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Instability and new phases of higher-dimensional rotating black holes

Óscar J. C. Dias,[1,*] Pau Figueras,[2,†] Ricardo Monteiro,[1,‡] Jorge E. Santos,[1,§] and Roberto Emparan[3,||]

[1]*DAMTP, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK*

[2]*Centre for Particle Theory and Department of Mathematical Sciences, University of Durham, Science Laboratories, South Road, Durham DH1 3LE, UK*

[3]*Institució Catalana de Recerca i Estudis Avançats (ICREA), Passeig Lluís Companys 23, E-08010 Barcelona, Spain and Departament de Física Fonamental, Universitat de Barcelona, Marti i Franquès 1, E-08028 Barcelona, Spain*

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It has been conjectured that higher-dimensional rotating black holes become unstable at a sufficiently large value of the rotation, and that new black holes with pinched horizons appear at the threshold of the instability. We search numerically and find the stationary axisymmetric perturbations of Myers-Perry black holes with a single spin that mark the onset of the instability and the appearance of the new black hole phases. We also find new ultraspinning Gregory-Laflamme instabilities of rotating black strings and branes.

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Black holes are the most basic and fascinating objects in general relativity and the study of their properties is essential for a better understanding of the dynamics of spacetime at its most extreme. In higher-dimensional spacetimes a vast landscape of novel black holes has begun to be uncovered [1]. Its layout—i.e., the connections between different classes of black holes in the space of solutions—hinges crucially on the analysis of their classical stability: most novel black hole phases are conjectured to branch off at the threshold of an instability of a known phase. Showing how this happens is an outstanding open problem that we address in this paper.

The best known class of higher-dimensional black holes, discovered by Myers and Perry (MP) in [2], appears in many respects as natural generalizations of the Kerr solution. In particular, their horizon is topologically spherical. However, the actual shape of the horizon can differ markedly from the four-dimensional one, which is always approximately round with a radius parametrically $\sim GM$. This is not so in $d \geq 6$. Considering for simplicity the case where only one spin J is turned on (of the $\lfloor \frac{d-1}{2} \rfloor$ independent angular momenta available), it is possible to have black holes with arbitrarily large J for a given mass M. The horizon of these *ultraspinning black holes* spreads along the rotation plane out to a radius $a \sim J/M$ much larger than the thickness transverse to this plane, $r_+ \sim (GM^3/J^2)^{1/(d-5)}$. This fact was picked out in [3] as an indication of an instability and a connection to novel black hole phases. In more detail, in the limit $a \to \infty$ with r_+ fixed, the geometry of the black hole in the region close to the rotation axis approaches that of a black membrane. Black branes are known to exhibit classical instabilities [4], at whose threshold a new branch of black branes with inhomogeneous horizons appears [5]. Reference [3] conjectured that this same phenomenon should be present for MP black holes at finite but sufficiently large rotation: they should become unstable beyond a critical value of a/r_+, and the marginally stable solution should admit a stationary, axisymmetric perturbation signaling a new branch of black holes pinched along the rotation axis. Simple estimates suggested that in fact $(a/r_+)_{\text{crit}}$ should not be much larger than 1. As a/r_+ increases, the MP solutions should admit a sequence of stationary perturbations, with pinches at finite latitude, giving rise to an infinite sequence of branches of "pinched black holes" (see Fig. 1).

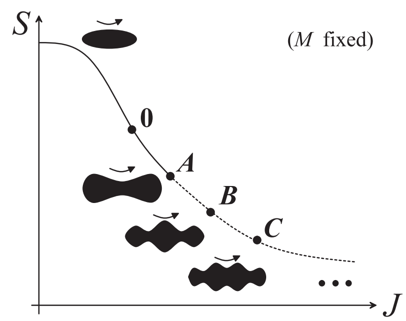

FIG. 1. Diagram of entropy vs spin, at fixed mass, for MP black holes in $d \geq 6$ illustrating the conjecture of [3] (see also [6]): at sufficiently large spin the MP solution becomes unstable, and at the threshold of the instability a new branch of black holes with a central pinch appears (A). As the spin grows new of branches of black holes with further axisymmetric pinches ($B, C, \ldots$) appear. We determine the points where the new branches appear, but it is not yet known in which directions they run. We also indicate that at the inflection point (0), where $\partial^2 S/\partial J^2 = 0$, there is a stationary perturbation that should not correspond to an instability nor a new branch but rather to a zero-mode that moves the solution along the curve of MP black holes.

*O.Dias@damtp.cam.ac.uk
†pau.figueras@durham.ac.uk
‡R.J.F.Monteiro@damtp.cam.ac.uk
§J.E.Santos@damtp.cam.ac.uk
||emparan@ub.edu

Reference [6] argued that this structure is indeed required in order to establish connections between MP black holes and the black ring and black Saturn solutions more recently discovered. Our main result is a numerical analysis that proves correct the conjecture illustrated in Fig. 1.

The solution for a MP black hole rotating in a single plane in d dimensions is [2]

$$ds^2 = -dt^2 + \frac{r_m^{d-3}}{r^{d-5}\Sigma}(dt + a\sin^2\theta d\phi)^2 + \Sigma\left(\frac{dr^2}{\Delta} + d\theta^2\right) + (r^2 + a^2)\sin^2\theta d\phi^2 + r^2\cos^2\theta d\Omega^2_{(d-4)}, \quad (1)$$

$$\Sigma = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 + a^2 - \frac{r_m^{d-3}}{r^{d-5}}. \quad (2)$$

The parameters here are the mass-radius r_m and the rotation-radius a,

$$r_m^{d-3} = \frac{16\pi GM}{(d-2)\Omega_{d-2}}, \qquad a = \frac{d-2}{2}\frac{J}{M}. \quad (3)$$

The event horizon lies at the largest real root $r = r_+$ of Δ.

The linearized perturbation theory of the Kerr black hole ($d = 4$) was disentangled in [7] using the Newman-Penrose formalism. Attempts to extend this formalism to decouple a master equation for the gravitational perturbations of (1) in $d \geq 5$ have failed so far. Moreover, even if some subsectors of the perturbations of some classes of MP black holes have been decoupled [8], none of them shows signs of any instability and indeed they do not contain the precise kind of perturbations we are interested in. Thus we take a more frontal numerical approach to a full set of coupled partial differential equations (PDE).

We intend to solve for a stationary linearized perturbation h_{ab} around the background (1). Choosing traceless-transverse (TT) gauge, $h^a{}_a = 0$ and $\nabla^a h_{ab} = 0$, the equations to solve are

$$(\triangle_L h)_{ab} = -\nabla_c\nabla^c h_{ab} - 2R_a{}^c{}_b{}^d h_{cd} = 0, \quad (4)$$

where $\triangle_L$ is the Lichnerowicz operator in the TT gauge. Actually, we solve the more general eigenvalue problem

$$(\triangle_L h)_{ab} = -k^2 h_{ab}, \quad (5)$$

which is known to appear in two contexts: Eqs. (5) determine the stationary perturbations of a black string in $d+1$ dimensions [obtained by adding a flat direction z to (1)] with a profile $e^{ikz}h_{ab}$. Thus such modes with $k > 0$ correspond to the threshold of the Gregory-Laflamme instability of black strings [4]. The same equations also describe the negative modes of quadratic quantum corrections to the gravitational Euclidean partition function [9]. A recent study of this problem for the Kerr black hole has shown the existence of a branch of solutions extending the negative Schwarzschild mode (with $k_{\rm Sch} \neq 0$) to finite rotation, with k growing as the rotation increases toward the Kerr bound [10].

Our reason to consider (5) instead of trying to solve directly for $k = 0$ is that there exist powerful numerical methods for eigenvalue problems that give the eigenvalues k together with the eigenvectors, i.e., the metric perturbations. If the ultraspinning instability is present for MP black holes in $d \geq 6$, then, in addition to the analogue of the branch studied in [10], a new branch of negative modes extending to $k = 0$ must appear. The eigenvalue $k = 0$ corresponds to a (perturbative) stationary solution with a slightly deformed horizon. In fact, as explained above, we expect an infinite sequence of such branches that reach $k = 0$ at increasing values of the rotation. The solutions for $k > 0$ imply new kinds of Gregory-Laflamme instabilities and inhomogeneous phases of ultraspinning black strings (see also [11]).

The modes we seek preserve the $SO(d-3) \times SO(2)$ rotational symmetries of the MP solution and depend only on the radial and polar coordinates, r and θ [3]. Thus we take the ansatz

$$ds^2 = -e^{2\nu_0}(dt - \omega d\phi)^2 + e^{2\nu_1}d\phi^2 + e^{2\eta}\sin^2\theta d\theta^2 + e^{2\gamma}(dr - \chi\sin\theta d\theta)^2 + e^{2\Phi}d\Omega^2_{d-4}. \quad (6)$$

We decompose a given quantity $Q = \{\nu_0, \nu_1, \eta, \gamma, \omega, \chi, \Phi\}$ as $Q = \bar{Q} + \delta Q$. The unperturbed contribution $\bar{Q}(r, \theta)$ describes (1). The perturbations $\delta Q(r, \theta)$ are determined solving the eigenvalue problem (5) subject to appropriate boundary conditions. After imposing TT gauge, Eq. (5) reduces to four coupled PDEs for $\delta\eta$, $\delta\gamma$, $\delta\chi$, and $\delta\Phi$ (the TT conditions then give $\delta\nu_0$, $\delta\nu_1$, and $\delta\omega$). The boundary conditions are that the perturbations are regular and finite at the horizon, $r = r_+$, at infinity, $r = \infty$, and at the poles $\theta = 0, \pi/2$. In addition, we impose $\delta\chi(r_+) = 0$. It is important to ensure that the eigenmodes we find are not pure gauge, $h_{ab} = \nabla_{(a}\xi_{b)}$. We can prove that in the TT gauge, pure gauge perturbations within our ansatz necessarily diverge at either the horizon or infinity. Thus, with our boundary conditions, the eigenmodes we obtain are never pure gauge.

We use a numerical approach successfully applied to the identification of the negative mode of Kerr and Kerr–anti-de Sitter (AdS) black holes [10]. It employs a Chebyshev spectral numerical method (see [10] for further details). We have carried out the calculations for $d = 7, 8, 9$. The cases $d = 5$ (where the heuristics of [3] do not allow one to predict any instability) and $d = 6$ present more difficult numerics. These, as well as a more detailed presentation of our numerical approach, will be discussed elsewhere.

The results for $d = 7$ are displayed in Fig. 2, the other two cases being qualitatively very similar. We plot the negative eigenvalue $-k^2$ as a function of the rotation parameter a. We normalize k and a relative to the mass-radius r_m, which is equivalent to plotting their values for fixed mass (or mass per unit length, in the black string interpretation). As described above, the left-most curve, which does not reach $k = 0$, is the higher-dimensional

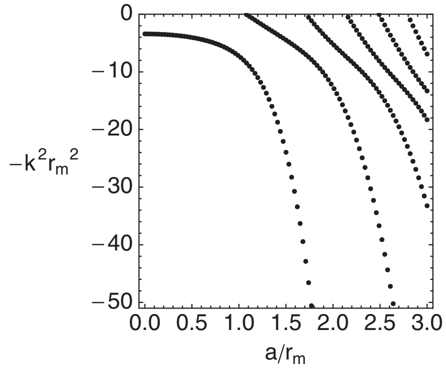

FIG. 2. Negative eigenvalues for the MP black hole in $d = 7$.

counterpart of the Kerr negative mode, and the eigenvalues k are the wave numbers of the Gregory-Laflamme threshold modes at rotation a. At larger rotation we find new branches of negative modes that intersect $k = 0$ at finite a/r_m. We label these successive branches with an integer $\ell = 1, 2, 3, \ldots$, and refer to them as "harmonics." The values of a/r_m at which the stationary perturbations appear are listed in Table I.

It is important to note that the $k = 0$ eigenmode of the harmonic $\ell = 1$ does *not* correspond to a new stationary solution. Instead it is a zero-mode that takes the solution to a nearby one along the family of MP black holes. The existence and location of this zero-mode is a consequence of the fact that if the Hessian of the Gibbs potential $G(T, \Omega_i) = M - TS - \sum_i J_i \Omega_i$, calculated along a family of solutions, has an eigenvalue changing sign for some particular solution, then there is a zero-mode perturbation of the gravitational (Euclidean) action $I = G/T$ that takes that solution to an infinitesimally nearby one along that family. That is, perturbing the solution with that zero-mode does not correspond to branching off into a new family of solutions.

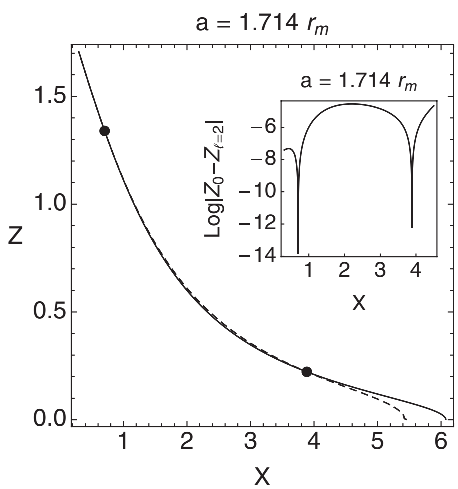

FIG. 3. Embedding diagram at $(a/r_m)_{\rm crit}$ of the $d = 7$ black hole horizon, unperturbed (solid), and with the first unstable harmonic perturbation ($\ell = 2$, $k = 0$) (dashed). The embedding Cartesian coordinates Z and X lie respectively along the rotation axis $\theta = 0$ and the rotation plane $\theta = \pi/2$. We also show the logarithmic difference between the embeddings of the perturbed ($Z_{\ell=2}$) and unperturbed (Z_0) horizons. The spikes represent the points where the two embeddings intersect. The perturbation has two nodes, so the horizon squeezes around the rotation axis, then bulges out, and squeezes again at the equator, as in the conjectured shape A in Fig. 1.

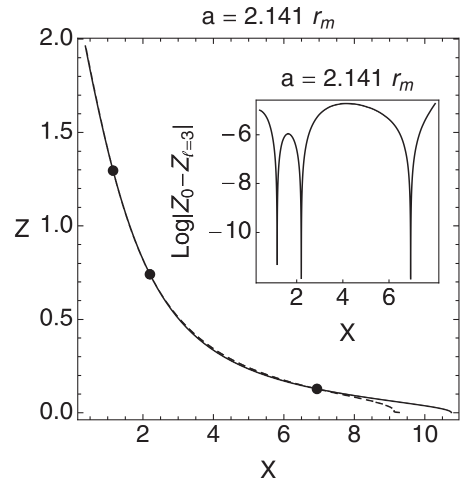

FIG. 4. Like Fig. 3, for $\ell = 3$: between the first two nodes of the perturbation the horizon has a pinch (shape B in Fig. 1).

One can easily check that the determinant of the Hessian of the Gibbs potential is inversely proportional to the determinant of the Hessian of the entropy with respect to only the angular momenta, i.e., to the determinant of

$$H_{ij} = \left(\frac{\partial^2 S}{\partial J_i \partial J_j}\right)_M. \tag{7}$$

Therefore, for solutions with a single spin, there must appear a stationary perturbation, in principle not associated to an instability of the black hole, at the inflection point of

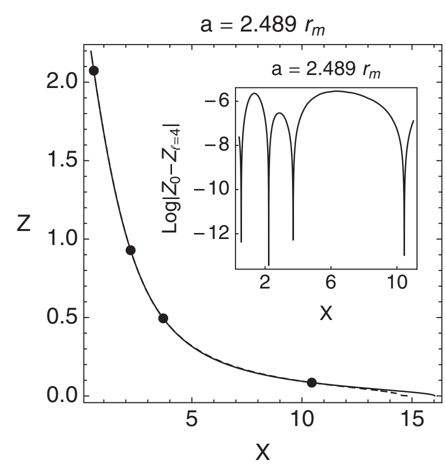

FIG. 5. Like Fig. 3, for $\ell = 4$: the four nodes deform the horizon into shape C of Fig. 1.

TABLE I. Values of the rotation a/r_m for the first three harmonics of stationary perturbation modes ($k = 0$). The estimated numerical error is $\pm 3 \times 10^{-3}$ in $d = 7$ and $\pm 5 \times 10^{-3}$ in $d = 8, 9$.

d	$(a/r_m)\|_{\ell=1}$	$(a/r_m)\|_{\ell=2}$	$(a/r_m)\|_{\ell=3}$
7	1.075	1.714	2.141
8	1.061	1.770	2.275
9	1.051	1.792	2.337

the curve $S(J)$ at fixed M (point 0 in Fig. 1). For the MP solutions this happens at

$$\left(\frac{a}{r_m}\right)_{\rm mem}^{d-3} = \frac{d-3}{2(d-4)}\left(\frac{d-3}{d-5}\right)^{(d-5)/2}. \tag{8}$$

The values of $(a/r_m)_{\rm mem}$ for $d = 7, 8, 9$ agree with the central values of the numerically determined rotations (a/r_m) for $\ell = 1$ (first column in Table I) up to the third decimal place. This is a very good check of the accuracy of our numerical methods.

The $k = 0$ eigenmodes of the higher harmonics, $\ell \geq 2$, do not admit this interpretation as perturbations along the MP family of solutions and thus correspond to genuinely new (perturbative) black hole solutions with deformed horizons. Their appearance conforms perfectly to the predictions in [3,6]. It is then natural to expect, although our approach does not prove it since it only captures zero-frequency perturbations, that the harmonic $\ell = 2$ signals the onset of the instability conjectured in [3]. The $k = 0$ eigenmodes for higher harmonics confirm the appearance of the sequence of new black hole phases as the rotation grows.

To visualize the effect on the horizon of the perturbations that give new solutions, and provide further confirmation of our interpretation, we draw an embedding diagram of the unperturbed MP horizon and compare it with the deformations induced by the ultraspinning harmonics $\ell \geq 2$. This is best done using the embedding proposed in [12], which has the advantage of allowing one to embed the horizon along the entire range $0 \leq \theta \leq \pi/2$ for any rotation, although at the cost of stretching the pole region, which acquires a conical profile. We do it for the $\ell = 2, 3, 4$ ultraspinning harmonics in Figs. 3–5. In spite of the distortion created by the embedding, the effect of the perturbations is clear: $\ell = 2$ modes create a pinch centered on the rotation axis $\theta = 0$; $\ell = 3$ modes have a pinch centered at finite latitude θ; $\ell = 4$ modes pinch the horizon twice: around the rotation axis and at finite latitude. These are the kind of deformations depicted in Fig. 1. To better identify the number of times that the perturbed horizon crosses the unperturbed solution, in these figures we also plot the logarithmic difference between the two embeddings.

Reference [3] gave several arguments to the effect that critical values a/r_m close to 1 were to be expected. In particular, it was pointed out that the change in the behavior of the black hole from "Kerr-like" to "black-membrane-like" could be pinpointed to the value of the spin where the temperature (i.e., surface gravity) reaches a minimum for fixed mass, which is the same, for solutions with a single spin, as the inflection point of $S(J)$. As we have argued, the zero-mode at this solution should not signal an instability. The $\ell = 2$ mode at the threshold of the actual instability instead appears at larger rotation, well within the regime of membrane-like behavior as conjectured in [6]. We expect this to be true in general: the ultraspinning instability of MP black holes should appear for angular momenta *strictly beyond* the (codimension 1) locus in the space of angular momenta where the Hessian H_{ij} has a zero eigenvalue.

In particular, in $d = 5$ this criterion does not allow any ultraspinning instability for any J_1, J_2, and in $d \geq 6$ with all the $N = \lfloor\frac{d-1}{2}\rfloor$ angular momenta J_i equal it predicts that the instability should appear at $a/r_m > 2^{-N/(d-3)}$. However we cannot predict the precise values of the rotation where the instability appears.

We have identified the points in the phase diagram where the new branches must appear, but we cannot determine in which direction these run. This requires calculating the area, mass, and spin of the perturbed solutions. However, for any $k \neq 0$—and numerically we can never obtain an exact zero—the linear perturbations decay exponentially in the radial direction, and so the mass and spin, measured at asymptotic infinity, are not corrected. It seems that in order to obtain the directions of the new branches one has to go beyond our level of approximation or adopt a different approach.

The new $\ell \geq 1$ branches extend to nonzero eigenvalues k. These imply a new ultraspinning Gregory-Laflamme instability for black strings, in which the horizon is deformed not only along the direction of the string, but also along the polar direction of the transverse sphere. Observe that, even if the $\ell = 1$, $k = 0$ mode is not an instability of the MP black hole, the modes $\ell = 1$, $k > 0$ are expected to correspond to thresholds of Gregory-Laflamme instabilities of MP black strings. At a given rotation, modes with larger ℓ have longer wavelength k^{-1} and so the branch $\ell = 1$ is expected to dominate the instability. The growth of k with a can be understood heuristically, since as a grows the horizon becomes thinner in directions transverse to the rotation plane and hence it can fit into a shorter compact circle.

To finish, we mention that pinched phases of rotating plasma balls, dual to pinched black holes in Scherk-Schwarz compactifications of AdS, have been found [13], as well as new kinds of deformations of rotating plasma tubes [14] and rotating plasma ball instabilities [15]. The relation of our results to these and other phenomena of rotating fluids will be discussed elsewhere.

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