PHYSICAL REVIEW D **80,** 105003 (2009)

Fermionic Casimir effect for parallel plates in the presence of compact dimensions with applications to nanotubes

S. Bellucci[1,*] and A. A. Saharian[2,†]

[1]*Laboratori Nazionali di Frascati, INFN, Via Enrico Fermi 40, 00044 Frascati, Italy*
[2]*Department of Physics, Yerevan State University, 1 Alex Manoogian Street, 0025 Yerevan, Armenia*
(Received 3 August 2009; published 6 November 2009)

We evaluate the Casimir energy and force for a massive fermionic field in the geometry of two parallel plates on background of Minkowski spacetime with an arbitrary number of toroidally compactified spatial dimensions. The bag boundary conditions are imposed on the plates and periodicity conditions with arbitrary phases are considered along the compact dimensions. The Casimir energy is decomposed into purely topological, single plate and interaction parts. With independence of the lengths of the compact dimensions and the phases in the periodicity conditions, the interaction part of the Casimir energy is always negative. In order to obtain the resulting force, the contributions from both sides of the plates must be taken into account. Then, the forces coming from the topological parts of the vacuum energy cancel out and only the interaction term contributes to the Casimir force. Applications of the general formulae to Kaluza-Klein-type models and carbon nanotubes are given. In particular, we show that for finite-length metallic nanotubes, the Casimir forces acting on the tube edges are always attractive, whereas for semiconducting-type ones, they are attractive for small lengths of the nanotube and repulsive for large lengths.

DOI: 10.1103/PhysRevD.80.105003 PACS numbers: 03.70.+k, 11.10.Kk, 61.46.Fg

I. INTRODUCTION

A key feature of most high energy theories of fundamental physics, including supergravity and superstring theories, is the presence of compact spatial dimensions. From an inflationary point of view, universes with compact spatial dimensions, under certain conditions, should be considered a rule rather than an exception [1]. The models of a compact universe with nontrivial topology may play an important role by providing proper initial conditions for inflation (for physical motivations of considering compact universes see, also, [2]). There has been a large activity to search for signatures of nontrivial topology by identifying ghost images of galaxies, clusters, or quasars. Recent progress in observations of the cosmic microwave background provides an alternative way to observe the topology of the Universe [3]. If the scale of periodicity is close to the particle horizon scale then the changed appearance of the microwave background sky pattern offers a sensitive probe of the topology. An interesting application of the field theoretical models with compact dimensions recently appeared in nanophysics [4]. The long-wavelength description of the electronic states in graphene can be formulated in terms of the Dirac-like theory in three-dimensional spacetime with the Fermi velocity playing the role of speed of light (see, e.g., Ref. [5]). Single-walled carbon nanotubes are generated by rolling up a graphene sheet to form a cylinder and the background spacetime for the corresponding Dirac-like theory has topology $R^2 \times S^1$.

In quantum field theory, the boundary conditions imposed on fields along compact dimensions change the spectrum of vacuum fluctuations. The resulting energies and stresses are known as topological Casimir effect (for the topological Casimir effect and its role in cosmology see [6–11] and references therein). In the Kaluza-Klein-type models, this effect has been used as a stabilization mechanism for moduli fields which parametrize the size and the shape of the extra dimensions. The Casimir energy can also serve as a model of dark energy needed for the explanation of the present accelerated expansion of the universe (see [12] and references therein). In addition to its fundamental interest the Casimir effect, it also plays an important role in the fabrication and operation of nano- and micro-scale mechanical systems (see, for instance, [13]).

The effects of the toroidal compactification of spatial dimensions on the properties of quantum vacuum for various spin fields have been discussed by several authors (see, for instance, [6–11,14–16] and references therein). The combined effect of extra compactified dimensions and boundaries on the Casimir energy in the classical configuration of two parallel plates has been recently considered in [17] for a scalar field and in [18] for the electromagnetic field. The Casimir energy and forces in braneworld models have been evaluated in Ref. [19] by using both dimensional and zeta function regularization methods. Local Casimir densities in these models were considered in Ref. [20]. The Casimir effect in higher dimensional generalizations of the Randall-Sundrum model with compact internal spaces has been investigated in [21]. In the present paper, we investigate the Casimir effect for a massive fermionic field in the geometry of two parallel plates on background of space-

*bellucci@lnf.infn.it
†saharian@ictp.it

 © 2009 The American Physical Society

time with an arbitrary number of toroidally compactified spatial dimensions. We will assume generalized periodicity conditions along the compact dimensions with arbitrary phases and Massachusetts Institute of Technology bag boundary conditions on the plates. This problem in background of four-dimensional Minkowski spacetime with trivial topology has been considered in [22] for a massless field and in [23] for the massive case (see, also, [6]). For arbitrary number of dimensions, the corresponding results are generalized in Refs. [24,25] for the massless and massive cases, respectively. The Casimir problem for fermions coupled to a static background field in one spatial dimension is investigated in [26]. The interaction energy density and the force are computed in the limit that the background becomes concentrated at two points. The fermionic Casimir effect for parallel plates with imperfect bag boundary conditions modeled by δ-like potentials is studied in [27].

This paper is organized as follows. In the next section, we specify the eigenfunctions and the eigenmodes for the Dirac equation in the region between the plates assuming the bag boundary conditions on them. In Sec. III, by using the Abel-Plana-type summation formula, we present the Casimir energy in the region between the plates as the sum of pure topological, single plate and interaction parts. In Sec. IV, we consider the Casimir force acting on the plates. In Sec. V, we evaluate the Casimir energy and forces by making use of the generalized zeta function technique. An alternative representation of the single plate part of the Casimir energy is also given. The special case of topology $R^{D-1}\times S^1$ is discussed in Sec. VI. In Sec. VII, we give applications of general formulae to the Casimir effect for electrons in finite-length carbon nanotubes within the framework of -3dimensional Dirac-like model. The main results of the paper are summarized in Sec. VIII.

II. EIGENFUNCTIONS AND EIGENMODES

We consider a quantum fermionic field ψ on background of $(D+1)$-dimensional flat spacetime with spatial topology $R^{p+1}\times(S^1)^q$, $p+q+1=D$. The corresponding line element has the form

$$ds^2 = dt^2 - \sum_{l=1}^{D}(dz^l)^2, \tag{1}$$

where $-\infty < z^l < \infty$, $l=1,\ldots,p+1$, and $0\le z^l\le L_l$ for $l=p+2,\ldots,D$. We assume that along the compact dimensions the field obeys boundary conditions

$$\psi(t,\mathbf{z}_p,z^{p+1},\mathbf{z}_q+L_l\mathbf{e}_l)=e^{2\pi i\alpha_l}\psi(t,\mathbf{z}_p,z^{p+1},\mathbf{z}_q), \tag{2}$$

with constant phases $0\le\alpha_l<1$. In (2), $\mathbf{z}_p=(z^1,\ldots,z^p)$ and $\mathbf{z}_q=(z^{p+2},\ldots,z^D)$ denote the coordinates along uncompactified and compactified dimensions, respectively, $\mathbf{e}_l$ is the unit vector along the direction of the coordinate z^l, $l=p+2,\ldots,D$. The periodicity conditions for untwisted and twisted fermionic fields are obtained from (2) as special cases with $\alpha_l=0$ and $\alpha_l=1/2$, respectively. As we will see below, special cases $\alpha_l=0,\ 1/3,\ 2/3$ are realized in nanotubes.

In this paper, we are interested in the Casimir effect for the geometry of two parallel plates placed at $z^{p+1}=0$ and $z^{p+1}=a$. On the plates the field obeys the MIT bag boundary condition:

$$(1+i\gamma^\mu n_\mu)\psi=0,\qquad z^{p+1}=0,a, \tag{3}$$

where γ^μ are the Dirac matrices and n_μ is the normal to the boundaries. In the $(D+1)$-dimensional spacetime, the Dirac matrices are $N_D\times N_D$ matrices with $N_D=2^{[(D+1)/2]}$, where the square brackets mean the integer part of the enclosed expression. We will assume that these matrices are given in the chiral representation:

$$\gamma^0=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\qquad \gamma^\mu=\begin{pmatrix}0&\sigma_\mu\\-\sigma_\mu^+&0\end{pmatrix},\qquad \mu=1,2,\ldots,D, \tag{4}$$

with the relation $\sigma_\mu\sigma_\nu^+ + \sigma_\nu\sigma_\mu^+ = 2\delta_{\mu\nu}$. In the discussion below, we consider the region between the plates $0\le z^{p+1}\le a$, where we have $n_\mu=-\delta_\mu^{p+1}$ for the plate at $z^{p+1}=0$ and $n_\mu=\delta_\mu^{p+1}$ for $z^{p+1}=a$.

The dynamics of the field is governed by the Dirac equation

$$i\gamma^\mu\partial_\mu\psi - m\psi = 0. \tag{5}$$

The positive- and negative-frequency solutions to this equation can be presented as

$$\psi_\beta^{(+)}=A_\beta e^{-i\omega t}\begin{pmatrix}\varphi\\ -i\boldsymbol{\sigma}^+\cdot\boldsymbol{\nabla}\varphi/(\omega+m)\end{pmatrix},\qquad \psi_\beta^{(-)}=A_\beta e^{i\omega t}\begin{pmatrix}i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\chi/(\omega+m)\\ \chi\end{pmatrix}, \tag{6}$$

where $\boldsymbol{\sigma}=(\sigma_1,\ldots,\sigma_D)$, $\omega=\sqrt{\mathbf{k}_p^2+k_{p+1}^2+\mathbf{k}_q^2+m^2}$. The spinors in (6) are given by the expressions

$$\varphi=e^{i\mathbf{k}_\parallel\cdot\mathbf{z}_\parallel}(\varphi_+e^{ik_{p+1}z^{p+1}}+\varphi_-e^{-ik_{p+1}z^{p+1}}),\qquad \chi=e^{-i\mathbf{k}_\parallel\cdot\mathbf{z}_\parallel}(\chi_+e^{ik_{p+1}z^{p+1}}+\chi_-e^{-ik_{p+1}z^{p+1}}), \tag{7}$$

with $\mathbf{k}_\parallel=(\mathbf{k}_p,\mathbf{k}_q)$ and $\mathbf{k}_p=(k_1,\ldots,k_p)$, $\mathbf{k}_q=(k_{p+2},\ldots,k_D)$. The eigenvalues for the components of the wave vector along the compactified dimensions are determined from the periodicity conditions (2):

$$\mathbf{k}_q=(2\pi(n_{p+2}+\alpha_{p+2})/L_{p+2},\ldots,2\pi(n_D+\alpha_D)/L_D),\qquad n_{p+2},\ldots,n_D=0,\pm1,\pm2,\ldots. \tag{8}$$

For the components along the uncompactified dimensions, one has $-\infty<k_l<\infty$, $l=1,\ldots,p$.

From the boundary condition on the plate at $z^{p+1}=0$, we find the following relations between the spinors in (7)

$$\varphi_+ = -\frac{m(\omega+m)+k_{p+1}^2-k_{p+1}\sigma_{p+1}\boldsymbol{\sigma}_\parallel^+\cdot\mathbf{k}_\parallel}{(m-ik_{p+1})(\omega+m)}\varphi_-,$$
$$\chi_- = -\frac{m(\omega+m)+k_{p+1}^2-k_{p+1}\sigma_{p+1}^+\boldsymbol{\sigma}_\parallel\cdot\mathbf{k}_\parallel}{(m+ik_{p+1})(\omega+m)}\chi_+, \tag{9}$$

where $\boldsymbol{\sigma}_\parallel=(\sigma_1,\ldots,\sigma_p,\sigma_{p+2},\ldots,\sigma_D)$. We will assume that they are normalized in accordance with $\varphi_-^+\varphi_-=\chi_+^+\chi_+=1$. As a set of independent spinors, we will take $\varphi_-=w^{(\sigma)}$ and $\chi_+=w^{(\sigma)\prime}$, where $w^{(\sigma)}$, $\sigma=1,\ldots,N_D/2$, are one-column matrices having $N_D/2$ rows with the elements $w_l^{(\sigma)}=\delta_{l\sigma}$, and $w^{(\sigma)\prime}=iw^{(\sigma)}$. Now the set of quantum numbers specifying the eigenfunctions (6) is $\beta=(\mathbf{k},\sigma)$. From the boundary condition at $z^{p+1}=a$, it follows that the eigenvalues of k_{p+1} are roots of the transcendental equation

$$ma\sin(k_{p+1}a)/(k_{p+1}a)+\cos(k_{p+1}a)=0. \tag{10}$$

All these roots are real. We will denote the positive solutions of Eq. (10) by $\lambda_n=k_{p+1}a$, $n=1,2,\ldots$. For a massless field we have $\lambda_n=\pi(n-1/2)$. Note that Eq. (10) does not contain the parameters of the compact subspace and is the same as in the corresponding problem on the topologically trivial Minkowski spacetime (see [6]).

The normalization coefficient A_β in (6) is determined from the orthonormalization condition

$$\int d\mathbf{z}_\parallel\int_0^a dz^{p+1}\psi_\beta^{(\pm)+}\psi_{\beta'}^{(\pm)}=\delta_{\beta\beta'}. \tag{11}$$

Here, the symbol $\delta_{\beta\beta'}$ is understood as the Dirac delta function for continuous indices and the Kronecker delta for discrete ones. Substituting the eigenfunctions (6) into this condition, one finds

$$A_\beta^2=\frac{\omega+m}{4(2\pi)^p\omega aV_q}\left[1-\frac{\sin(2k_{p+1}a)}{2k_{p+1}a}\right]^{-1}, \tag{12}$$

where $V_q=L_{p+2}\cdots L_D$ is the volume of the compact subspace.

III. CASIMIR ENERGY

For the spatial topology $R^{p+1}\times(S^1)^q$, the vacuum energy (per unit volume along the directions $z^1,\ldots,z^p$) in the region between the plates is given by the following mode sum:

$$E_{p+1,q}=-\frac{N_D}{2}\int\frac{d\mathbf{k}_p}{(2\pi)^p}\sum_{\mathbf{n}_q\in\mathbf{Z}^q}\sum_{n=1}^{\infty}\omega, \tag{13}$$

where $\mathbf{n}_q=(n_{p+1},\ldots,n_D)$ and

$$\omega^2=\mathbf{k}_p^2+k_{\mathbf{n}_q}^2+\lambda_n^2/a^2+m^2,$$
$$k_{\mathbf{n}_q}^2=\sum_{l=p+2}^{D}[2\pi(n_l+\alpha_l)/L_l]^2. \tag{14}$$

Of course, the expression on the right-hand side of Eq. (13) is divergent. We will assume that some cutoff function is present, without writing it explicitly. For the further evaluation of the Casimir energy, we apply to the sum over n in Eq. (13) the Abel-Plana-like summation formula

$$\sum_{n=1}^{\infty}\frac{\pi f(\lambda_n)}{1-\sin(2\lambda_n)/(2\lambda_n)}=-\frac{\pi maf(0)}{2(ma+1)}+\int_0^\infty dzf(z)-i\int_0^\infty dt\frac{f(it)-f(-it)}{\frac{t+ma}{t-ma}e^{2t}+1}. \tag{15}$$

This formula is obtained as a special case of the summation formula derived in Ref. [28] by using the generalized Abel-Plana formula (see, also, Ref. [29]). Note that we have the relation

$$1-\frac{\sin(2\lambda_n)}{2\lambda_n}=1+\frac{ma}{(ma)^2+\lambda_n^2}. \tag{16}$$

By taking into account Eq. (16), we apply the summation formula (15) with the function

$$f(z)=\sqrt{z^2+\mathbf{k}_p^2a^2+m_{\mathbf{n}_q}^2a^2}\left[1+\frac{ma}{(ma)^2+z^2}\right], \tag{17}$$

where we have introduced the notation

$$m_{\mathbf{n}_q}^2=k_{\mathbf{n}_q}^2+m^2. \tag{18}$$

This allows one to present the Casimir energy in the decomposed form

$$E_{p+1,q}=aE_{p+1,q}^{(0)}+2E_{p+1,q}^{(1)}+\Delta E_{p+1,q}, \tag{19}$$

where

$$E_{p+1,q}^{(0)}=-\frac{N_D}{2}\int\frac{d\mathbf{k}_{p+1}}{(2\pi)^{p+1}}\sum_{\mathbf{n}_q\in\mathbf{Z}^q}\sqrt{\mathbf{k}_{p+1}^2+m_{\mathbf{n}_q}^2} \tag{20}$$

is the Casimir energy (per unit volume along the directions $z^1,\ldots,z^{p+1}$) in the topology $R^{p+1}\times(S^1)^q$ when the boundaries are absent. The part

$$E_{p+1,q}^{(1)}=-\frac{N_D}{4\pi}\int\frac{d\mathbf{k}_p}{(2\pi)^p}\sum_{\mathbf{n}_q\in\mathbf{Z}^q}\left(-\frac{\pi}{2}\sqrt{k_\parallel^2+m^2}+m\int_0^\infty dz\frac{\sqrt{z^2+k_\parallel^2+m^2}}{m^2+z^2}\right) \tag{21}$$

is the Casimir energy for a single plate (when the other plate is absent) in the half space. The last term in Eq. (19)

$$\Delta E_{p+1,q} = -\frac{N_D}{\pi}\int\frac{d\mathbf{k}_p}{(2\pi)^p}\sum_{\mathbf{n}_q\in\mathbf{Z}^q}\int_{\sqrt{\mathbf{k}_p^2+m_{\mathbf{n}_q}^2}}^{\infty}dz \times\frac{\sqrt{z^2-\mathbf{k}_p^2-m_{\mathbf{n}_q}^2}}{(z+m)e^{2az}+z-m}\left[a(z-m)-\frac{m}{z+m}\right] \tag{22}$$

is the interaction part. This term is finite for all nonzero distances between the plates and the cutoff function can be removed safely. Note that the single plate part of the Casimir energy does not depend on the separation between the plates and hence will not contribute to the Casimir force.

The pure topological part (20) is investigated in our previous paper [16]. After the renormalization, this part is presented in the form

$$E_{p+1,q}^{(0)} = 2N_D V_q\sum_{j=p+2}^{D}\frac{(2\pi)^{-(j+1)/2}}{V_{D-j+1}L_j^j}\times\sum_{n=1}^{\infty}\frac{\cos(2\pi n\alpha_j)}{n^{j+1}}\sum_{\mathbf{n}_{D-j}\in\mathbf{Z}^{D-j}}f_{(j+1)/2}(nL_j m_{\mathbf{n}_{D-j}}), \tag{23}$$

where we have defined

$$m_{\mathbf{n}_{D-j}}^2 = \sum_{l=j+1}^{D}[2\pi(n_l+\alpha_l)/L_l]^2+m^2. \tag{24}$$

Here, and in the discussion below, we use the notation

$$f_\nu(x) = x^\nu K_\nu(x). \tag{25}$$

An alternative expression for the topological part is obtained by using the zeta function technique (see below). In particular, the topological part of the Casimir energy is positive for untwisted fields ($\alpha_l = 0$) and is negative for twisted fields ($\alpha_l = 1/2$).

A. Single plate part

Now, let us consider the single plate part in the Casimir energy, given by formula (21). First, we note that this part vanishes for a massless field. This is directly seen by taking into account that in the limit $m \to 0$, the second term in braces of (21) gives nonzero contribution which cancels the first term. For the further evaluation of the single plate part for a massive field, we apply to the sum over n_{p+2} in Eq. (21) the Abel-Plana summation formula in the form [30]

$$\sum_{n_{p+2}=-\infty}^{+\infty}f(|n_{p+2}+\alpha_{p+2}|) = 2\int_0^\infty dx f(x)+i\int_0^\infty dx\sum_{\lambda=\pm1}\frac{f(ix)-f(-ix)}{e^{2\pi(x+i\lambda\alpha_{p+2})}-1}. \tag{26}$$

The part with the first term on the right of this formula gives the Casimir energy for a single plate in the case of topology $R^{p+2}\times(S^1)^{q-1}$. As a result, we obtain the following recurrence formula

$$\varepsilon_{p+1,q}^{(1)} = \varepsilon_{p+2,q-1}^{(1)}+\Delta_{p+2}\varepsilon_{p+1,q}^{(1)}, \tag{27}$$

where $m_{\mathbf{n}_{q-1}} = \sqrt{k_{\mathbf{n}_{q-1}}^2+m^2}$ and we have introduced the vacuum energy per unit volume of the compact subspace $\varepsilon_{p+1,q}^{(1)} = E_{p+1,q}^{(1)}/V_q$. In Eq. (27), the term

$$\Delta_{p+2}\varepsilon_{p+1,q}^{(1)} = -\frac{2N_D L_{p+2}}{(2\pi)^{p/2+2}V_q}\sum_{n=1}^{\infty}\frac{\cos(2\pi n\alpha_{p+2})}{(nL_{p+2})^{p+2}}\sum_{\mathbf{n}_{q-1}\in\mathbf{Z}^{q-1}}\times\left[\frac{\pi}{2}f_{p/2+1}(nL_{p+2}m_{\mathbf{n}_{q-1}})-\int_{m_{\mathbf{n}_{q-1}}}^{\infty}dx\frac{m}{x^2-k_{\mathbf{n}_{q-1}}^2}\frac{xf_{p/2+1}(nL_{p+2}x)}{\sqrt{x^2-m_{\mathbf{n}_{q-1}}^2}}\right] \tag{28}$$

is the part induced by the compactness of the direction z^{p+2}. In deriving this formula, we have used the integration formula

$$\int d\mathbf{k}_p\int_{\sqrt{\mathbf{k}_p^2+c^2}}^{\infty}dz(z^2-\mathbf{k}_p^2-c^2)^{(s+1)/2}f(z) = \frac{\pi^{p/2}\Gamma((s+3)/2)}{\Gamma((p+s+3)/2)}\int_c^\infty dx x(x^2-c^2)^{(p+s+1)/2}f(x). \tag{29}$$

This formula is obtained by integrating over the angular part of $\mathbf{k}_p$, changing the integration variable to $y = \sqrt{z^2-\mathbf{k}_p^2-c^2}$, and introducing polar coordinates in the $(|\mathbf{k}_p|, y)$-plane.

After the recurring application of formula (27), the Casimir energy for a single plate is presented in the form

$$E_{p+1,q}^{(1)} = V_q E_{D,0}^{(1)}+E_{p+1,q}^{(1,c)}. \tag{30}$$

Here, $E_{D,0}^{(1)}$ is the Casimir energy per unit volume along the directions $z^1,\ldots,z^{D-1}$ for a single plate in Minkowski spacetime with trivial topology and the second term

$$E_{p+1,q}^{(1,c)} = V_q\sum_{j=p+2}^{D}\Delta_j\varepsilon_{j-1,D+1-j}^{(1)} \tag{31}$$

is the topological part. The latter is finite and in the corresponding expression, the cut-off function can be removed. The renormalization is needed for the term $E_{D,0}^{(1)}$ only.

B. Interaction part

By using Eq. (29), the interaction part of the Casimir energy is presented in the form

$$\Delta E_{p+1,q} = -\frac{(4\pi)^{-(p+1)/2}N_D}{\Gamma((p+3)/2)}\sum_{\mathbf{n}_q\in\mathbf{Z}^q}\int_{m_{\mathbf{n}_q}}^{\infty}dz \times\frac{(z^2-m_{\mathbf{n}_q}^2)^{(p+1)/2}}{(z+m)e^{2az}+z-m}\left[a(z-m)-\frac{m}{z+m}\right]. \tag{32}$$

From here it follows that this part is always negative and it is a monotonically increasing function of a. Equation (32) can also be written in the following equivalent form

$$\Delta E_{p+1,q} = -\frac{(4\pi)^{-(p+1)/2}N_D}{\Gamma((p+1)/2)} \times\sum_{\mathbf{n}_q\in\mathbf{Z}^q}\int_{m_{\mathbf{n}_q}}^{\infty}dz z(z^2-m_{\mathbf{n}_q}^2)^{(p-1)/2} \times\ln\left(1+\frac{z-m}{z+m}e^{-2az}\right). \tag{33}$$

For a massless fermionic field from here we find

$$\Delta E_{p+1,q} = \frac{(2\pi)^{-p/2-1}N_D}{(2a)^{p+1}}\sum_{\mathbf{n}_q\in\mathbf{Z}^q}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{p+2}}f_{p/2+1}(2ank_{\mathbf{n}_q}), \tag{34}$$

where the function $f_\nu(x)$ is defined by Eq. (25).

Let us consider the asymptotic behavior of the interaction part in the Casimir energy at small and large separations between the plates. In the limit $L_l \gg a$, the main contribution comes from large values of n_l, $l = p+2,\cdots,D$, and we can replace the summation by the integration: $\sum_{\mathbf{n}_q\in\mathbf{Z}^q}\to\int d\mathbf{n}_q$. By making use of the integration formula (29) with $p\to q$, we find

$$\Delta E_{p+1,q} \approx V_q\Delta E_{D,0} = -V_q\frac{(4\pi)^{-D/2}N_D}{\Gamma(D/2)}\int_m^{\infty}dz z(z^2-m^2)^{D/2-1} \times\ln\left(1+\frac{z-m}{z+m}e^{-2az}\right). \tag{35}$$

In this formula, $\Delta E_{D,0}$ is the interaction part of the fermionic Casimir energy per unit volume along the directions $z^1,\ldots,z^{D-1}$ for two parallel plates in D-dimensional space with trivial topology (see Refs. [6,23] for the case $D=3$ and Ref. [25] for general D). Note, that for a massless field we have

$$\Delta E_{D,0} = -\frac{N_D(1-2^{-D})}{(4\pi)^{(D+1)/2}a^D}\Gamma((D+1)/2)\zeta(D+1), \tag{36}$$

where $\zeta(x)$ is the Riemann zeta function.

Now, let us consider the limit $L_l \ll a$. In this case, and for $\alpha_l = 0$, the main contribution comes from the zero mode with $\mathbf{n}_q = 0$ and $\Delta E_{p+1,q}/N_D$ coincides with the corresponding result for the Casimir effect in topologically trivial $(p+1)$-dimensional space: $\Delta E_{p+1,q} \approx N_D\Delta E_{p+1,0}/N_{p+1}$, where $\Delta E_{p+1,0}$ is given by formula (35) with the replacement $D\to p+1$. In this case, the contribution of the nonzero modes is exponentially suppressed. For $\alpha_l \neq 0$, the zero mode is absent. Assuming that am is fixed, to the leading order we have

$$\Delta E_{p+1,q} \approx -\frac{N_D e^{-2ac_0}}{2(4\pi a)^{(p+1)/2}}c_0^{(p+1)/2}, \tag{37}$$

where

$$c_0^2 = \sum_{l=p+2}^{D}(2\pi\beta_l a/L_l)^2, \qquad \beta_l = \min(\alpha_l, 1-\alpha_l). \tag{38}$$

In this case, the interaction part of the Casimir energy is exponentially suppressed.

In the discussion above, we have considered the region between the plates. The plates divide the background space into three regions: $z^{p+1}<0$, $0<z^{p+1}<a$, and $z^{p+1}>a$. The vacuum energy in the regions $z^{p+1}<0$ and $z^{p+1}>a$ is obtained from the results given above in the limit $a\to\infty$. In this limit, the interaction part vanishes and we have

$$E_{p+1,q} = aE_{p+1,q}^{(0)} + E_{p+1,q}^{(1)}, \qquad \begin{matrix} z^{p+1}<0, \\ z^{p+1}>a, \end{matrix} \tag{39}$$

with the topological and single plate parts given by Eqs. (23) and (30).

IV. THE CASIMIR FORCE

The total vacuum energy in the region $0\le z^l\le c_l$, $l=1,\ldots,p$, $0\le z^{p+1}\le a$ will be $E_{p+1,q}c_1\cdots c_p$ and the volume of this region is $V = c_1\cdots c_p a V_q$. The vacuum stress at $z^{p+1}=0+$ is given by

$$P_{p+1,q}(0+) = -\frac{\partial}{\partial V}E_{p+1,q}c_1\cdots c_p = P_{p+1,q}^{(0)} + \Delta P_{p+1,q}, \tag{40}$$

where we have introduced the notations

$$P_{p+1,q}^{(0)} = -\frac{E_{p+1,q}^{(0)}}{V_q}, \qquad \Delta P_{p+1,q} = -\frac{1}{V_q}\frac{\partial}{\partial a}\Delta E_{p+1,q}. \tag{41}$$

The vacuum stress at $z^{p+1}=a-$ is given by the same expression. The term $P_{p+1,q}^{(0)}$ does not depend on the separation between the plates and is the pure topological part of the vacuum force. The term $\Delta P_{p+1,q}$ is induced by the presence of the second plate and determines the interaction

force between the plates. Using the formula for $\Delta E_{p+1,q}$, for this part we find

$$\Delta P_{p+1,q} = -\frac{2(4\pi)^{-(p+1)/2}N_D}{\Gamma((p+1)/2)V_q}\sum_{\mathbf{n}_q\in\mathbf{Z}^q}\int_{m_{\mathbf{n}_q}}^{\infty} dz \times \frac{z^2(z^2-m_{\mathbf{n}_q}^2)^{(p-1)/2}}{\frac{z+m}{z-m}e^{2az}+1}. \quad (42)$$

Now, we see that $\Delta P_{p+1,q} < 0$, independent of the boundary conditions imposed on the field along the compactified dimensions and hence the interaction forces between the plates, are always attractive. For a massless fermionic field we have

$$\Delta P_{p+1,q} = -\frac{2N_D}{(2\pi)^{p/2+1}V_q}\sum_{\mathbf{n}_q\in\mathbf{Z}^q}\sum_{n=1}^{\infty} \times \frac{f_{p/2+1}(2ank_{\mathbf{n}_q}) - f_{p/2+2}(2ank_{\mathbf{n}_q})}{(-1)^n(2an)^{p+2}}. \quad (43)$$

For small separations between the plates $L_l \gg a$, we replace the summation over $\mathbf{n}_q$ by the integration. In the way similar to that we have used for the Casimir energy, it can be seen that in the leading order the interaction force coincides with the corresponding result for parallel plates on background of D-dimensional space with trivial topology:

$$\Delta P_{p+1,q} \approx \Delta P_{D,0} = -\frac{2N_D}{(4\pi)^{D/2}\Gamma(D/2)}\int_m^{\infty} dz\frac{z^2(z^2-m^2)^{D/2-1}}{\frac{z+m}{z-m}e^{2az}+1}. \quad (44)$$

The contribution of the nonzero modes is exponentially small. For the massless field we have

$$\Delta P_{D,0} = -\frac{ND(1-2^{-D})}{(4\pi)^{(D+1)/2}a^{D+1}}\Gamma((D+1)/2)\zeta(D+1). \quad (45)$$

This result can also be directly obtained from Eq. (36).

For large interplate separations $L_l \ll a$ and for $\alpha_l = 0$, the main contribution comes from the zero mode $\mathbf{n}_q = 0$. In this limit, $V_q\Delta P_{p+1,q}/N_D$ coincides with the corresponding result for the Casimir effect in $(p+1)$-dimensional space: $\Delta P_{p+1,q} \approx N_D\Delta P_{p+1,0}/(N_{p+1}V_q)$, where $\Delta P_{p+1,0}$ is given by formula (44) with $D \to p+1$. If $\alpha_l \neq 0$ and am is fixed, the interaction force is exponentially suppressed

$$\Delta P_{p+1,q} \approx -\frac{N_D c_0^{(p+3)/2}e^{-2ac_0}}{(4\pi a)^{(p+1)/2}V_q}, \quad (46)$$

with c_0 defined by Eq. (38).

If the quantum field lives in all regions, in considering the total forces acting on the plate, we should also take into account the force acting on the sides $z^{p+1} = 0-$ and $z^{p+1} = a+$. The corresponding forces per unit surface are equal to $P^{(0)}_{p+1,q}$ and they are directed along the positive/negative direction of the axis z^{p+1} in the case $P^{(0)}_{p+1,q} > 0/P^{(0)}_{p+1,q} < 0$. Now, we see that the topological parts of the force acting from the left-hand and right-hand sides of the plate compensate and the resulting force is determined by (42). This force is attractive for all interplate separations. There are physical situations [bag model, finite-length carbon nanotubes (see below)], where the quantum field is confined to the interior of some region and there is no field outside. For the problem under consideration, if the quantum field is confined in the region between the plates, the total Casimir force acting per unit surface of the plate is determined by Eq. (40) and the pure topological part contributes as well. At large distances, this part dominates and the corresponding forces tend to increase/decrease the distance between the plates when $P^{(0)}_{p+1,q} > 0/P^{(0)}_{p+1,q} < 0$. In particular, $P^{(0)}_{p+1,q} < 0$ for untwisted fields and $P^{(0)}_{p+1,q} > 0$ for twisted fields. Hence, if the quantum field is confined in the region between the plates, for untwisted fields the Casimir forces are attractive for all separations. For twisted fields, these forces are attractive for small distances and they are repulsive at large distances.

V. ZETA FUNCTION APPROACH

In this section, for the evaluation of the vacuum energy in the region $0 \le z^{p+1} \le a$, we will use the zeta function technique [7,31]. This allows one to obtain alternative representations for the pure topological and single plate parts in the Casimir effect. Instead of the divergent expression on the right of Eq. (13), we consider the finite quantity

$$E_{p+1,q}(\mu, s) = -\mu^{2s+1}\frac{N_D}{2}\int\frac{d\mathbf{k}_p}{(2\pi)^p}\sum_{\mathbf{n}_q\in\mathbf{Z}^q}\sum_{n=1}^{\infty} \times (\mathbf{k}_p^2 + m_{\mathbf{n}_q}^2 + \lambda_n^2/a^2)^{-s}, \quad (47)$$

where the arbitrary mass scale μ is introduced in order to keep the dimensionality of the expression. Performing the integration over $\mathbf{k}_p$, we find

$$E_{p+1,q}(\mu, s) = -\mu^{2s+1}\frac{N_D\Gamma(s-p/2)}{2(4\pi)^{p/2}\Gamma(s)}\sum_{\mathbf{n}_q\in\mathbf{Z}^q}\sum_{n=1}^{\infty} \times (m_{\mathbf{n}_q}^2 + \lambda_n^2/a^2)^{p/2-s}. \quad (48)$$

The computation of the Casimir energy requires the analytic continuation of $E_{p+1,q}(\mu, s)$ to the value $s = -1/2$. The starting point of our consideration is the representation of the partial zeta function as a contour integral in the complex plane z:

$$\sum_{n=1}^{\infty}(m_{\mathbf{n}_q}^2+\lambda_n^2/a^2)^{p/2-s}=\frac{1}{2\pi i}\int_C dz(z^2/a^2+m_{\mathbf{n}_q}^2)^{p/2-s}\times\frac{d}{dz}\ln\left[\frac{(ma/z)\sin z+\cos z}{1+ma}\right], \tag{49}$$

where C denotes a closed counterclockwise contour enclosing all zeros λ_n. We assume that the contour C is made of a large semicircle (with radius tending to infinity) centered at the origin and placed to its right, plus a straight part overlapping the imaginary axis and avoiding the points $\pm i m_{\mathbf{n}_q} a$ by small semicircles in the right half plane. When the radius of the large semicircle tends to infinity, the corresponding contribution vanishes for $\mathrm{Re}\, s > (p+1)/2$. Assuming that $(p+1)/2 < \mathrm{Re}\, s < p/2+1$, from (49) we find the following integral representation for the regularized vacuum energy:

$$E_{p+1,q}(\mu,s)=-\frac{(4\pi)^{-p/2}\mu^{2s+1}N_D}{2\Gamma(s)\Gamma(1-s+p/2)}\sum_{\mathbf{n}_q\in\mathbf{Z}^q}\times\int_{m_{\mathbf{n}_q}}^{\infty}dz(z^2-m_{\mathbf{n}_q}^2)^{p/2-s}\times\frac{d}{dz}\ln\left[\frac{(m/z)\sinh(az)+\cosh(az)}{1+ma}\right]. \tag{50}$$

Now, we decompose the logarithmic term in this expression as

$$\frac{d}{dz}\ln\left[\frac{(m/z)\sinh(az)+\cosh(az)}{1+ma}\right]=a+\frac{d}{dz}\ln(1+m/z)+\frac{d}{dz}\ln\left(1+\frac{z-m}{z+m}e^{-2az}\right). \tag{51}$$

As a result, we have the following decomposition of the generalized zeta function:

$$E_{p+1,q}(\mu,s)=aE_{p+1,q}^{(0)}(\mu,s)+2E_{p+1,q}^{(1)}(\mu,s)+\Delta E_{p+1,q}(\mu,s), \tag{52}$$

where the first, second, and third terms on the right-hand side come from the corresponding terms in Eq. (51). The interaction term $\Delta E(\mu,s)$ in Eq. (52) is finite at the physical point $s=-1/2$ and gives the result (33): $\Delta E_{p+1,q}(\mu,-1/2)=\Delta E_{p+1,q}$. Below, we will be focused on the pure topological and single plate parts.

First, let us consider the term $E_{p+1,q}^{(0)}(\mu,s)$. This term is the regularized vacuum energy in the topology $R^{p+1}\times(S^1)^q$ without boundaries. In this term, the integration over z is done explicitly and we find

$$E_{p+1,q}^{(0)}(\mu,s)=-\frac{\mu^{2s+1}N_D\Gamma(s-(p+1)/2)}{2(4\pi)^{(p+1)/2}\Gamma(s)}\sum_{\mathbf{n}_q\in\mathbf{Z}^q}m_{\mathbf{n}_q}^{p+1-2s}. \tag{53}$$

Further analytic continuation of this expression to the physical point $s=-1/2$ is done by using the extended Chowla-Selberg formula [32]. The corresponding result is given by the expression [16]:

$$E_{p+1,q}^{(0)}=\frac{N_D m^{D+1}V_q}{(2\pi)^{(D+1)/2}}\sum_{\mathbf{m}_q\in\mathbf{Z}^q}{}'\cos(2\pi\mathbf{m}_q\cdot\boldsymbol{\alpha}_q)\times\frac{f_{(D+1)/2}(mg(\mathbf{L}_q,\mathbf{m}_q))}{(mg(\mathbf{L}_q,\mathbf{m}_q))^{D+1}}, \tag{54}$$

where we have used the notation

$$g(\mathbf{L}_q,\mathbf{m}_q)=\left(\sum_{i=p+2}^{D}L_i^2m_i^2\right)^{1/2}. \tag{55}$$

The prime on the summation sign in Eq. (54) means that the term $\mathbf{m}_q=0$ should be excluded from the sum.

Now, we turn to the part $E_{p+1,q}^{(1)}(\mu,s)$ which is the regularized vacuum energy in the half-space induced by a single plate. In the corresponding integral representation, we expand $\ln(1+m/z)$ in powers of m/z and integrate over z explicitly. This leads to the result

$$E_{p+1,q}^{(1)}(\mu,s)=-\frac{\mu^{2s+1}N_D}{8(4\pi)^{p/2}\Gamma(s)}\times\sum_{l=1}^{\infty}m^l\frac{\Gamma((l-p)/2+s)}{(-1)^l\Gamma(l/2+1)}\sum_{\mathbf{n}_q\in\mathbf{Z}^q}m_{\mathbf{n}_q}^{p-2s-l}. \tag{56}$$

The application to the multiseries over $\mathbf{n}_q$ of the extended Chowla-Selberg formula allows one to present $E_{p+1,q}^{(1)}(\mu,s)$ as the sum of two parts:

$$E_{p+1,q}^{(1)}(\mu,s)=V_qE_{R^D}^{(1)}(\mu,s)+E_{p+1,q}^{(1,c)}(\mu,s). \tag{57}$$

Here, the term

$$E_{R^D}^{(1)}(\mu,s)=-\frac{\mu^{2s+1}N_Dm^{D-2s-1}}{8(4\pi)^{(D-1)/2}\Gamma(s)}\sum_{l=1}^{\infty}(-1)^l\times\frac{\Gamma(s+(l+1-D)/2)}{(-1)^l\Gamma(l/2+1)} \tag{58}$$

is the corresponding quantity in the case of trivial topology R^D. The topological term $E_{p+1,q}^{(1,c)}(\mu,s)$ is finite at the physical point $s=-1/2$ and the topological part of the vacuum energy for a single plate has the form

$$E_{p+1,q}^{(1,c)} = E_{p+1,q}^{(1,c)}(\mu, -1/2) = \frac{N_D m^D V_q}{4(2\pi)^{D/2}} \sum_{l=1}^{\infty} \frac{2^{-l/2}(-1)^l}{\Gamma(l/2+1)} \times \sum_{\mathbf{m}_q \in \mathbf{Z}^q}{}' \cos(2\pi \mathbf{m}_q \cdot \boldsymbol{\alpha}_q) f_{(l-D)/2}(mg(\mathbf{L}_q, \mathbf{m}_q)), \tag{59}$$

where we have used the relation $f_{-\nu}(x) = x^{-2\nu} f_{\nu}(x)$. Note, that we can write the function $\cos(2\pi \mathbf{m}_q \cdot \boldsymbol{\alpha}_q)$ on the right of formula (59) in the form of the product $\prod_{i=p+2}^{D} \cos(2\pi m_i \alpha_i)$. The equivalence of two representations (30) and (59) for the topological part in the Casimir energy for a single plate can be seen by making use of the relation [16]

$$\sum_{\mathbf{m}_{q-1} \in Z^{q-1}} \cos(2\pi \mathbf{m}_{q-1} \cdot \boldsymbol{\alpha}_{q-1}) f_{(l-D)/2}(mg(\mathbf{L}_q, \mathbf{m}_q)) = \frac{(2\pi)^{(q-1)/2} L_{p+2}}{V_q m^{D-l}} \sum_{\mathbf{n}_{q-1} \in \mathbf{Z}^{q-1}} \frac{f_{(p-l)/2+1}(m_{p+2} L_{p+2} m_{\mathbf{n}_{q-1}})}{(m_{p+2} L_{p+2})^{p-l+2}}, \tag{60}$$

and the formula

$$\frac{2}{\pi} \int_1^{\infty} dx \frac{c}{x^2 - 1 + c^2} \frac{x f_{p/2+1}(bx)}{\sqrt{x^2-1}} = \sum_{l=0}^{\infty} \frac{2^{-l/2}(-1)^l}{\Gamma(l/2+1)} (bc)^l f_{(p-l)/2+1}(b), \tag{61}$$

valid for $0 \le c \le 1$.

VI. SPECIAL CASE OF TOPOLOGY

By taking into account the importance of special case $p = D - 2$, $q = 1$ in Kaluza-Klein models and in carbon nanotubes, in this section we consider it separately. For the later convenience, the parameters of the compactified dimension we will denote by $L_D = L$ and $\alpha_D = \alpha$. The corresponding formulae for the separate parts in the Casimir energy take the form

$$E_{D-1,1}^{(0)} = \frac{2N_D L^{-D}}{(2\pi)^{(D+1)/2}} \sum_{n=1}^{\infty} \frac{\cos(2\pi n\alpha)}{n^{D+1}} f_{(D+1)/2}(mnL),$$

$$E_{D-1,1}^{(1,c)} = \frac{N_D m^D L}{2(2\pi)^{D/2}} \sum_{n=1}^{\infty} \cos(2\pi n\alpha) \sum_{l=1}^{\infty} \frac{2^{-l/2}(-1)^l}{\Gamma(l/2+1)} \times f_{(l-D)/2}(mnL),$$

$$\Delta E_{D-1,1} = -\frac{(4\pi)^{-(D-1)/2} N_D}{\Gamma((D-1)/2)} \sum_{l=-\infty}^{+\infty} \times \int_{m_l}^{\infty} dz z (z^2 - m_l^2)^{(D-3)/2} \times \ln\left(1 + \frac{z-m}{z+m} e^{-2az}\right), \tag{62}$$

where we have introduced the notation

$$m_l^2 = [2\pi(l+\alpha)/L]^2 + m^2. \tag{63}$$

An equivalent representation of the single plate part is obtained from Eqs. (28) and (31).

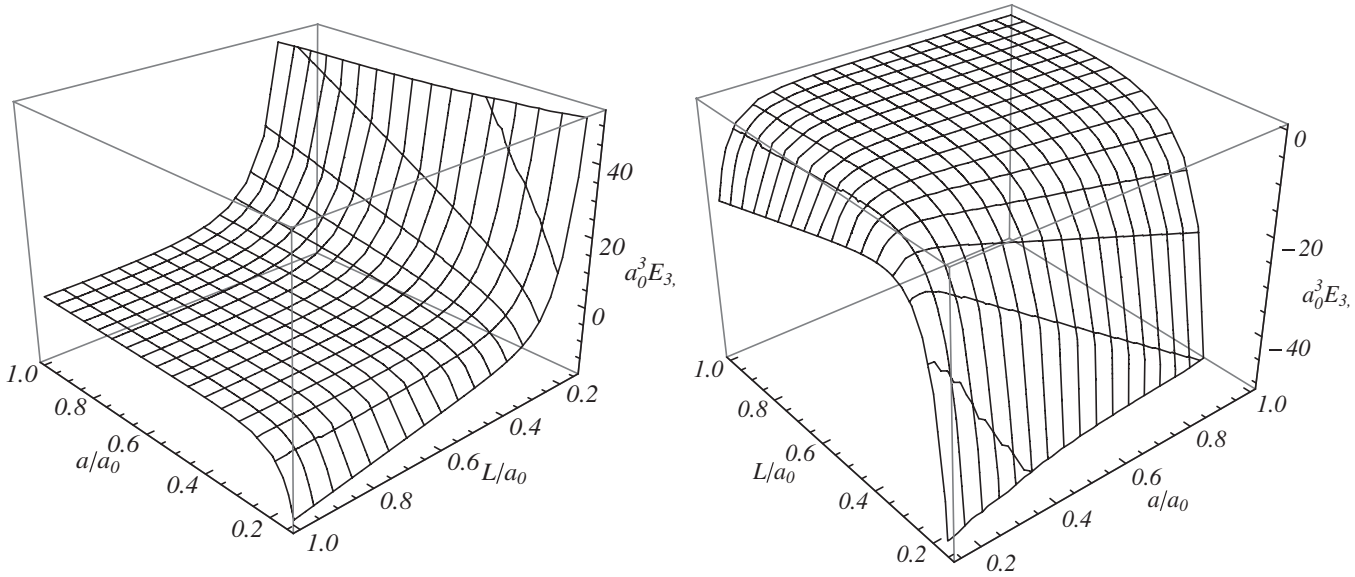

FIG. 1. The Casimir energy of a massless fermionic field in four-dimensional space with topology $R^3 \times S^1$ as a function of the interplate distance and the length of the compact dimension. The left-hand/right-hand panel corresponds to untwisted/twisted fields, respectively.

For the massless case, these formulae are simplified to

$$E^{(0)}_{D-1,1} = \frac{N_D L^{-D}}{\pi^{(D+1)/2}}\Gamma((D+1)/2)\sum_{n=1}^{\infty}\frac{\cos(2\pi n\alpha)}{n^{D+1}},$$
$$\Delta E_{D-1,1} = \frac{(2\pi)^{-D/2}N_D}{(2a)^{D-1}}\sum_{l=-\infty}^{+\infty}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^D} \times f_{D/2}(4\pi n|l+\alpha|a/L), \quad (64)$$

and the single plate part vanishes. In Fig. 1, we have presented the Casimir energy $E_{D-1,1}$ for a massless fermionic field in the simplest Kaluza-Klein model with $D = 4$ as a function of the interplate distance and the length of the internal space measured in units of a fixed length a_0. The left-hand panel corresponds to the untwisted field ($\alpha = 0$) and the right one is for the twisted field ($\alpha = 1/2$). For large interplate separations, the pure topological part dominates and the Casimir energy is a linear function of a. At small distances, the interaction part is dominant and the Casimir energy behaves as a^{-D}.

In the special case under consideration for the interaction part of the Casimir force, we have the formula

$$\Delta P_{D-1,1} = -\frac{2(4\pi)^{-(D-1)/2}N_D}{\Gamma((D-1)/2)L_D} \times \sum_{l=-\infty}^{+\infty}\int_{m_l}^{\infty}dz\frac{z^2(z^2-m_l^2)^{(D-3)/2}}{\frac{z+m}{z-m}e^{-2az}+1}. \quad (65)$$

In the massless case, this formula takes the form

$$\Delta P_{D-1,1} = -\frac{2N_D}{(2\pi)^{D/2}L}\sum_{l=-\infty}^{+\infty}\sum_{n=1}^{\infty} \times \frac{f_{D/2}(y)-f_{D/2+1}(y)}{(-1)^n(2an)^D}\Big|_{y=4\pi n|l+\alpha|a/L}. \quad (66)$$

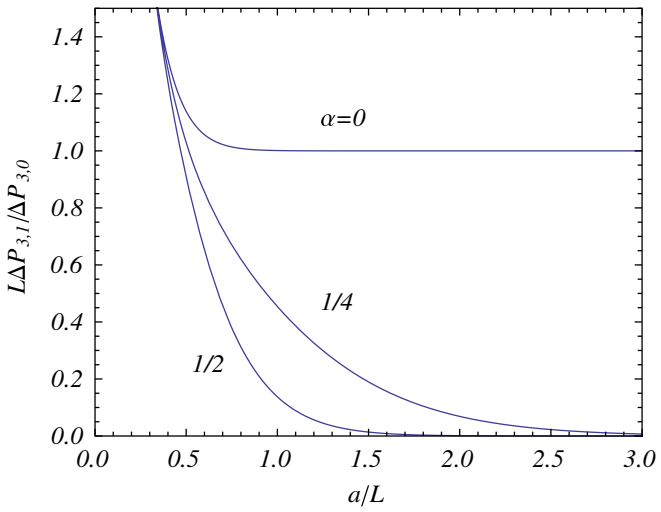

FIG. 2 (color online). The ratio of the fermionic Casimir force for two parallel plates in the space with topology $R^3 \times S^1$ to the standard Casimir force in R^3, for a massless field, as a function of a/L. The values on each of the curves correspond to those of the parameter α.

In Fig. 2, we have plotted the ratio $L\Delta P_{3,1}/\Delta P_{3,0}$ versus a/L for different values of the parameter α. As it already has been explained before, only in the case of untwisted field the Casimir force at large separations tends to the corresponding force (up to the factor related to the number of polarizations) for the model where the compactified dimensions are absent. For other cases, the force is exponentially suppressed at large separations which is clearly seen in Fig. 2.

VII. APPLICATIONS TO FINITE-LENGTH NANOTUBES

For a number of planar condensed matter systems, the fermionic excitations in the long-wavelength regime are described by the Dirac-like model. A well-known example is the graphene. In this section, we specify the general results given above for the electrons on a graphene sheet rolled into a cylindrical shape (carbon nanotube). The carbon nanotube is characterized by its chiral vector $\mathbf{C}_h = n_w\mathbf{a}_1 + m_w\mathbf{a}_2$, where $\mathbf{a}_1$ and $\mathbf{a}_2$ are the basis vectors of the hexagonal lattice of graphene and n_w, m_w are integers. The circumference length of the nanotube is given by $L = |\mathbf{C}_h| = a_g\sqrt{n_w^2 + m_w^2 + n_w m_w}$, with $a_g = |\mathbf{a}_1| = |\mathbf{a}_2| = 2.46$Å being the lattice constant. Zigzag nanotubes correspond to the special case $\mathbf{C}_h = (n_w, 0)$, and for armchair nanotubes one has $\mathbf{C}_h = (n_w, n_w)$. All other cases correspond to chiral nanotubes. The electron properties of carbon nanotubes can be either metallic or semiconductorlike depending on the manner the cylinder is obtained from the graphene sheet. In the case $n_w - m_w = 3q_w$, $q_w \in Z$, the nanotube will be metallic. When $n_w - m_w \neq 3q_w$, the nanotube will be semiconductor with an energy gap inversely proportional to the diameter. In particular, the armchair nanotube is metallic and the $(n_w, 0)$ zigzag nanotube is metallic if and, only if, n_w is an integer multiple of 3.

The electronic band structure of a carbon nanotube close to the Dirac points shows a conical dispersion $E(\mathbf{k}) = v_F|\mathbf{k}|$, where $\mathbf{k}$ is the momentum measured relative to the Dirac points and v_F represents the Fermi velocity. The latter plays the role of speed of light. The corresponding low-energy excitations can be described by a pair of two-component spinors, which are composed of the Bloch states residing on the two different sublattices of the honeycomb lattice of the graphene sheet. The corresponding Fermi velocity is given by $v_F = 3ta/2$ ($v_F \approx 10^8$ cm/s in graphene), where t is the nearest neighbor hopping energy. The Dirac-like model is valid, provided that the cylinder circumference is much larger than the interatomic spacing.

For typical nanotubes, the corresponding ratio can be between 10 and 20 and this approximation is adequate [4,5].

For a carbon nanotube, $D = 2$ and we have the spatial topology $R^1 \times S^1$ with the compactified dimension of the length L. We will assume that the nanotube has finite-length a. As the $D = 2$ Dirac field lives on the cylinder surface, it is natural to impose bag boundary conditions (3) on the cylinder edges which ensure the zero fermion flux through these edges. The additional confinement of the fermionic field along the tube axis leads to the change of the ground state energy. The corresponding expressions for the Casimir energy and force are obtained from the formulae of the previous section taking $D = 2$. Here, by taking into account that in the presence of an external magnetic field an effective mass term is generated for the fermionic excitations, we consider the general case of massive spinor field. The formulae for a massless case, appropriate for carbon nanotubes in the absence of external fields, will be given separately.

In order to specify the boundary condition along the compactified dimension, we note that for the (n_w, m_w) nanotube the phase factor in the wave function has the form $e^{i[m_1+(n_w-m_w)/3]\varphi}$, where φ is the angular coordinate along the compact dimension and m_1 is an integer. From here it follows that for metallic nanotubes, the periodic boundary condition ($\alpha = 0$) is realized. For semiconductor nanotubes, depending on the chiral vector, there are two classes of inequivalent boundary conditions corresponding to $\alpha = 1/3$ ($n_w - m_w = 3q_w + 2$) and $\alpha = 2/3$ ($n_w - m_w = 3q_w + 1$). In the expressions for the pure topological parts of the Casimir energy and force the phase, α appears in the form $\cos(2\pi n\alpha)$. Hence, these quantities are the same for $\alpha = 1/3$ and $\alpha = 2/3$. As the boundary induced parts have the structure $\sum_{l=-\infty}^{+\infty} f(|l + \alpha|)$, the same property holds for these parts.

In the case $D = 2$, the general formulae for the separate parts of the Casimir energy from the previous section take the form ($N = 2$)

$$E_{1,1}^{(0)} = \frac{1}{\pi L^2}\sum_{n=1}^{\infty}(1 + mnL)\cos(2\pi n\alpha)\frac{e^{-mnL}}{n^3},$$

$$E_{1,1}^{(1,c)} = \frac{m^2 L}{2\pi}\sum_{n=1}^{\infty}\cos(2\pi n\alpha)\sum_{l=1}^{\infty}\frac{2^{-l/2}(-1)^l}{\Gamma(l/2+1)}f_{l/2-1}(nLm),$$

$$\Delta E_{1,1} = -\frac{1}{\pi}\sum_{l=-\infty}^{+\infty}\int_0^{\infty}dz \times \ln\left(1 + \frac{\sqrt{z^2+m_l^2}-m}{\sqrt{z^2+m_l^2}+m}e^{-2a\sqrt{z^2+m_l^2}}\right). \quad (67)$$

For the Casimir force acting on the edges of the tube, we have

$$P_{1,1} = -\frac{1}{\pi L^3}\sum_{n=1}^{\infty}(1 + mnL)\cos(2\pi n\alpha)\frac{e^{-mnL}}{n^3} - \frac{2}{\pi L}\sum_{l=-\infty}^{+\infty}\int_0^{\infty}dzz \times \left(\frac{\sqrt{z^2+m_l^2}+m}{\sqrt{z^2+m_l^2}-m}e^{2a\sqrt{z^2+m_l^2}}+1\right)^{-1}. \quad (68)$$

In the massless case for the total Casimir energy and the stresses, we find the formulae

$$E_{1,1} = \frac{a}{\pi L^2}\sum_{n=1}^{\infty}\frac{\cos(2\pi n\alpha)}{n^3} - \frac{1}{2\pi a}\sum_{l=-\infty}^{+\infty}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}f_1(4\pi n|l+\alpha|a/L),$$

$$P_{1,1} = -\frac{1}{\pi L^3}\sum_{n=1}^{\infty}\frac{\cos(2\pi n\alpha)}{n^3} - \frac{1}{2\pi a^2 L}\sum_{l=-\infty}^{+\infty}\sum_{n=1}^{\infty}\frac{f_1(y)-f_2(y)}{(-1)^n n^2}\Big|_{y=4\pi n|l+\alpha|a/L}. \quad (69)$$

The corresponding expressions for the Casimir energy and force in finite-length cylindrical nanotubes are obtained from (69) with additional factor 2 which takes into account the presence of two sublattices. In standard units, the factor $\hbar v_F$ appears as well. So, for the Casimir force acting per unit length of the edge of a carbon nanotube, one has $P^{(\mathrm{CN})} = 2\hbar v_F P_{1,1}$, where $P_{1,1}$ is given by Eq. (69). For long tubes $a/L \gg 1$, the first term on the right is dominant and we have $P^{(\mathrm{CN})} \approx -0.765\hbar v_F/L^3$ for metallic nanotubes and $P^{(\mathrm{CN})} \approx 0.34\hbar v_F/L^3$ for semiconducting ones. In the limit $a/L \ll 1$, the interaction part is dominant. In the leading order, the Casimir force does not depend on the chirality and one has $P^{(\mathrm{CN})} \approx -0.144\hbar v_F/a^3$. In Fig. 3, we have plotted the Casimir forces acting on the edges of metallic (left-hand panel) and semiconducting-type (right-hand panel) carbon nanotube as functions of the tube length for different values of the fermion mass. As it is seen, for metallic nanotubes, these forces are always attractive, whereas for semiconducting-type ones, they are attractive for small lengths and repulsive for large lengths.

In the discussion above, we have considered bag boundary conditions on the edges of the nanotube. The periodicity conditions along the axis correspond to the toroidal compactification of the carbon nanotubes. The Casimir energies in toroidal nanotubes are investigated in Ref. [16]. It was shown that the toroidal compactification of a cylindrical nanotube along its axis increases the Casimir energy for periodic boundary conditions and decreases the Casimir energy for the semiconducting-type compactifications. Recently, in the last paper of Ref. [13], the Casimir interaction between two plates resulting from the quantum fluctuations of the bulk electromagnetic field

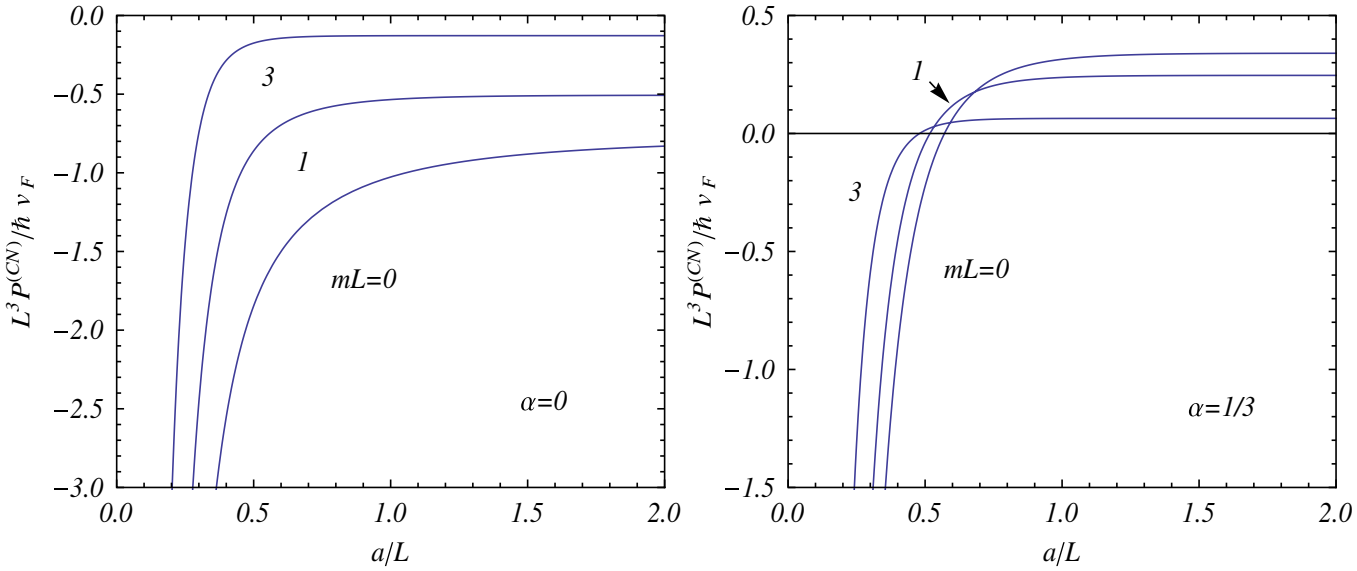

FIG. 3 (color online). The fermionic Casimir forces acting on the edges of the metallic (left-hand panel) and semiconducting-type (right-hand panel) nanotubes as functions of the tube length for different values of the field mass.

is investigated with one plate being graphene described in the Dirac model and the other one being an ideal conductor. The interaction of the electromagnetic field with the fermion field confined on the graphene sheet is equivalent to imposing boundary condition for the electromagnetic field. At large separations, the corresponding force is proportional to the fine structure constant and falls off as the inverse cube of distance between the plates.

VIII. CONCLUSION

We have investigated the effect of compact spatial dimensions on the Casimir energy and force for a massive fermionic field in the geometry of two parallel plates on which the field obeys MIT bag boundary condition. Along the compact dimensions, we have assumed periodicity conditions (2) with constant phases α_l. The eigenvalues of the wave-vector component normal to the plates are roots of transcendental Eq. (10). By applying the Abel-Plana-type summation formula to the corresponding series in the mode sum for the vacuum energy, we have explicitly extracted, in a cutoff independent way, the pure topological part and the contributions induced by the single plates. The surface divergences in the Casimir energy are contained in the single plate components only and the remaining interaction part is finite for all nonzero interplate distances. The latter is given by Eq. (33) for a massive field and by Eq. (34) in the massless case. The interaction part of the Casimir energy is always negative. We have decomposed the single plate part in the vacuum energy into two terms: the first one is the Casimir energy for a single plate in the trivial topology R^D and the second one is the topological part. The second term is cutoff independent and, in this way, the renormalization procedure is reduced to that for the plate in topology R^D.

The Casimir forces between the plates have been considered in Sec. IV. Single plate parts in the Casimir energy do not depend on the plates separation and do not contribute to the force. For the region between the plates, the forces are presented as the sum of topological and interaction parts. In the situations where the quantum field lives on both sides of the plate, the topological parts are the same on the left-hand and right-hand sides and the effective force is determined by the interaction part only. The latter is given by formulae (42) and (43) for the massive and massless fields, respectively. With independence of the lengths of compact dimensions and the phases in the periodicity conditions, the corresponding force is attractive and is a monotonic function of the distance. When the field is confined in the region between the plates, only the topological part contributes to the resulting force and it dominates at large separations between the plates. In dependence of the phases in the periodicity conditions, the corresponding forces can be either attractive or repulsive. In particular, for untwisted fields, the Casimir forces are attractive for all separations and for twisted fields these forces are attractive for small distances and repulsive at large distances. For small separations, the interaction part dominates and the Casimir force is attractive. For small values of the size of the compact subspace and in models where the zero mode along the internal space is present, the main contribution to the Casimir force comes from this mode and the contributions of the nonzero modes are exponentially suppressed. In this limit, to leading order, we recover the standard result for the Casimir force between two plates in $(p+2)$-dimensional Minkowski spacetime. When the zero mode is absent, the Casimir forces are exponentially suppressed in the limit of small size of the internal space.

In Sec. V, we have evaluated the Casimir energy by using an alternative method based on the generalized zeta function technique. With the combination of the extended Chowla-Selberg formula, this allowed us to present the topological part for the geometry of a single plate in an alternative form given by formula (59). As an illustration of the general results, in Sec.. VI we have considered a special model with a single compact dimensions. In Sec. VII, we specify the general formulae for the model with $D = 2$. This model may be used for the evaluation of the Casimir energy and force within the framework of the Dirac-like theory for the description of the electronic states in carbon nanotubes where the role of speed of light is played by the Fermi velocity. The pure topological part of the Casimir energy is positive for metallic cylindrical nanotubes and is negative for semiconducting ones. For finite-length carbon nanotubes, the Casimir forces acting on the tube edges are always attractive for metallic nanotubes, whereas for semiconducting-type ones they are attractive for small lengths and repulsive for large lengths.

ACKNOWLEDGMENTS

A. A. S. was supported by the Armenian Ministry of Education and Science Grant No. 119.

[1] A. Linde, J. Cosmol. Astropart. Phys. 10 (2004) 004.
[2] G. D. Starkman, Classical Quantum Gravity **15**, 2529 (1998); N. J. Cornish, D. N. Spergel, and G. D. Starkman, Classical Quantum Gravity **15**, 2657 (1998).
[3] J. Levin, Phys. Rep. **365**, 251 (2002); N. J. Cornish, D. N. Spergel, G. D. Starkman, and E. Komatsu, Phys. Rev. Lett. **92**, 201302 (2004).
[4] R. Saito, G. Dresselhaus, and M. S. Dresselhaus, *Physical Properties of Carbon Nanotubes* (Imperial College Press, London, 1998); C. Dupas, P. Houdy, and M. Lahmani, *Nanoscience: Nanotechnologies and Nanophysics* (Springer, Berlin, 2007).
[5] G. W. Semenoff, Phys. Rev. Lett. **53**, 2449 (1984); D. P. Di Vincenzo and E. J. Mele, Phys. Rev. B **29**, 1685 (1984); J. Gonzàlez, F. Guinea, and M. A. H. Vozmediano, Nucl. Phys. **B406**, 771 (1993); Phys. Rev. B **63**, 134421 (2001); H.-W. Lee and D. S. Novikov, Phys. Rev. B **68**, 155402 (2003); S. G. Sharapov, V. P. Gusynin, and H. Beck, Phys. Rev. B **69**, 075104 (2004); K. S. Novoselov *et al.*, Nature (London) **438**, 197 (2005); D. S. Novikov and L. S. Levitov, Phys. Rev. Lett. **96**, 036402 (2006); E. Perfetto, J. González, F. Guinea, S. Bellucci, and P. Onorato, Phys. Rev. B **76**, 125430 (2007); A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov, and A. K. Geim, Rev. Mod. Phys. **81**, 109 (2009).
[6] V. M. Mostepanenko and N. N. Trunov, *The Casimir Effect and Its Applications* (Clarendon, Oxford, 1997).
[7] E. Elizalde, S. D. Odintsov, A. Romeo, A. A. Bytsenko, and S. Zerbini, *Zeta Regularization Techniques with Applications* (World Scientific, Singapore, 1994); E. Elizalde, *Ten Physical Applications of Spectral Zeta Functions*, Lecture Notes in Physics (Springer-Verlag, Berlin, 1995).
[8] M. Bordag, U. Mohidden, and V. M. Mostepanenko, Phys. Rep. **353**, 1 (2001).
[9] K. A. Milton, *The Casimir Effect: Physical Manifestation of Zero-Point Energy* (World Scientific, Singapore, 2002).
[10] M. Bordag, G. L. Klimchitskaya, U. Mohideen, and V. M. Mostepanenko, *Advances in the Casimir Effect* (Oxford University Press, Oxford, 2009).
[11] M. J. Duff, B. E. W. Nilsson, and C. N. Pope, Phys. Rep. **130**, 1 (1986); A. A. Bytsenko, G. Cognola, L. Vanzo, and S. Zerbini, Phys. Rep. **266**, 1 (1996).
[12] K. A. Milton, Gravitation Cosmol. **9**, 66 (2003); E. Elizalde, S. Nojiri, and S. D. Odintsov, Phys. Rev. D **70**, 043539 (2004); E. Elizalde, J. Phys. A **39**, 6299 (2006); B. Greene and J. Levin, J. High Energy Phys. 11 (2007) 096; P. Burikham, A. Chatrabhuti, P. Patcharamaneepakorn, and K. Pimsamarn, J. High Energy Phys. 07 (2008) 013.
[13] Y. Srivastava, A. Widom, and M. H. Friedman, Phys. Rev. Lett. **55**, 2246 (1985); E. Buks and M. L. Roukes, Phys. Rev. B **63**, 033402 (2001); H. B. Chan, V. A. Aksyuk, R. N. Kleiman, D. J. Bishop, and F. Capasso, Science **291**, 1941 (2001); Phys. Rev. Lett. **87**, 211801 (2001); G. Barton, J. Phys. A **38**, 2997 (2005); E. V. Blagov, G. L. Klimchitskaya, and V. M. Mostepanenko, Phys. Rev. B **71**, 235401 (2005); M. Bordag, J. Phys. A **39**, 6173 (2006); M. Bordag, B. Geyer, G. L. Klimchitskaya, and V. M. Mostepanenko, Phys. Rev. B **74**, 205431 (2006); G. L. Klimchitskaya, E. V. Blagov, and V. M. Mostepanenko, J. Phys. A **41**, 164012 (2008); C. Genet, A. Lambrecht, and S. Reynaud, Eur. Phys. J. Special Topics **160**, 183 (2008); T. Emig, arXiv:0901.4568; M. Bordag, I. V. Fialkovsky, D. M. Gitman, and D. V. Vassilevich, arXiv:0907.3242.
[14] J. S. Dowker and R. Critchley, J. Phys. A **9**, 535 (1976); R. Banach and J. S. Dowker, J. Phys. A **12**, 2545 (1979); B. S. DeWitt, C. F. Hart, and C. J. Isham, Physica A (Amsterdam) **96**, 197 (1979); S. G. Mamayev and N. N. Trunov, Russ. Phys. J. **22**, 766 (1979); **23**, 551 (1980); L. H. Ford, Phys. Rev. D **21**, 933 (1980); J. Ambjørn and S. Wolfram, Ann. Phys. (N.Y.) **147**, 1 (1983); S. G. Mamayev and V. M. Mostepanenko, In *Proceedings of the Third Seminar on Quantum Gravity* (World Scientific, Singapore, 1985); Yu. P. Goncharov and A. A. Bytsenko, Phys. Lett. **160B**, 385 (1985); Nucl. Phys. **B271**, 726 (1986); Classical Quantum Gravity **4**, 555 (1987); E. Elizalde, Z. Phys. C **44**, 471 (1989); E. Ponton and E. Poppitz, J. High Energy Phys. 06 (2001) 019; H. Queiroz, J. C. da Silva, F. C. Khanna, J. M. C. Malbouisson, M. Revzen, and A. E. Santana, Ann. Phys. (N.Y.) **317**, 220 (2005).
[15] A. A. Saharian and M. R. Setare, Phys. Lett. B **659**, 367

(2008); S. Bellucci and A. A. Saharian, Phys. Rev. D **77**, 124010 (2008); A. A. Saharian, Classical Quantum Gravity **25**, 165012 (2008); E. R. Bezerra de Mello and A. A. Saharian, J. High Energy Phys. 12 (2008) 081.

[16] S. Bellucci and A. A. Saharian, Phys. Rev. D **79**, 085019 (2009).

[17] H. B. Cheng, Phys. Lett. B **643**, 311 (2006); **668**, 72 (2008); A. Edery and I. MacDonald, J. High Energy Phys. 09 (2007) 005; S. A. Fulling and K. Kirsten, Phys. Lett. B **671**, 179 (2009); K. Kirsten and S. A. Fulling, Phys. Rev. D **79**, 065019 (2009); L. P. Teo, Phys. Lett. B **672**, 190 (2009); Nucl. Phys. **B819**, 431 (2009); L. P. Teo, J. High Energy Phys. 06 (2009) 076; E. Elizalde, S. D. Odintsov, and A. A. Saharian, Phys. Rev. D **79**, 065023 (2009).

[18] K. Poppenhaeger, S. Hossenfelder, S. Hofmann, and M. Bleicher, Phys. Lett. B **582**, 1 (2004); H. Alnes, K. Olaussen, F. Ravndal, and I. K. Wehus, Phys. Rev. D **74**, 105017 (2006); A. Edery and V. N. Marachevsky, Phys. Rev. D **78**, 025021 (2008); J. High Energy Phys. 12 (2008) 035; F. Pascoal, L. F. A. Oliveira, F. S. S. Rosa, and C. Farina, Braz. J. Phys. **38**, 581 (2008); L. Perivolaropoulos, Phys. Rev. D **77**, 107301 (2008).

[19] S. Nojiri, S. D. Odintsov, and S. Zerbini, Classical Quantum Gravity **17**, 4855 (2000); W. Goldberger and I. Rothstein, Phys. Lett. B **491**, 339 (2000); A. Flachi and D. J. Toms, Nucl. Phys. **B610**, 144 (2001); J. Garriga, O. Pujolàs, and T. Tanaka, Nucl. Phys. **B605**, 192 (2001); E. Elizalde, S. Nojiri, S. D. Odintsov, and S. Ogushi, Phys. Rev. D **67**, 063515 (2003); A. A. Saharian and M. R. Setare, Phys. Lett. B **584**, 306 (2004); M. Frank, I. Turan, and L. Ziegler, Phys. Rev. D **76**, 015008 (2007); H. Cheng, arXiv:0904.4183; A. Flachi and T. Tanaka, arXiv:0906.2898.

[20] A. Knapman and D. J. Toms, Phys. Rev. D **69**, 044023 (2004); A. A. Saharian, Phys. Rev. D **70**, 064026 (2004); Nucl. Phys. **B712**, 196 (2005).

[21] A. Flachi, J. Garriga, O. Pujolàs, and T. Tanaka, J. High Energy Phys. 08 (2003) 053; A. Flachi and O. Pujolàs, Phys. Rev. D **68**, 025023 (2003); A. A. Saharian, Phys. Rev. D **73**, 044012 (2006); **73**, 064019 (2006); **74**, 124009 (2006); R. Linares, H. A. Morales-Técotl, and O. Pedraza, Phys. Rev. D **77**, 066012 (2008); M. Frank, N. Saad, and I. Turan, Phys. Rev. D **78**, 055014 (2008); L. P. Teo, arXiv:0907.2989.

[22] K. Johnson, Acta Phys. Pol. B **6**, 865 (1975).

[23] S. G. Mamaev and N. N. Trunov, Sov. Phys. J. **23**, 551 (1980).

[24] R. D. M. De Paola, R. B. Rodrigues, and N. F. Svaiter, Mod. Phys. Lett. A **14**, 2353 (1999).

[25] E. Elizalde, F. C. Santos, and A. C. Tort, Int. J. Mod. Phys. A **18**, 1761 (2003).

[26] P. Sundberg and R. L. Jaffe, Ann. Phys. (N.Y.) **309**, 442 (2004).

[27] C. D. Fosco and E. L. Losada, Phys. Rev. D **78**, 025017 (2008).

[28] A. Romeo and A. A. Saharian, J. Phys. A **35**, 1297 (2002).

[29] A. A. Saharian, *The Generalized Abel-Plana Formula with Applications to Bessel Functions and Casimir Effect* (Yerevan State University Publishing House, Yerevan, 2008); Report No. ICTP/2007/082; arXiv:0708.1187.

[30] N. Inui, J. Phys. Soc. Jpn. **72**, 1035 (2003); E. R. Bezerra de Mello and A. A. Saharian, Phys. Rev. D **78**, 045021 (2008).

[31] K. Kirsten, *Spectral Functions in Mathematics and Physics* (CRC Press, Boca Raton, FL, 2001).

[32] E. Elizalde, Commun. Math. Phys. **198**, 83 (1998); J. Phys. A **34**, 3025 (2001).