# Asymptotically free four-Fermi theory in 4 dimensions at the $z = 3$ Lifshitz-like fixed point

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We show that a Nambu-Jona-Lasinio type four-fermion coupling at the $z = 3$ Lifshitz-like fixed point in $3 + 1$ dimensions is asymptotically free and generates a mass scale dynamically. This result is nonperturbative in the limit of a large number of fermion species. The theory is ultraviolet complete and at low energies exhibits Lorentz invariance as an emergent spacetime symmetry. Many of our results generalize to $z = d$ in odd $d$ spatial dimensions; $z = d = 1$ corresponds to the Gross-Neveu model. The above mechanism of mass generation has potential applications to the fermion mass problem and to dynamical electroweak symmetry breaking. We present a scenario in which a composite Higgs field arises from a condensate of these fermions, which then couples to quarks and leptons of the standard model. Such a scenario could eliminate the need for the Higgs potential and the associated hierarchy problem. We also show that the axial anomaly formula at $z = 3$ coincides with the usual one in the relativistic domain.

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## I. INTRODUCTION AND SUMMARY

A fundamental problem of particle physics is the question of mass generation of elementary particles in $3 + 1$ dimensions. Early attempts in this direction were made in [1,2] based on an analogy with the theory of superconductivity. In the standard model (SM) this problem is addressed by introducing the Higgs mechanism and Yukawa couplings. The technicolor models were invented to generate particle masses dynamically. However these have not been phenomenologically viable for a number of reasons (see, e.g., [3,4]).

In this paper we make an observation which has a bearing on this question. We show that if we are willing to give up Lorentz invariance in the ultraviolet then it is possible to have a renormalizable model involving a Nambu-Jona-Lasinio type [2] four-fermi interaction in $3 + 1$ dimensions. In fact, it turns out that this model is asymptotically free and has dynamical mass generation.[1] Moreover, the relativistic Dirac theory emerges at low energies. Our calculations are nonperturbative in the limit of a large number of fermion species.

The idea that a relativistic theory at low energies may have a Lorentz noninvariant uv completion has been suggested recently in [6,7], where the theory at high energy is characterized by an anisotropic scaling exponent $z$ which describes a different scaling of space and time: $x \to bx$, $t \to b^z t$. Quantum critical systems with anisotropic scaling are known in condensed matter physics (see, e.g., [8–10]). Recently these theories have been discussed in the context of AdS/(non)-CFT duality; see, e.g. [11–14]. The idea of relinquishing relativistic invariance at high energies has also appeared in cosmology, e.g. as an explanation of ultrahigh energy cosmic rays above the Greisen-Zatsepin-Kuzmin cutoff [15]. In a somewhat different approach to the subject, Lorentz symmetry breaking has also been used as a regulator for quantum field theories; see [16] for a recent reference; see also [17]. Currently there is a lot of interest in the application of such ideas to gravity; however, in this paper we will only focus on nongravitational theories.

The plan of this paper is as follows. In Sec. II we present the four-fermi model with $z = 3$ scaling in 3 spatial dimensions. The fermions carry a species index $i$ which takes $N$ different values. We use the large $N$ limit and compute the nonperturbative ground state characterized by a fermion condensate. A mass scale is dynamically generated and the four-fermi coupling, in this vacuum, exhibits asymptotic freedom. This result can be extended to $z = d$ in any odd $d$ spatial dimensions. Calculations in this section are similar to those of the Gross-Neveu model [18], which can be regarded as the $z = d = 1$ case. In Sec. III we consider $1/N$ fluctuations around the condensate and show that the phase of the condensate appears as a Nambu-Goldstone boson. When the broken symmetry is gauged, the Nambu-Goldstone boson is "eaten up" by the dynamical gauge field, as in the usual Higgs mechanism. In Sec. IV we add a relevant coupling to the $z = 3$ model and discuss how a Lorentz-invariant theory emerges at low energies. In Sec. V we discuss the structure of axial anomalies in this theory and compute the anomaly coefficient. In Sec. VI we briefly discuss the application of this mechanism to dynamical electroweak symmetry breaking. We conclude in

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[1]It is important to note that in 4D theories involving relativistic fermions, it is impossible to achieve asymptotic freedom without dynamical gauge fields [5]. We are able to circumvent this theorem here by working with a Lorentz noninvariant theory.

Sec. VII with some discussions. Appendix A provides some details of the gap equation while Appendix B computes one-loop propagators for the bosonic fluctuations.

## II. ASYMPTOTIC FREEDOM

Our model consists of 2N species of fermions $\psi_{ai}(t, \vec{x})$, $a = 1, 2$; $i = 1, \ldots, N$ which carry representations of $SU(N)$ and a flavor group $U(1)_1 \times U(1)_2$, as follows:

$$\psi_{ai} \to U_{ij}\psi_{aj} \qquad \psi_{ai} \to e^{i\alpha_a}\psi_{ai}, \qquad a = 1, 2. \tag{1}$$

Each of these fermions is an $SU(2)_s$ spinor, where $SU(2)_s$ is the double cover of the spatial rotation group $SO(3)$.

An action which is consistent with the above symmetries is

$$S = \int d^3\vec{x}dt[\psi^\dagger{}_{1i}(i\partial_t + i\vec{\partial}\cdot\vec{\sigma}\partial^2)\psi_{1i} + \psi^\dagger{}_{2i}(i\partial_t - i\vec{\partial}\cdot\vec{\sigma}\partial^2)\psi_{2i} + g^2\psi^\dagger{}_{1i}\psi_{2i}\psi^\dagger{}_{2j}\psi_{1j}], \tag{2}$$

where $\{\vec{\sigma}\}$ are the Pauli matrices. We will study the dynamics of this action in the large $N$ limit in which $\lambda = g^2N$, the 'tHooft coupling, is held fixed. Note the sign flip of the spatial derivative term between the two flavors $a = 1$ and $a = 2$; this ensures that the Lagrangian is invariant under a parity operation under which $\psi_{1i}(t, \vec{x}) \to \psi_{2i}(t, -\vec{x})$.

Note that if we assign scaling dimensions according to $z = 3$, i.e. $[L] = -1$, $[T] = -3$, then $[\psi] = 3/2$. In this case, all the three terms appearing in the above action are of dimension 6 and hence marginal.

It is important that the four-fermion interaction term is marginal at $z = 3$. Recall that in the usual context of a 3 + 1 dimensional Lorentz-invariant theory, any interaction involving four fermions represents an irrelevant operator and so must be understood as a low-energy effective interaction. By contrast, here the marginality of the interaction leads one to hope that the theory (2) is perhaps uv complete. We will show below that this is indeed the case since the four-fermi coupling turns out to be asymptotically free.

A more general $z = 3$ action which considers all relevant and marginal couplings, and is consistent with the symmetries of (2), can also be written down:

$$\begin{aligned} S = \int d^3\vec{x}dt[&\psi^\dagger{}_{1i}(i\partial_t - i\vec{\partial}\cdot\vec{\sigma}((-i\partial)^2 + g_1) + g_2\partial^2)\psi_{1i} \\ &+ \psi^\dagger{}_{2i}(i\partial_t + i\vec{\partial}\cdot\vec{\sigma}((-i\partial)^2 + g_1) + g_2\partial^2)\psi_{2i} \\ &+ g_3(\psi^\dagger{}_{1i}\psi_{1i} + \psi^\dagger{}_{2i}\psi_{2i}) + g_4^2((\psi^\dagger_{1i}\psi_{1i})^2 \\ &+ (\psi^\dagger_{21}\psi_{2i})^2) + g_5^2(\psi^\dagger_{1i}\psi_{1i}\psi^\dagger{}_{2j}\psi_{2j}) \\ &+ g_6^2(\psi^\dagger_{1i}\psi_{2i}\psi^\dagger{}_{2j}\psi_{1j})]. \end{aligned} \tag{3}$$

The action (2) corresponds to putting all the couplings $g_1, \ldots, g_5 = 0$ and setting $g_6 = g$. We will treat some of these other couplings in Secs. II C and IV.

One can eliminate the four-fermi interaction in (2) by using a standard Gaussian trick:

$$\begin{aligned} &\exp\Big[i\Big(g^2\int\psi^\dagger{}_{1i}\psi_{2i}\psi^\dagger{}_{2j}\psi_{1j}\Big)\Big] \\ &= \int\mathcal{D}\phi\exp\Big[i\int\phi^*\psi^\dagger{}_{1i}\psi_{2i} + \phi\psi^\dagger{}_{2i}\psi_{1i} - \frac{1}{g^2}\phi^*\phi\Big]. \end{aligned}$$

This gives us the following action, which is entirely equivalent to (2):

$$\begin{aligned} S = \int d^3\vec{x}dt\Big[&\psi^\dagger{}_{1i}(i\partial_t + i\vec{\partial}\cdot\vec{\sigma}\partial^2)\psi_{1i} \\ &+ \psi^\dagger{}_{2i}(i\partial_t - i\vec{\partial}\cdot\vec{\sigma}\partial^2)\psi_{2i} \\ &+ \phi^*\psi^\dagger{}_{1i}\psi_{2i} + \phi\psi^\dagger{}_{2i}\psi_{1i} - \frac{1}{g^2}\phi^*\phi\Big]. \end{aligned} \tag{4}$$

The scalar field $\phi$ is an $SU(N)$ singlet and is charged under the axial $U(1)$ parametrized by $\exp[i(\alpha_1 - \alpha_2)]$.

### A. The gap equation

Since the action (4) is quadratic in fermions, one can integrate them out, leading to the following effective action for the boson:

$$S_{\rm eff}[\phi] = -iN\,{\rm Tr}\ln\tilde{D} - \frac{1}{g^2}\int\phi^*\phi, \tag{5}$$

where $\tilde{D}$ is defined in (A2). Here Tr represents a trace over space-time as well as the flavor and spinor indices.

In the large $N$ limit, the classical equation of motion $\delta S_{\rm eff}/\delta\phi = 0$ is exact, leading to (see Appendix A for details)

$$i\int\frac{d^4k}{(2\pi)^4}\frac{1}{k_0^2 - |\vec{k}|^6 - \phi^*\phi + i\epsilon} = \frac{1}{2\lambda}, \qquad \lambda = g^2N. \tag{6}$$

This gap equation determines only the absolute value of $\phi$. The phase of $\phi$ will be identified with a Nambu-Goldstone mode $\pi$ in the next section where we consider fluctuations.

The left-hand side of the gap equation is logarithmically divergent by $z = 3$ power counting (both numerator and denominator have dimension 6). Rotating the contour from $k_0 \in (-\infty, \infty)$ to $k_0 \in (-i\infty, i\infty)$ (this is an anticlockwise rotation in the complex $k_0$ plane by $\pi/2$ and can be done without touching the poles of the Feynman propagator), we get

$$\int\frac{dk_0d^3k}{(2\pi)^4}\frac{1}{k_0^2 + k^6 + \phi^*\phi} = \frac{1}{2\lambda}. \tag{7}$$

It is easy to do the angular integration. Then, using the variable $w = k^3$ and extending the range of $w$ integral to the entire real line (possible because the integrand has $w \leftrightarrow -w$ symmetry), we get

$$\frac{2\pi/3}{(2\pi)^4}\int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} dw \frac{1}{k_0^2 + w^2 + \phi^*\phi} = \frac{1}{2\lambda}. \qquad (8)$$

The above integral has an $SO(2)$ rotational symmetry between $k_0$ and $w$. In particular, if we parametrize

$$(k_0, w) = K(\cos\theta, \sin\theta), \qquad \theta \in [0, \pi], \qquad (9)$$

then the angle $\theta$ can be integrated out. Using the $SO(2)$-invariant cutoff $K \leq \Lambda^3$ and discarding a finite piece,[2] we get

$$\ln\left(\frac{\Lambda}{m}\right) = \frac{2\pi^2}{\lambda}, \qquad (10)$$

where $\Lambda$ has momentum dimension 1 and we have introduced the parameter $m$ of momentum dimension 1 by defining $m^6 \equiv \phi^*\phi$. In the large $N$ limit, fluctuations of $\phi$ are suppressed and this solution of the gap equation becomes exact.

We see from (4) that, around this symmetry broken vacuum, the term involving the parameter $m$ is like a mass term for the fermions. When we perturb this model by adding a relevant term that takes it at low energies to the relativistic fixed point at $z = 1$, this term goes over to the familiar mass term for relativistic fermions, with a mass proportional to $m$. This is discussed further in Sec. IV.

Equation (10) determines $m$ in terms of $\lambda$ and the cutoff $\Lambda$. We will demand that $\lambda$ must be assigned an appropriate $\Lambda$ dependence such that the fermion mass $m^3 = |\langle\phi\rangle|$ is kept invariant. From (10) this gives us

$$\lambda(\Lambda) = \frac{2\pi^2}{\ln(\Lambda/m)}. \qquad (11)$$

We see that $\lambda$ is an asymptotically free coupling. The theory generates a mass scale analogous to $\Lambda_{\mathrm{QCD}}$, given by

$$m = \Lambda \exp\left[-\frac{2\pi^2}{\lambda}\right].$$

The $\beta$ function is easy to compute and it is negative:

$$\beta(\lambda) = \Lambda\frac{d\lambda}{d\Lambda} = -\frac{\lambda^2}{2\pi^2}.$$

The calculation presented above is similar to that for the Gross-Neveu model [18]. Indeed, we will show in the next subsection that the results presented above generalize to all odd $d$ spatial dimensions at $z = d$. The Gross-Neveu model, from this viewpoint, is simply the $d = 1$, $z = 1$ example. Unlike in the higher dimensional examples, however, the fermion condensate in the Gross-Neveu model breaks only a discrete $Z_2$ symmetry and there is no Nambu-Goldstone mode.

We should point out that the condensate is generated here for arbitrarily weak coupling $g$. This is in contrast with what happens in the usual relativistically invariant Nambu-Jona-Lasinio model at the $z = 1$ fixed point [2,19–21], where the symmetry breaking phase occurs only beyond a certain critical value $g_c$ of the coupling.

It is useful to express the above result in terms of an effective potential for the homogeneous mode of $\phi$. This is essentially the negative of (5), evaluated for constant $\phi$, and is given by

$$V_{\mathrm{eff}}(\phi) = \frac{|\phi|^2}{g^2}\left(1 - \frac{\lambda}{12\pi^2}\left(\ln\frac{\Lambda^6}{|\phi|^2} + 1\right)\right). \qquad (12)$$

A plot of $V$ as a function of $|\phi|$ can be found in Fig 1. It shows a minimum at

$$|\phi| = m^3 = \Lambda^3 \exp[-6\pi^2/\lambda], \qquad (13)$$

as found above. We emphasize that the treatment of the effective potential and the renormalization group (RG) flow presented above is exact in the strict $N = \infty$ limit.

### B. Other dimensions and $z = d$

In this subsection we show that the above conclusion generalizes to $z = d$ in $d = 2n + 1$ spatial dimensions. We will again consider fermions $\psi_{ai}$ which transform in the fundamental representation of $SU(N)$ and a flavor group $U(1)_1 \times U(1)_2$; each fermion transforms as a spinor of (an appropriate covering group of) the spatial rotation group $SO(2n + 1)$. The action (4) now reads

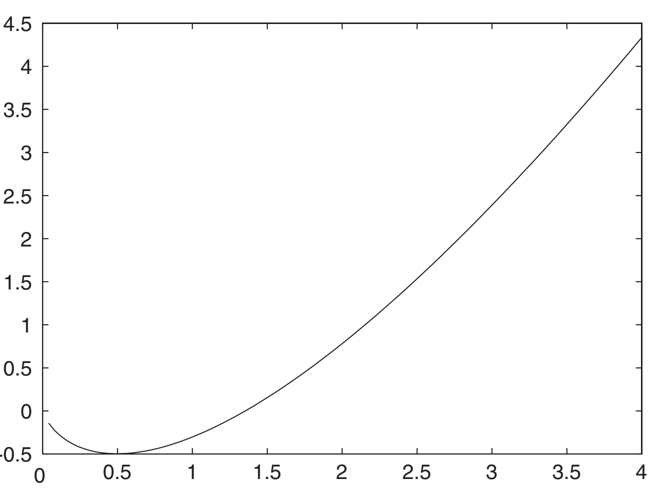



FIG. 1. The effective potential, $V_{\mathrm{eff}}(\phi)$, as a function of $|\phi|$.

[2] The actual result for the left-hand side using this cutoff is $\ln(1 + \Lambda^6/|\phi|^2)$. The finite pieces depend on the cutoff scheme, e.g. if, in (8), we integrate $k_0$ first from $-\infty$ to $\infty$ and then $w$ from 0 to $\Lambda^3$, the left-hand side of (10) would be $\ln(\sqrt{1 + \Lambda^6/|\phi|^2} + \Lambda^6/|\phi|^2)$.

$$S = \int d^{2n+1}\vec{x}dt[\psi^\dagger{}_{1i}(i\partial_t + i\vec{\partial}\cdot\vec{\Gamma}\partial^{2n})\psi_{1i} + \psi^\dagger{}_{2i}(i\partial_t - i\vec{\partial}\cdot\vec{\Gamma}\partial^{2n})\psi_{2i} + g^2\psi^\dagger{}_{1i}\psi_{2i}\psi^\dagger{}_{2j}\psi_{1j}]. \tag{14}$$

Here $\Gamma^i$, $i = 1, 2, \ldots, 2n$ are the gamma matrices in $2n$ Euclidean dimensions. For $z = d$, the dimension of the fermion is $[\psi] = d/2$. Hence the four-fermi coupling is marginal for any $d$.

The gap equation now reads

$$2^{n+1}\int\frac{dk_0 d^{2n+1}k}{(2\pi)^{2n+2}}\frac{1}{k_0^2 - k^{2n+2} - \phi^*\phi + i\epsilon} = \frac{2}{\lambda}, \tag{15}$$

from which we get

$$\lambda(\Lambda) = \frac{A}{\ln(\Lambda/m)}, \qquad A = 2\pi^{n+1}(2n-1)!!,$$

showing asymptotic freedom of the coupling. Here $(2n-1)!! = (2n-1)(2n-3)\ldots 1$ for $n \geq 1$ and $= 1$ for $n = 0$. The beta-function is given by

$$\beta(\lambda) = -\frac{1}{A}\lambda^2.$$

Note that the $\beta$ function vanishes exponentially as $d \to \infty$.

### C. The space of marginal couplings

In the bulk of this section, we have dealt with the action (2), or equivalently with (4). It is not difficult to generalize our results to include the marginal couplings $g_4$ and $g_5$ in (3). We postpone the details to a forthcoming publication [22] and simply quote the results here. As before, we can use a Gaussian trick to replace the new quartic interactions by introducing additional auxiliary scalar fields. A general effective potential involving these fields can be obtained along the lines of (12). It turns out that (a) the new terms in the effective potential do not depend on the cutoff, and (b) at the minimum of the potential $\langle\phi\rangle$ is still given by (13) whereas the other condensates vanish. Furthermore, only the coupling $g_6 = g$, considered before, has a nontrivial beta function and the theory remains asymptotically free. If we do include the relevant coupling $g_3$, then the other condensates acquire nonvanishing vacuum expectation values (vev); however, even then it is only $g_6$ which has a nontrivial RG flow and the theory is asymptotically free. We will discuss the consequence of including the relevant coupling $g_1$ in Sec. IV.

## III. QUANTUM FLUCTUATIONS

In the previous section, we considered the classical solution of $S_{\text{eff}}(\phi)$ [Eq. (5)], which is exact in the large $N$ limit. In this section we will go beyond this approximation and consider fluctuations of the scalar field $\phi$. It is convenient to parametrize the fluctuations in terms of a radial field (sigma) and a phase (pion):

$$\phi(x) = \rho(x)e^{ig\pi(x)}, \qquad \rho(x) = m^3 + \frac{g}{\sqrt{2}}\sigma(x). \tag{16}$$

It is convenient to use the notation of Dirac matrices and rewrite the action (4) in the form given by (A1) and (A3). Substituting (16) in these equations, we get the following action for the fluctuations:

$$S = \int d^4x\bigg[\bar{\Psi}_i(i\gamma^0\partial_t + i(\vec{\gamma}\cdot\vec{\partial})(i\vec{\partial})^2)\Psi_i + \bar{\Psi}_i\bigg(\bigg(m^3 + \frac{g}{\sqrt{2}}\sigma(x)\bigg)e^{ig\pi(x)}P_L + \bigg(m^3 + \frac{g}{\sqrt{2}}\sigma(x)\bigg) \times e^{-ig\pi(x)}P_R\bigg)\Psi_i - \frac{1}{g^2}\bigg(m^3 + \frac{g}{\sqrt{2}}\sigma(x)\bigg)^2\bigg], \tag{17}$$

where $P_{L,R} = \frac{1}{2}(1 \pm \gamma^5)$. The action has the following global $U(1)$ symmetry:

$$\Psi_i \to e^{ig\alpha\gamma^5}\Psi_i, \qquad \pi \to \pi - \alpha. \tag{18}$$

In terms of the original $U(1)_1 \times U(1)_2$ symmetry of the action, this is the off-diagonal (axial) $U(1)$. The fermion condensate breaks this symmetry, with the pion $\pi(x)$ as a Nambu-Goldstone boson.

The masslessness of the pion can be argued as follows. By making a local phase rotation $\Psi_i \to e^{-ig\gamma^5\pi(x)/2}\Psi_i$ in the fermion functional integral, the pion field can be eliminated from the Yukawa coupling terms, with the replacements

$$\partial_t \to \partial_t + \frac{ig}{2}\partial_t\pi, \qquad \partial_i \to \partial_i + \frac{ig}{2}\partial_i\pi, \tag{19}$$

in the fermion kinetic terms. This shows that the effective action (5) contains the pion field only through its derivatives, which, therefore, rules out a mass term.

The above argument relies on the invariance of the fermionic measure under an axial phase rotation and could be potentially invalidated by the appearance of anomalies. We will discuss this issue in the next subsection which deals with coupling to gauge fields. For the present, we note that in the absence of gauge fields any potential anomaly vanishes and the argument about the masslessness of the pion goes through.

In Appendix B, further evidence for the masslessness of the $\pi$ field is provided by an explicit computation of the one-loop propagator for the bosonic fluctuations.

### Coupling to gauge fields

If we gauge the axial $U(1)$ by appropriately coupling the fermions to a dynamical gauge field, then the effect of the phase rotation $\exp[-ig\gamma^5\pi(x)/2]$ on the fermions will be to replace the gauge-covariant derivatives in a manner analogous to (19). The pion field and the gauge field will then appear in an extended covariant derivative of the form

$$\tilde{D}_t = \partial_t + ie\left(A_t + \frac{g}{2e}\partial_t\pi\right),$$
$$\tilde{D}_i = \partial_i + ie\left(A_i + \frac{g}{2e}\partial_i\pi\right). \tag{20}$$

This shows that the gauge field effectively absorbs the pion field, as in the standard Higgs mechanism, and becomes massive. This mechanism for the generation of gauge field mass terms is familiar from technicolor theories. This is discussed further in Sec. VI.

As in the previous subsection, we have assumed invariance of the fermion functional integral under a local axial phase rotation. This argument is valid only if there are no anomalies of the axial current. In Sec. V we will calculate this anomaly and show that it is the same as for relativistic fermions. The absence of this anomaly therefore imposes the usual requirement on the spectrum of fermions coupled to the axial gauge field.

## IV. EMERGENCE OF LORENTZ INVARIANCE

In this section we will consider the effect of adding the relevant coupling $g_1$, defined in (3), to the theory (17). According to $z = 3$ scaling, the momentum dimension of $g_1$ is 2. Denoting $g_1 \equiv M^2$, the action reads

$$S = \int d^4x\Big[\bar{\Psi}_i(i\gamma^0\partial_t + i(\vec{\gamma}\cdot\vec{\partial})(M^2 + (i\vec{\partial})^2))\Psi_i + \bar{\Psi}_i\Big(\Big(m^3 + \frac{g}{\sqrt{2}}\sigma(x)\Big)e^{ig\pi(x)}P_L + \Big(m^3 + \frac{g}{\sqrt{2}}\sigma(x)\Big) \times e^{-ig\pi(x)}P_R\Big)\Psi_i - \frac{1}{g^2}\Big(m^3 + \frac{g}{\sqrt{2}}\sigma(x)\Big)^2\Big]. \tag{21}$$

The mass-shell condition of the fermion is

$$k_0^2 - k^2(M^2 + k^2)^2 - m^6 = 0. \tag{22}$$

For the momentum range

$$k \ll M, \tag{23}$$

we get

$$k_0^2 - M^4(k^2 + m_*{}^2) = 0, \qquad m_* = \frac{m^3}{M^2}.$$

Let us introduce a rescaled time and energy

$$t' = tM^2, \qquad k_0' = k_0/M^2, \tag{24}$$

so that $t'$ is of mass dimension $-1$ and $k_0'$ of mass dimension 1. The mass-shell condition becomes the standard form dictated by the Lorentz invariance:

$$(k_0')^2 = (k^2 + m_*{}^2). \tag{25}$$

In the momentum range (23) and in terms of the rescaled time (24), the action (21) becomes

$$S = \int d^3x dt'\Big[\bar{\Psi}_i(i\gamma^0\partial_{t'} + i\vec{\gamma}\cdot\vec{\partial})\Psi_i + \bar{\Psi}_i\Big(\Big(m_* + \frac{g}{\sqrt{2}}\sigma'(x)\Big) \times e^{ig\pi(x)}P_L + \Big(m_* + \frac{g}{\sqrt{2}}\sigma'(x)\Big)e^{-ig\pi(x)}P_R\Big)\Psi_i - \frac{M^2}{g^2}\Big(m_* + \frac{g}{\sqrt{2}}\sigma'(x)\Big)^2\Big], \tag{26}$$

where we have defined the rescaled bosonic field $\sigma' = \sigma/M^2$. In view of (23), the above action should be understood with an effective cutoff $M$.

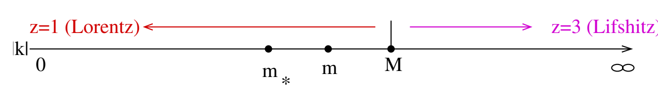
The following points are worth noting:

(1) In principle, the coupling $g_1$ in (3) flows under RG. This is because the fermion propagator receives corrections from the diagrams shown in Appendix B 3. Hence, strictly speaking, we should make a distinction between the coefficient $M$ appearing in (21) and the constant $M$ appearing in (22) . The former should be regarded as a cutoff dependent coupling $M(\Lambda)$, whereas the latter should be regarded as an RG-invariant mass scale $M$ entering in the mass-shell condition (22). However, the correction is of order $\lambda/N$, as can be seen from (B5). Since it is suppressed by $1/N$, the correction can be neglected in the large $N$ limit. Therefore, in the present case, the $M$ appearing in the two equations (21) and (22) can be taken to be the same.

(2) Although the coupling $\lambda$ is asymptotically free and grows logarithmically at low $k$, the coupling constant relevant to quantum fluctuations around the fermion condensate is $g = \sqrt{\lambda/N}$ which remains weak at large $N$.

(3) The $\psi^\dagger\psi\sigma$ coupling $g$, remarkably, is marginal both at $z = 3$ and at $z = 1$. The reason is that in (21) $\sigma$ has dimension 3, hence $\psi^\dagger\psi\sigma$ has dimension 6 (which is marginal at $z = 3$); on the other hand in (26) the rescaled bosonic field $\sigma' = \sigma/M^2$ has dimension 1, hence the $\psi^\dagger\psi\sigma'$ coupling has dimension 4 (which is marginal at $z = 1$).

The various mass scales which appeared above can be schematically represented on the momentum line:




Here $m$ is the mass scale generated by the condensate, $m = |\phi|^{1/3}$. $M$ is the momentum scale below which Lorentz symmetry appears. $m_*$ is the rest mass of the emergent Dirac fermion. The three masses are related by $m^3 = m_* M^2$. Here we have chosen the order $M \gg m \gg m_*$ for potential applications to weak interaction (see Sec. VI). Theoretically, the other order $M \ll m \ll m_*$ is also allowed; however, with that choice, the "Higgs vev" is higher than the scale of violation of Lorentz symmetries, which is unrealistic.

## V. ANOMALIES

In this section we present a brief discussion of axial anomalies when our model (2) is coupled to gauge fields. We will first consider the case where the diagonal $U(1)$ is gauged and will look for possible anomalies in the axial $U(1)$ current.

We will compute the anomaly adapting the method of Fujikawa [23] (see also [19,20]) to our problem. The path integral for the gauged model is given by [using the form of the fermion action given in (21)][3]

$$Z = \int \prod_i \mathcal{D}\Psi_i \mathcal{D}\bar{\Psi}_i e^{iS},$$
$$S = \int d^3x dt \bar{\Psi}_i i\not{\mathbf{D}}\Psi_i \not{\mathbf{D}} = \gamma^\mu \mathbf{D}_\mu, \qquad \mathbf{D}_t = D_t, \tag{27}$$
$$\mathbf{D}_i = D_i(-(\vec{D})^2 + M^2), \qquad D_\mu = \partial_\mu + ieA_\mu.$$

Here we have ignored the four-fermion term since it does not play a role in the calculation of anomalies. We will treat the gauge field as an external field. If we make a local axial rotation

$$\delta\Psi_i = i\alpha(x)\gamma^5\Psi_i, \qquad \delta\bar{\Psi}_i = \bar{\Psi}_i i\alpha(x)\gamma^5, \tag{28}$$

we generate a term proportional to $\int \alpha(x)\partial_\mu J^{\mu 5}$ in the action and if the path integral measure is invariant under (28) then the axial current is conserved. However, it was shown in [23] that under (28), the measure picks up a Jacobian. Taking this into account in the present case, we get[4]

$$\partial_\mu J^{\mu 5} = 2\langle x|\mathrm{tr}(\gamma^5 \exp[i\not{\mathbf{D}}^2/\Lambda^6])|x\rangle,$$

where the gauge-invariant exponential operator is introduced as a regulator, as in [23], except that the cutoff $\Lambda$ follows $z = 3$ scaling (cf. Sec. II A). Here "tr" refers to a Dirac trace as well as a sum over the species index $i = 1, \ldots, N$. To evaluate the right-hand side, we will expand the exponential in powers of the charge $e$, by using

$$(\not{\mathbf{D}})^2 = -\mathbf{D}_\mu \mathbf{D}^\mu + \frac{i}{2}\Sigma^{\mu\nu}[\mathbf{D}_\mu, \mathbf{D}_\nu].$$

The second term is proportional to $e^2$. In order for the Dirac trace to survive, we must bring down two powers of this term. Using $\mathrm{tr}(\gamma^5\Sigma^{0i}\Sigma^{jk}) = 4i\epsilon^{ijk}$ and ignoring terms which would drop out at large $\Lambda$ (these include all terms involving $M$) we find

$$\partial_\mu J^{\mu 5} = -\frac{2ie^2N}{\Lambda^{12}}\epsilon^{ijk}\langle x|\exp[(-\partial_0^2 + (\vec{\partial})^6)/\Lambda^6] \times \{E_i(\vec{\partial})^2 + E_p\partial_p\partial_i, F_{jk}(\vec{\partial})^4 + 2F_{jl}\partial_l\partial_k(\vec{\partial})^2 + 2F_{lk}\partial_l\partial_j(\vec{\partial})^2\}|x\rangle.$$

The terms inside $\langle x|\ldots|x\rangle$ give rise to various powers of $\Lambda$, the highest being $\Lambda^{12}$ which is exactly cancelled by the $1/\Lambda^{12}$ outside. Terms involving derivatives of the electric or magnetic field give rise to lower powers of $\Lambda$ and eventually drop out. The final result, at the end of a long calculation, is

$$\partial_\mu J^{\mu 5} = -\frac{e^2N}{16\pi^2}\epsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma}, \tag{29}$$

which is identical to the usual relativistic calculation of the axial anomaly.[5] The robustness of the anomaly coefficient with respect to different $z$ values is likely to be related to its topological character. The result is also gratifying from the viewpoint of model building since we do not want to impose different requirements on the fermion spectra at different energy scales.

The axial anomaly for chiral fermions (only $\psi_{1i}$ and no $\psi_{2i}$) as well as chiral anomalies for chiral gauge theories can be obtained by simple generalizations of the computation presented above.

## VI. APPLICATION TO LOW-ENERGY PHENOMENOLOGY

In this section we will consider a simple extension of the fermion model (4) which can describe electroweak symmetry breaking. The extension consists of an additional $SU(2)$ group, under which the $a = 1$ fermions transform as a doublet and the $a = 2$ fermions transform as a singlet. Using the Dirac spinor notation employed in the previous section, let us denote the $a = 1$ fermions as $\psi_L$ (these satisfy $\gamma^5 = 1$) and $a = 2$ fermions as $\psi_R$ (these satisfy $\gamma^5 = -1$). The fermion fields will then be denoted as $\psi_{Li\alpha}$, $\psi_{Ri}$ where $\alpha = 1, 2$ is the new $SU(2)$ index.[6] We then couple the fermions to $SU(2)$ gauge fields.[7]

The scalar field, $\phi_\alpha$, which is classically equivalent to the fermion bilinear $g\psi_{Ri}\psi_{Li\alpha}$, now carries the additional $SU(2)$ index $\alpha$ and transforms as a doublet. This will play the role of the composite Higgs field.

In addition to the above fermions, we will have the usual quark and lepton degrees of freedom. These do not carry the species index $i$, but they do have quartic interaction terms with the above fermions, similar to those in (2). These quartic interactions are designed to respect the $SU(2)$ gauge symmetry and the global symmetries of the action. An example is

[3]We use $i$ to label the fermion as well as spatial coordinates; the specific usage should be clear from the context.

[4]Note that the spatial component of the current, $\vec{J}^5$, is much more complicated than its relativistic counterpart.

[5]We can, in fact, recover (29) in the $z = 1$ limit from (27) by making the replacement $-(\vec{D})^2 + M^2 \rightarrow M^2$.

[6]In order to generate masses for all the $\psi$ fermions, we need to double the number of right-handed fermions as well, still keeping them singlets under the above $SU(2)$. In this more general model, the four-fermi couplings can be arranged such that one still has only a doublet of Higgs in the broken phase

[7]We can also add a gauge field to gauge the vector part of $U(1) \times U(1)$.

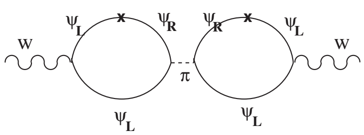
$$(\psi^{\dagger}{}_{Li\alpha}\psi_{Ri})(q_R^{\dagger}q_{L\alpha}), \tag{30}$$

where the $q$'s denote quarks. This interaction will generate the Yukawa couplings after the $\psi$'s have condensed, as we show below.

We can now repeat the analysis of Secs II and III to show that $\phi_\alpha$ develops a vev, thereby dynamically breaking the gauge symmetry. By parametrizing $\phi = \exp(i\vec{\pi}(x)\cdot\vec{\tau})\rho$, we can show, as in Sec. , that $\vec{\pi}(x)$'s combine with the $SU(2)$ gauge fields to give them their longitudinal components. The fluctuation $\sigma_\alpha(x)$ of the radial field $\rho_\alpha(x)$ becomes the massive Higgs field, in terms of which (30) gives the usual coupling of quarks to the Higgs field:

$$q_R^{\dagger}\sigma_\alpha q_{L\alpha}. \tag{31}$$

The gauge field masses arise from their gauge-invariant interactions with the $\psi$'s. The relevant diagram is shown in the following diagram:




The crosses on fermion propagators indicate insertions of the dynamically generated mass. The main point is the exchange of the massless Nambu-Goldstone "pion." which is responsible for generating the gauge boson masses. This well-known mechanism was originally discovered in the context of the Meissner effect (see [1] and references therein).

We conclude this section with the following observations:

(1) The introduction of the fermions $(\psi_{Li\alpha}, \psi_{Ri})$ with the marginal four-fermi coupling gives rise to a composite Higgs field and hence eliminates the need for the Higgs potential. Consequently, the hierarchy problem of the standard model is avoided.

(2) We used a large number $N$ of these fermions in order to make the point that the model exists nonperturbatively and that the fluctuations around the nonperturbative vacuum are weakly coupled. However, this is not an in-principle requirement if we can tackle the four-fermion coupling nonperturbatively by some other method.

(3) For realistic applications, we need to ensure that the mass $m_*$ of the extra fermions $(\psi_{Li\alpha}, \psi_{Ri})$ is beyond the presently observed energy scales while the Higgs mass and the Higgs vev are not too high. We wish to come back to a detailed analysis of this constraint in a later paper.

(4) The mechanism of mass generation presented here is much simpler than in technicolor theories [3,4].

(5) The spectrum and quantum number of fermions is constrained by the requirement of vanishing of gauge anomalies, the computation of which is described in Sec. V.

(6) For applications to the SM, one must appropriately couple the fermions $\psi_{Li\alpha}$ to the gauge fields of the SM. Since these couplings are not suppressed in the large $N$ limit, the fermion propagator will receive corrections from the exchange of gauge fields which are of order one in $N$. Thus, in this case the coupling $g_1 = M^2$ will get renormalized, even to leading order in large $N$. In this case, then, a scale dependent $g_1$ will enter in the fermion mass-shell condition, and, more generally, in the effective "speed of light" parameter for each particle. For phenomenologically consistent applications, one then needs to make sure that the variation of the "speed of light" is compatible with existing bounds from experiments. Finally, one must find a consistent framework to couple such a model to gravity. Detailed work on the various aspects outlined above is crucial to establish a consistent framework for applications to the real world, but this is beyond the scope of the present paper.

## VII. DISCUSSION

In this paper, we have shown that at the $z = 3$ fixed point, an Nambu-Jona-Lasinio–like four-fermi coupling in $3 + 1$ dimensions is asymptotically free, thus providing an uv completion of the low-energy four-fermion coupling at the $z = 1$ fixed point. The price to pay is Lorentz non-invariance in the ultraviolet. Our work provides a novel composite Higgs mechanism for dynamical gauge symmetry breaking and a generation of fermion masses.

The asymmetry in the ultraviolet cutoff corresponding to space and time directions may be a fundamental feature of our world. If true, this feature could have important consequences for low-energy particle physics and model building. It would be interesting to see if the $z = 3$ model described here for the electroweak sector can be extended to include strong and gravitational interactions, especially in the framework of string theory. For this reason, it is important to explore the formulation of string theory itself which incorporates Lorentz violation in the ultraviolet. For example, we observe that in the exact formulation of 2-dimensional string theory in terms of matrix quantum mechanics, one naturally arrives at a $z = 2$ theory of nonrelativistic fermions [24–28]. The theory becomes relativistic ($z = 1$) only for low-energy fluctuations around the fermi surface.

## ACKNOWLEDGMENTS

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*Note added.*—After the first version of this paper was submitted to the archive, we became aware of the work [29] which has some overlap with the contents of this paper.

## APPENDIX A: SOME STEPS FOR THE GAP EQUATION

Let us combine the flavor and spinor indices to write a four-component fermion

$$\Psi_i = \begin{pmatrix} \psi_{1i} \\ \psi_{2i} \end{pmatrix}.$$

In this notation, (4) reads

$$\mathcal{L} = \int d^3\vec{x} dt \Psi_i^\dagger \tilde{D} \Psi_i, \tag{A1}$$

where

$$\tilde{D} \equiv i\partial_t \mathbf{1} \otimes \mathbf{1} + i\partial^2 \partial_i \sigma_3 \otimes \sigma_i + (\phi^* \sigma^+ + \phi \sigma^-) \otimes \mathbf{1}. \tag{A2}$$

We find that subsequent calculations get considerably simplified if we write the operator $\tilde{D}$ in terms of Dirac's gamma matrices $\gamma^0$, $\gamma^i$

$$\begin{aligned} &\tilde{D} = \gamma^0 D, \\ &D = i\gamma^0 \partial_t + i\gamma^i \partial_i (i\partial)^2 + (\phi_R - i\phi_I \gamma^5). \end{aligned} \tag{A3}$$

Here $\phi = \phi_R + i\phi_I$. In our convention

$$\gamma^0 = \sigma_1 \otimes \mathbf{1}, \qquad \gamma^i = i\sigma_2 \otimes \sigma_i,$$
$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \sigma_3 \otimes \mathbf{1}.$$

We emphasize that although we find it expedient to use the gamma matrices, the operator $D$ above is *not* the Dirac operator. For instance, the coefficient of $\gamma^i$ has three powers of momenta, as appropriate for a $z = 3$ theory.

It is obvious that integrating the fermions out from (A1) leads to the effective action (5). Let us consider the equation of motion $\delta S_{\text{eff}}/\delta\phi_R = 0$. This gives

$$\frac{2}{g^2}\phi_R = -iN\,\text{Tr}(\tilde{D}^{-1}\gamma^0) = -iN\,\text{Tr}(D^{-1}). \tag{A4}$$

The operator $iD^{-1}$ is simply the propagator. In the momentum basis it is given by

$$D^{-1} = \frac{k_0\gamma^0 + k^2 k_i \gamma^i - (\phi_R + i\phi_I\gamma^5)}{k_0^2 - k^6 - \phi^*\phi + i\epsilon}.$$

Equation (6) now simply follows from (A4).

In $d = 2n + 1$ spatial dimensions, the propagator is $iD^{-1}$, with

$$\begin{aligned} &D = i\gamma^0\partial_t + i\partial^{2n}\partial_i\gamma^i + (\phi_R - i\phi_I\gamma^{d+2}) \\ &\gamma^0 = \sigma_1 \otimes \mathbf{1}, \qquad \gamma^i = i\sigma_2 \otimes \Gamma_i, \\ &\gamma^5 = i^n\gamma^0\gamma^1 \ldots \gamma^d = \sigma_3 \otimes \mathbf{1}. \end{aligned} \tag{A5}$$

## APPENDIX B: ONE-LOOP BOSON PROPAGATOR

In this section we will show the masslessness of the pion by an explicit one-loop computation.

We will find it convenient, for the purpose of this calculation, to expand the scalar field $\phi$ as

$$\phi = m^3 + g\eta, \qquad \eta = \frac{\tilde{\sigma} + i\tilde{\pi}}{\sqrt{2}}.$$

To this order, the $\tilde{\sigma}$ and $\tilde{\pi}$ fields are simply the $\sigma$ and $\pi$ fields of Sec. III, up to constant factors.

Using the form of the action as given by (A1) and (A3), we get

$$\begin{aligned} S = \int d^4x\Big[&\bar{\Psi}_i(i\gamma^0\partial_t + i(\vec{\gamma}\cdot\vec{\partial})(i\vec{\partial})^2 + m^3)\Psi_i \\ &+ \frac{g}{\sqrt{2}}\bar{\Psi}_i\Psi_i\tilde{\sigma} + \frac{g}{\sqrt{2}}\bar{\Psi}_i\gamma^5\Psi_i\tilde{\pi} \\ &- \frac{1}{2}\Big(\Big(\frac{m^3\sqrt{2}}{g} + \tilde{\sigma}\Big)^2 + \tilde{\pi}^2\Big)\Big]. \end{aligned} \tag{B1}$$

### 1. Summary of results

The tree-level propagator for $\tilde{\sigma}$ and $\tilde{\pi}$ fields are nondynamical. However, the propagators develop a nontrivial correction through fermion loops. We present the summary of our results here and defer details of the computation to the next subsection. To leading order in $1/N$, we find the following results for the propagators $G_{\tilde{\sigma}}(p)$ and $G_{\tilde{\pi}}(p)$ for the $\tilde{\sigma}$ and $\tilde{\pi}$ fields, respectively:

$$\begin{aligned} &G_{\tilde{\sigma}}(p) = \frac{-i}{1 + i\Gamma^{(2)}_{\tilde{\sigma}}(p)}, \\ &\Gamma^{(2)}_{\tilde{\sigma}}(p) = i\Big(1 - \frac{\lambda}{6\pi^2}\Big) + o(p^2), \\ &G_{\tilde{\pi}}(p) = \frac{-i}{1 + i\Gamma^{(2)}_{\tilde{\pi}}(p)}, \\ &\Gamma^{(2)}_{\tilde{\pi}}(p) = i + o(p^2). \end{aligned}$$

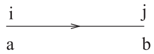
In the small $p$ limit,

$$G_{\tilde{\sigma}}(p) = \frac{1}{\lambda/(6\pi^2) + o(p^2)}, \qquad G_{\tilde{\pi}}(p) = \frac{-i}{o(p^2)}. \tag{B2}$$

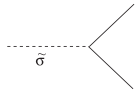
Therefore, the pion propagator has a massless pole, whereas the $\tilde{\sigma}$ field is massive.

### 2. Details

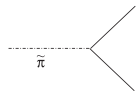
The Feynman rules that follow from (B1) are given by

Fermion propagator:  $= \frac{i\delta_{ab}\delta_{ij}}{\gamma^0 p_0+\vec{\gamma}\cdot\vec{p}\ p^2+m^3} = \Delta_F(p)$



Yukawa couplings:  $= \frac{ig}{\sqrt{2}}$

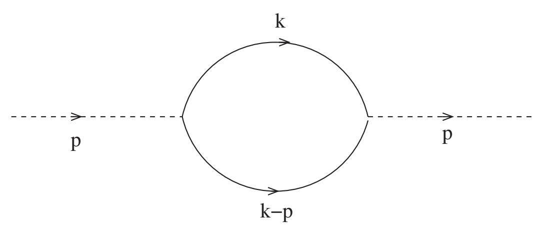


 $= \frac{-g\gamma^5}{\sqrt{2}}$



The propagators for $\tilde{\sigma}$ and $\tilde{\pi}$ are simply given by $-i$.

We will first compute the one-loop two-point function of $\tilde{\sigma}$. To order $g^2$, it is represented by the following Feynman diagram:





which evaluates to

$$\Gamma^{(2)}_{\tilde{\sigma}}(p) = (-1)\,\mathrm{Tr}\int\frac{d^4k}{(2\pi)^4}\frac{ig}{\sqrt{2}}\Delta_F(k)\frac{ig}{\sqrt{2}}\Delta_F(k-p)$$
$$= -2\lambda\int\frac{d^4k}{(2\pi)^4}$$
$$\times\frac{k_0(k_0-p_0)-\vec{k}.(\vec{k}-\vec{p})(\vec{k})^2(\vec{k}-\vec{p})^2+m^6}{(k_0^2-\vec{k}^6-m^6)((k_0-p_0)^2-(\vec{k}-\vec{p})^6-m^6)}. \tag{B3}$$

The full propagator for $\tilde{\sigma}$ at momentum $p$ is obtained by summing over an infinite series of such diagrams, and we obtain

$$G_{\tilde{\sigma}}(p) = -i + (-i)\Gamma^{(2)}_{\tilde{\sigma}}(p)(-i) + \ldots = \frac{-i}{1+i\Gamma^{(2)}_{\tilde{\sigma}}(p)}.$$

Note that by changing $k_0 \to -k_0$ and $\vec{k}\to-\vec{k}$ we can prove that $\Gamma^{(2)}_{\tilde{\sigma}}(-p) = \Gamma^{(2)}_{\tilde{\sigma}}(-p)$. Note also that

$$\Gamma^{(2)}_{\tilde{\sigma}}(0) = -2\lambda\left(-\frac{i}{2\lambda}+2m^6\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k_0^2-k^6-m^6)^2}\right)$$
$$= i - 4\lambda m^6\frac{\partial}{\partial m^6}\left(\frac{-i}{2\lambda}\right)\Gamma^{(2)}_{\tilde{\sigma}}(0) = i\left(1-\frac{\lambda}{6\pi^2}\right).$$

Hence, at $p\to 0$,

$$G_s(0) = \frac{-i}{\lambda/6\pi^2},$$

which shows that $\tilde{\sigma}$ is a massive particle.

The one-loop two-point function for $\tilde{\pi}$ is represented by a Feynman diagram similar to the above, and is given by

$$\Gamma^{(2)}_{\tilde{\pi}}(p) = (-1)\,\mathrm{Tr}\int\frac{d^4k}{(2\pi)^4}\frac{-g\gamma^5}{\sqrt{2}}\Delta_F(k)\frac{-g\gamma^5}{\sqrt{2}}\Delta_F(k-p)$$
$$= -2\lambda\int\frac{d^4k}{(2\pi)^4}$$
$$\times\frac{k_0(k_0-p_0)-\vec{k}.(\vec{k}-\vec{p})(\vec{k})^2(\vec{k}-\vec{p})^2-m^2}{(k_0-\vec{k}^6-m^6)((k_0-p_0)^2-(\vec{k}-\vec{p})^6-m^6)}. \tag{B4}$$

As for $\tilde{\sigma}$, the full propagator for $\tilde{\pi}$ at momentum $p$ is given by the sum

$$G_{\tilde{\pi}}(p) = -i + (-i)\Gamma^{(2)}_{\tilde{\pi}}(p)(-i) + \ldots = \frac{-i}{1+i\Gamma^{(2)}_{\tilde{\pi}}(p)}.$$

Note, like before, that $\Gamma^{(2)}_{\tilde{\pi}}(-p) = \Gamma^{(2)}_{\tilde{\pi}}(-p)$. Also $\Gamma^{(2)}_{\tilde{\pi}}(0) = -2\lambda\frac{-i}{2\lambda} = i$. Thus, $\Gamma^{(2)}_{\tilde{\pi}}(p) = i + o(p^2)$.

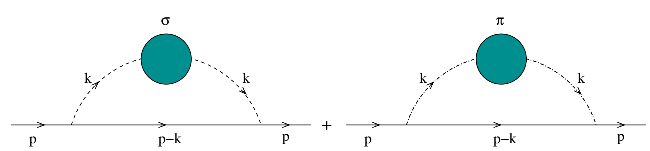
Hence, as $p\to 0$,

$$G_{\tilde{\pi}}(p) = \frac{-i}{o(p^2)}.$$

Thus, the $\tilde{\pi}$ propagator has a pole at $p^2 = 0$. Hence the pion is massless.

### 3. Fermion two-point function

In this subsection we present an expression for the fermion 2-point function $\Gamma^{(2)}_F(p)$. To $o(g^2)$, it is given by the following diagram [the blobs represent the full propagators $G_{\tilde{\sigma}}(p)$ and $G_{\tilde{\pi}}(p)$, respectively]:





The diagram evaluates to

$$\Gamma^{(2)}_F(p) = \int\frac{d^4k}{(2\pi)^4}\left[\frac{ig}{\sqrt{2}}iG_{\tilde{\sigma}}(k)\frac{ig}{\sqrt{2}}\Delta_F(p-k)\right.$$
$$\left.+\frac{-g}{\sqrt{2}}iG_{\tilde{\sigma}}(k)\frac{-g}{\sqrt{2}}\gamma^5\Delta_F(p-k)\gamma^5\right]$$
$$= \frac{-\lambda}{2N}\int\frac{d^4k}{(2\pi)^4}\left[\frac{G_{\tilde{\sigma}}(k+p)}{\gamma^0k_0+\vec{\gamma}\cdot\vec{k}(\vec{k})^2-m^3}\right.$$
$$\left.+\frac{G_{\tilde{\pi}}(k+p)}{\gamma^0k_0+\vec{\gamma}\cdot\vec{k}(\vec{k})^2+m^3}\right]. \tag{B5}$$

The expression, at least formally, contains terms involving $\vec{p}\cdot\vec{\gamma}$, which renormalize the relevant coupling $g_1$ in (3). We postpone a detailed analysis of this diagram to future work.

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