^{(2)\prime}) + B_i^{,i(2)} + 2(3C^{(1)\prime} + B_j^{,j(1)})(2C^{(1)} - A^{(1)}) - 9\mathcal{H}(A^{(1)})^2 - 2C^{,i(1)}B_i^{(1)} + 3\mathcal{H}B_i^{(1)}B^{i(1)}$$
$$- 2C^{ij(1)}B_{j,i}^{(1)} - 2B^{i(1)}C_{ij}^{,j(1)} + C^{ij(1)}C_{ij}^{(1)\prime}],$$

$$\tilde{K}_{ij}^{(2)} = \pm a\left(-\frac{C_{ij}^{(2)\prime}}{4} + \frac{1}{2}B_{(i,j)}^{(2)} + \frac{1}{2}A^{(1)}C_{ij}^{(1)\prime} + \frac{1}{2}B^{k(1)}C_{ij,k}^{(1)} - C^{(1)\prime}C_{ij}^{(1)} - \frac{1}{3}\partial^2B^{(1)}C_{ij}^{(1)} - A^{(1)}B_{(i,j)}^{(1)} + 2C_{,(i}^{(1)}B_{j)}^{(1)} - B_{k,(i}^{(1)}C_{j)}^{k(1)}\right.$$
$$\left.- B_k^{(1)}C_{(i,j)}^{k(1)} + B_{,(i}^{k(1)}C_{j)k}^{(1)}\right) \mp a\frac{\delta_{ij}}{3}\left(\frac{B_k^{k(2)}}{2} - A^{(1)}B_k^{,k(1)} + 2C^{,k(1)}B_k^{(1)} - B^{k,l(1)}C_{kl}^{(1)} - B_k^{(1)}C_l^{k,l(1)} + \frac{1}{2}C^{kl(1)}C_{kl}^{(1)\prime}\right). \tag{95}$$

The remaining matching conditions are then

$$[\tilde{K}_{ij}^{(2)}]_-^+ = 0, \qquad [K^{(2)}]_-^+ = 0. \tag{96}$$

D. Matching conditions for second-order tensors from first-order scalars

1. Adiabatic matching

For adiabatic perturbations we match on a uniform-density hypersurface, $\delta\rho = 0$. Matching the induced metric on the transition surface then implies, from Eq. (91), that

$$[h_{ij}^{(2)}{}_{\mathrm{UD}}]_-^+ = 0. \tag{97}$$

Setting $E^{(1)} = 0$ and neglecting first-order vector and tensor perturbations, so that $C_{ij}^{(1)} = 0$, the matching condition (96) for the transverse and traceless part of the extrinsic curvature (95) gives

$$[h_{ij}^{(2)\prime}{}_{\mathrm{UD}}]_-^+ = -4[(A_{\mathrm{UD}}^{(1)}B_{\mathrm{UD},ij}^{(1)} + 2\zeta_{,(i}B_{\mathrm{UD},j)}^{(1)})]_-^{+TT}. \tag{98}$$

Continuity of ζ and $B_{\mathrm{UD}}^{(1)}$ across the matching hypersurface is enforced by the first-order scalar matching conditions (69) and (74). In addition, in the uniform-density gauge, from (73) we have

$$[A_{\mathrm{UD}}^{(1)}]_-^+ = \left[\frac{\zeta'}{\mathcal{H}}\right]_-^+. \tag{99}$$

Thus, for adiabatic perturbations on large scales, for which $\zeta' = 0$, we find

$$[h_{ij}^{(2)\prime}{}_{\mathrm{UD}}]_-^+ = 0. \tag{100}$$

Thus the tensor metric perturbation and its first derivative are continuous in the uniform-density gauge for adiabatic perturbations on large scales.

The second-order gauge transformation from the uniform-density gauge to the Poisson gauge yields

$$h^{(2)}_{ij\ \mathrm{UD}} = h^{(2)}_{ij\ P} - 2(B^{(1)}_{\mathrm{UD},(i}B^{(1)}_{\mathrm{UD},j)})^{TT}, \qquad (101)$$

$$h^{(2)'}_{ij\ \mathrm{UD}} = h^{(2)'}_{ij\ P} - 4(B^{(1)}_{\mathrm{UD},(i}B^{(1)'}_{\mathrm{UD},j)})^{TT}. \qquad (102)$$

For adiabatic perturbations on large scales, we have

$$B^{(1)'}_{\mathrm{UD}} = \zeta + 2\Psi. \qquad (103)$$

Thus, from Eqs. (101) and the continuity of $B^{(1)}_{\mathrm{UD}}$, ζ and Ψ across the matching surface, we have

$$[h^{(2)}_{ij\ P}]^{+}_{-} = 0, \qquad [h^{(2)'}_{ij\ P}]^{+}_{-} = 0. \qquad (104)$$

The tensor metric perturbation $h^{(2)}_{ij}$ and its time derivative are thus continuous across the matching surface in either uniform-density or Poisson gauge for adiabatic perturbations on large scales.

2. Nonadiabatic matching

In this subsection, we will consider the nonadiabatic case where the transition surface is determined by a fixed value of the field χ. Therefore, we match on a uniform-χ hypersurface which, for nonadiabatic perturbations, need not coincide with a uniform-density hypersurface.

Continuity of the induced metric on the uniform-χ hypersurface implies

$$[h^{(2)}_{ij\ \chi}]^{+}_{-} = 0. \qquad (105)$$

Because we choose $E^{(1)}_{\chi} = 0$, Eq. (87) implies that $B^{(1)}_{\chi}$ is continuous across the transition surface. This fact simplifies the matching conditions for the extrinsic curvature, Eqs. (95) and (96), to give

$$[h^{(2)'}_{ij\ \chi}]^{+}_{-} = -4[(A^{(1)}_{\chi}B^{(1)}_{\chi,ij})]^{+TT}_{-}. \qquad (106)$$

The second-order gauge transformation for the tensor metric perturbation from a general gauge to the uniform-χ gauge is given by Eq. (37), where from the first-order gauge transformation for a scalar (21) gives

$$\alpha^{(1)}_{\chi} = -\frac{\delta\chi^{(1)}}{\chi^{(0)}}. \qquad (107)$$

Thus, the matching condition (105) can be written in terms of the metric perturbations in an arbitrary gauge as

$$[h^{(2)}_{ij}]^{+}_{-} = 2[(2B^{(1)} - \alpha^{(1)}_{\chi})\alpha^{(1)}_{\chi,ij}]^{+TT}_{-}. \qquad (108)$$

Matching the derivative (106) in an arbitrary gauge we obtain

$$[h^{(2)'}_{ij}]^{+}_{-} = 4[B^{(1)'}\alpha^{(1)}_{\chi,ij} - (B^{(1)} - \alpha^{(1)}_{\chi})(A^{(1)}_{,ij} + \mathcal{H}\alpha^{(1)}_{\chi,ij})]^{+TT}_{-}. \qquad (109)$$

In the Poisson gauge $B^{(1)}_P = 0$, $A^{(1)}_P = \Psi$, and we have

$$[h^{(2)}_{ij\ P}]^{+}_{-} = -2[\alpha^{(1)P}_{\chi}\alpha^{(1)P}_{\chi,ij}]^{+TT}_{-}, \qquad (110)$$

$$[h^{(2)'}_{ij\ P}]^{+}_{-} = -4[\alpha^{(1)P}_{\chi}(\Psi_{,ij} + \mathcal{H}\alpha^{(1)P}_{\chi,ij})]^{+TT}_{-}, \qquad (111)$$

where

$$\alpha^{(1)P}_{\chi} = -\frac{\delta\chi^{(1)}_P}{\chi^{(0)'}} = \frac{\zeta_\chi + \Psi}{\mathcal{H}}. \qquad (112)$$

Equations (87), (89), and (62) show that ζ_χ, Ψ, and $\mathcal{H}$ are continuous, and thus $\alpha^{(1)P}_{\chi}$ is also continuous. Thus, Eqs. (110) and (111) show that the tensor perturbation, $h^{(2)}_{ij\ P}$, and its derivative, $h^{(2)'}_{ij\ P}$, are continuous in the Poisson gauge, even in the nonadiabatic matching case.

In the uniform-density gauge the situation is rather different. $B^{(1)}_{\mathrm{UD}}$ is given by Eq. (28) and $A^{(1)}_{\mathrm{UD}} = \zeta'/\mathcal{H}$ on large scales, and we have

$$\alpha^{(1)\mathrm{UD}}_{\chi} = -\frac{\delta\chi^{(1)}_{\mathrm{UD}}}{\chi^{(0)'}} = \frac{\zeta_\chi - \zeta}{\mathcal{H}}. \qquad (113)$$

Allowing for the fact that ζ_χ and Ψ are continuous, the matching conditions (108) and (109) reduce to

$$[h^{(2)}_{ij\ \mathrm{UD}}]^{+}_{-} = \frac{2}{\mathcal{H}^2}[(\zeta + 2\Psi)\zeta_{,ij}]^{+TT}_{-}, \qquad (114)$$

$$[h^{(2)'}_{ij\ \mathrm{UD}}]^{+}_{-} = \frac{4}{\mathcal{H}^2}[(\zeta' - \mathcal{H}\zeta - 2\mathcal{H}\Psi)(\zeta + \Psi)_{,ij}]^{+TT}_{-}. \qquad (115)$$

If ζ is not continuous across the transition then the tensor perturbation in the uniform-density gauge, and its time derivative, will not be continuous.

Assuming the evolution is piece-wise adiabatic (that is, $\zeta' = 0$ before and after the transition) and using Eq. (89) for the jump in ζ due to the nonadiabatic transition, we have

$$[h^{(2)}_{ij\ \mathrm{UD}}]^{+}_{-} = \frac{2}{\mathcal{H}^2}\left(\frac{w_+ - w_-}{1 + w_+}\right)\left(\left\{\frac{w_+ - w_-}{1 + w_+}\mathcal{S}_{\chi-} + \frac{4}{5 + 3w_-}\zeta_-\right\}\mathcal{S}_{\chi-,ij}\right)^{TT}, \qquad (116)$$

$$[h^{(2)'}_{ij\ \mathrm{UD}}]^{+}_{-} = -\frac{4}{\mathcal{H}}\left(\frac{w_+ - w_-}{1 + w_+}\right)\left(\left\{\frac{w_+ - w_-}{1 + w_+}\mathcal{S}_{\chi-} + \frac{1 - 3w_-}{5 + 3w_-}\zeta_-\right\}\mathcal{S}_{\chi-,ij}\right)^{TT}. \qquad (117)$$

V. GRAVITATIONAL WAVES FROM INFLATIONARY PERTURBATIONS ON LARGE SCALES

As an application of the matching conditions for tensor perturbations we will consider the generation of gravitational waves on super-Hubble scales after the end of inflation. During slow-roll inflation we have $\epsilon \ll 1$ where the slow-roll parameter is defined as $\epsilon \equiv -\dot{H}/H^2 = 3(1+w)/2$.

Using the inflationary solution to set the initial conditions for the tensor metric perturbation on super-Hubble scales we can use the solution (46) with the equation of state $w_- = -1 + (2/3)\epsilon$. During inflation the conformal time η decreases, and we neglect the decaying solutions proportional to Z_{ijP-} and W_-. Then the large-scale solution in the Poisson gauge, Eq. (46), is

$$h^{(2)}_{ij\ P-} = Y_{ijP-} - 4\epsilon\eta_-^2[\partial_i V_- \partial_j V_-]^{TT}. \tag{118}$$

The constant solution on large scales, Y_{ijP-}, is the usual free part of the gravitational wave solution whose amplitude is determined by the quantum vacuum on small scales, assuming inflation is sufficiently long lived. We are specifically interested in the production of second-order tensor perturbations from first-order scalar perturbations, ζ or Ψ, during inflation. Hence, we choose to set Y_{ijP-} equal to zero and study the solution

$$\begin{aligned} h^{(2)}_{ij\ P-} &= -\frac{4\epsilon}{\mathcal{H}^2}[\partial_i V_- \partial_j V_-]^{TT}, \\ h^{(2)\prime}_{ij\ P-} &= \frac{8\epsilon}{\mathcal{H}}[\partial_i V_- \partial_j V_-]^{TT}. \end{aligned} \tag{119}$$

The tensor metric perturbations produced by the first-order scalar metric perturbations during inflation are thus suppressed in the Poisson gauge, and vanish in the limit $\epsilon \longrightarrow 0$.

One can use the gauge transformation Eq. (51) during inflation to find the inflationary initial conditions in the uniform-density gauge from the Poisson gauge initial conditions as

$$\begin{aligned} h^{(2)}_{ij\ \mathrm{UD}-} &= -\frac{2}{\mathcal{H}^2}[\partial_i V_- \partial_j V_-]^{TT}, \\ h^{(2)\prime}_{ij\ \mathrm{UD}-} &= \frac{4}{\mathcal{H}}(1-\epsilon)[\partial_i V_- \partial_j V_-]^{TT}. \end{aligned} \tag{120}$$

We see that the tensor metric perturbations during inflation produced from first-order metric perturbations are not slow-roll suppressed in the uniform-density gauge, in contrast to the Poisson gauge result (119).

Equations (119) or (120) together with the matching conditions obtained in Sec. IV then set the initial conditions on large scales at the start of the radiation era that follows the inflationary period.

A. Adiabatic matching

First-order scalar metric perturbations can be calculated in any gauge, but the curvature perturbation in the uniform-density gauge, ζ, remains constant after inflation for adiabatic perturbations and on large scales [35], and hence we have $V_+ = V_-$. The curvature perturbation in the longitudinal gauge, Ψ, is also continuous for adiabatic perturbations, as shown by the matching conditions Eq. (78), but the continuity of ζ implies from Eq. (29) that Ψ' is discontinuous at a sudden change in the equation of state. Hence, Ψ becomes time dependent after inflation.

The adiabatic matching conditions for the scalar metric perturbations (78) determine the amplitude of the decaying mode (81) in a radiation era, with equation of state $w_+ = 1/3$, following slow-roll inflation

$$W_+ = \frac{2}{3}\left(1 - \frac{3}{2}\epsilon\right)\eta_{+*}^3 V_-. \tag{121}$$

During the radiation era the general solution for tensor metric perturbations in the Poisson gauge, Eq. (46), reduces to

$$\begin{aligned} h^{(2)}_{ij\ P+} = {} & Y_{ijP+} + Z_{ijP+}\eta_+^{-1} + \frac{8}{9}[\partial_i V_+ \partial_j V_+]^{TT}\eta_+^2 \\ & + 2[\partial_i W_+ \partial_j W_+]^{TT}\eta_+^{-4}. \end{aligned} \tag{122}$$

Using the adiabatic matching conditions for the second-order tensor perturbation in the Poisson gauge (104), which require that both the tensor perturbation and its first derivative are continuous at the end of inflation, we find that during the radiation era following inflation the integration constants are given by

$$\begin{aligned} Y_{ijP+} &= -4\epsilon\eta_{+*}^2[\partial_i V_- \partial_j V_-]^{TT}, \\ Z_{ijP+} &= -\frac{16}{9}\left(1 - \frac{3}{2}\epsilon\right)\eta_{+*}^3[\partial_i V_- \partial_j V_-]^{TT}. \end{aligned} \tag{123}$$

The general solution for the tensor perturbation in the uniform-density gauge (50) for $w_+ = 1/3$ becomes

$$h^{(2)}_{ij\ \mathrm{UD}+} = Y_{ij\mathrm{UD}+} + Z_{ij\mathrm{UD}+}\eta_+^{-1} + \frac{2}{3}[\partial_i V_+ \partial_j V_+]^{TT}\eta_+^2. \tag{124}$$

Either from the adiabatic matching conditions for the tensor perturbations in the uniform-density gauge (97) and (100), or from the gauge transformation Eqs. (52) of the Poisson gauge results (123), one can obtain the uniform-density integration constants as

$$\begin{aligned} Y_{ij\mathrm{UD}+} &= -4\epsilon\eta_{+*}^2[\partial_i V_- \partial_j V_-]^{TT}, \\ Z_{ij\mathrm{UD}+} &= -\frac{8}{3}\left(1 - \frac{3}{2}\epsilon\right)\eta_{+*}^3[\partial_i V_- \partial_j V_-]^{TT}. \end{aligned} \tag{125}$$

Thus, the full solution for the induced tensor perturbations, $h^{(2)}_{ij}$, on large scales during a radiation era after inflation, is given in the Poisson gauge (122) and uniform-density gauge (124) by

$$h^{(2)}_{ij\ P+} = \left[-4\epsilon\eta^2_{+*} + \frac{8}{9}\eta^2_+ - \frac{16}{9}\left(1-\frac{3}{2}\epsilon\right)\eta^3_{+*}\eta^{-1}_+ + \frac{8}{9}(1-3\epsilon)\eta^6_{+*}\eta^{-4}_+\right][\partial_i V_- \partial_j V_-]^{TT}, \quad (126)$$

$$h^{(2)}_{ij\ \mathrm{UD}+} = \left[-4\epsilon\eta^2_{+*} + \frac{2}{3}\eta^2_+ - \frac{8}{3}\left(1-\frac{3}{2}\epsilon\right)\eta^3_{+*}\eta^{-1}_+\right] \times [\partial_i V_- \partial_j V_-]^{TT}. \quad (127)$$

We see that although the time-dependent parts of the tensor perturbation in the two gauges are different, the constant mode after inflation in both gauges is the same and it is slow-roll suppressed, $Y_{ij\,P+} = Y_{ij\,\mathrm{UD}+} = \mathcal{O}(\epsilon)$, even in the uniform-density gauge where the tensor perturbation is not suppressed during slow-roll inflation.

B. Nonadiabatic matching

In the presence of nonadiabatic perturbations the scalar curvature perturbation ζ needs no longer be constant on large scales and may be discontinuous, as shown in Eq. (89). As a simple model of nonadiabatic perturbations we consider the case where the transition from slow-roll inflation to radiation occurs on a uniform-χ hypersurface with curvature $\zeta_\chi \neq \zeta_-$. From the nonadiabatic matching condition for scalar metric perturbations, Eq. (90), we obtain the amplitude of the constant mode and the decaying mode of Ψ in the radiation era following inflation

$$V_+ = \left(1-\frac{\epsilon}{2}\right)\zeta_{\chi-} + \frac{\epsilon}{2}V_-, \quad (128)$$

$$W_+ = \frac{1}{3}((2-\epsilon)\zeta_{\chi-} - 2\epsilon V_-)\eta^3_{+*}. \quad (129)$$

For $\zeta_\chi = V_-$ we recover the adiabatic case (121).

The tensor perturbation in the Poisson gauge, and its time derivative remain continuous for a nonadiabatic transition. Matching the inflationary solution (119) to the radiation era solution (122) with the integration constant V_+ and Z_+ given by Eqs. (128) and (129) determines the two remaining integration constants in the Poisson gauge

$$Y_{ij\,P+} = 4\epsilon\eta^2_{+*}[\partial_i V_- \partial_j V_- - 2\partial_i\zeta_{\chi-}\partial_j V_-]^{TT}, \quad (130)$$

$$Z_{ij\,P+} = -8\eta^3_{+*}\left[\epsilon\partial_i V_- \partial_j V_- + \frac{2}{9}(1-\epsilon)\partial_i\zeta_{\chi-}\partial_j\zeta_{\chi-} - \frac{10}{9}\epsilon\partial_i\zeta_{\chi-}\partial_j V_-\right]^{TT}. \quad (131)$$

The tensor metric perturbation in the uniform-density gauge can be discontinuous, with the jump in h_{ij} and its time derivative being given by (116) and (117), where the incoming nonadiabatic perturbation is given by

$$S_{\chi-} = \zeta_{\chi-} - V_-. \quad (132)$$

As a result, matching the inflationary solution (120) to the radiation era solution (124) with the integration constant V_+ and Z_+ given by Eqs. (128) and (129) gives the two remaining integration constants

$$Y_{ij\,UD+} = Y_{ijP+}, \quad (133)$$

$$Z_{ij\,\mathrm{UD}+} = -8\eta^3_{+*}\left[\epsilon\partial_i V_- \partial_j V_- + \frac{1}{3}(1-\epsilon)\partial_i\zeta_{\chi-}\partial_j\zeta_{\chi-} - \frac{7}{6}\epsilon\partial_i\zeta_{\chi-}\partial_j V_-\right]^{TT}. \quad (134)$$

Therefore, the full solution for the induced tensor perturbations, $h^{(2)}_{ij}$, on large scales during a radiation era after inflation in this nonadiabatic model for the transition, is given in either the Poisson gauge (122) and uniform-density gauge (124) by

$$h^{(2)}_{ij\ P+} = \frac{8}{9}\eta^2_+[(1-\epsilon)\partial_i\zeta_{\chi-}\partial_j\zeta_{\chi-} + \epsilon\partial_i\zeta_{\chi-}\partial_j V_-]^{TT} + 4\epsilon\eta^2_{+*}[\partial_i V_- \partial_j V_- - 2\partial_i\zeta_{\chi-}\partial_j V_-]^{TT} - 8\eta^3_{+*}\eta^{-1}_+\left[\epsilon\partial_i V_- \partial_j V_- + \frac{2}{9}(1-\epsilon)\partial_i\zeta_{\chi-}\partial_j\zeta_{\chi-} - \frac{10}{9}\epsilon\partial_i\zeta_{\chi-}\partial_j V_-\right]^{TT} + \frac{8}{9}\eta^6_{+*}\eta^{-4}_+[(1-\epsilon)\partial_i\zeta_{\chi-}\partial_j\zeta_{\chi-} - 2\epsilon\partial_i\zeta_{\chi-}\partial_j V_-]^{TT}, \quad (135)$$

$$h^{(2)}_{ij\ \mathrm{UD}+} = \frac{2}{3}\eta^2_+[(1-\epsilon)\partial_i\zeta_{\chi-}\partial_j\zeta_{\chi-} + \epsilon\partial_i\zeta_{\chi-}\partial_j V_-]^{TT} + 4\epsilon\eta^2_{+*}[\partial_i V_- \partial_j V_- - 2\partial_i\zeta_{\chi-}\partial_j V_-]^{TT} - 8\eta^3_{+*}\eta^{-1}_+\left[\epsilon\partial_i V_- \partial_j V_- + \frac{1}{3}(1-\epsilon)\partial_i\zeta_{\chi-}\partial_j\zeta_{\chi-} - \frac{7}{6}\epsilon\partial_i\zeta_{\chi-}\partial_j V_-\right]^{TT}. \quad (136)$$

In the adiabatic case where $\zeta_{\chi-} = V_-$ and hence $S_{\chi-} = 0$, this reduces to Eqs. (126) and (127). The presence of nonadiabatic perturbations changes the amplitude of the decaying mode, proportional to Z_{ij}, and the amplitude of the growing mode proportional to η^2_+. But the free part of the gravitational field, which is gauge-independent, remains constant on super-Hubble scales and remains suppressed in the slow-roll limit

$$Y_{ij\,\mathrm{UD}+} = Y_{ijP+} = \mathcal{O}(\epsilon). \qquad (137)$$

VI. CONCLUSION

Tensor perturbations of the Friedman-Robertson-Walker metric become gauge dependent at second and higher order [19]. It is however possible to construct gauge-invariant tensor perturbations by eliminating the gauge degrees of freedom [24]. As examples we have given gauge-invariant definitions of the tensor perturbation at second order in the Poisson gauge and the uniform-density gauge, and the (gauge-invariant) difference between the two. All of this should be familiar from the study of gauge-dependent scalar perturbations which are already gauge dependent at first order, and one can choose to work in terms of the curvature perturbation in the Poisson (or longitudinal) gauge, Ψ, or in the uniform-density gauge, ζ, or both.

We have focussed on the question of matching conditions for the tensor perturbation. At first order, the Israel junction conditions require that the tensor metric perturbation and its time derivative are always continuous across a spacelike hypersurface in any gauge. But at second order we need to consider the behavior in different gauges. For adiabatic perturbations the matching hypersurface should coincide with a uniform-density hypersurface. The Israel junction conditions then imply the tensor perturbation in the uniform-density gauge, and its time derivative, will be continuous. For the case of nonadiabatic matching the hypersurface need not coincide with a uniform-density surface and so the tensor perturbation in the uniform-density gauge may be discontinuous. However, in both cases (adiabatic or nonadiabatic) we show that the second-order tensor perturbation in the Poisson gauge, and its time-derivative, are continuous.

As an application we consider the generation of gravitational waves during inflation. We give expressions for the general solution for the tensor perturbation on large (super-Hubble) scales in both the Poisson and uniform-density gauges when the barotropic index, w is constant. This is a good description of the evolution during the radiation dominated era, when $w = 1/3$, and during a preceding era of slow-roll inflation, when $w = -1 + (2/3)\epsilon$ and $\epsilon \ll 1$ is a slow-roll parameter. It has previously been shown [26,27] that the tensor metric perturbation on large-scales *during* inflation is slow-roll suppressed in the Poisson gauge, but not suppressed in the uniform-density gauge. This reflects the fact that the scalar metric perturbations are slow-roll suppressed during inflation in the Poisson gauge with respect to those in the uniform-density gauge. The question is then, whether the amplitude of gravitational waves *after* inflation is slow-roll suppressed, or indeed whether large tensor perturbation can be generated at second order by the transition to a radiation dominated universe.

Continuity of the tensor perturbation and its first derivative is sufficient to show that the tensor perturbation remains slow-roll suppressed after inflation in the Poisson gauge. In the uniform-density gauge tensor perturbations are not slow-roll suppressed, reflecting the gauge dependence of the second-order tensor perturbations. However, the constant mode on large scales, Y_{ij}, corresponding to the free part of the tensor perturbations is the same in the Poisson or uniform-density gauge in the radiation era. It is this constant mode that describes the gravitational waves produced by inflation on large scales. In the case of nonadiabatic perturbations this amplitude can change in the uniform-density gauge due to nonadiabatic matching condition, but even in this case its amplitude vanishes in the slow-roll limit $\epsilon \to 0$.

In addition to the constant mode there is a growing mode proportional to η^2 on large scales in the radiation era. This is not the free part of the gravitational field, i.e., it vanishes in the absence of first-order scalar perturbations. But, like the scalar metric perturbations during the radiation era, it is not slow-roll suppressed in either Poisson or uniform-density gauge. This becomes the dominant term in the tensor perturbations on large scales and corresponds to the production of gravitational waves from density perturbations in the radiation dominated era. Ananda *et al.* [8] showed that the amplitude of tensor perturbations reaches a maximum at Hubble-crossing, $k\eta \approx 1$ (where our large-scale approximation breaks down), and thereafter follows the usual behavior for free gravitational waves, $|h| \propto a^{-1}$, as the scalar metric perturbations oscillate and decay on sub-Hubble scales, $k\eta \gg 1$. The first-order scalar perturbation, Φ, decays rapidly on sub-Hubble scales leaving an unambiguous prediction for the amplitude of gravitational waves.

Thus, we conclude that the power spectrum of the stochastic gravitational wave background generated from scalar metric perturbations from inflation is due primarily to the production of tensor perturbations around Hubble entry during the radiation era, and the second-order gravitational waves from inflation are suppressed in all models of slow-roll inflation.

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APPENDIX: INVERSE METRIC UP TO SECOND ORDER

From $g_{\mu\lambda}g^{\lambda\nu} = \delta_\mu^\nu$ we can obtain the inverse metric order by order. At zeroth order we get

$$g^{00(0)} = -a^{-2}, \qquad g^{0i(0)} = 0, \qquad g^{ij(0)} = a^{-2}\delta^{ij}. \tag{A1}$$

At first order, it gives

$$g^{00(1)} = 2a^{-2}A^{(1)}, \qquad g^{0i(1)} = a^{-2}B^{i(1)}, \qquad g^{ij(1)} = a^{-2}(2C^{(1)}\delta^{ij} - C^{ij(1)}). \tag{A2}$$

The second-order components of the inverse metric are

$$g^{00(2)} = a^{-2}[B_i^{(1)}B^{i(1)} - 4(A^{(1)})^2 + A^{(2)}], \tag{A3}$$

$$g^{0i(2)} = a^{-2}\left[2(C^{(1)} - A^{(1)})B^{i(1)} - B_j^{(1)}C^{ij(1)} + \frac{B^{i(2)}}{2}\right], \tag{A4}$$

$$g^{ij(2)} = -a^{-2}\left[B^{i(1)}B^{j(1)} - 4(C^{(1)})^2\delta^{ij} + 4C^{(1)}C^{ij(1)} - C_k^{i(1)}C^{kj(1)} - C^{(2)}\delta^{ij} + \frac{1}{2}C^{ij(2)}\right], \tag{A5}$$

where the indices were raised with δ^{ij}.

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