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Quantum corrections to the Larmor radiation formula in scalar electrodynamics

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We use the semiclassical approximation in perturbative scalar quantum electrodynamics to calculate the quantum correction to the Larmor radiation formula to first order in Planck's constant in the nonrelativistic approximation, choosing the initial state of the charged particle to be a momentum eigenstate. We calculate this correction in two cases: in the first case the charged particle is accelerated by a time-dependent but space-independent vector potential whereas in the second case it is accelerated by a time-independent vector potential which is a function of one spatial coordinate. We find that the corrections in these two cases are different even for a charged particle with the same classical motion. The correction in each case turns out to be nonlocal in time in contrast to the classical approximation.

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I. INTRODUCTION

A well-known result in classical electrodynamics, discovered during the burst of activity in the late nineteenth century, is that an accelerated charge emits radiation. In particular, the formula which gives the amount of energy radiated by the charge was found by Larmor in this period. The relativistic generalization of this formula is

$$E_{\text{em}}^{(0)} = -\frac{e^2}{6\pi c^3}\int dt \frac{d^2x^\mu}{d\tau^2}\frac{d^2x_\mu}{d\tau^2}, \tag{1.1}$$

where e is the charge of the particle and c is the speed of light. Here and below the metric signature is $+---$ and τ is the proper time along the world line of the particle, $x^\mu(\tau)$, with $x^0 = ct$. (See Ref. [1], Sec. 14.2, for a derivation of this result.)

Since classical electrodynamics is an approximation to quantum electrodynamics (QED), one expects that the Larmor formula should be reproduced in the latter theory in the limit $\hbar \to 0$ (at order e^2). Indeed it has been shown that this formula is recovered in QED for a scalar charged particle moving on a straight line in the limit $\hbar \to 0$ [2]. Furthermore it has been shown [3,4] that the Lorentz-Dirac radiation-reaction force [5–7] is obtained in the limit $\hbar \to 0$ in QED for a charged scalar particle in three-dimensional motion under the influence of a vector potential depending only on one spacetime coordinate. (For other approaches for studying the Lorentz-Dirac force in the context of QED, see Refs. [8–12].) This work indirectly shows that the Larmor formula is reproduced in the limit $\hbar \to 0$ for a charged scalar particle in three-dimensional motion under the conditions specified because the Lorentz-Dirac force and energy-momentum conservation imply the Larmor formula.

Although the Larmor formula correctly gives the amount of energy emitted as radiation in the limit $\hbar \to 0$, it is clearly not exact. For example, in Ref. [13] a model with a charged scalar particle which is soluble to order e^2 in QED was studied and the exact result for the energy emitted was shown to differ from the Larmor formula. It will be interesting, therefore, to estimate the correction of order $\hbar$ to the Larmor formula for general motion of the charged particle. The purpose of this paper is to carry out this task in the simple setting used in Refs. [2–4] where the scalar particle is accelerated by a vector potential that depends only on one spacetime coordinate under the additional condition that the initial state of the charged particle is a momentum eigenstate.

One might hope that there would be a universal expression for this correction which depended only on the motion of the corresponding classical particle, but we find that the correction depends on how the particle is accelerated. For this reason we calculate the quantum correction to the Larmor formula at order $\hbar$ in two cases: in the first case the charged particle is accelerated by a time-dependent but space-independent vector potential whereas in the second case it is accelerated by a time-independent vector potential which is a function of one spatial coordinate. We also use the nonrelativistic approximation because a fully relativistic calculation would be too complicated for the purpose of this paper, which is to show how the quantum correction to the Larmor formula can be found in simple examples.

The rest of the paper is organized as follows. In Sec. II, we show directly that the Larmor formula is reproduced in scalar QED in the limit $\hbar \to 0$. We then proceed in Secs. III and IV to calculate the correction to this formula at order $\hbar$ in the two cases mentioned above. Finally, in Sec. V we provide a summary and concluding remarks. Throughout this paper we retain $\hbar$ explicitly but let $c = 1$ except where it is convenient not to do so.

II. THE LARMOR FORMULA IN QED

In this section we derive the Larmor formula from QED for a charged scalar particle accelerated by a vector poten-

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tial V^μ which depends only on t. We follow Refs. [2,3] closely. (The derivation for the case with a potential which depends on one space coordinate will not be presented, but it is very similar to the case treated here.) We assume that the variation in $V^\mu(t)$ occurs only over a bounded interval $[-T, T]$, $T > 0$. (This assumption is needed to make the transition amplitude in the QED calculation finite.) We let $V^\mu(t) = 0$ for $t < -T$ without loss of generality and $V^\mu(t)$ for $t > T$ be a constant which is not necessarily zero.[1] We also use a gauge transformation to impose the condition $V_0(t) = 0$ for all t.

The Lagrangian density of our model is

$$\mathcal{L} = [(D_\mu + ieA_\mu)\phi]^\dagger[(D^\mu + ieA^\mu)\phi] - \frac{m^2}{\hbar^2}\phi^\dagger\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2, \tag{2.1}$$

where $D_\mu \equiv \partial_\mu + \frac{i}{\hbar}V_\mu$. We have adopted the Feynman gauge, in which the noninteracting field equation, i.e. the field equation with $e = 0$, for A_μ is $\partial_\nu\partial^\nu A^\mu = 0$. We can therefore expand it in terms of momentum modes,

$$A_\mu(x) = \int \frac{d^3\mathbf{k}}{2k(2\pi)^3}[a_\mu(\mathbf{k})e^{-ik\cdot x} + a_\mu{}^\dagger(\mathbf{k})e^{ik\cdot x}], \tag{2.2}$$

where $k = |\mathbf{k}|$. The operators $a_\mu(\mathbf{k})$ and $a_\mu{}^\dagger(\mathbf{k})$ obey the usual commutation relations,

$$[a_\mu(\mathbf{k}), a_\nu{}^\dagger(\mathbf{k}')] = -2\hbar k(2\pi)^3 g_{\mu\nu}\delta^3(\mathbf{k} - \mathbf{k}'). \tag{2.3}$$

We can use the Fourier expansion for the scalar field as well. Thus we write

$$\phi(x) = \hbar\int \frac{d^3\mathbf{p}}{2p_0(2\pi\hbar)^3}[A(\mathbf{p})\Phi_{\mathbf{p}}(x) + B^\dagger(\mathbf{p})\bar{\Phi}^*_{\mathbf{p}}(x)], \tag{2.4}$$

where $p_0 = \sqrt{|\mathbf{p}|^2 + m^2}$. The mode functions $\Phi_{\mathbf{p}}(x)$ are different from the standard "free" mode functions, $e^{-ip\cdot x/\hbar}$. [We do not need to consider the antiparticle modes $\bar{\Phi}_{\mathbf{p}}(x)$ though their relation to $\Phi_{\mathbf{p}}(x)$ is very simple.] This is because the equation of motion for the scalar field with $e = 0$ is not the free field equation, but rather,

$$(\hbar^2 D_\mu D^\mu + m^2)\Phi_{\mathbf{p}}(x) = 0. \tag{2.5}$$

Since the potential $V_\mu(t)$ depends only on t, these mode functions can be written in the following form:

$$\Phi_{\mathbf{p}}(x) = \sqrt{p_0}\,\phi_{\mathbf{p}}(t)\exp\left(\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}\right). \tag{2.6}$$

Since we are interested in the limit $\hbar \to 0$, we use the WKB approximation, which gives

$$\phi_{\mathbf{p}}(t) = \frac{1}{\sqrt{\sigma_{\mathbf{p}}(t)}}\exp\left[-\frac{i}{\hbar}\int_0^t \sigma_{\mathbf{p}}(\zeta)d\zeta\right]\psi_{\mathbf{p}}(t), \tag{2.7}$$

where

$$\sigma_{\mathbf{p}}(t) \equiv \sqrt{|\mathbf{p} - \mathbf{V}(t)|^2 + m^2} \tag{2.8}$$

is the kinetic energy of a scalar particle with momentum $\mathbf{p}$. The function $\psi_{\mathbf{p}}(t)$ contains the corrections of higher order in $\hbar$, i.e.

$$\psi_{\mathbf{p}}(t) = 1 + i\hbar g_{\mathbf{p}}(t) + O(\hbar^2). \tag{2.9}$$

It can readily be shown that $g_{\mathbf{p}}(t)$ is real. The nontrivial commutation relations among annihilation and creation operators are

$$[A(\mathbf{p}), A^\dagger(\mathbf{p}')] = [B(\mathbf{p}), B^\dagger(\mathbf{p}')] = 2p_0(2\pi\hbar)^3\delta^3(\mathbf{p} - \mathbf{p}'). \tag{2.10}$$

The operators $A^\dagger(\mathbf{p})$ and $B^\dagger(\mathbf{p})$ create a particle and an antiparticle, respectively.

The initial state with one charged scalar particle and no photon can be given in general as

$$|i\rangle = \int \frac{d^3\mathbf{p}}{\sqrt{2p_0}(2\pi\hbar)^3}f(\mathbf{p})A^\dagger(\mathbf{p})|0\rangle. \tag{2.11}$$

This state is normalized so that $\langle i|i\rangle = 1$. This condition implies

$$\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}|f(\mathbf{p})|^2 = 1. \tag{2.12}$$

It is sufficient to assume that the function $f(\mathbf{p})$ is peaked about a given momentum with width of order $\hbar$ to derive the Larmor formula. However, this assumption will not be sufficient when we come to consider its quantum correction. For this reason we assume that $f(\mathbf{p})$ is sharply peaked with an arbitrary accuracy and take the limit such that $|f(\mathbf{p})|^2$ is proportional to a delta function at an appropriate stage. This procedure amounts to the condition that the initial state is a momentum eigenstate.

An initial state with one charged particle evolves in general to order e^2 as

$$A^\dagger(\mathbf{p})|0\rangle \mapsto [1 + i\hbar^{-1}\mathcal{F}(\mathbf{p})]A^\dagger(\mathbf{p})|0\rangle + \frac{i}{\hbar}\int \frac{d^3\mathbf{k}}{2k(2\pi)^3}\mathcal{A}^\mu(\mathbf{p}, \mathbf{k})a_\mu{}^\dagger(\mathbf{k})A^\dagger(\mathbf{P})|0\rangle, \tag{2.13}$$

where $\mathbf{P} = \mathbf{p} - \hbar\mathbf{k}$ is the outgoing momentum of the scalar particle when a photon is emitted, $\mathcal{A}^\mu(\mathbf{p}, \mathbf{k})$ is the amplitude for the emission of one photon, and $\mathcal{F}(\mathbf{p})$ is the forward-scattering amplitude, which plays no role in this paper. Thus, the initial state $|i\rangle$ evolves to

$$|f\rangle = |f_0\rangle + |f_1\rangle, \tag{2.14}$$

[1] In Refs. [2,3] the convention was slightly different in that V^μ was chosen to satisfy $V^\mu(t) = 0$ for positive t.

where

$$|f_0\rangle = \int \frac{d^3\mathbf{p}}{\sqrt{2p_0}(2\pi\hbar)^3}[1 + i\hbar^{-1}\mathcal{F}(\mathbf{p})]f(\mathbf{p})A^\dagger(\mathbf{p})|0\rangle, \tag{2.15}$$

$$|f_1\rangle = \frac{i}{\hbar}\int \frac{d^3\mathbf{p}}{\sqrt{2p_0}(2\pi\hbar)^3} \times \int \frac{d^3\mathbf{k}}{2k(2\pi)^3} f(\mathbf{p})\mathcal{A}^\mu(\mathbf{p},\mathbf{k})a_\mu{}^\dagger(\mathbf{k})A^\dagger(\mathbf{P})|0\rangle. \tag{2.16}$$

The emission probability in the limit where $f(\mathbf{p})$ is arbitrarily sharply peaked can be found using the commutation relations (2.3) and (2.10) as

$$\Gamma = \langle f_1|f_1\rangle = \frac{1}{\hbar}\int \frac{d^3\mathbf{k}}{2k(2\pi)^3}\frac{P_0}{p_0}\left|\frac{\partial\mathbf{P}}{\partial\mathbf{p}}\right|^{-1}|\mathcal{A}(\mathbf{p},\mathbf{k})|^2, \tag{2.17}$$

where $|\mathcal{A}(\mathbf{p},\mathbf{k})|^2 \equiv -\mathcal{A}^*_\mu(\mathbf{p},\mathbf{k})\mathcal{A}^\mu(\mathbf{p},\mathbf{k})$ and where

$$\frac{\partial\mathbf{P}}{\partial\mathbf{p}} \equiv \det\left(\frac{\partial P^i}{\partial p^j}\right) \tag{2.18}$$

is the Jacobian determinant. The momentum $\mathbf{p}$ is now the peak value of the momentum distribution of the initial state. The energy emitted is obtained by multiplying the integrand in Eq. (2.17) by the photon energy, $\hbar k$. We have $\partial\mathbf{P}/\partial\mathbf{p} = 1$ because $\mathbf{P} = \mathbf{p} - \hbar\mathbf{k}$. Hence, the 4-momentum of the radiation emitted is

$$\mathcal{P}^\mu = \int \frac{d^3\mathbf{k}}{16\pi^3}\frac{P_0}{p_0}n^\mu|\mathcal{A}(\mathbf{p},\mathbf{k})|^2, \tag{2.19}$$

where $n^\mu \equiv k^\mu/k$. It can be shown [3,4] that

$$\mathcal{A}_\mu(\mathbf{p},\mathbf{k}) = -ie\hbar\int\frac{d^3\mathbf{p}'}{2p'_0(2\pi\hbar)^3} \times \int d^4x e^{ik\cdot x}[\Phi^*_{\mathbf{p}'}(x)D_\mu\Phi_{\mathbf{p}}(x) - (D_\mu\Phi_{\mathbf{p}'}(x))^*\Phi_{\mathbf{p}}(x)]. \tag{2.20}$$

Since $\Phi_{\mathbf{p}}(x) = \sqrt{p_0}\phi_{\mathbf{p}}(t)e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}$, the exponential factors in the integrand of Eq. (2.20) result in $(2\pi\hbar)^3\delta^3(\mathbf{p} - \hbar\mathbf{k} - \mathbf{p}')$ upon integration over $\mathbf{x}$. Thus, we find

$$\mathcal{A}_i(\mathbf{p},\mathbf{k}) = -\frac{e}{2}\sqrt{\frac{p_0}{P_0}}\int dte^{ikt}\phi^*_{\mathbf{P}}(t)\phi_{\mathbf{p}}(t) \times [p_i + P_i - 2V_i(t)], \tag{2.21}$$

$$\mathcal{A}_0(\mathbf{p},\mathbf{k}) = -\frac{ie\hbar}{2}\sqrt{\frac{p_0}{P_0}}\int dte^{ikt}\left[\phi^*_{\mathbf{P}}(t)\frac{d\phi_{\mathbf{p}}(t)}{dt} - \frac{d\phi^*_{\mathbf{P}}(t)}{dt}\phi_{\mathbf{p}}(t)\right]. \tag{2.22}$$

Now we use the WKB approximation (2.7) and find

$$\phi^*_{\mathbf{P}}(t)\phi_{\mathbf{p}}(t) = \frac{1}{\sqrt{\sigma_{\mathbf{P}}(t)\sigma_{\mathbf{p}}(t)}} \times \exp\left\{-\frac{i}{\hbar}\int_0^t[\sigma_{\mathbf{p}}(\zeta) - \sigma_{\mathbf{P}}(\zeta)]d\zeta\right\}\psi^*_{\mathbf{P}}(t)\psi_{\mathbf{p}}(t). \tag{2.23}$$

To lowest order in $\hbar$ we have

$$-\frac{i}{\hbar}\int_0^t[\sigma_{\mathbf{p}}(\zeta) - \sigma_{\mathbf{P}}(\zeta)]d\zeta \approx -i\mathbf{k}\cdot\int_0^t\frac{\mathbf{p} - \mathbf{V}(\zeta)}{\sigma_{\mathbf{p}}(\zeta)}d\zeta, \tag{2.24}$$

where the relation $\mathbf{P} = \mathbf{p} - \hbar\mathbf{k}$ has been used. If $\mathbf{x}(t)$ is the position of a classical particle corresponding to the state $A^\dagger(\mathbf{p})|0\rangle$, i.e. with momentum $\mathbf{p}$ under the influence of the vector potential $\mathbf{V}(t)$, then

$$m\frac{d\mathbf{x}}{d\tau} = \mathbf{p} - \mathbf{V}(t), \tag{2.25}$$

$$m\frac{dt}{d\tau} = \sigma_{\mathbf{p}}(t). \tag{2.26}$$

These relations imply $[\mathbf{p} - \mathbf{V}(t)]/\sigma_{\mathbf{p}}(t) \approx d\mathbf{x}/dt$ to lowest order in $\hbar$. Using this approximation in Eq. (2.24) and substituting the result into Eq. (2.23) and requiring $\mathbf{x}(0) = \mathbf{0}$, we find to lowest order in $\hbar$ that

$$\phi^*_{\mathbf{P}}(t)\phi_{\mathbf{p}}(t) \approx \frac{1}{\sigma_{\mathbf{p}}(t)}e^{-i\mathbf{k}\cdot\mathbf{x}}. \tag{2.27}$$

Also it can readily be shown that

$$i\hbar\frac{d\phi_{\mathbf{p}}(t)}{dt} \approx \sigma_{\mathbf{p}}(t)\phi_{\mathbf{p}}(t). \tag{2.28}$$

Substituting Eqs. (2.27) and (2.28) into Eqs. (2.21) and (2.22), and using Eqs. (2.25) and (2.26), we find

$$\mathcal{A}^0(\mathbf{p},\mathbf{k}) = -e\int dte^{ik\cdot x}, \tag{2.29}$$

$$\mathcal{A}^i(\mathbf{p},\mathbf{k}) = -e\int dt\frac{dx^i}{dt}e^{ik\cdot x}, \tag{2.30}$$

which can be combined as

$$\mathcal{A}^\mu(\mathbf{p},\mathbf{k}) = -e\int d\xi\frac{dx^\mu}{d\xi}e^{ik\xi}, \tag{2.31}$$

where $\xi \equiv n\cdot x$.

Equation (2.31) is ill-defined because $dx^\mu/d\xi$ is finite for arbitrarily large values of $|\xi|$. We therefore introduce a compactly supported cutoff factor, $\chi(a\xi)$, $0 < a \le 1$, which is 1 on a compact interval including the region where the acceleration takes place, and smoothly varies between 0 and 1. Then the emission amplitude becomes

$$\mathcal{A}^\mu(\mathbf{p},\mathbf{k}) = -e\int d\xi \frac{dx^\mu}{d\xi}\chi(a\xi)e^{ik\xi} \tag{2.32}$$

$$= -\frac{ie}{k}\int d\xi\left[\frac{d^2x^\mu}{d\xi^2}\chi(a\xi) + a\frac{dx^\mu}{d\xi}\chi'(a\xi)\right]e^{ik\xi}. \tag{2.33}$$

By substituting this equation into Eq. (2.19) and taking the limit $a \to 0$, we find the 4-momentum of the radiation emitted to lowest order in $\hbar$ and e as

$$\mathcal{P}^\mu = -\frac{e^2}{16\pi^2}\int d\Omega\int d\xi n^\mu \frac{d^2x^\nu}{d\xi^2}\frac{d^2x_\nu}{d\xi^2}, \tag{2.34}$$

where $d\Omega$ is the solid-angle for the unit vector $\mathbf{n} = \mathbf{k}/k$. We convert the ξ derivative to the t derivative by using the formula $d\xi/dt = n_\mu dx^\mu/dt$ as

$$\frac{d^2x^\mu}{d\xi^2} = \left(\frac{dt}{d\xi}\right)^3\left(\frac{d\xi}{dt}\frac{d^2x^\mu}{dt^2} - \frac{d^2\xi}{dt^2}\frac{dx^\mu}{dt}\right). \tag{2.35}$$

The result is

$$\mathcal{P}^\mu = -\frac{e^2}{16\pi^2}\int dt\int d\Omega\,\dot{\xi}^{-5}n^\mu n_\sigma n_\rho \times[\dot{x}^\sigma\dot{x}^\rho\ddot{x}^\nu\ddot{x}_\nu - 2\dot{x}^\sigma\ddot{x}^\rho\ddot{x}^\nu\dot{x}_\nu + \ddot{x}^\sigma\ddot{x}^\rho\dot{x}^\nu\dot{x}_\nu], \tag{2.36}$$

where the dot indicates the t derivative. The integration over the solid angle can be carried out by using (see Ref. [3])

$$\int d\Omega\,\dot{\xi}^{-5}n^\mu n_\sigma n_\rho = \tfrac{4}{3}\pi[6\gamma^8\dot{x}^\mu\dot{x}_\sigma\dot{x}_\rho - \gamma^6(\delta^\mu_\sigma\dot{x}_\rho + \dot{x}^\mu g_{\sigma\rho} + \delta^\mu_\rho\dot{x}_\sigma)], \tag{2.37}$$

where $\gamma \equiv dt/d\tau = (\dot{x}^\mu\dot{x}_\mu)^{-1/2}$. Thus we obtain

$$\mathcal{P}^\mu = -\frac{e^2}{6\pi}\int dt\dot{x}^\mu[\gamma^4\ddot{x}\cdot\ddot{x} - \gamma^6(\dot{x}\cdot\ddot{x})^2]. \tag{2.38}$$

By converting the t derivative to the τ derivative, we find

$$\mathcal{P}^\mu = -\frac{e^2}{6\pi c^4}\int d\tau\frac{dx^\mu}{d\tau}\frac{d^2x^\nu}{d\tau^2}\frac{d^2x_\nu}{d\tau^2}, \tag{2.39}$$

which is a well-known result in classical electrodynamics, and the component $\mathcal{P}^0 c$ gives the Larmor formula (1.1).

III. QUANTUM CORRECTION WITH TIME-DEPENDENT VECTOR POTENTIAL

In this section we calculate the correction to the Larmor formula at order $\hbar$ to lowest order in the nonrelativistic approximation in the case where the charged scalar particle is accelerated by a time-dependent but space-independent vector potential.

From Eq. (2.19) we find the energy emitted as

$$E_{\text{em}} = -\int\frac{d^3\mathbf{k}}{16\pi^3}\frac{P_0}{p_0}\mathcal{A}^*_\mu(\mathbf{p},\mathbf{k})\mathcal{A}^\mu(\mathbf{p},\mathbf{k}), \tag{3.1}$$

where $\mathcal{A}_i(\mathbf{p},\mathbf{k})$ and $\mathcal{A}_0(\mathbf{p},\mathbf{k})$ are given by Eqs. (2.21) and (2.22), respectively. By substituting the WKB expression for $\phi_\mathbf{p}(t)$ given by Eq. (2.7) into these equations we find

$$\begin{aligned}\mathcal{A}^0(\mathbf{p},\mathbf{k}) = &-\frac{ec}{2}\sqrt{\frac{p_0}{P_0}}\int dte^{i\omega t}\frac{1}{\sqrt{\sigma_\mathbf{p}(t)\sigma_\mathbf{P}(t)}}\\ &\times\left\{\sigma_\mathbf{p}(t) + \sigma_\mathbf{P}(t) + i\hbar c\left[\frac{\sigma'_\mathbf{P}(t)}{2\sigma_\mathbf{P}(t)} - \frac{\sigma'_\mathbf{p}(t)}{2\sigma_\mathbf{p}(t)}\right.\right.\\ &\left.\left.+\frac{\psi'_\mathbf{p}(t)}{\psi_\mathbf{p}(t)} - \frac{\psi^{*\prime}_\mathbf{P}(t)}{\psi^*_\mathbf{P}(t)}\right]\right\}\exp\left[-\frac{ic}{\hbar}\int_0^t(\sigma_\mathbf{p}(\zeta)\right.\\ &\left.-\sigma_\mathbf{P}(\zeta))d\zeta\right]\psi^*_\mathbf{P}(t)\psi_\mathbf{p}(t),\end{aligned} \tag{3.2}$$

$$\begin{aligned}\mathcal{A}^i(\mathbf{p},\mathbf{k}) = &-\frac{ec}{2}\sqrt{\frac{p_0}{P_0}}\int dte^{i\omega t}\frac{1}{\sqrt{\sigma_\mathbf{p}(t)\sigma_\mathbf{P}(t)}}\\ &\times[p^i - V^i(t) + P^i - V^i(t)]\\ &\times\exp\left\{-\frac{ic}{\hbar}\int_0^t[\sigma_\mathbf{p}(\zeta) - \sigma_\mathbf{P}(\zeta)]d\zeta\right\}\psi^*_\mathbf{P}(t)\psi_\mathbf{p}(t),\end{aligned} \tag{3.3}$$

where $\sigma_\mathbf{p} = \sqrt{|\mathbf{p}-\mathbf{V}|^2 + m^2c^2}$ and $\omega = kc$. We have restored factors of c by dimensional analysis, anticipating the use of the nonrelativistic approximation. It is straightforward to calculate the amplitude to order $\hbar$ using $\mathbf{P} = \mathbf{p} - \hbar\mathbf{k}$. The result is

$$\begin{aligned}\mathcal{A}^0(\mathbf{p},\mathbf{k}) = &-ec\sqrt{\frac{p_0}{P_0}}\int dte^{i\omega t - i\mathbf{k}\cdot\mathbf{x}}\\ &\times\exp\left\{i\frac{\hbar c}{2}\int_0^t\left[\frac{k^2}{\sigma_\mathbf{p}(t)} - \frac{(\mathbf{k}\cdot\dot{\mathbf{x}})^2}{\sigma_\mathbf{p}(t)c^2}\right]d\zeta\right\},\end{aligned} \tag{3.4}$$

$$\begin{aligned}\mathcal{A}^i(\mathbf{p},\mathbf{k}) = &-ec\sqrt{\frac{p_0}{P_0}}\int dte^{i\omega t - i\mathbf{k}\cdot\mathbf{x}}\left\{\frac{\dot{x}^i}{c} - \frac{\hbar}{2\sigma_\mathbf{p}(t)}\right.\\ &\left.\times\left[k^i - \frac{\dot{x}^i(\mathbf{k}\cdot\dot{\mathbf{x}})}{c^2}\right]\right\}\exp\left\{i\frac{\hbar c}{2}\int_0^t\left[\frac{k^2}{\sigma_\mathbf{p}(\zeta)}\right.\right.\\ &\left.\left.-\frac{(\mathbf{k}\cdot\dot{\mathbf{x}}(\zeta))^2}{\sigma_\mathbf{p}(\zeta)c^2}\right]d\zeta\right\}.\end{aligned} \tag{3.5}$$

Note, in particular, that there is no contribution from the factor $\psi^*_\mathbf{P}(t)\psi_\mathbf{p}(t) \approx 1 + i\hbar(g_\mathbf{p}(t) - g_\mathbf{P}(t))$ at order $\hbar$.

One could write down a formal expression for the expected amount of energy emitted to order $\hbar$ by substituting these formulas into Eq. (3.1). Instead of doing so, we use the nonrelativistic approximation in order to find an expression in terms of the classical trajectory of the particle in closed form. We calculate the correction from the exponential factor common to both $\mathcal{A}^i$ and A^0 and that from the additional term in $\mathcal{A}^i$ separately and add them up.

Denoting the correction due to the exponential factor by ΔE_1, we have

$$\Delta E_1 = \frac{ie^2\hbar}{32\pi^3 c^3}\int d\Omega \int_0^\infty d\omega\omega^4 \int dtdt' \times e^{i\omega(t-t')-i\omega\mathbf{n}\cdot[\mathbf{x}(t)-\mathbf{x}(t')]/c}[\dot{\mathbf{x}}(t)\cdot\dot{\mathbf{x}}(t') - c^2] \times \int_{t'}^{t}\left[\frac{1}{\sigma_{\mathbf{p}}(\zeta)c} - \frac{(\mathbf{n}\cdot\dot{\mathbf{x}}(\zeta))^2}{\sigma_{\mathbf{p}}(\zeta)c^3}\right]d\zeta. \tag{3.6}$$

We use the nonrelativistic approximation to order c^{-5}. Thus, we expand the factor $e^{-i\omega\mathbf{n}\cdot[\mathbf{x}(t')-\mathbf{x}(t)]/c}$ with respect to ω/c to order c^{-2}. [Notice that $\sigma_{\mathbf{p}}(t) \approx mc$ to lowest order in c^{-1}.] Then we integrate over ω, regularizing the integral by changing $e^{i\omega(t-t')}$ to $e^{i\omega(t-t'+i\varepsilon)}$ and using the formula

$$\int_0^\infty \omega^n e^{i\omega(t-t'+i\varepsilon)}d\omega = i^{n+1}\frac{\partial^n}{\partial t'^n}\frac{1}{t-t'+i\varepsilon}. \tag{3.7}$$

Thus, we obtain

$$\Delta E_1 = -\frac{e^2\hbar}{32\pi^3 c^3}\int d\Omega\int dtdt'\left[\frac{4!}{(t-t'+i\varepsilon)^5} + \frac{6!\{\mathbf{n}\cdot[\mathbf{x}(t')-\mathbf{x}(t)]\}^2}{2(t-t'+i\varepsilon)^7 c^2}\right][\dot{\mathbf{x}}(t)\cdot\dot{\mathbf{x}}(t') - c^2] \times \int_{t'}^{t}\left[\frac{1}{\sigma_{\mathbf{p}}(\zeta)c} - \frac{(\mathbf{n}\cdot\dot{\mathbf{x}}(\zeta))^2}{mc^4}\right]d\zeta. \tag{3.8}$$

This integral is ill-defined since the integrand remains finite if we let $|t+t'|$ be arbitrarily large while keeping $t-t'$ finite. For this reason we insert a cutoff factor $\chi(at)\chi(at')$, $0 < a \le 1$, such that $\chi(at)$ is smooth and compactly supported, and that $\chi(at) = 1$ for $t \in [-T, T]$, i.e. while $V^\mu(t)$ is not constant. Then, we find that this integral is the sum of terms of the form $A_1^{(1)}$ and $A_1^{(3)}$ as defined in Eq. (A1). Therefore, as is shown in Appendix A, we can formally integrate by parts with respect to t and t' to reduce the power of $t'-t+i\varepsilon$ in the denominator.[2] Then we find

$$-c^2\int dtdt'\frac{4!}{(t-t'+i\varepsilon)^5}\int_{t'}^{t}\left[\frac{1}{\sigma_{\mathbf{p}}(\zeta)c} - \frac{(\mathbf{n}\cdot\dot{\mathbf{x}}(\zeta))^2}{mc^4}\right]d\zeta = 0 \tag{3.9}$$

by integrating by parts with respect to t and t'. This means that, to find ΔE_1 to order c^{-5}, we can let

$$\int_{t'}^{t}\left[\frac{1}{\sigma_{\mathbf{p}}(\zeta)c} - \frac{(\mathbf{n}\cdot\dot{\mathbf{x}}(\zeta))^2}{mc^4}\right]d\zeta \approx \frac{1}{mc^2}(t-t'). \tag{3.10}$$

Hence we have

$$\Delta E_1 = -\frac{e^2\hbar}{8\pi^3 mc^5}\int d\Omega\int dtdt'\left[\frac{3!\dot{\mathbf{x}}(t)\cdot\dot{\mathbf{x}}(t')}{(t-t'+i\varepsilon)^4} - \frac{3\cdot 5!\{\mathbf{n}\cdot[\mathbf{x}(t')-\mathbf{x}(t)]\}^2}{4(t-t'+i\varepsilon)^6}\right]. \tag{3.11}$$

Integrating the second term by parts with respect to t and t' and carrying out the $\mathbf{n}$ integration, we find

$$\Delta E_1 = -\frac{e^2\hbar}{4\pi^2 mc^5}\int dtdt'\frac{3!\dot{\mathbf{x}}(t)\cdot\dot{\mathbf{x}}(t')}{(t-t'+i\varepsilon)^4} = -\frac{e^2\hbar}{8\pi^2 mc^5}\int dtdt'\dot{\mathbf{x}}(t)\cdot\dot{\mathbf{x}}(t')\left(\frac{\partial^3}{\partial t^2\partial t'} - \frac{\partial^3}{\partial t'^2\partial t}\right) \times \frac{1}{t-t'+i\varepsilon}. \tag{3.12}$$

By integrating by parts, we find

$$\Delta E_1 = \frac{e^2\hbar}{8\pi^2 mc^5}\int dtdt'\left(\frac{d^3\mathbf{x}}{dt^3}\cdot\frac{d^2\mathbf{x}'}{dt'^2} - \frac{d^2\mathbf{x}}{dt^2}\cdot\frac{d^3\mathbf{x}'}{dt'^3}\right)\frac{1}{t-t'}, \tag{3.13}$$

where we have defined $x^i \equiv x^i(t)$ and $x'^i \equiv x^i(t')$.

We move now to the correction which comes from the other multiplicative factor in $\mathcal{A}^i(\mathbf{p},\mathbf{k})$. Since we only need this quantity to order c^{-2}, we find from Eq. (3.5)

$$\mathcal{A}^i(\mathbf{p},\mathbf{k})|_{\mathrm{nonex}} \approx -e\sqrt{\frac{p_0}{P_0}}\int dt e^{i\omega t - i\omega\mathbf{n}\cdot\mathbf{x}/c}\left[\dot{x}^i(t) - \frac{\hbar\omega n^i}{2mc}\right], \tag{3.14}$$

where we have dropped the correction to the exponential factor. We find the corresponding correction in the Larmor formula by substituting this formula in Eq. (3.1) as

$$\Delta E_2 = -\frac{e^2\hbar}{32\pi^3 mc^4}\int d\Omega\int_0^\infty d\omega\omega^3 \times \int dtdt' e^{i\omega(t-t')-i\omega\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}')/c}\mathbf{n}\cdot(\dot{\mathbf{x}}'+\dot{\mathbf{x}}). \tag{3.15}$$

By expanding the factor $e^{-i\omega\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}')/c}$ to first order in ω/c and integrating over $\mathbf{n}$ and ω we find

$$\Delta E_2 = \frac{e^2\hbar}{24\pi^2 mc^5}\int dtdt'\frac{4!(\mathbf{x}'-\mathbf{x})\cdot(\dot{\mathbf{x}}+\dot{\mathbf{x}}')}{(t-t'+i\varepsilon)^5} \times \chi(at)\chi(at'). \tag{3.16}$$

This integral is of the form $A_1^{(1)}$ in Eq. (A1). Therefore one can integrate by parts, twice with respect to t and twice with respect to t', neglecting the cutoff factor $\chi(at)\chi(at')$. Then, we find

[2] In some cases formal integration by parts leads to the wrong results [14]. Therefore one needs to justify it by introducing the cutoff factor as we do here.

$$\Delta E_2 = \tfrac{1}{3}\Delta E_1, \tag{3.17}$$

where ΔE_1 is given by Eq. (3.13). By adding ΔE_1 and ΔE_2, we find the total correction to the Larmor formula at order $e^2\hbar$ to be

$$\Delta E = \frac{e^2\hbar}{6\pi^2 mc^5}\int dt dt' \left(\frac{d^3\mathbf{x}}{dt^3}\cdot\frac{d^2\mathbf{x}'}{dt'^2} - \frac{d^2\mathbf{x}}{dt^2}\cdot\frac{d^3\mathbf{x}'}{dt'^3}\right)\frac{1}{t-t'}. \tag{3.18}$$

Let us estimate the size of this correction in a simple situation where the acceleration is linear and given by $a(t) = a_0(1 - t^2/t_0^2)$ for $|t| \le t_0$ and $a(t) = 0$ otherwise. We find

$$\Delta E = -\frac{4e^2\hbar a_0^2}{3\pi^2 mc^5}. \tag{3.19}$$

On the other hand, the energy of radiation emitted according to the Larmor formula can be found from Eq. (1.1) as $E_{\rm em}^{(0)} = 8a_0^2 t_0/45\pi c^3$. Hence we have

$$\frac{|\Delta E|}{E_{\rm em}^{(0)}} = \frac{15\hbar}{2\pi mc^2 t_0}. \tag{3.20}$$

Therefore, the Larmor formula is expected to be a good approximation as long as $t_0 \gg \hbar/mc^2$, which is the time for a light ray to traverse a Compton wavelength of the charged scalar particle. Since the probability distribution for the frequency of the photon emitted is given by the square of the Fourier transform of $a(t)$, the typical energy of the photon emitted will be of order $\hbar/t_0$ (though the probability of emission can be made small by letting a_0 be small). This energy will be comparable to mc^2 if $t_0 \sim \hbar/mc^2$. Then the scattered charged scalar particle will be relativistic, and it is not surprising that the nonrelativistic approximation will break down. It is interesting that the classical (nonrelativistic) Larmor formula seems to remain a good approximation as long as the scattered state remains nonrelativistic even if its momentum may be much different from that of the initial state, in the case where the particle is accelerated by a time-dependent but space-independent vector potential.

IV. QUANTUM CORRECTION WITH SPACE-DEPENDENT VECTOR POTENTIAL

In this section we treat the case in which the potential varies in a space coordinate, taken to be z, but is independent of t. As in the previous section we assume further that the external vector potential $V_\mu(z)$ is constant except in the interval $[-Z, Z]$, $Z > 0$, with $V_\mu(z) = 0$ for $z < -Z$. We do not assume the constant value of $V_\mu(z)$ for $z > Z$ to be 0. We further let $V_z(z) = 0$ by a gauge transformation.

The mode functions for the scalar particle can be chosen to be proportional to $\exp[(-ip_0 t + i\mathbf{p}_\perp\cdot\mathbf{x}_\perp)/\hbar]$ with $\mathbf{x}_\perp = (x, y)$ and $\mathbf{p}_\perp = (p_x, p_y)$, where p_x and p_y are the x and y components of the *contravariant* vector $\mathbf{p}$. (Below we also write the z component of a contravariant vector $\mathbf{b}$ as b_z in general.) We use the WKB approximation for the ordinary differential equation which determines the z dependence of the mode functions. The particle (as opposed to antiparticle) solution to the field equation thus obtained that is moving in the positive z direction is

$$\Phi_{\mathbf{p}}(t,\mathbf{x}) = \sqrt{\frac{p}{\kappa_{\mathbf{p}}(z)}}\exp\left[\frac{i}{\hbar}\int_0^z \kappa_{\mathbf{p}}(\zeta)d\zeta\right] \times \exp\left[\frac{i}{\hbar}(\mathbf{p}_\perp\cdot\mathbf{x}_\perp - p_0 t)\right], \tag{4.1}$$

where the function analogous to the varying energy $\sigma_{\mathbf{p}}(t)$ in the time-dependent case is now a varying z component of the momentum,

$$\kappa_{\mathbf{p}}(z) = \sqrt{[p_0 - V_0(z)]^2 - |\mathbf{p}_\perp - \mathbf{V}_\perp(z)|^2 - m^2}, \tag{4.2}$$

and where $p = \sqrt{p_0{}^2 - |\mathbf{p}_\perp|^2 - m^2}$. [We assume that $\kappa_{\mathbf{p}}(z)$ is real for all z.] As in the case with a time-dependent vector potential, it can be shown that higher-order corrections to Eq. (4.1) do not contribute to the energy emitted at order $\hbar$.

The Jacobian determinant defined by Eq. (2.18) is

$$\frac{\partial\mathbf{P}}{\partial\mathbf{p}} = \frac{dP}{dp}, \tag{4.3}$$

where the derivative of $P = \sqrt{P_0{}^2 - |\mathbf{P}_\perp|^2 - m^2}$, with $P_0 = p_0 - \hbar k$, $\mathbf{P}_\perp = \mathbf{p}_\perp - \hbar\mathbf{k}_\perp$, is taken with $\mathbf{p}_\perp$ and $\mathbf{k}$ fixed. Hence, the energy emitted is given, in the limit where the momentum distribution is arbitrarily sharply peaked, by

$$E_{\rm em} = -\int\frac{d^3\mathbf{k}}{16\pi^3}\frac{P_0}{p_0}\frac{dp}{dP}\mathcal{A}_\mu^*(\mathbf{p},\mathbf{k})\mathcal{A}^\mu(\mathbf{p},\mathbf{k}), \tag{4.4}$$

where $\mathbf{p}$ is the peak value of the momentum distribution. Many of the details of the calculation which follows find, as one might expect, direct analogues in the time-dependent case. Although occasional mention will be made of these details, many will be left unremarked.

The formula for the emission amplitude, Eq. (2.20), remains the same. After integrating over t, $\mathbf{x}_\perp$ and $\mathbf{p}'$, we find

$$\mathcal{A}_0(\mathbf{p},\mathbf{k}) = \frac{e}{2}\sqrt{\frac{p}{P}}\int dz e^{-ik_z z}\frac{1}{\sqrt{\kappa_{\mathbf{p}}(z)\kappa_{\mathbf{P}}(z)}}[2V_0(z) - (p_0 + P_0)]\exp\left\{\frac{i}{\hbar}\int_0^z[\kappa_{\mathbf{p}}(\zeta) - \kappa_{\mathbf{P}}(\zeta)]d\zeta\right\}, \tag{4.5}$$

$$\mathcal{A}_\perp(\mathbf{p},\mathbf{k}) = \frac{e}{2}\sqrt{\frac{p}{P}}\int dz e^{-ik_z z}\frac{1}{\sqrt{\kappa_{\mathbf{p}}(z)\kappa_{\mathbf{P}}(z)}} \times [2\mathbf{V}_\perp(z) - (\mathbf{p}_\perp + \mathbf{P}_\perp)] \times \exp\left\{\frac{i}{\hbar}\int_0^z[\kappa_{\mathbf{p}}(\zeta) - \kappa_{\mathbf{P}}(\zeta)]d\zeta\right\}, \tag{4.6}$$

and

$$\mathcal{A}_z(\mathbf{p},\mathbf{k}) = \frac{ie\hbar}{2}\sqrt{\frac{p}{P}}\int dz e^{-ik_z z}\frac{1}{\sqrt{\kappa_{\mathbf{p}}(z)\kappa_{\mathbf{P}}(z)}} \times \left\{-\frac{1}{2}\left[\frac{\kappa'_{\mathbf{p}}(z)}{\kappa_{\mathbf{p}}(z)} - \frac{\kappa'_{\mathbf{P}}(z)}{\kappa_{\mathbf{P}}(z)}\right] + \frac{i}{\hbar}[\kappa_{\mathbf{p}}(z) + \kappa_{\mathbf{P}}(z)]\right\} \times \exp\left\{\frac{i}{\hbar}\int_0^z[\kappa_{\mathbf{p}}(\zeta) - \kappa_{\mathbf{P}}(\zeta)]d\zeta\right\}. \tag{4.7}$$

Thus, to order $\hbar$, i.e. letting $\kappa'_{\mathbf{p}}(z)/\kappa_{\mathbf{p}}(z) \approx \kappa'_{\mathbf{P}}(z)/\kappa_{\mathbf{P}}(z)$ in Eq. (4.7), we have

$$|\mathcal{A}(\mathbf{p},\mathbf{k})|^2 = \frac{e^2}{4}\frac{p}{P}\int dz dz' \frac{e^{ik_z(z'-z)}}{\sqrt{\kappa_{\mathbf{p}}(z)\kappa_{\mathbf{P}}(z)\kappa_{\mathbf{p}}(z')\kappa_{\mathbf{P}}(z')}} \times \exp\left\{\frac{i}{\hbar}\int_{z'}^z[\kappa_{\mathbf{p}}(\zeta) - \kappa_{\mathbf{P}}(\zeta)]d\zeta\right\}\{-[2V_0(z) - (p_0 + P_0)][2V_0(z') - (p_0 + P_0)] + [2\mathbf{V}_\perp(z) - (\mathbf{p}_\perp + \mathbf{P}_\perp)]\cdot[2\mathbf{V}_\perp(z') - (\mathbf{p}_\perp + \mathbf{P}_\perp)] + [\kappa_{\mathbf{p}}(z) + \kappa_{\mathbf{P}}(z)] \times [\kappa_{\mathbf{p}}(z') + \kappa_{\mathbf{P}}(z')]\}. \tag{4.8}$$

We find the energy emitted, $E_{\rm em}$, by substituting this formula into Eq. (4.4). We can simplify $E_{\rm em}$ by noting that

$$\frac{P_0}{p_0}\frac{dp}{dP}\frac{p}{P} = 1, \tag{4.9}$$

which can readily be proved by using $dP/dP_0 = P_0/P$, $dp/dp_0 = p_0/p$ and $dP_0/dp_0 = 1$. The following formulas are crucial in expressing the energy emitted in terms of the motion of the corresponding classical particle:

$$\frac{\mathbf{p}_\perp - \mathbf{V}_\perp}{\kappa_{\mathbf{p}}(z)} = \left.\frac{d\mathbf{x}_\perp}{dz}\right|_{\mathbf{p}}, \tag{4.10}$$

$$\frac{p_0 - V_0}{\kappa_p(z)} = \left.\frac{dx^0}{dz}\right|_{\mathbf{p}}, \tag{4.11}$$

where $x^\mu(z)$ is the world line of the classical particle under the potential $V^\mu(z)$, and where "$|_{\mathbf{p}}$" indicates that the quantity is evaluated with the initial momentum $\mathbf{p}$. We obtain

$$E_{\rm em} = -\frac{e^2}{8}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\int dz dz' e^{ik_z(z'-z)} \times \exp\left\{\frac{i}{\hbar}\int_{z'}^z[\kappa_{\mathbf{p}}(\zeta) - \kappa_{\mathbf{P}}(\zeta)]d\zeta\right\} \times \left[\sqrt{\frac{\kappa_{\mathbf{p}}(z)}{\kappa_{\mathbf{P}}(z)}}\left.\frac{dx^\mu}{dz}\right|_{\mathbf{p}} + \sqrt{\frac{\kappa_{\mathbf{P}}(z)}{\kappa_{\mathbf{p}(z)}}}\left.\frac{dx^\mu}{dz}\right|_{\mathbf{P}}\right] \times \left[\sqrt{\frac{\kappa_{\mathbf{p}}(z')}{\kappa_{\mathbf{P}}(z')}}\left.\frac{dx'_\mu}{dz'}\right|_{\mathbf{p}} + \sqrt{\frac{\kappa_{\mathbf{P}}(z')}{\kappa_{\mathbf{p}(z')}}}\left.\frac{dx'_\mu}{dz'}\right|_{\mathbf{P}}\right]. \tag{4.12}$$

The correction to the energy emitted to first order in $\hbar$ again can be attributed to two sources: the exponential factor and the other multiplicative factor. We shall examine these separately but combine the intermediate results to simplify the calculation.

To consider the contribution from the exponential factor, we need to find the expansion of $\kappa_{\mathbf{P}}(z)$ to second order in $\hbar$, which is analogous to that of $\sigma_{\mathbf{P}}(t)$ in the time-dependent case. We have

$$\kappa_{\mathbf{P}}(z) = \kappa_{\mathbf{p}}(z)\left\{1 - \frac{\hbar\omega}{\kappa_{\mathbf{p}}}\left(\frac{dt}{dz} - \frac{\mathbf{n}_\perp}{c}\cdot\frac{d\mathbf{x}_\perp}{dz}\right) + \frac{\hbar^2\omega^2}{2\kappa_{\mathbf{p}}^2}\left[-\left(\frac{dt}{dz} - \frac{\mathbf{n}_\perp}{c}\cdot\frac{d\mathbf{x}_\perp}{dz}\right)^2 + \frac{n_z^2}{c^2}\right]\right\}. \tag{4.13}$$

Therefore, the correction to the energy emitted coming from the exponential factor is

$$\Delta E_1 = \frac{ie^2\hbar}{32\pi^3 m}\int d\Omega\int_0^\infty d\omega\omega^4\int dt dt' \times e^{i\omega(t-t') - i\omega\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}')/c}(\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}' - c^2) \times \int_{t'}^t\left(1 - 2\frac{\mathbf{n}_\perp}{c}\cdot\frac{d\mathbf{x}_\perp}{dT}\right)\left(\frac{dz}{dT}\right)^{-2}dT, \tag{4.14}$$

where $(\mathbf{x}_\perp(T), z(T))$ is the position of the corresponding classical particle at time T with $(\mathbf{x}_\perp(0), z(0)) = (\mathbf{0}, 0)$. After inserting the cutoff factor $\chi(at)\chi(at')$, we again find that the integral is of the form $A_1^{(n)}$, $n = 1, 2, 3$, in Eq. (A1). Hence, one may integrate the ill-defined integral (4.14) formally by parts. This means that

$$\int_0^\infty d\omega\omega^4\int dt dt' e^{i\omega(t-t')}(-c^2) \times \int_{t'}^t\left(1 - 2\frac{\mathbf{n}_\perp}{c}\cdot\frac{d\mathbf{x}_\perp}{dT}\right)\left(\frac{dz}{dT}\right)^{-2}dT = 0. \tag{4.15}$$

By expanding the factor $e^{-i\omega\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}')/c}$ to second order in ω/c and integrating over $\mathbf{n}$, we obtain to lowest order in c^{-1}

$$\Delta E_1 = \frac{ie^2\hbar}{8\pi^2 m}\int_0^\infty d\omega \int dt dt' \omega^4 e^{i\omega(t-t'+i\varepsilon)}\chi(at)\chi(at') \times\bigg[\bigg(\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}' + \frac{\omega^2}{6}|\mathbf{x}-\mathbf{x}'|^2\bigg)\int_{t'}^t \dot{z}^{-2}dT - \frac{2i\omega}{3}(\mathbf{x}_\perp - \mathbf{x}'_\perp)\cdot\int_{t'}^t \dot{\mathbf{x}}_\perp \dot{z}^{-2}dT\bigg]. \quad (4.16)$$

Again we have replaced $e^{i\omega(t-t')}$ by $e^{i\omega(t-t'+i\varepsilon)}$ to regularize the integral over ω.

We now turn to the contribution from the nonexponential factor in Eq. (4.12). For $\mu = m \neq 3$, we have

$$\frac{dx^m}{dz}\bigg|_{\mathbf{P}} = \frac{\kappa_{\mathbf{p}}(z)}{\kappa_{\mathbf{P}}(z)}\frac{dx^m}{dz}\bigg|_{\mathbf{p}} - \frac{\hbar k^m}{\kappa_{\mathbf{P}}(z)}, \quad (4.17)$$

where $x^0 \equiv ct$ and $k^0 \equiv \omega/c$. Since we are only looking for corrections at first order in $\hbar$, we find from Eq. (4.13)

$$\frac{dx^m}{dz}\bigg|_{\mathbf{P}} = \bigg[1 + \frac{\hbar k_n}{\kappa_{\mathbf{p}}(z)}\frac{dx^n}{dz}\bigg]\frac{dx^m}{dz}\bigg|_{\mathbf{p}} - \frac{\hbar k^m}{\kappa_{\mathbf{p}}(z)}, \quad (4.18)$$

where the summations over Roman indices exclude the z component. Here we have used

$$k_n\frac{dx^n}{dz} = \omega\bigg(\frac{dt}{dz} - \frac{\mathbf{n}_\perp}{c}\cdot\frac{d\mathbf{x}_\perp}{dz}\bigg). \quad (4.19)$$

Thus, denoting the contribution from the nonexponential factor by ΔE_2, we have the following result:

$$E_{\rm em}^{(0)} + \Delta E_2 = -\frac{e^2}{16\pi^3 c^3}\int d\Omega \int_0^\infty d\omega\omega^2 \int dz dz' \times e^{i\omega(t-t')-i\omega\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}')/c} \times\bigg\{\sqrt{\frac{\kappa_{\mathbf{p}}(z)}{\kappa_{\mathbf{P}}(z)}}\sqrt{\frac{\kappa_{\mathbf{p}}(z')}{\kappa_{\mathbf{P}}(z')}}\frac{dx^m}{dz}\bigg|_{\mathbf{P}}\frac{dx'_m}{dz'}\bigg|_{\mathbf{P}} - \frac{\hbar k^m}{2}\bigg[\frac{1}{\kappa_{\mathbf{p}}(z')}\frac{dx_m}{dz}\bigg|_{\mathbf{p}} + \frac{1}{\kappa_{\mathbf{p}}(z)}\frac{dx'_m}{dz'}\bigg|_{\mathbf{p}}\bigg] - 1\bigg\}. \quad (4.20)$$

Therefore, using Eqs. (4.13) and (4.19), we can write

$$\Delta E_2 = -\frac{e^2\hbar}{32\pi^3 c^3}\int d\Omega \int_0^\infty d\omega\omega^2 \int dz dz' \times e^{i\omega(t-t')-i\omega\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}')/c} \times\bigg\{\bigg[\frac{k_n}{\kappa_{\mathbf{p}}(z)}\frac{dx^n}{dz} + \frac{k_n}{\kappa_{\mathbf{p}}(z')}\frac{dx'^n}{dz'}\bigg]\frac{dx^m}{dz}\frac{dx'_m}{dz'} - \bigg[\frac{k_n}{\kappa_{\mathbf{p}}(z')}\frac{dx^n}{dz} + \frac{k_n}{\kappa_{\mathbf{p}}(z)}\frac{dx'^n}{dz'}\bigg]\bigg\}. \quad (4.21)$$

Collecting only the terms up to order c^0 in the integrand, we have

$$\Delta E_2 = -\frac{e^2\hbar}{32\pi^3 c^3}\int d\Omega \int_0^\infty d\omega\omega^3 \int dt dt' \times e^{i\omega(t-t')-i\omega\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}')/c}\bigg\{\bigg[\frac{1}{\dot{z}\kappa_{\mathbf{p}}(z)}\bigg(1 - \frac{\mathbf{n}_\perp}{c}\cdot\dot{\mathbf{x}}_\perp\bigg) + \frac{1}{\dot{z}'\kappa_{\mathbf{p}}(z')}\bigg(1 - \frac{\mathbf{n}_\perp}{c}\cdot\dot{\mathbf{x}}'_\perp\bigg)\bigg](c^2 - \dot{\mathbf{x}}_\perp\cdot\dot{\mathbf{x}}'_\perp) - \frac{2}{m}\bigg\}, \quad (4.22)$$

where we have used $\kappa_{\mathbf{p}}(z) \approx m dz/dt$ to lowest order in c^{-1}. The argument that led to Eq. (4.15) can be used to conclude that

$$c^2\int_0^\infty d\omega\omega^3 \int dt dt' e^{i\omega(t-t')}\bigg[\frac{1}{\dot{z}\kappa_{\mathbf{p}}(z)} + \frac{1}{\dot{z}'\kappa_{\mathbf{p}}(z')}\bigg] = 0. \quad (4.23)$$

Expanding the factor $e^{-i\omega\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}')/c}$ to order ω^2/c^2 and carrying out the $\mathbf{n}$ integration, we find

$$\Delta E_2 = \frac{e^2\hbar}{8\pi^2 mc^3}\int_0^\infty d\omega\omega^3 \int dt dt' e^{i\omega(t-t'+i\varepsilon)}\chi(at)\chi(at') \times\bigg[\bigg(\dot{\mathbf{x}}_\perp\cdot\dot{\mathbf{x}}'_\perp + \frac{\omega^2}{6}|\mathbf{x}-\mathbf{x}'|2\bigg)(\dot{z}^{-2} + \dot{z}'^{-2}) - \frac{i\omega}{3}(\mathbf{x}_\perp - \mathbf{x}'_\perp)\cdot(\dot{\mathbf{x}}_\perp\dot{z}^{-2} + \dot{\mathbf{x}}'_\perp\dot{z}'^{-2})\bigg]. \quad (4.24)$$

It is convenient to combine the two integrals ΔE_1 and ΔE_2 at this point. After integrating over ω, $\Delta E \equiv \Delta E_1 + \Delta E_2$ to order c^{-3} becomes

$$\Delta E = \frac{e^2\hbar}{8\pi^2 mc^3}\int dt dt' \chi(at)\chi(at') \times [F_1(t,t') + F_2(t,t') + F_3(t,t')], \quad (4.25)$$

where

$$F_1(t,t') \equiv -\frac{4!\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}'}{(t-t'+i\varepsilon)^5}\int_{t'}^t \dot{z}^{-2}dT + \frac{3!\dot{\mathbf{x}}_\perp\cdot\dot{\mathbf{x}}'_\perp(\dot{z}^{-2} + \dot{z}'^{-2})}{(t-t'+i\varepsilon)^4}, \quad (4.26)$$

$$F_2(t,t') \equiv \frac{1}{6}\bigg[\frac{6!|\mathbf{x}-\mathbf{x}'|^2}{(t-t'+i\varepsilon)^7}\int_{t'}^t \dot{z}^{-2}dT - \frac{5!|\mathbf{x}-\mathbf{x}'|^2(\dot{z}^{-2} + \dot{z}'^{-2})}{(t-t'+i\varepsilon)^6}\bigg], \quad (4.27)$$

$$F_3(t,t') \equiv \frac{1}{3}\bigg[\frac{2\cdot 5!(\mathbf{x}_\perp - \mathbf{x}'_\perp)}{(t-t'+i\varepsilon)^6}\cdot\int_{t'}^t \dot{\mathbf{x}}_\perp\dot{z}^{-2}dT + \frac{4!(\mathbf{x}_\perp - \mathbf{x}'_\perp)\cdot(\dot{\mathbf{x}}_\perp\dot{z}^{-2} + \dot{\mathbf{x}}'_\perp\dot{z}'^{-2})}{(t-t'+i\varepsilon)^5}\bigg]. \quad (4.28)$$

Note that all terms in Eq. (4.25) are of the form $A_1^{(n)}$ in Eq. (A1). For example, the integral in Eq. (4.25) involving the first term of $F_2(\omega, t, t')$ can be seen to be of the form $A_1^{(3)}$ with $f(t, t') = 1$, $g_1(t) = g_2(t) = x_i(t)$ and $g_3(t) = \int_0^t \dot{z}^{-2} dT$. Thus, we integrate by parts to reduce the denominator to $(t - t' + i\varepsilon)^3$ in each term in Eqs. (4.26), (4.27), and (4.28). We integrate the terms proportional to $\int_{t'}^t \dot{z}^{-2} dT$ so that the coefficient functions are differentiated with respect to each of t and t' twice, and for the rest we choose to integrate by parts so that there is no derivative on either $\dot{z}^{-2}$ or $\dot{z}'^{-2}$. Thus, we find

$$F_1(t, t') \sim \frac{2}{(t - t' + i\varepsilon)^3}\left(\ddot{\mathbf{x}} \cdot \ddot{\mathbf{x}}' \int_{t'}^t \dot{z}^{-2} dT - \ddot{z}\dot{z}'^{-1} + \ddot{z}'\dot{z}^{-1}\right), \tag{4.29}$$

$$F_2(t, t') \sim \frac{2}{3(t - t' + i\varepsilon)^3}\left(-\ddot{\mathbf{x}} \cdot \ddot{\mathbf{x}}' \int_{t'}^t \dot{z}^{-2} dT + \ddot{\mathbf{x}} \cdot \dot{\mathbf{x}}'\dot{z}'^{-2} - \ddot{\mathbf{x}}' \cdot \dot{\mathbf{x}}\dot{z}^{-2}\right), \tag{4.30}$$

$$F_3(t, t') \sim -\frac{2}{3(t - t' + i\varepsilon)^3}(\ddot{\mathbf{x}}_\perp \cdot \dot{\mathbf{x}}'_\perp \dot{z}'^{-2} - \ddot{\mathbf{x}}'_\perp \cdot \dot{\mathbf{x}}_\perp \dot{z}^{-2}), \tag{4.31}$$

where $\sim$ indicates equivalence under integration over t and t'. Adding these three terms and integrating some terms further by parts, we find

$$\Delta E = \frac{e^2\hbar}{12\pi^2 mc^3}\int dt dt'\left[\frac{2\ddot{\mathbf{x}}_\perp \cdot \ddot{\mathbf{x}}'_\perp}{(t - t' + i\varepsilon)^3} - \frac{\dddot{z}\,\dddot{z}'}{t - t' + i\varepsilon}\right] \times \int_{t'}^t \dot{z}^{-2} dT. \tag{4.32}$$

A form more convenient for concrete calculations can be found by integrating the first term by parts further as

$$\Delta E = \frac{e^2\hbar}{12\pi^2 mc^3}\int \frac{dt dt'}{t - t'}\left(-\dddot{\mathbf{x}} \cdot \dddot{\mathbf{x}}' \int_{t'}^t \dot{z}^{-2} dT + \dddot{\mathbf{x}}_\perp \cdot \ddot{\mathbf{x}}'_\perp \dot{z}'^{-2} - \dddot{\mathbf{x}}'_\perp \cdot \ddot{\mathbf{x}}_\perp \dot{z}^{-2}\right). \tag{4.33}$$

This correction is of the same order in c^{-1} as the Larmor formula, though it is of course of higher order in $\hbar$, in contrast to the correction (3.18) for a time-dependent vector potential, which is of higher order in c^{-1}.

To estimate the size of this correction, we consider a charged particle moving at a constant speed v_z in the z direction and accelerated in the x direction with acceleration given by $a(t) = a_0(1 - t^2/t_0^2)$ for $|t| \leq t_0$. It is possible to arrange the vector potential to realize this motion as shown in Appendix B. Here, the vector potential is z-dependent and varies only for $|z| \leq v_z t_0$. The first term in brackets in Eq. (4.33) gives a vanishing contribution. From the remaining terms we find

$$\Delta E = -\frac{2e^2\hbar a_0{}^2}{3\pi^2 m v_z^2 c^3}, \tag{4.34}$$

and

$$\frac{|\Delta E|}{E_{\rm em}^{(0)}} = \frac{15\hbar}{4\pi m v_z^2 t_0}, \tag{4.35}$$

where $E_{\rm em}^{(0)} = 8a_0^2 t_0/45\pi c^3$ is the energy emitted according to the Larmor formula as before. Thus, the correction is small and expected to be reliable as long as the kinetic energy associated with the motion in the z direction is much larger than the typical energy of the photon emitted, $\hbar/t_0$.

V. SUMMARY AND OUTLOOK

In this paper we showed that the energy and momentum of radiation emitted by a charged scalar particle in QED agree with the classical result (2.39) at order e^2 in the limit $\hbar \to 0$ and then went on to study the correction (at first order in $\hbar$) to the energy emitted in the nonrelativistic limit in two cases: one with a time-dependent but space-independent vector potential and the other with a time-independent vector potential which depends on one space coordinate, z. Both corrections were found to arise entirely due to the fact that the momenta of the initial and final scalar wave functions are different in the emission amplitude. The results are given by Eqs. (3.18) and (4.33). They are expressed in terms of the classical trajectory and are different from each other. Thus, the quantum correction is sensitive to how the particle is accelerated as well as to the motion of the corresponding classical particle. Another notable feature of these corrections is that they are nonlocal in time unlike the classical approximation.

In a fully quantum-mechanical calculation to all orders in $\hbar$, the expectation value of the power of emission obviously cannot be expressed in terms of the corresponding classical trajectory in a simple manner [13]. Therefore, it is not surprising that the correction to the Larmor formula at first order in $\hbar$ expressed in terms of the classical trajectory is nonlocal in time. The fact that the corrections to the Larmor formula at first order in $\hbar$ are different in the two cases studied in this paper also reflects the difference in these cases in a fully quantum-mechanical calculation.

We estimated the size of the correction in each case for a given acceleration of simple form. For the time-dependent potential the correction is much smaller than the classical result unless the typical energy of the photon emitted is comparable to the rest mass energy of the particle, with the nonrelativistic approximation itself breaking down. On the other hand, for the z-dependent potential, the correction is small compared to the classical result if the typical energy of the photon emitted is much smaller than the kinetic energy of the particle in the z direction.

It would be interesting to test the quantum corrections to the Larmor formula obtained in this paper though it would be difficult to realize the conditions in which our results can directly be compared with experimental results. One quantum system to which the Larmor formula and other classical results are applicable is a Rydberg atom, i.e. an atom with an electron with a very high principal quantum number up to a few hundred. Indeed the Larmor formula is known to give a very good approximation to the lifetime of states with high principal and angular-momentum quantum numbers [15,16]. It would be interesting to calculate the quantum correction to this approximation by extending our calculations to cases with a charged particle in a radially varying potential and possibly to extend our results to the case with a vector potential varying in a more general way. It would also be interesting to study the quantum corrections to the Larmor formula in the ultrarelativistic limit and determine whether the difference between the two cases studied in this paper persists in this limit.

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APPENDIX A: CUTOFF INDEPENDENCE OF INTEGRALS IN SECS. III AND IV

In this appendix we show that formal integration by parts used in this paper to find the quantum corrections is justified. Let $I \equiv [-T, T]$, $T > 0$. We choose this interval such that the acceleration of the particle is nonzero only if $t \in I$. Let $f(t, t')$ be a smooth function such that the support of $\partial_t f(t, t')$ [resp. $\partial_{t'} f(t, t')$] is a subset of $I \times \mathbb{R}$ (resp. $\mathbb{R} \times I$). Then, one can show that f, $\partial_t f$ and $\partial_{t'} f$ are all bounded. Let $g_i(t)$, $i = 1, 2, \ldots, n$, be smooth functions such that the support of $g_i''(t)$ is a subset of I. Then, it can readily be seen that g_i' are bounded. We let $\chi(t)$ be a smooth function such that it is compactly supported with $\chi(t) = 1$ for $t \in I$. We use $\chi(at)$, $0 < a \leq 1$, as our cutoff factor, with the limit $a \to 0$ taken at the end. Note that $\lim_{a\to 0} a^2 \int_{-\infty}^{\infty} [\chi'(at)]^2 dt = 0$. This property was necessary for the cutoff factor for deriving the Larmor formula in Sec. II. All integrals in Secs. III and IV that are ill-defined without the cutoff factor take the form

$$A_1^{(n)} = \int dt dt' \frac{f(t, t')}{(t - t' + i\varepsilon)^{n+4}} \times \left\{ \prod_{i=1}^{n} [g_i(t) - g_i(t')] \right\} \chi(at)\chi(at'). \quad \text{(A1)}$$

What we show in this appendix is that this integral can be reduced to the sum of integrals with no derivatives on the cutoff factor and convergent without them and those which tend to zero as $a \to 0$. This implies that one can use formal integration by parts for this integral until it is convergent, as we did in Secs. III and IV.

We first prove that the integral of the following form is convergent without the cutoff factor:

$$A_2^{(n)} = \int dt dt' \frac{\partial_t f(t, t')}{(t - t' + i\varepsilon)^{n+3}} \times \left\{ \prod_{i=1}^{n} [g_i(t) - g_i(t')] \right\} \chi(at)\chi(at'). \quad \text{(A2)}$$

Since $\partial_t f(t, t') = 0$ for $t \notin I$, the t integral is effectively over interval I. We have noted that $\partial_t f(t, t')$ is bounded. Since $g_i''(t)$ are nonzero only for $t \in I$, we have $g_i(t') = \alpha_i^- t' + \beta_i^-$ for $t' < -T$ and $g_i(t') = \alpha_i^+ t' + \beta_i^+$ for $t' > T$ for some constants $\alpha_i^{\pm}$ and $\beta_i^{\pm}$. Then it is clear that the t' integral is convergent without the cutoff factor.

Next we prove that the following integral tends to zero as $a \to 0$:

$$A_3^{(n)} = a \int dt dt' \frac{f(t, t')}{(t - t' + i\varepsilon)^{n+3}} \times \left\{ \prod_{i=1}^{n} [g_i(t) - g_i(t')] \right\} \chi'(at)\chi(at'). \quad \text{(A3)}$$

To this end it is useful to prove that the following integrals tend to zero as $a \to 0$:

$$A_4^{(n)} = a \int dt dt' \frac{\partial_{t'} f(t, t')}{(t - t' + i\varepsilon)^{n+2}} \times \left\{ \prod_{i=1}^{n} [g_i(t) - g_i(t')] \right\} \chi'(at)\chi(at'), \quad \text{(A4)}$$

$$A_5^{(n)} = a^2 \int dt dt' \frac{f(t, t')}{(t - t' + i\varepsilon)^{n+2}} \times \left\{ \prod_{i=1}^{n} [g_i(t) - g_i(t')] \right\} \chi'(at)\chi'(at'). \quad \text{(A5)}$$

The t' integral in Eq. (A4) is over the interval I, and hence we can drop the cutoff factor $\chi(at')$. Furthermore, $\partial_{t'} f(t, t') \equiv F(t')$ is t-independent where $\chi'(at) \neq 0$. Then, by letting $t = \eta/a$, we have

$$\begin{aligned} A_4^{(n)} &= a^2 \int_{-\infty}^{-T} d\eta \int_{-T}^{T} dt' \frac{F(t')}{(\eta - at')^{n+2}} \times \left\{ \prod_{i=1}^{n} [\alpha_i^- \eta + a(\beta_i^- - g_i(t'))] \right\} \chi'(\eta) \\ &\quad + a^2 \int_{T}^{\infty} d\eta \int_{-T}^{T} dt' \frac{F(t')}{(\eta - at')^{n+2}} \times \left\{ \prod_{i=1}^{n} [\alpha_i^+ \eta + a(\beta_i^+ - g_i(t'))] \right\} \chi'(\eta). \end{aligned} \quad \text{(A6)}$$

These integrals have finite limits as $a \to 0$. Hence, $A_4^{(n)} \to 0$ as $a \to 0$. To show that $A_5^{(n)} \to 0$ as $a \to 0$, we note that $f(t, t')$ is constant if $|t|$, $|t'| > T$. The integral $A_5^{(n)}$ has nonzero contributions only from the four disjoint regions $(T, \infty) \times (T, \infty)$, $(-\infty, -T) \times (T, \infty)$, $(-\infty, -T) \times (-\infty, -T)$ and $(T, \infty) \times (-\infty, -T)$ on the tt' plane because of the factor $\chi'(at)\chi'(at')$. Let $f(t, t') = f_{++}$ in the first region. Then the contribution from the first region to $A_5^{(n)}$ can be written, after the change of variables $t = \eta/a$, $t' = \eta'/a$,

$$A_5^{(n)}|_{++} = a^2 f_{++} \prod_{i=1}^{n} \alpha_i^+ \int_T^\infty d\eta \int_T^\infty d\eta' \frac{\chi'(\eta)\chi'(\eta')}{(\eta - \eta' + i\varepsilon)^2}. \tag{A7}$$

The contribution from $(-\infty, -T) \times (-\infty, -T)$ has a similar expression. The contribution from $(T, \infty) \times (-\infty, -T)$ with $f(t, t') = f_{+-}$ is

$$A_5^{(n)}|_{\pm\mp} = a^2 f_{+-} \int_T^\infty d\eta \int_{-\infty}^{-T} d\eta' \frac{\chi'(\eta)\chi'(\eta')}{(\eta - \eta')^{n+2}} \times \prod_{i=1}^{n} (\alpha_i^+ \eta + a\beta_i^+ - \alpha_i^- \eta' - a\beta_i^-), \tag{A8}$$

and that from $(-\infty, -T) \times (T, \infty)$ is similar. Hence we find that $A_5^{(n)} \to 0$ as $a \to 0$.

To show that $A_3^{(n)} \to 0$ as $a \to 0$, we first integrate by parts and find

$$\begin{aligned} A_3^{(n)} &= -\frac{a}{n+2} \int dt dt' \frac{1}{(t - t' + i\varepsilon)^{n+2}} \times \frac{\partial}{\partial t'} \left\{ f(t, t') \prod_{i=1}^{n} [g_i(t) - g_i(t')] \chi(at') \right\} \chi'(at) \\ &= -\frac{A_4^{(n)}}{n+2} - \frac{A_5^{(n)}}{n+2} + \frac{a}{n+2} \sum_{k=1}^{n} \int dt dt' \times \frac{f(t, t') g_k'(t')}{(t - t' + i\varepsilon)^{n+2}} \left\{ \prod_{i \neq k} [g_i(t) - g_i(t')] \right\} \chi'(at)\chi(at'). \end{aligned} \tag{A9}$$

Now, the first and second terms tend to zero as $a \to 0$. Each of the remaining terms is of the form $A_3^{(n-1)}$ because the partial derivatives of $f(t, t') g_k'(t')$ with respect to t and t' have support in $I \times \mathbb{R}$ and $\mathbb{R} \times I$, respectively. Therefore, if $A_3^{(n-1)}$ tends to zero as $a \to 0$, so does $A_3^{(n)}$. This means that all we need to show is $A_3^{(0)} \to 0$ as $a \to 0$, which is true because

$$A_3^{(0)} = -\tfrac{1}{2} A_4^{(0)} - \tfrac{1}{2} A_5^{(0)}. \tag{A10}$$

Now, we are ready to turn to the integrals $A_1^{(n)}$, which are the ones we encounter in our calculations. By integrating by parts with respect to t we find

$$\begin{aligned} A_1^{(n)} &= \frac{1}{n+3} \int dt dt' \frac{1}{(t - t' + i\varepsilon)^{n+3}} \times \frac{\partial}{\partial t} \left\{ f(t, t') \prod_{i=1}^{n} [g_i(t) - g_i(t')] \chi(at) \right\} \chi(at') \\ &= \frac{1}{n+3} \left[A_2^{(n)} + A_3^{(n)} + \sum_{k=1}^{n} \int dt dt' \frac{f(t, t') g_k'(t)}{(t - t' + i\varepsilon)^{n+3}} \times \left\{ \prod_{i \neq k} [g_i(t) - g_i(t')] \right\} \chi(at)\chi(at') \right]. \end{aligned} \tag{A11}$$

As we have seen, the term $A_2^{(n)}$ is cutoff independent and $A_3^{(n)} \to 0$ as $a \to 0$. The remaining terms are of the form $A_1^{(n-1)}$. Thus, all we need to show is that $A_1^{(0)}$ can written in a cutoff independent form. Indeed we have

$$A_1^{(0)} = \tfrac{1}{3} A_2^{(0)} + \tfrac{1}{3} A_3^{(0)} \to \tfrac{1}{3} A_2^{(0)} \quad \text{as } a \to 0. \tag{A12}$$

Thus, we have shown that we may use integration by parts with respect to t for integrals of the form $A_1^{(n)}$ disregarding the cutoff factor until it is convergent without them. It is clear that this statement holds for integration by parts with respect to t' as well.

APPENDIX B: THE VECTOR POTENTIAL FOR THE MOTION USED IN SEC. IV

We recall that the local momentum of the particle is given by

$$m \frac{d\mathbf{x}_\perp}{d\tau} = \mathbf{p}_\perp - \mathbf{V}_\perp(z), \tag{B1}$$

$$m \frac{dz}{d\tau} = \sqrt{[p_0 - V_0(z)/c]^2 - |\mathbf{p}_\perp - \mathbf{V}_\perp(z)|^2 - m^2 c^2}, \tag{B2}$$

where we have restored the factors of c in Eq. (4.2) letting V_0 and $\mathbf{V}_\perp$ have the dimensions of energy and momentum, respectively. Here $\mathbf{p}_\perp$ and p_0 are constants. Thus, it is clear that any motion in the perpendicular direction can be realized by adjusting $\mathbf{V}_\perp(z)$ appropriately while maintaining the condition $dz/dt \approx dz/d\tau = v_z$ by adjusting $V_0(z)$.

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