

Relativistic stars in $f(R)$ gravity

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We study the strong gravity regime in viable models of so-called $f(R)$ gravity that account for the observed cosmic acceleration. In contrast with recent works suggesting that very relativistic stars might not exist in these models, we find numerical solutions corresponding to static star configurations with a strong gravitational field. The choice of the equation of state for the star is crucial for the existence of solutions. Indeed, if the pressure exceeds 3 times the energy density in a large part of the star, static configurations do not exist. In our analysis, we use a polytropic equation of state, which is not plagued with this problem and, moreover, provides a better approximation for a realistic neutron star.

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One of the most challenging tasks for cosmology and fundamental physics today is to try to understand the apparent acceleration of the Universe. Among the many models which have been proposed, a class which has attracted a lot of attention is the so-called $f(R)$ gravity theories where the standard Einstein-Hilbert gravitational Lagrangian, proportional to the scalar curvature R , is replaced by a function of R while the matter part of the Lagrangian is left unchanged (see, e.g., [1] for a recent review). After many detours, it has been realized that viable $f(R)$ theories must satisfy stringent conditions in order to avoid instabilities and to be compatible with the present laboratory and astrophysical constraints. A few models [2–4] have been carefully constructed to meet these requirements, using, in particular, the chameleon mechanism [5] to satisfy the solar system constraints (the binary pulsar constraints are, so far, weaker for these models [6]).

However, recent works [7,8] studying the strong gravity limit of these models have questioned their viability by suggesting that, inside neutron stars, where the effects of general relativity are strong, one would easily reach the singularity where R becomes infinite. In particular, in [8], this impression seemed reinforced by the impossibility to construct numerically relativistic stars beyond some critical value of the gravitational field.

In this paper, we reexamine this question and, unlike these previous works, find that highly relativistic stars can be obtained numerically. We also give qualitative arguments to understand the existence of these solutions. An important conclusion of our work is that the equation of state must satisfy $\rho - 3P > 0$ in most of the star. Otherwise, tachyonic instabilities associated with a negative effective squared mass develop and prevent the existence of a static star configuration. This problem affects, in particular, the highly relativistic constant energy density stars. This is why we have used a polytropic equation of state, which has the additional advantage to be a better approximation to a realistic neutron star.

Our starting point is the action

$$S = \frac{M_P^2}{2} \int d^4x \sqrt{-\tilde{g}} f(\tilde{R}) + S_m[\Phi_m; \tilde{g}_{\mu\nu}], \quad (1)$$

where S_m is the action for the matter, which is minimally coupled to the so-called Jordan metric $\tilde{g}_{\mu\nu}$; \tilde{R} is the scalar curvature associated with $\tilde{g}_{\mu\nu}$ and we have defined $M_P^{-2} \equiv 8\pi G$. In contrast with [7,8], we reexpress the model as a scalar-tensor theory in the so-called Einstein frame. The two formulations are of course equivalent (at least on the classical level), but the Einstein frame is useful to see that the behavior of $f(R)$ theories is quite analogous to chameleonlike scalar-tensor theories.

By introducing the scalar field $\phi = \sqrt{\frac{3}{2}} M_P \ln f_{,\tilde{R}}$ and the metric $g_{\mu\nu} = \Omega^{-2} \tilde{g}_{\mu\nu}$, with $\Omega^{-2} = f_{,\tilde{R}} = \exp[\sqrt{\frac{2}{3}} \phi / M_P] \equiv \exp[-2Q\phi / M_P]$, the action (1) becomes

$$S = \int d^4x \sqrt{-g} \left[\frac{M_P^2}{2} R - \frac{1}{2} (\nabla\phi)^2 - V(\phi) \right] + S_m[\Phi_m; \Omega^2 g_{\mu\nu}], \quad (2)$$

with the potential

$$V = M_P^2 \frac{\tilde{R} f_{,\tilde{R}} - f}{2f_{,\tilde{R}}^2},$$

which can be expressed in terms of ϕ by inverting the definition of ϕ as a function of \tilde{R} .

The first and second derivatives of the potential V in terms of ϕ are

$$\begin{aligned} \frac{dV}{d\phi} &= \sqrt{\frac{2}{3}} M_P \frac{2f - \tilde{R} f_{,\tilde{R}}}{2f_{,\tilde{R}}^2}, \\ \frac{d^2V}{d\phi^2} &= \frac{1}{3f_{,\tilde{R}}\tilde{R}} \left[1 + \frac{\tilde{R} f_{,\tilde{R}\tilde{R}}}{f_{,\tilde{R}}} - \frac{4f f_{,\tilde{R}\tilde{R}}}{f_{,\tilde{R}}^2} \right]. \end{aligned} \quad (3)$$

The function $f(R)$ is severely constrained by observations. Here we will consider the model suggested by Starobinsky [4],

$$f(\tilde{R}) = R_0[x - \lambda(1 - (1 + x^2)^{-n})], \quad x \equiv \frac{\tilde{R}}{R_0}. \quad (4)$$

For this model the potential $V(\phi)$ is shown in Fig. 1. Substituting this expression into (3) yields the minimum of V corresponding to the asymptotic de Sitter solution $\tilde{R}_\infty \equiv x_\infty R_0$. It is convenient to express the parameter λ in terms of x_∞ :

$$\lambda = \frac{x_\infty(1 + x_\infty^2)^{n+1}}{2[(1 + x_\infty^2)^{n+1} - 1 - (n+1)x_\infty^2]}.$$

In the strong gravity limit, corresponding to $x \gg 1$, we have

$$f(\tilde{R}) \approx R_0\left(x - \lambda + \frac{\lambda}{x^{2n}}\right), \quad \frac{\phi}{M_P} \approx -\frac{\sqrt{6}\lambda n}{x^{2n+1}}. \quad (5)$$

The curvature singularity thus corresponds to a finite value $\phi = 0$ for the scalar field, as well as a finite value for the potential $V = (\lambda/2)M_P^2 R_0$. However, the derivative $dV/d\phi$ becomes infinite in the limit $\phi \rightarrow 0^-$.

For matter we consider a perfect fluid characterized by the energy density $\tilde{\rho}$ and pressure \tilde{P} , defined in the Jordan frame (i.e., with respect to the metric $\tilde{g}_{\mu\nu}$). In the Einstein frame, the corresponding energy density and pressure are respectively $\rho = \Omega^4 \tilde{\rho}$ and $P = \Omega^4 \tilde{P}$. Inside the star, ϕ will be extremely close to zero and therefore, the quantities defined in the two frames will be numerically indistinguishable.

We consider a static and spherically symmetric metric (in the Einstein frame)

$$ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Introducing the radial function $m(r)$ so that $e^{-\lambda} = 1 - 2m/r$, the time and radial components of Einstein's equations yield

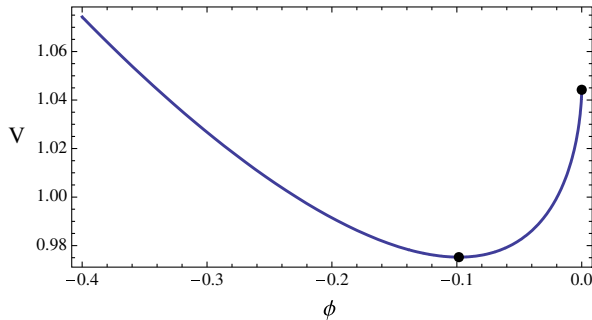


FIG. 1 (color online). Potential V (in units of $M_P^2 R_0$) as a function of ϕ (in Planck units) for $n = 1$ and $x_\infty = 3.6$. The lower black dot corresponds to the de Sitter attractor while the upper-right dot shows the curvature singularity.

$$m' = \frac{r^2}{2M_P^2} \left[\Omega^4 \tilde{\rho} + \frac{1}{2} e^{-\lambda} \phi'^2 + V(\phi) \right], \quad (6)$$

$$\nu' = e^\lambda \left[\frac{2m}{r^2} + \frac{r}{M_P^2} \left(\frac{1}{2} e^{-\lambda} \phi'^2 - V(\phi) \right) + \frac{r\Omega^4 \tilde{P}}{M_P^2} \right], \quad (7)$$

where a prime denotes a derivative with respect to the radial coordinate, and one recognizes the total energy density in the brackets on the right-hand side of the first equation. Instead of the angular component of Einstein's equations, one can use the conservation of the energy-momentum conservation, $\tilde{\nabla}_\mu \tilde{T}^\mu{}_\nu = 0$, which reads

$$\tilde{P}' = -\frac{1}{2}(\tilde{\rho} + \tilde{P})\left(\nu' + 2\frac{\Omega_\phi}{\Omega}\phi'\right), \quad (8)$$

where $\Omega_\phi \equiv d\Omega/d\phi$. The Klein-Gordon equation for the scalar field reads,

$$\phi'' + \left(\frac{2}{r} + \frac{1}{2}(\nu' - \lambda')\right)\phi' = e^\lambda \left[\frac{dV}{d\phi} + \Omega^3 \Omega_\phi (\tilde{\rho} - 3\tilde{P}) \right]. \quad (9)$$

In order to close the system of Eqs. (6)–(9), one should finally specify the equation of state for the matter. In [8], relativistic stars with constant energy density were considered. The advantage is that the pressure can be easily determined analytically in general relativity (i.e., ignoring the backreaction of the scalar field) and is given by

$$P(r) = \rho_0 \frac{(1 - \frac{2GM}{r_*})^{1/2} - (1 - \frac{2GM r^2}{r_*^3})^{1/2}}{(1 - \frac{2GM r^2}{r_*^3})^{1/2} - 3(1 - \frac{2GM}{r_*})^{1/2}},$$

where M is the mass of the star and r_* its radius. However, this implies that, in the innermost regions of the star, the quantity $\rho - 3P$, i.e., the opposite of the trace of the energy-momentum tensor, becomes negative as soon as

$$-\Phi_* \equiv \frac{GM}{r_*} > \frac{5}{18}.$$

In stars with a large region where $\tilde{\rho} - 3\tilde{P} < 0$, we were unable to find numerically solutions for the scalar field. This is not related to the presence of the curvature singularity, since it occurs in chameleon type models as well [9]. The reason is the presence of instabilities, because the matter contribution to the effective squared mass is proportional to $\tilde{\rho} - 3\tilde{P}$ and thus becomes negative. Note that a similar effect has been observed in [10] for another class of scalar-tensor theories, where $\ln\Omega$ depends quadratically on the scalar field.

The equation of state deep inside a neutron star is not known but most realistic equations of state satisfy the condition $\tilde{\rho} - 3\tilde{P} > 0$. This is why we have chosen for the present analysis a polytropic equation of state, given by

$$\tilde{\rho}(\tilde{n}) = m_B \left(\tilde{n} + K \frac{\tilde{n}^2}{n_0} \right), \quad \tilde{P}(\tilde{n}) = K m_B \frac{\tilde{n}^2}{n_0}, \quad (10)$$

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with $m_B = 1.66 \times 10^{-27}$ kg, $n_0 = 0.1 \text{ fm}^{-1}$, and $K = 0.1$. In Fig. 2, the radial profile of the energy density and pressure, together with $\tilde{\rho} - 3\tilde{P}$, is plotted for a star with the central particle number density $\tilde{n}_c = 0.4 \text{ fm}^{-3}$. This corresponds to $|\Phi_*| \approx 0.25$, which is more than twice higher than the value reached in [8].

We have integrated numerically the system of Eqs. (6)–(9), together with (10), to obtain the profile of the scalar field inside and outside the star in the $f(R)$ model (4) with $n = 1$. A key parameter is the ratio between the energy density at infinity (i.e., the cosmological energy density) and the energy density at the center of the star, which can be parametrized by $\nu_0 = M_p^2 R_0 / \tilde{\rho}_c$. Realistic values of this parameter are extremely small and are numerically challenging to explore, because the scalar field value at the center is proportional to ν_0^3 , as we will see below. The smallest value we considered is $\nu_0 = 10^{-4}$, which is much higher than the realistic one. However we believe that the situation is qualitatively similar for smaller values. Instead of the shooting method used in [8], we have resorted to a relaxation method.

As is clear from Fig. 3, the scalar field tends to interpolate between an extremely high density regime, inside the star, and a very low density regime, outside the star. In the very high density regime, the scalar field is, numerically, very close to the singularity. This behavior is quite analogous to that of the “chameleon” mechanism [5], as we will discuss in more details in [9].

In order to understand intuitively our results, it is useful to introduce the effective potential

$$V_{\text{eff}} = V + \frac{1}{4}\Omega^4(\tilde{\rho} - 3\tilde{P}), \quad (11)$$

which is obtained by integrating, with respect to ϕ , the terms between brackets on the right-hand side of the Klein-Gordon Eq. (9) (and we have set the constant of integration to zero). The local minimum of the effective potential, which we will denote $\phi_{\text{min}}(r)$, is characterized, if it exists,

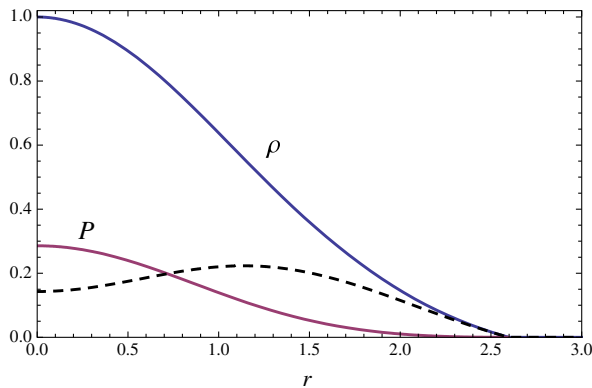


FIG. 2 (color online). Energy density $\tilde{\rho}$ (upper solid line), pressure \tilde{P} (lower solid line) and the combination $\tilde{\rho} - 3\tilde{P}$ (black dotted line), in units of the central density ρ_c , as functions of the radial coordinate r (in units of $M_p \tilde{\rho}_c^{-1/2}$).

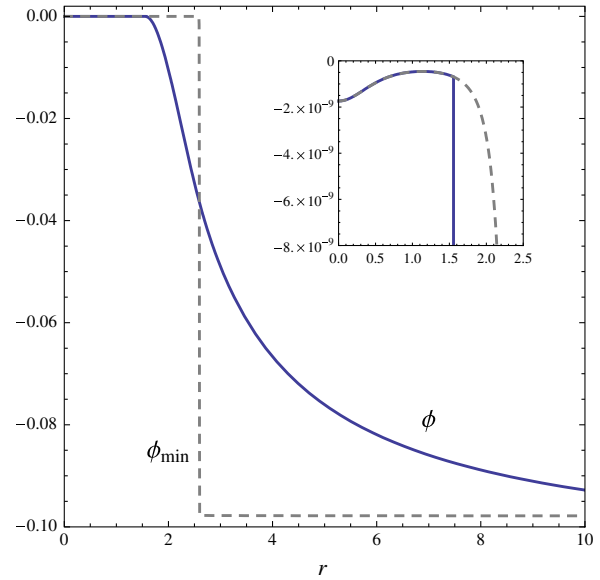
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FIG. 3 (color online). Profile of the scalar field ϕ (in Planck units), shown by the solid (blue) line, as a function of the radius (in units of $M_p \tilde{\rho}_c^{-1/2}$), for the model (4) with $n = 1$, $x_\infty = 3.6$, and $\nu_0 = 10^{-4}$. The value ϕ_{min} for the minimum of the effective potential is plotted by the dashed (gray) line.

by

$$\frac{dV_{\text{eff}}}{d\phi} = \frac{dV}{d\phi} + \frac{Q}{M_p} e^{4Q\phi/M_p} (\tilde{\rho} - 3\tilde{P}) = 0.$$

At high curvature ($x \gg 1$), $dV/d\phi \approx M_p R_0 x / \sqrt{6}$, according to (5), and the minimum corresponds to

$$x_{\text{min}} \approx \frac{\tilde{\rho} - 3\tilde{P}}{M_p^2 R_0},$$

which is inversely proportional to ν_0 (the scalar field value ϕ_{min} is thus proportional to ν_0^3). This minimum exists only if the matter term $\tilde{\rho} - 3\tilde{P}$ is positive. This implies that, for a very compact constant density star, the effective potential in the central layers of the star does not have any minimum.

One can also define a corresponding effective squared mass as

$$m_{\text{eff}}^2 \equiv \frac{d^2V}{d\phi^2} + 4 \frac{Q^2}{M_p^2} e^{4Q\phi/M_p} (\tilde{\rho} - 3\tilde{P}), \quad (12)$$

which can become negative, as discussed earlier, if the second term dominates and $\tilde{\rho} - 3\tilde{P} < 0$. In the regime $x \gg 1$, if the minimum (12) exists, one finds by evaluating (12) at the minimum, using (5),

$$m_{\text{eff}}^2 \approx \frac{R_0}{6\lambda n(2n+1)} x_{\text{min}}^{2n+2}.$$

In view of our results, which seem contradictory with the conclusions of [7,8], it is worth reexamining in detail their arguments against the existence of very relativistic stars. Frolov’s argument is based on the nonrelativistic limit of

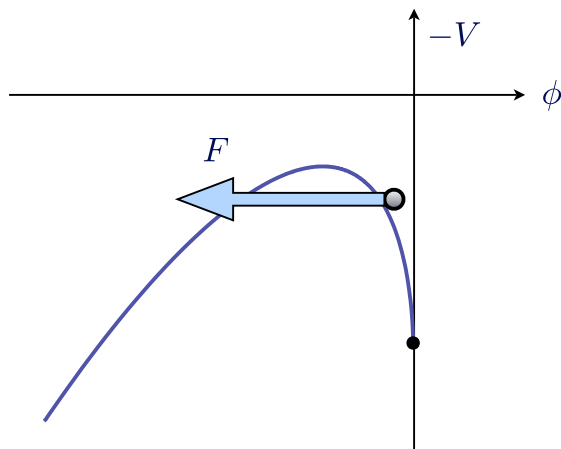


FIG. 4 (color online). Effective potential $U(\phi) = -V(\phi)$ in which the scalar field ϕ “moves” under the action of the force \mathcal{F} , in the picture of the classical mechanics analogy.

the Klein-Gordon equation which can be written in the form

$$\Delta\psi = -\frac{8\pi G}{3}\tilde{\rho} - 2\frac{Q}{M_p}\frac{dV}{d\phi},$$

for $\psi = f_{\tilde{r}} - 1 \approx -2Q\phi/M_p$. If one neglects the second term on the right-hand side, ψ then satisfies an equation analogous to the Poisson equation $\Delta\Phi = 4\pi G\rho$ and is thus related to the gravitational potential Φ by $\psi = \psi_\infty - 2\Phi/3$. Since ψ_∞ is small, this suggests that the curvature singularity can be easily reached with a sufficiently strong gravitational potential. However, inside the star, the energy density is much higher than at infinity and this corresponds precisely to the high curvature regime, where $dV/d\phi$ becomes very high and cannot be neglected.

One can also understand the scalar field profile, as in [8], in terms of a kinematic analogy with the motion of a particle in a potential $U(\phi)$ which is submitted to a force $\mathcal{F} = -(8\pi G/3)(\tilde{\rho} - 3\tilde{P})$, where the radius r is considered to be a “fictive” time parameter (see Fig. 4). The radial profile of the scalar field can then be interpreted as the “motion” of the particle from $r = 0$ to $r = \infty$. At $r = 0$, the particle is at rest since $\phi' = 0$, but is submitted to a leftward force that makes it roll uphill. The initial value for ϕ must be such that the scalar field reaches the top of the potential with vanishing velocity. We find a stronger gravitational potential in more compact stars with high energy density. The amplitude of \mathcal{F} then tends to become stronger, but there does not seem to be any critical value beyond which the particle would necessarily fall into the singularity or overshoot the top of the hill. As the density of the star increases, the minimum ϕ_{\min} of the effective potential

becomes closer to the singularity $\phi = 0$, according to (12). One thus expects that the initial value of ϕ will also be closer and closer to the singularity $\phi = 0$. This is confirmed by our numerical analysis, even if it becomes more and more difficult numerically to distinguish the extremely small value of ϕ from zero.

Moreover, as the numerical solution plotted in Fig. 3 shows explicitly, the motion of the particle can be subtle. In this case, it first starts to approach the singularity, it then stops and moves in the opposite direction towards the top of the potential. This complicated motion is a direct consequence of the nonmonotonous dependence of $\tilde{\rho} - 3\tilde{P}$, which reaches its maximum value at some nonzero radius of the star, as can be seen in Fig. 2.

To conclude, our work shows that it is possible to construct highly relativistic stars in $f(R)$ theories, despite recent indications to the contrary. Numerically, the task can be challenging as the scalar field value is extremely close to the singularity in the center of the star. We have also studied relativistic stars in chameleonlike models and found a very similar behavior [9]. In both cases, a crucial requirement for obtaining a static configuration is that the equation of state satisfies the condition $\tilde{\rho} - 3\tilde{P} > 0$ in most of the star. Otherwise, the effective squared mass of the scalar field becomes negative and the associated instabilities prevent the construction of a static configuration. This is problematic for constant energy density stars, which have been used in several recent works because of their analytical simplicity, since $\tilde{\rho} - 3\tilde{P}$ becomes negative in the central part of very massive stars of this type. This might explain why, beyond some critical value for the gravitational field, no configuration was found in [11] for chameleonlike models. Realistic neutron stars do not, however, suffer from this problem. Moreover, once the scalar field configuration is stable, the (hydrodynamical) stability of the stars should be very similar to that of their general relativistic counterparts because the backreaction of the scalar field on the star is extremely small. An open question, which is far beyond the scope of this work, is whether the static configurations described here can be reached dynamically during the collapse of a very massive star into a neutron star. It would also be interesting to study whether and how black holes would form during the collapse of matter, i.e., such has been done for instance in [12] in the context of scalar-tensor theories without potential.

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