PHYSICAL REVIEW D **80,** 107502 (2009)

# More on the covariant retarded Green's function for the electromagnetic field in de Sitter spacetime

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(Received 24 July 2009; published 13 November 2009)

In a recent paper [2] it was shown in examples that the covariant retarded Green's functions in certain gauges for electromagnetism and linearized gravity can be used to reproduce field configurations correctly in spite of the spacelike nature of past infinity in de Sitter spacetime. In this paper we extend the work of Ref. [2] concerning the electromagnetic field and show that the covariant retarded Green's function with an arbitrary value of the gauge parameter reproduces the electromagnetic field from two opposite charges at antipodal points of de Sitter spacetime.

DOI: 10.1103/PhysRevD.80.107502 PACS numbers: 04.62.+v

It was claimed by some authors that the causal structure, in particular, the spacelike nature of past infinity, of de Sitter spacetime invalidates the covariant retarded Green's functions in electromagnetism and linearized gravity in de Sitter spacetime (see, e.g. Ref. [1]). In electromagnetism, for example, they claimed that, since a charge following a geodesic can influence only part of the space, the field generated by the causal retarded Green's function violates Gauss's law, and that, as a consequence, the covariant retarded Green's function is wrong.

Now, the classical initial value problem for electromagnetism with a gauge-fixing term and a source term is certainly well defined and causal in de Sitter spacetime, and the causal retarded Green's function is simply a mathematical tool that gives the field in the future of the initial surface for a given source term. Therefore, the claim mentioned above is rather puzzling. As was pointed out in Ref. [2], in a spacetime with spacelike past infinity, such as de Sitter spacetime, one needs to take into account the contribution from the initial data on past infinity, which are necessarily nonzero if the charge density does not vanish there [3,4], when calculating the field configuration using the retarded Green's function. This issue does not arise in Minkowski spacetime, and that is why some authors consider only the contribution from the charge and erroneously conclude that the retarded Green's function does not work in de Sitter spacetime.

In Ref. [2] one example for showing how the retarded Green's function should be used in de Sitter spacetime was the electromagnetic field produced by two charges with opposite signs placed at antipodal points. This example was studied in the Feynman gauge. The purpose of this paper is to extend this example by using the covariant gauge with an arbitrary value of the gauge parameter.

We consider the free electromagnetic field described by the Lagrangian

$$\mathcal{L} = \sqrt{-g}\Big[-\frac{1}{4}F_{ab}F^{ab} - \frac{1}{2\zeta}(\nabla_a A^a)^2\Big], \tag{1}$$

where $g$ is the determinant of the background de Sitter metric and where $F_{ab} = \nabla_a A_b - \nabla_b A_a$. Let $J^a(x)$ be a current coupled to the electromagnetic potential $A_a(x)$ and let $\Sigma$ be a Cauchy surface. Then, as was explained in Ref. [2], the field $A_a$ in the future domain of dependence of $\Sigma$, denoted $D^+(\Sigma)$, is given in terms of the retarded Green's function $G_{ab'}(x, x')$ by

$$A_a(x) = A_a^{(S)}(x) + A_a^{(I)}(x), \tag{2}$$

where

$$A_a^{(S)}(x) = \int_{D^+(\Sigma)} d^4x'\sqrt{-g(x')}G_{ab'}(x, x')J^{b'}(x'), \tag{3}$$

$$A_a^{(I)}(x) = \int_\Sigma d\Sigma_{a'}[\pi^{a'b'}(x')G_{ab'}(x, x') - A_{b'}(x') \times (L_\pi G)_a{}^{a'b'}(x, x')], \tag{4}$$

with

$$\pi^{a'b'} = \frac{1}{\sqrt{-g}}\frac{\partial\mathcal{L}}{\partial(\nabla_{a'}A_{b'})} = -F^{a'b'} - \zeta^{-1}g^{a'b'}\nabla_{c'}A^{c'}, \tag{5}$$

$$(L_\pi\tilde{G})_a{}^{a'b'} = -2\nabla^{[a'}\tilde{G}_a^{b']} - \zeta^{-1}g^{a'b'}\nabla^{c'}\tilde{G}_{ac'}. \tag{6}$$

As in Ref. [2] we call the fields $A_a^{(S)}(x)$ and $A_a^{(I)}(x)$ the *source field* and *initial field*, respectively. In de Sitter spacetime the initial field with past infinity adopted as $\Sigma$ is crucial in reproducing the field in terms of the retarded Green's function. This fact was overlooked in the erroneous claim that the retarded Green's function in the covariant gauge is invalid in de Sitter spacetime.

A metric of de Sitter spacetime that covers the whole spacetime is

$$ds^2 = H^{-2}\sec^2\tau(-d\tau^2 + d\chi^2 + \sin^2\chi d\Omega^2), \tag{7}$$

where $|\tau| < \pi/2$ and $0 \le \chi \le \pi$, and where $d\Omega^2$ is the

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metric on the unit 2-sphere ($S^2$). The electromagnetic field from charges $q$ at the North Pole ($\chi = 0$) and $-q$ at the South Pole ($\chi = \pi$) can be

$$A_\tau = -\frac{q}{4\pi}\frac{\cos\tau\cot\chi}{\cos\tau + \sin\chi}, \tag{8}$$

$$A_\chi = -\frac{q}{4\pi}\frac{\sin\tau}{\cos\tau + \sin\chi}. \tag{9}$$

In Ref. [2] this field was reproduced using Eq. (2) from the initial data on past infinity and the current $J^a(x)$ corresponding to the charges at the North and South Poles in the Feynman gauge ($\zeta = 1$).

We first consider the source field due to a charge $q$ at the North Pole. The causal future of the North Pole, with the condition $\tau > \chi - \pi/2$, can be covered by the coordinates $(\lambda, r)$ (and the angular coordinates on $S^2$) given by

$$H\lambda = \frac{\cos\tau}{\cos\chi + \sin\tau}, \tag{10}$$

$$Hr = \frac{\sin\chi}{\cos\chi + \sin\tau}. \tag{11}$$

The metric in these coordinates is

$$ds^2 = \frac{1}{H^2\lambda^2}(-d\lambda^2 + dr^2 + r^2 d\Omega^2). \tag{12}$$

The conformal time $\lambda$ decreases toward the future from $\infty$ to 0. The North Pole is at $r = 0$. The electromagnetic potential given by Eqs. (8) and (9) in this coordinate system is

$$A_\lambda = \frac{q}{4\pi}\left(\frac{1}{r} - \frac{1}{\lambda + r}\right), \tag{13}$$

$$A_r = -\frac{q}{4\pi(\lambda + r)}. \tag{14}$$

Following Ref. [5], we define for spacelike points $x$ and $x'$ that can be connected by a spacelike geodesic

$$z \equiv [1 + \cos H\mu(x, x')]/2, \tag{15}$$

where $\mu(x, x')$ is the geodesic distance between $x$ and $x'$. With the notation $x = (\lambda, \mathbf{x})$, $x' = (\lambda', \mathbf{x}')$ we have (see, e.g. Eq. (4.28) of Ref. [2])

$$\cos H\mu(x, x') = \frac{\lambda^2 + \lambda'^2 - \|\mathbf{x} - \mathbf{x}'\|^2}{2\lambda\lambda'}. \tag{16}$$

This formula allows one to extend the variable $z$ to the cases where $x$ and $x'$ cannot be connected by a spacelike geodesic.

One can readily find the Feynman propagator for the electromagnetic potential using the method of Allen and Jacobson [5] as

$$Q_{ab'}(x, x') = Q^{FG}_{ab'}(x, x') + (\zeta - 1)\nabla_a\nabla_{b'}\tilde{\Delta}(x, x'), \tag{17}$$

where

$$\frac{\partial}{\partial z}\tilde{\Delta}(x, x') = \frac{H^2}{16\pi^2}\left[\frac{1}{1 - z + i\epsilon} - \frac{1}{3z} - \frac{2z + 1}{3z^2}\log(1 - z + i\epsilon)\right]. \tag{18}$$

(Since $\nabla_a\tilde{\Delta} = \nabla_a z\partial\tilde{\Delta}/\partial z$, we only need the $z$ derivative of $\tilde{\Delta}$.) The retarded Green's function is found as the discontinuity across the cuts from $z = -\infty$ to $z = -1$ and from $z = 1$ to $z = \infty$ on the complex $z$ plane (see, e.g. Eq. (4.10) of Ref. [2]) as

$$G_{ab'}(x, x') = G^{FG}_{ab'}(x, x') + (\zeta - 1)\tilde{G}_{ab'}(x, x'), \tag{19}$$

where $G^{FG}_{ab'}(x, x')$ is the retarded Green's function in the Feynman gauge [2,5] and

$$\tilde{G}_{ab'}(x, x') = \theta(\lambda' - \lambda)[\sigma(z)H^2 g_{ab'} + 4\gamma(z)\nabla_a z\cdot\nabla_{b'}z], \tag{20}$$

with

$$\sigma(z) = \frac{1}{16\pi}\left[\delta(1 - z) + \frac{2z + 1}{3z^2}\theta(z - 1)\right], \tag{21}$$

$$\gamma(z) = \frac{1}{16\pi}\left[\delta(1 - z) - \frac{1}{2}\delta'(1 - z) - \frac{1}{6z^3}\theta(z - 1)\right]. \tag{22}$$

The bivector $g_{ab'}(x, x')$ [5] is expressed as [2]

$$g_{ab'} = \frac{1}{H^2}\left(\partial_a\partial_{b'}\cos H\mu - \frac{1}{2z}\partial_a\cos H\mu\cdot\partial_{b'}\cos H\mu\right). \tag{23}$$

If points $x$ and $x'$ can be connected by a spacelike geodesic, then the bivector $g_{ab'}$ is the parallel propagator along the geodesic between these points.

Now, we use the retarded Green's function (19) to find the source field $A^{(S)}(x)$ given by Eq. (3) due the charge $q$ at the North Pole $r = 0$. We let the charge be present only for $\lambda < \lambda_0$ and let $\lambda_0 \to \infty$ at the end. The contribution from $G^{FG}_{ab'}(x, x')$ in Eq. (19) is given by Eqs. (4.34) and (4.35) of Ref. [2]. Here, we calculate the contribution from the second term in Eq. (19) denoted by $\tilde{A}^{(S)}_a(x)$. If we write

$$\tilde{G}_{ab'} = \tilde{G}^{(\delta)}_{ab'}\delta(1 - z) + \tilde{G}^{(\delta')}_{ab'}\delta'(1 - z) + \tilde{G}^{(\theta)}_{ab'}\theta(z - 1), \tag{24}$$

we have

$$\tilde{A}^{(S)}_a(x) = -q(\zeta - 1)\theta(\lambda_0 - \lambda - r) \times\left\{\frac{2\lambda\lambda'}{r}\left(\tilde{G}^{(\delta)}_{a\lambda'} + \frac{\partial}{\partial\lambda'}\left[\left(\frac{\partial z}{\partial\lambda'}\right)^{-1}\tilde{G}^{(\delta')}_{a\lambda'}\right]\right)\Bigg|_{\lambda'=\lambda+r} + \int_{\lambda+r}^{\infty} d\lambda'\tilde{G}^{(\theta)}_{a\lambda'}\right\}_{r'=0}. \tag{25}$$

We find that the only nonvanishing component is

$$\tilde{A}^{(S)}_{\lambda} = \frac{q}{12\pi\lambda}(\zeta - 1)\theta(\lambda_0 - \lambda - r). \tag{26}$$

Next, we calculate the initial field using the coordinates covering the causal *past* of the North Pole with the condition $\tau < \pi/2 - \chi$. Note that its past boundary is past infinity $\tau = -\pi/2$. This part of the spacetime is covered by the coordinates $(\hat{\lambda}, \hat{r})$ defined by

$$H\hat{\lambda} = \frac{\cos\tau}{\cos\chi - \sin\tau} = \frac{\lambda}{H(\lambda^2 - r^2)}, \tag{27}$$

$$H\hat{r} = \frac{\sin\chi}{\cos\chi - \sin\tau} = \frac{r}{H(\lambda^2 - r^2)}. \tag{28}$$

The equalities relating $(\hat{\lambda}, \hat{r})$ and $(\lambda, r)$ are valid only in the overlapping region with $\chi - \pi/2 < \tau < \pi/2 - \chi$. The metric in coordinates $(\hat{\lambda}, \hat{r})$ is given by Eq. (12) with $\lambda$ and $r$ replaced by $\hat{\lambda}$ and $\hat{r}$, respectively. (The North Pole is again at $\hat{r} = 0$.) The conformal time $\hat{\lambda}$ increases from 0 to $\infty$, and the surface $\hat{\lambda} = 0$ is past infinity. In finding the initial field, we let the initial surface $\Sigma$ be the $\hat{\lambda} = \hat{\lambda}'$ ($=$ const) surface and let $\hat{\lambda}' \to 0$ at the end. The field given by Eqs. (8) and (9) is expressed in coordinates $(\hat{\lambda}, \hat{r})$ on the initial surface $\hat{\lambda} = \hat{\lambda}'$ (with $\hat{r} = \hat{r}'$) as

$$A_{\hat{\lambda}'} = -\frac{q}{4\pi}\left(\frac{1}{\hat{r}'} - \frac{1}{\hat{r}' + \hat{\lambda}'}\right), \tag{29}$$

$$A_{\hat{r}'} = \frac{q}{4\pi}\frac{1}{\hat{r}' + \hat{\lambda}'}, \tag{30}$$

with the angular components vanishing (see Eqs. (4.39) and (4.40) in Ref. [2]).

Now, it can be shown that $\nabla^{[a'}\tilde{G}_a{}^{b']} = 0$, and $\nabla^{c'}G^{FG}_{ac'} = \nabla^{c'}\tilde{G}_{ac'}$. (That is, the retarded Green's function is divergence-free in the Landau gauge $\zeta = 0$.) This equality can be used to show that

$$(L_\pi G)_a{}^{a'b'} = (L_\pi G^{FG})_a{}^{a'b'}|_{\zeta=1}. \tag{31}$$

Thus, the second term in Eq. (4) is $\zeta$ independent. This implies that the additional contribution to the initial field is

$$\tilde{A}^{(I)}_a(x) = (\zeta - 1)\int_\Sigma d\Sigma_{a'}\pi^{a'b'}(x')\tilde{G}_{ab'}(x, x'). \tag{32}$$

The field $\pi^{a'b'}(x')$ is defined by Eq. (5). The additional contribution to the initial field in the causal past of the North Pole thus obtained is

$$\tilde{A}^{(I)}_{\hat{\lambda}} = \frac{q}{12\pi}(\zeta - 1)\left(\frac{1}{\hat{\lambda} + \hat{r}} + \frac{1}{\hat{\lambda} - \hat{r}} - \frac{1}{\hat{\lambda}}\right)\theta(\hat{\lambda} - \hat{\lambda}' - \hat{r}), \tag{33}$$

$$\tilde{A}^{(I)}_{\hat{r}} = \frac{q}{12\pi}(\zeta - 1)\left(\frac{1}{\hat{\lambda} + \hat{r}} - \frac{1}{\hat{\lambda} - \hat{r}}\right)\theta(\hat{\lambda} - \hat{\lambda}' - \hat{r}), \tag{34}$$

and all other components vanish. Notice that $\tilde{A}^{(I)}_a = 0$ if $\hat{\lambda} - \hat{r} < \hat{\lambda}'$, i.e. if the point is not in the causal future of the charge at the North Pole. Note also that the region covered by the coordinates $(\hat{\lambda}, \hat{r})$ does not intersect the causal future of the South Pole, and hence that in this region the source field from the charge at the South Pole vanishes.

Let us present the source and initial fields by adding together those in the Feynman gauge found in Ref. [2] and the additional contributions due to the change in the gauge parameter found in this paper. The source field is

$$A^{(S)}_\lambda = \frac{q}{4\pi}\left(\frac{1}{r} - \frac{1}{\lambda + r} + \frac{\zeta}{3\lambda}\right), \tag{35}$$

$$A^{(S)}_r = -\frac{q}{4\pi(\lambda + r)}, \tag{36}$$

where we have let $\lambda_0 \to \infty$. The initial field is

$$A^{(I)}_{\hat{\lambda}} = -\frac{q}{4\pi}\left(\frac{1}{\hat{r}} - \frac{1}{\hat{r} + \hat{\lambda}}\right)\theta(\hat{r} - \hat{\lambda}) - \frac{q\zeta}{12\pi}\left(\frac{1}{\hat{\lambda}} - \frac{1}{\hat{\lambda} + \hat{r}} - \frac{1}{\hat{\lambda} - \hat{r}}\right)\theta(\hat{\lambda} - \hat{r}), \tag{37}$$

$$A^{(I)}_{\hat{r}} = \frac{q}{4\pi(\hat{r} + \hat{\lambda})}\theta(\hat{r} - \hat{\lambda}) + \frac{q\zeta}{12\pi}\left(\frac{1}{\hat{\lambda} + \hat{r}} - \frac{1}{\hat{\lambda} - \hat{r}}\right)\theta(\hat{\lambda} - \hat{r}), \tag{38}$$

where we have let $\hat{\lambda}' \to 0$. The initial field in the overlapping region can be written in coordinates $(\lambda, r)$, using Eqs. (27) and (28), as

$$\tilde{A}^{(I)}_\lambda|_{\hat{\lambda}>\hat{r}} = -\frac{q\zeta}{12\pi\lambda}, \tag{39}$$

$$\tilde{A}^{(I)}_r|_{\hat{\lambda}>\hat{r}} = 0. \tag{40}$$

Thus, the sum $A^{(S)}_a + A^{(I)}_a$ is $\zeta$ independent and reproduces the field configuration given by Eqs. (8) and (9) in the region with $\tau < \pi/2 - \chi$, i.e. in the causal past of the North Pole. It is clear that this conclusion holds for the causal past of the South Pole with $\tau < \chi - \pi/2$. Hence, we can conclude that Eq. (2) reproduces the correct field configuration for $\tau < 0$. Then by using the uniqueness of the solution to the field equations for the electromagnetic potential for given initial data on the Cauchy surface $\tau = -\pi/4$, say, we conclude that Eq. (2) reproduces the correct field over the whole spacetime for any $\zeta$, extending the result for $\zeta = 1$ in Ref. [2].

The work of J. R. N. was supported in part by Nuffield Science Bursary under Contract No. URB/35537 in July 2008.

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