Information on the structure of the rho meson from the pion form factor

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The electromagnetic pion form factor is calculated in a Bethe-Salpeter approach which accounts for pion rescattering. In the scattering kernel the pion-pion contact interaction from lowest-order chiral perturbation theory is considered together with an optional vector meson in the s channel. Correspondingly the virtual photon couples to a two-pion state and optionally to the vector meson. It is shown that for reasonable ranges of input parameters the experimentally observed pion form factor cannot be described by the iterated pion-pion contact interaction alone, i.e. without an elementary vector meson. The inclusion of an elementary vector meson allows for an excellent description. This completes a recent study ("Information on the structure of the a_1 from tau decay") where it has been shown that the a_1 meson can be well understood as a rescattering process of the ρ meson and the pion. Here it is demonstrated that within the same formalism the ρ meson cannot be understood as a pion-pion rescattering process. This suggests that the chiral partners a_1 and ρ are not only different in mass, but also different in nature.

I. INTRODUCTION AND SUMMARY

The ρ meson and the a_1 meson are chiral partners in the following sense: As deduced, e.g., from tau decays [1], the ρ/a_1 meson couples strongly to the isovector current of a vector/axial-vector type. On the other hand, vector and axial-vector currents are related by chiral transformations. The fact that the masses of the ρ and a_1 mesons are very different is explained by the spontaneous breaking of chiral symmetry. According to common lore the ρ and the a_1 meson are both quark-antiquark states. In this quark-model picture the two mesons have the same nature, but differ in mass due to chiral symmetry breaking.

Recently, this picture has been cast into doubt. As has been shown in [2] the spectrum of lowest-lying axial-vector mesons—including the a_1 meson—can be understood as being dynamically generated from coupled-channel scattering processes of Goldstone bosons on vector mesons. Concerning the a_1 meson this has been reconfirmed in a technically somewhat different approach in [3]. In these works [2,3] axial-vector meson masses have been deduced from the scattering amplitudes. The latter, however, are not observable quantities since the vector mesons are not asymptotic states.[1]

An analysis of a real observable, namely, the semihadronic tau decay into three pions, has been studied in [4,5]. In this decay process the a_1 meson appears as a prominent bump in the spectrum. As demonstrated in [4,5] this bump can be well explained by a π-ρ rescattering process without the need for an elementary (quark-antiquark) axial-

vector meson. This immediately raises the question whether also the ρ meson, as the chiral partner of the a_1 meson, can be seen as a meson-meson rescattering process without the need for an elementary (quark-antiquark) vector meson. In principle, such a question has already been addressed in [6]. Based on the N/D method, it has been shown that pion-pion rescattering without an elementary vector meson is not sufficient to describe the elastic pion-scattering phase shifts.

In the present paper we will add three aspects to the analysis of the ρ meson. First, to obtain a consistent picture with our a_1 analysis [4,5], we will use a Bethe-Salpeter approach, i.e. the very same method which explained the a_1 bump in the tau decay data by meson-meson rescattering without an elementary axial-vector meson [4,5]. Second, we do not concentrate on the pion-scattering phase shift but rather on the electromagnetic pion form factor as this probes the isovector-vector channel just like the tau decay into three pions probes the isovector–axial-vector channel. As an intermediate step we will also obtain the elastic pion-scattering phase shift for the p-wave isovector channel.

We will study here two scenarios to describe the pion form factor: one with and one without an elementary vector meson. In both cases we take into account the pion-pion and pion-photon interactions from lowest-order chiral perturbation theory. It is an important aspect that these interactions are fixed model independently by chiral [2,3,6] or charge symmetry, respectively. In full qualitative agreement with [6] we will find that an elementary vector meson is needed to describe the pion form factor. Thus, together with our previous analysis of the a_1 meson [4,5] it is suggestive that ρ and a_1 are not only different in mass, but also in nature (a_1 = dominantly π-ρ state, ρ = dominantly quark-antiquark state). Such a finding is in line with the hadrogenesis conjecture [2,7,8] which

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[1]Strictly speaking also the Goldstone bosons are not asymptotic states, but, e.g., the pions are stable with respect to the strong interaction. Therefore the pions in practice live long enough to allow for pion beams and for pions reaching the detectors. In contrast, vector mesons only show up in scattering phase shifts and not as scattering partners.

implies that the lowest-lying pseudoscalar and vector mesons (and in the baryon sector the nucleon octet and the Delta decuplet) are dominantly quark-antiquark (three-quark) states while other low-lying hadrons are dynamically generated.

There is a third, technical, aspect why we present this analysis in spite of the fact that the qualitative outcome is fully in line with the findings of [6]: Typically the spectral information of a resonance is determined in a Dyson-Schwinger approach, i.e. by determining a one-body quantity, the self-energy. In contrast, a Bethe-Salpeter approach starts from the objects which form the resonance, i.e. from a two-body quantity, the scattering amplitude. It is interesting to work out the connections between the two approaches, in particular, for the nontrivial case where there is not only the resonance in the scattering kernel, but also an additional nonresonant, here pointlike, interaction.

For the pion rescattering we will restrict ourselves in the following to the elastic channel, as will be discussed in more detail below. In principle, it is straightforward to include the $K\bar{K}$ channel as intermediate states. However, there is a second channel with about the same threshold, namely, the $\pi\omega$ channel which then should also be included. It can also be seen as one, presumably important, representative of the four-pion channel. The inclusion of the $\pi\omega$ channel is more demanding since the scattering kernel of $\pi\pi \to \omega\pi$ is not in the realm of chiral perturbation theory. The power counting for pseudoscalar and vector mesons as developed in [8,9] can be used for such an extension. However, the interactions which play a role there are not restricted to point interactions and s-channel resonance formation while they are in our present approach as we will discuss in more detail below. Rather t- and u-channel exchange diagrams come into play there which calls for a proper treatment of the appearing left-hand cuts within a resummation scheme like the Bethe-Salpeter approach (cf. also the discussion in [6,10]). Without a proper handling of the left-hand cuts analyticity is spoiled and statements concerning the possibility to dynamically generate a state become questionable. Such an extension is beyond the scope of the present work. We therefore restrict ourselves to elastic pion rescattering and leave the inclusion of the channels $K\bar{K}$ and $\pi\omega$ for future work.

The paper is structured in the following way: In the next section we present the formalism to calculate the pion form factor. After some general considerations we will formulate the Bethe-Salpeter equation for pion rescattering in Sec. II B. The pion form factor is addressed in Sec. II C. Section II D is devoted to a comparison of the Dyson-Schwinger and the Bethe-Salpeter equations. In Sec. III the results are presented and compared to the experimental data for the pion form factor and the pion-scattering phase shift. The two Secs. III A and III B concern the two scenarios without and with an elementary vector meson. Finally an Appendix is added for some technical details.

II. FORMALISM

A. General considerations

The electromagnetic pion form factor is defined via

$$\langle \pi^+(p')|j^\mu_{\rm em}|\pi^+(p)\rangle = (p+p')^\mu F_\pi(q^2), \qquad (1)$$

with $q = p - p'$ and the electromagnetic current $j^\mu_{\rm em}$. The normalization is chosen such that the form factor is unity for the pure QED process where only the charge of the pion is probed. The timelike region of the pion form factor is accessible by the reaction $e^+e^- \to \pi^+\pi^-$ [11–13]. The pion form factor has been studied by many groups; see, e.g., [14–18] and references therein. However, a Bethe-Salpeter approach has rarely been used and the question whether one can understand the pion form factor by pure rescattering of pions without an elementary vector meson has not been addressed. For the scenario where an elementary vector meson is included our approach is technically very similar to the one of [17]. Only the renormalization issue is treated somewhat differently where we take into account the considerations of [2,19], as will be explained below. For a Bethe-Salpeter approach, i.e. including rescattering, one needs at least the scattering kernels for the following reactions: $e^+e^- \to \pi^+\pi^-$ and $\pi^+\pi^- \to \pi^+\pi^-$. In principle, one also needs the elastic channel $e^+e^- \to e^+e^-$, but as a nonhadronic channel its contribution is negligibly small (cf. also the Appendix). One can also imagine considering other intermediate states like $K\bar{K}$ or 4π. In the following we restrict ourselves to center-of-mass energies below 1 GeV. The threshold for $K\bar{K}$ production lies at about 1 GeV and it opens up as a p wave. Thus it should be reasonable to neglect this channel. If one included $K\bar{K}$, one should presumably also consider $\pi\omega$ which is a representative of the four-pion channel. The nominal threshold for $\pi\omega$ is at about 900 MeV. For uncorrelated pions the 4π threshold is even below 600 MeV. Experimentally, however, it turns out that the four-pion channel is not very important below 1 GeV [20]. Therefore, we keep things as simple as possible and consider only two-pion intermediate states for the reaction of interest, $e^+e^- \to \pi^+\pi^-$.

Finally, a comment is in order concerning the three-pion intermediate state. This state is forbidden by G parity, but since isospin is not an exact symmetry, such an intermediate state is possible. In particular, the three pions might be correlated to an ω meson. Thus the small probability to violate isospin can be overcompensated by the large probability to create a sharp resonance. Indeed, one observes the ω meson in the reaction $e^+e^- \to \pi^+\pi^-$ ("ρ-ω mixing"). However, if one stays outside of the isolated ω peak there is no further visible trace of the three-pion intermediate state. The present work is not concerned with the isospin violating mixing to the ω meson. It is well known how to include this mixing on a phenomenological level [17,21]. For the present work we ignore this aspect and demand exact isospin symmetry.

Under this assumption additional experimental information is available: In the limit where the isospin symmetry is exact one can relate the pion form factor to the decay of the tau lepton into neutrino plus two pions. In this limit the hadronic spectral information extracted from the latter process is related to the pion form factor by [1]

$$v_1(s) = \frac{1}{12}\left(1 - \frac{4m_\pi^2}{s}\right)^{3/2}|F_\pi(s)|^2, \qquad (2)$$

where m_π denotes the pion mass. In reality, isospin symmetry is explicitly broken by the difference between up- and down-quark masses and by electromagnetic effects. This breaking induces differences between the "neutral" pion form factor defined in (1) and the "charged" pion form factor which can be extracted from (2). Therefore one should keep in mind that an excellent description of the (neutral) pion form factor (1) does not necessarily imply a reproduction of v_1 with the very same quality, and vice versa. In the present work we will ignore isospin breaking effects. Concerning this issue for low-energy reactions where chiral perturbation theory applies, we refer to [22] and references therein.

For the construction of the scattering kernels we follow the logic of [4,5]: If there is an elementary resonance in the kinematical region of interest we include the corresponding s-channel diagram in the kernel. All other interactions are smooth functions of the center-of-mass energy $\sqrt{s}$. We approximate these nonresonant interactions by the respective lowest-order term of chiral perturbation theory [23–25]. Of course, one has to make sure that there is no double counting in this procedure. We will come back to this point later.

For our case of interest we have the following contributions to the kernels:

$$k_{e^+e^-\to\pi^+\pi^-} = k^{\text{QED}}_{e^+e^-\to\gamma^*\to\pi^+\pi^-} + k^{\text{res}}_{e^+e^-\to\gamma^*\to\rho^0\to\pi^+\pi^-}, \qquad (3)$$

$$k_{\pi^+\pi^-\to\pi^+\pi^-} = k^{\text{point}}_{\pi^+\pi^-\to\pi^+\pi^-} + k^{\text{res}}_{\pi^+\pi^-\to\rho^0\to\pi^+\pi^-}. \qquad (4)$$

The nonresonant contributions k^{QED} and k^{point} emerge from chiral perturbation theory in lowest, i.e. second, order [23–25]. k^{QED} is just the QED-type process where the virtual photon couples to the charge of the pion. k^{point} is the pion four-point interaction of the nonlinear sigma model.

As already announced we study two scenarios: One with and one without an elementary vector meson. In the latter case we set $k^{\text{res}} = 0$ in (3) and (4). For the former case we follow [8,17,26] and use the tensor realization of the vector mesons. If the vector mesons were integrated out for low energies, they would only contribute to the fourth-order Lagrangian of chiral perturbation theory, not to the second-order one [26,27]. Thus there is no double counting in (3) and (4) on the level of the kernels.

The Bethe-Salpeter equation iterates the kernels to infinite order. In that way loops emerge which require renormalization. In general, the renormalization of the resummed series is not as clear-cut as the renormalization of a perturbation theory. Ambiguities arise which can also influence the issue of double counting [19]. We will come back to that point later when discussing the Bethe-Salpeter equation in more detail.

B. Pion-scattering amplitude

We discuss in the following the more general scenario which includes an elementary vector meson. The other scenario can easily be obtained by putting the appropriate coupling constants (e_V and h_P, see below) to zero. We specify first the scattering amplitude for elastic pion-pion scattering: In the center-of-mass system the Feynman scattering amplitude $\mathcal{M}$ is decomposed into amplitudes t_l with fixed orbital angular momentum l:

$$\mathcal{M}(s,\cos\theta) = \sum_l (2l+1)t_l(s)\mathrm{P}_l(\cos\theta), \qquad (5)$$

with the Legendre polynomials P_l. The phase shift is given by

$$\cot\delta_l = \frac{\mathrm{Re}t_l}{\mathrm{Im}t_l}, \qquad (6)$$

and the optical theorem reads

$$\mathrm{Im}(t_l^{-1}) = -\frac{p_{\text{c.m.}}}{8\pi\sqrt{s}}, \qquad (7)$$

with the center-of-mass momentum $p_{\text{c.m.}}$.

The final process of interest, $e^+e^- \to \pi^+\pi^-$, proceeds via a virtual photon in the s channel. Consequently the orbital angular momentum of the two-pion system is fixed to $l = 1$. The isospin is then fixed to $I = 1$ to allow for an overall symmetric two-pion state.

The scattering amplitude can be decomposed into two-particle reducible and two-particle irreducible parts. The latter constitute the kernel of the Bethe-Salpeter equation while the former are automatically generated by this equation. As a first step we need an approximation for the kernel. As already spelled out we use lowest-order chiral perturbation theory plus a (bare) vector-meson s-channel diagram. Note that the width of the vector meson is generated by the Bethe-Salpeter equation. Hence both parts are tree-level contributions.

With the conventions of [8] we find

$$k_{\pi^+\pi^-\to\pi^+\pi^-} = k_{l=1,I=1}(s)$$
$$= \frac{2}{3f^2}p_{\text{c.m.}}^2\left(1 - \frac{m_V^2h_P^2}{8f^2}\frac{s}{s-m_{\rho,\text{bare}}^2}\right). \qquad (8)$$

Here $f = 90$ MeV denotes the pion-decay constant in the chiral limit and $m_{\rho,\text{bare}}$ the mass of the elementary vector meson. The combination $m_V h_P$ parametrizes the coupling

of the vector meson to the pions. The dimensionful quantity m_V is conveniently chosen as $m_V = 776$ MeV. In the following we use the dimensionless quantity h_P as a free input parameter which will be adjusted to the data for the scenario including an elementary vector meson. For the alternative scenario, i.e. where there is no elementary vector meson, we simply set h_P to zero. Finally we note for completeness: $p_{\text{c.m.}} = \frac{1}{2}\sqrt{s - 4m_\pi^2}$ with the pion mass m_π.

With the formalism of [2] the Bethe-Salpeter equation reads

$$t_{l=1,I=1}^{-1}(s) = k_{l=1,I=1}^{-1}(s) - I(s) + \text{Re}I(\mu^2). \qquad (9)$$

Here $I(s)$ is the loop function

$$I(s = p^2) = -i \int \frac{d^d q}{(2\pi)^d}$$
$$\times \frac{1}{[q^2 - m_\pi^2 + i\eta][(q - p)^2 - m_\pi^2 + i\eta]}, \qquad (10)$$

regularized by dimensional regularization with $d = 4 + 2\epsilon$. Several remarks are in order here:

(1) Unitarity (7) is exactly fulfilled by (9) since $\text{Im}k = 0$ (two-particle irreducibility) and $\text{Im}I(s) = p_{\text{c.m.}}/(8\pi\sqrt{s})$. This is in contrast to any perturbation theory which satisfies unitarity only perturbatively, but not exactly. On the other hand, exact unitarity is an important requirement in the resonance region [10], i.e. for energies larger than the region of applicability of strict chiral perturbation theory.

(2) Analyticity is also fulfilled by (9) since the kernel k as a rational function—given in (8)—does not have any cuts. This property is actually a necessary requirement to write down the Bethe-Salpeter equation in an analytic way as given by (9).

(3) Crossing symmetry is not fulfilled by any Bethe-Salpeter equation since processes are iterated in the s channel, but not in the t and u channels. As pointed out in [2] *approximate* crossing symmetry can be ensured by a proper choice of the renormalization point μ (see the next item).

(4) The loop function is renormalized by the replacement

$$I(s) \to I(s) - \text{Re}I(\mu^2) = J(s) - \text{Re}J(\mu^2), \qquad (11)$$

where we have introduced the finite function

$$J(s) = \frac{1}{16\pi^2}\left(2 + \sigma(s)\log\frac{\sigma(s) - 1}{\sigma(s) + 1}\right), \qquad (12)$$

and the phase space $\sigma(s) = 2p_{\text{c.m.}}/\sqrt{s}$. By the replacement (11) a new parameter, the renormalization point μ, is introduced. Approximate crossing symmetry is ensured if the full scattering amplitude t reduces to the perturbative amplitude k below and

close to threshold [2]. For the scenario with an elementary vector meson we will follow this requirement and choose

$$\mu \approx m_\pi. \qquad (13)$$

We will study the impact of moderate changes. For the scenario without an elementary vector meson we will allow for arbitrary changes of the renormalization point. But we will keep in mind that a drastic deviation from (13) is physically questionable. As pointed out in [19] an improper choice of μ is even related to the double-counting problem raised above. We will come back to that point later when we discuss the results for our two scenarios.

C. Pion form factor

The process $e^+ e^- \to \pi^+ \pi^-$ can be treated within the Bethe-Salpeter approach as a two-channel problem. A simplification arises, however, if one treats the electromagnetic processes on a perturbative level. This issue is worked out in the Appendix. The result is

$$t_{e^+e^-\to\pi^+\pi^-}(s) \approx k_{e^+e^-\to\pi^+\pi^-}(s)$$
$$\times [1 + (I(s) - \text{Re}I(\tilde{\mu}^2))t_{l=1,I=1}(s)], \qquad (14)$$

with the elastic pion-scattering amplitude $t_{l=1,I=1}$ introduced already in (9).

It is worth discussing the renormalization issue, i.e. the replacement $I(s) \to I(s) - \text{Re}I(\tilde{\mu}^2)$ which led from (A2) to (14). Note, in particular, that we have introduced a new renormalization point $\tilde{\mu}$ here which we kept distinct from the renormalization point μ of pion rescattering appearing in (9). In principle, the coupled-channel Bethe-Salpeter approach [2] demands that the renormalization points μ and $\tilde{\mu}$ should be the same:

$$\tilde{\mu} = \mu \approx m_\pi. \qquad (15)$$

For the scenario with an elementary vector meson in the kernels we will follow this demand (15). For the case without an elementary vector meson we will explore the consequences of a free choice for μ and $\tilde{\mu}$ independent of each other.

It is interesting to figure out which loop is actually renormalized in (14): The first contribution to $t_{e^+e^-\to\pi^+\pi^-}$ (coming from the 1) is just the emission of two pions without rescattering. The second contribution is the process where the pions rescatter. Thus the loop function I appearing in (14) emerges from the process with an incoming virtual photon, two pions in the loop and two outgoing pions. In contrast, the loop function appearing in (9) emerges from the process with two incoming pions instead of the virtual photon. One could imagine that in principle both processes are renormalized in different ways. Thus we feel legitimated to explore the consequences of different renormalization points μ and $\tilde{\mu}$.

Nonetheless, we recall the argument from [2] that the choice (15) ensures approximate crossing symmetry. We will come back to that point later.

The pion form factor emerges from the scattering amplitude $t_{e^+e^- \to \pi^+\pi^-}$ by normalizing to the QED process, i.e. the coupling of the virtual photon to the charge of the pion, cf. (3):

$$
\begin{aligned}
F_\pi(s) &= \frac{t_{e^+e^- \to \pi^+\pi^-}(s)}{k^{\mathrm{QED}}_{e^+e^- \to \gamma^* \to \pi^+\pi^-}(s)} \\
&= \frac{k_{e^+e^- \to \pi^+\pi^-}(s)}{k^{\mathrm{QED}}_{e^+e^- \to \gamma^* \to \pi^+\pi^-}(s)} [1 + (I(s) - \mathrm{Re}I(\tilde{\mu}^2))t_{l=1,I=1}(s)] \\
&= \left(1 + \frac{k^{\mathrm{res}}_{e^+e^- \to \gamma^* \to \rho^0 \to \pi^+\pi^-}(s)}{k^{\mathrm{QED}}_{e^+e^- \to \gamma^* \to \pi^+\pi^-}(s)}\right) \\
&\quad \times [1 + (I(s) - \mathrm{Re}I(\tilde{\mu}^2))t_{l=1,I=1}(s)] \\
&=: F^{\mathrm{tree}}_\pi(s)[1 + (I(s) - \mathrm{Re}I(\tilde{\mu}^2))t_{l=1,I=1}(s)].
\end{aligned}
\tag{16}
$$

With the conventions of [8] and ignoring the electron mass one gets

$$
F^{\mathrm{tree}}_\pi(s) = 1 - \frac{m_V^2 h_P e_V}{16 e f^2} \frac{s}{s - m_{\rho,\mathrm{bare}}^2},
\tag{17}
$$

with the electromagnetic charge e of the pion π^+. The dimensionless quantity e_V parametrizes the coupling of the photon to the elementary vector meson.

We have argued above that it is reasonable to explore the consequences of choosing renormalization points μ and $\tilde{\mu}$ different from each other and different from (15). It is important to add, however, that this line of reasoning does not hold for processes with an elementary vector meson. In this case the pion loop starts at the three-point vertex with the vector meson, no matter whether the vector meson has emerged from a photon or from incoming pions. Thus one has to renormalize always the same type of loop. Consequently, for the scenario with an elementary vector meson one should keep at least $\mu = \tilde{\mu}$. Also from a technical point of view the equality of μ and $\tilde{\mu}$ is necessary: The tree-level singularity of $F^{\mathrm{tree}}_\pi(s)$ at $s = m_{\rho,\mathrm{bare}}^2$ is only dressed for the full $F_\pi(s)$ if both renormalization points coincide. Only in this case one finds

$$
\begin{aligned}
F_\pi(s) &= F^{\mathrm{tree}}_\pi(s)[1 + (I(s) - \mathrm{Re}I(\mu^2))t_{l=1,I=1}(s)] \\
&= \frac{F^{\mathrm{tree}}_\pi(s)}{k_{l=1,I=1}(s)} \frac{1}{k^{-1}_{l=1,I=1}(s) - (I(s) - \mathrm{Re}I(\mu^2))}.
\end{aligned}
\tag{18}
$$

The tree-level pion form factor $F^{\mathrm{tree}}_\pi(s)$ and the tree-level pion-scattering amplitude $k_{l=1,I=1}(s)$ both diverge at $s = m_{\rho,\mathrm{bare}}^2$, but their ratio remains finite. The rewriting of (18) cannot be achieved for $\mu \neq \tilde{\mu}$.

To summarize the renormalization issue: For the scenario with an elementary vector meson we have only one renormalization point. To ensure approximate crossing symmetry we will use (15), but explore moderate deviations from this relation. For the scenario without an elementary vector meson we will study the consequences of arbitrary choices for the two renormalization points.

D. Resummations in the Bethe-Salpeter and the Dyson-Schwinger equations

The discussion in the present section will concern, of course, only the scenario with an elementary vector meson. We start out slightly more general than the case considered in the rest of the paper. At the same time we will be somewhat more schematic (e.g., by concentrating on scalar quantities for the Dyson-Schwinger equation and disregarding renormalization issues). Suppose that a scattering process happens via a resonant and a nonresonant subprocess, i.e. the scattering kernel is given by

$$
K = g_1^2(s) - \frac{g_2^2(s)}{s - m_{\mathrm{bare}}^2} = g_1^2(s) - g_2^2(s)D_{\mathrm{bare}}(s),
\tag{19}
$$

cf. (8). Here g_1^2 can be regarded as coming from a point interaction between the scattering partners while g_2 is the coupling of the scattering partners to the resonance. The scattering amplitude as obtained from the Bethe-Salpeter equation, cf. (9) and (10), is given by

$$
T = \frac{K}{1 - KI}.
\tag{20}
$$

The graphical version is displayed in Fig. 1. For convenience we also introduce the modified scattering amplitude which emerges if there was only the nonresonant part in the kernel:

$$
T' = \frac{g_1^2}{1 - g_1^2 I}.
\tag{21}
$$

For illustration we refer to Fig. 2.

Next we turn to the Dyson-Schwinger equation. It is given by
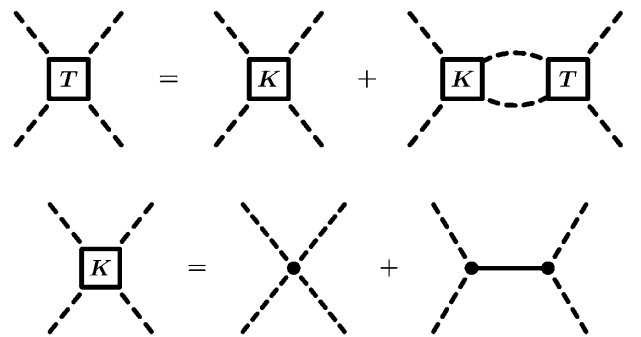

FIG. 1. Bethe-Salpeter equation (top) and its kernel (bottom) from (19). The dashed lines correspond to the scattering partners, the solid line to the (bare) resonance propagator.
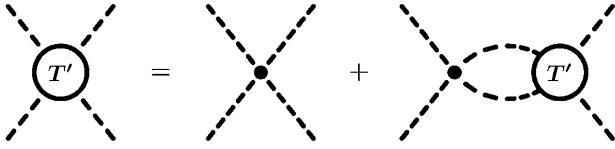
FIG. 2.   Bethe-Salpeter equation for a purely pointlike kernel.

$$D_{\text{full}}^{-1} = D_{\text{bare}}^{-1} - \Pi \tag{22}$$

and displayed in Fig. 3. All nontrivial information is contained in the self-energy Π which appears in (22). The standard approach is to approximate the self-energy by the one-loop expression. For our case at hand this is given by

$$\Pi_{\text{one loop}} = -g_2^2 I. \tag{23}$$

Within the same spirit one calculates the scattering amplitude in resonance approximation:

$$T \stackrel{?}{\approx} T_{\text{res}} := -g_2^2 D_{\text{full}}. \tag{24}$$

Of course, in the absence of the point interaction $\sim g_1^2$ in (19)—or in practice if all nonresonant interactions are sufficiently small—the result of the Bethe-Salpeter equation is identical to the result of (24), if the full propagator is obtained from the Dyson-Schwinger equation (22) within the one-loop approximation (23) for the self-energy. In this case one gets

$$T|_{g_1^2=0} = \frac{1}{K^{-1}-I}\bigg|_{g_1^2=0} = \frac{-g_2^2}{D_{\text{bare}}^{-1}+g_2^2 I} = -g_2^2 D_{\text{full}}$$
$$= T_{\text{res}}. \tag{25}$$

However, for the more general case $g_1^2 \neq 0$ it should be clear that the result from the Bethe-Salpeter equation (20)

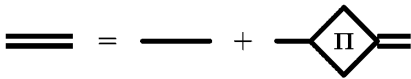FIG. 3.   The Dyson-Schwinger equation. The solid line denotes the bare and the double line the full resonance propagators.
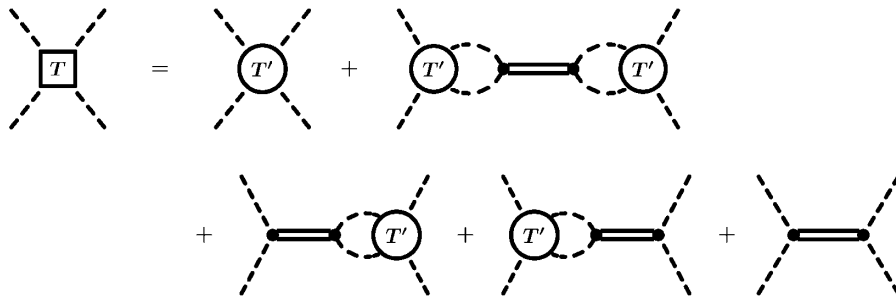
resums more processes than the one-loop plus resonance approximation, (23) and (24), respectively. Therefore we ask how we have to improve (23) and (24) such that they contain the same information as (20). The demand is that the scattering amplitude can be obtained from the full propagator plus nonresonant terms. It is natural to expect the following relation:

$$T = T' - T'Ig_2 D_{\text{full}}g_2 IT' - g_2 D_{\text{full}}g_2 IT' - T'Ig_2 D_{\text{full}}g_2$$
$$- g_2 D_{\text{full}}g_2, \tag{26}$$

which is graphically displayed in Fig. 4. Note that here the modified scattering amplitude (21) shows up which resums the nonresonant scattering kernel. It is a straightforward exercise to show that (26) is equivalent to (20) if the self-energy is given by

$$\Pi = -g_2^2 I - g_2^2 I^2 T' = -g_2^2 I \frac{1}{1-g_1^2 I}. \tag{27}$$

The graphical version of this relation is shown in Fig. 5. Note that the appearance of T' instead of T in (27) is very natural: The self-energy contains only one-particle irreducible diagrams. This would be spoiled by the appearance of any intermediate resonance propagator. T contains such diagrams while T' does not.

The physics point we want to make is the following: In resonance models one typically uses (23) and (24). However, from a more general effective-field-theory point of view one should not disregard point interactions—or more general nonresonant interactions—without checking their importance. Here the Bethe-Salpeter equation provides a natural tool. Alternatively, one can use the Dyson-Schwinger equation together with the improved relations (26) and (27). Even if one includes point interactions it is common practice to treat them on a perturbative level while for the resonant part a Dyson-Schwinger resummation is performed. The reason is clear from a practical point of view: A bare resonance propagator needs a width to provide sensible, nondivergent results. For the nonresonant interactions it is not obvious that something is missing if one does not resum them. However, from the point of view of exact unitarity one should resum all interactions, just like the Bethe-Salpeter equation automatically does.
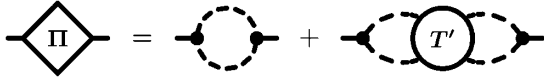

FIG. 4.   Representation of the scattering amplitude via the full propagator and the nonresonant terms. For details see the captions of Figs. 1 and 3.

FIG. 5.   The improved self-energy. For details see the captions of Figs. 1 and 3.
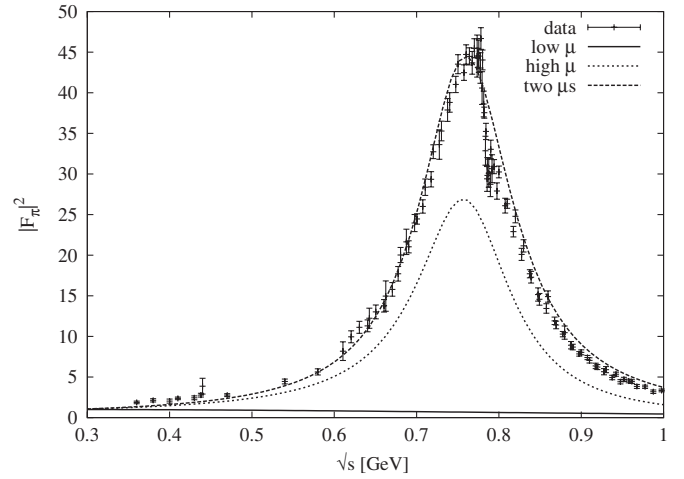

FIG. 6.   Description of the modulus square of the pion form factor in the Bethe-Salpeter approach without an elementary vector meson, i.e. including in the kernels only the interactions from lowest-order chiral perturbation theory. The solid line, labeled "low μ," denotes the physically reasonable case where the renormalization points are chosen according to (15). The other lines denote the technically possible, but physically questionable, cases where the renormalization points are chosen in the TeV range. The dotted line, labeled "high μ," is obtained for $\mu = \tilde{\mu} = 1.1$ TeV and the dashed line, labeled "two μs," for $\mu = 1.1$ TeV, $\tilde{\mu} = 10$ TeV. See the main text for details. Data taken from [11–13].

Finally it is interesting to study how the high-energy behavior is modified when changing from the one-loop self-energy (23) to the improved version (27). For point interactions $g_1^2(s)$ is a polynomial in s and is real. The same holds for $g_2(s)$. After renormalization the loop function $I(s)$ diverges logarithmically with s [cf. (12)]. Its imaginary part reaches a constant. Therefore the real part of the quantity $\Pi_{\text{one loop}}/g_2^2$ diverges logarithmically while its imaginary part becomes constant. In contrast, the improved quantity Π/g_1^2 has a real part which converges $\sim 1/g_1^2$ and an imaginary part which drops to zero $\sim 1/(g_1^4 \log^2 s)$.

After these general considerations how a Bethe-Salpeter resummation compares to a Dyson-Schwinger resummation we return to the main subject of the present work. We recall that the task is to figure out whether one needs an elementary vector meson to describe the experimental data for the electromagnetic pion form factor or whether it is sufficient to iterate the point interaction of lowest-order chiral perturbation theory.

III. RESULTS

The pion form factor is given in (16). With the ingredients specified in (8), (9), (11), (12), and (17) one obtains an expression which in general depends on the resonance parameters h_P, e_V, and $m_{\rho,\text{bare}}$ of the bare vector meson and on the renormalization points μ and $\tilde{\mu}$. In the following we will discuss the two scenarios where an elementary (bare) vector meson is not included or is included, respectively.

A. Scenario without an elementary vector meson

The case where there is no elementary vector meson is easily obtained by setting $h_P = 0$ in (8) and (17). In this case the values for e_V and $m_{\rho,\text{bare}}$ are irrelevant. The only parameters left are the renormalization points. Demanding in addition approximate crossing symmetry for the scattering amplitudes resulting from the Bethe-Salpeter equation, i.e. fixing the renormalization points according to (15), one is left with a parameter-free result. This result is compared to data in Fig. 6, with the solid line labeled "low μ." Obviously, the pion four-point interaction of leading-order chiral perturbation theory, iterated by the Bethe-Salpeter equation, is insufficient to create the peak seen in the data. This confirms the findings of [6]. Hence, one needs in addition an elementary vector meson and in the following section we will demonstrate that then an excellent description of the pion form factor can be obtained.

The physics case is closed with the previous remarks. Nonetheless, it is interesting to explore whether it is *technically* possible at all to come close to the data in an approach without an elementary vector meson. Therefore, we abandon the constraints (15) in the following and study the dependence of our result on the renormalization points μ and $\tilde{\mu}$. It turns out that for choices below the TeV range no appreciable peak is obtained. (We also refer to [6] for a similar discussion concerning the pion-scattering amplitude.) If one keeps the two renormalization points the same, one can generate a peak for large values of $\mu = \tilde{\mu}$, but not a decent description of the data. An example is provided in Fig. 6 by the dotted line labeled "high μ." A (technically but not physically) satisfying description of the data is obtained, if the renormalization points are allowed to differ from each other. This is demonstrated in Fig. 6 by the dashed line labeled "two μs." Here we have chosen $\mu = 1.1$ TeV, $\tilde{\mu} = 10$ TeV.

The technical success of this approach is nicely explained in [19]: It is shown there that an improper choice of the renormalization point(s), i.e. a choice which deviates significantly from (15), mimics an elementary resonance. Therefore, such an improper choice can lead to the wrong conclusion. The naive interpretation of the good agreement between the dashed line in Fig. 6 and the data would be that one can describe the pion form factor without an elementary vector meson. The correct interpretation, however, is that one has introduced an elementary resonance through

the back door by the (improper) choice for the renormalization points.

B. Scenario including an elementary vector meson

The previous considerations also tell us something for the case to which we turn now, namely, the one *with* an elementary vector meson. Since an improper choice for the renormalization points mimics an elementary resonance, one generates double counting in such a case. Both the included elementary state and the wrongly chosen renormalization point would introduce one and the same resonance. This is avoided by the constraint (15) which we use from now on more or less. Still we will study the consequences of small deviations from (15).

Having fixed the renormalization points, we are left with three parameters: The coupling of the virtual photon to the elementary vector meson $\sim e_V$, the coupling of the pions to this vector meson $\sim h_P$, and the (bare) mass of this vector meson, $m_{\rho,\text{bare}}$. It should not be too surprising that these parameters can be used to fix the height, the position, and the width of the peak and obtain in this way a good description of the data. This is demonstrated in Fig. 7 by the solid line. We note in passing that the largest deviation of our curve from the data happens at the small additional narrow peak slightly to the right of the peak position of the broad main peak. This is just the appearance of the ω meson due to the already mentioned isospin violating ρ-ω mixing.

As already pointed out in Sec. II A, in the limit of exact isospin symmetry the pion form factor is related to the hadronic spectral information v_1 extracted from tau decays. The relation is expressed in (2). For v_1 the comparison between data [1] and theory (including an elementary vector meson) is shown in Fig. 8.

We have chosen the following parameter sets to produce the theory curves displayed in Figs. 7 and 8:

$$\text{para 1: } h_P = 0.304, \quad e_V = 0.228,$$
$$m_{\rho,\text{bare}} = 0.711 \text{ GeV};$$
$$\text{para 2: } h_P = 0.310, \quad e_V = 0.228, \quad (28)$$
$$m_{\rho,\text{bare}} = 0.711 \text{ GeV}.$$

Obviously, the first parameter set yields a very good description of the pion form factor (except for ρ-ω mixing) while the second parameter set better fits the spectral information v_1. There are two reasons why we have decided to present two parameter sets instead of an intermediate one which would lie on the lower edge of the error bars for the pion form factor and on the upper edge for v_1. First, as already pointed out in Sec. II A, there is no need that the data fully agree: Electromagnetic corrections and other isospin breaking effects yield experimentally different neutral and charged pion form factors. In [1] it has been suggested that there is disagreement between the data even if these effects are accounted for. Second, we used the opportunity to display the sensitivity of the results on the chosen resonance parameters: The deviation between the values for h_P from the first and from the second parameter set is only on the 2% level, but one already sees visible differences in Figs. 7 and 8. Clearly it is the high quality of the data which enables a discrimination between the two parameter sets for a given observable. We also note that
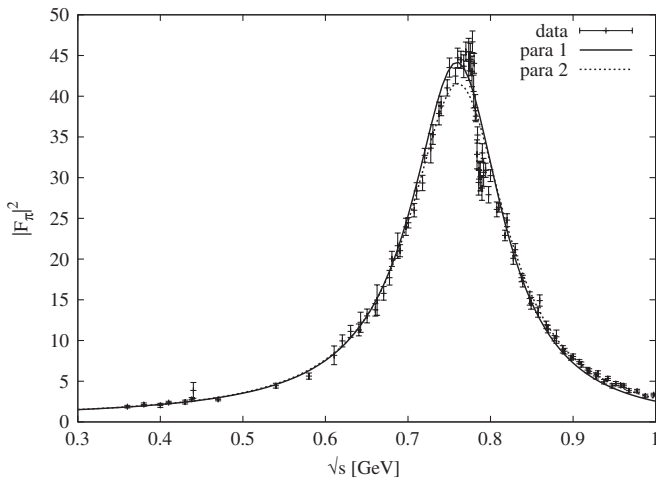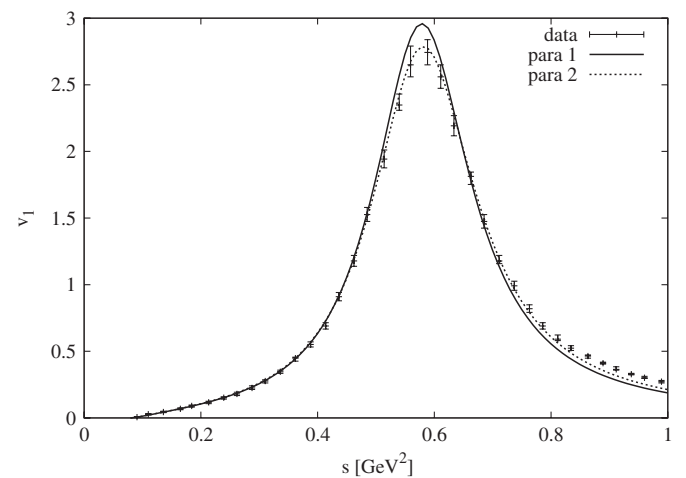

FIG. 7. Description of the modulus square of the pion form factor in the Bethe-Salpeter approach including in the kernels an elementary vector-meson resonance together with the interactions from lowest-order chiral perturbation theory. For parameter set "para 1" the resonance parameters are chosen such that a good description of the pion form factor is obtained concerning height (mainly controlled by parameter e_V), peak position ($m_{\rho,\text{bare}}$), and width (h_P). For "para 2" a good description of the spectral information v_1 is achieved, cf. Figure 8. The renormalization points are always chosen according to (15). See the main text for details. Data taken from [11–13].

FIG. 8. Description of the hadronic spectral information contained in the process $\tau^- \to \nu_\tau \pi^0 \pi^-$. The Bethe-Salpeter approach is used including an elementary vector meson, cf. Figure 7 and the main text for details. Data taken from [1].

variations of the other parameters e_V and $m_{\rho,\text{bare}}$ on a comparable level would worsen the agreement with the data. It remains to be seen in future work if both observables of Figs. 7 and 8 can be simultaneously described even better if electromagnetic corrections and isospin breaking effects are taken into account in the Bethe-Salpeter approach. In view of the initial assumptions—isospin limit, contact interactions from lowest-order chiral perturbation theory—we can be very satisfied with the quality of the theory curves and with the possibility to pin down the resonance parameters at the few-percent level.

Finally we note that the parameters h_P and e_V have also been determined in [8] from the partial decay widths of the two-body decays of vector mesons. Our results (28) are more precise, but in full agreement with the determination of [8]. The precise value for h_P from "para 1" in (28) has already been used in [9] with good success in the description of the three-pion decay of the ω meson.

For completeness we also show the pion-scattering phase shift introduced in (6). The result is displayed in Fig. 9. Note that our intention was to achieve a good description of the pion form factor. Hence our resonance parameters (28) have not been tuned to describe the pion-scattering phase shift. In view of that we can be rather satisfied with the solid and the dotted lines of Fig. 9 which use (15) and (28). We have also displayed the result for the phase shift if no elementary vector meson is included using (15), the dashed line in Fig. 9.

Finally we explore the consequence of a small deviation from (15): Varying $\mu = \tilde{\mu}$ between the electron mass (i.e.
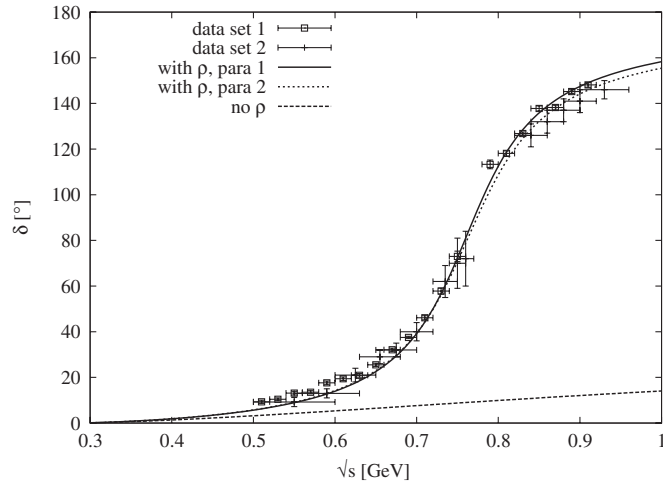

FIG. 9. Description of the pion-scattering phase shift in the Bethe-Salpeter approach. The dashed line labeled "no ρ": including in the kernel only the interaction from lowest-order chiral perturbation theory. Solid and dotted lines labeled "with ρ": including in addition an elementary vector-meson resonance. For the differences between the parameter sets "para 1" (solid line) and "para 2" (dotted line), see the caption of Fig. 7 and the main text for details. The data set 1/2 is taken from [28,29].

essentially zero) and twice the pion mass and keeping the resonance parameters unchanged from (28) does not lead to visible changes in Fig. 7. Even for larger deviations from (15) one can argue that a change in the renormalization point can be largely compensated by a change in the resonance parameters. After all, what determines the pion form factor to a large extent is the elementary vector meson. In other words, the pion form factor is more or less fixed by its total height, the peak position, and the peak width. These quantities are dominantly influenced by the vector-meson–photon coupling e_V, the bare vector-meson mass $m_{\rho,\text{bare}}$, and the vector-meson–pion coupling h_P, respectively.

ACKNOWLEDGMENTS

APPENDIX: TWO-CHANNEL BETHE-SALPETER EQUATION WITH A PERTURBATIVE CHANNEL

In this Appendix we keep the discussion more general than in the rest of the paper, but also more schematic. We consider a two-channel Bethe-Salpeter equation

$$T^{-1} = K^{-1} - I, \tag{A1}$$

with symmetric 2×2 matrices T, K, and I where I is a diagonal matrix. We want to determine the off-diagonal matrix element T_{12} for the case in which all matrix elements of K except for K_{11} are very small. A straightforward exercise yields

$$
\begin{aligned}
T_{12} &= \frac{K_{12}}{1 - I_{11}K_{11} - I_{22}K_{22} - I_{11}I_{22}(K_{12}^2 - K_{11}K_{22})} \\
&\approx \frac{K_{12}}{1 - I_{11}K_{11}} \approx K_{12}(1 + I_{11}T_{11}).
\end{aligned}
\tag{A2}
$$

In the last step we have used the fact that T_{11} can be obtained in lowest-order approximation from the one-channel Bethe-Salpeter equation

$$T_{11}^{-1} \approx K_{11}^{-1} - I_{11}, \tag{A3}$$

if all other entries of K are perturbatively small:

$$
\begin{aligned}
T_{11} &= \frac{K_{11} + I_{22}(K_{12}^2 - K_{11}K_{22})}{1 - I_{11}K_{11} - I_{22}K_{22} - I_{11}I_{22}(K_{12}^2 - K_{11}K_{22})} \\
&\approx \frac{K_{11}}{1 - I_{11}K_{11}}.
\end{aligned}
\tag{A4}
$$

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