

**An analytic Lifshitz black hole**

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A Lifshitz point is described by a quantum field theory with anisotropic scale invariance (but not Galilean invariance). In [8], gravity duals were conjectured for such theories. We construct analytically a black hole that asymptotes to a vacuum Lifshitz solution; this black hole solves the equations of motion of some simple (but somewhat strange) extensions of the models of [8]. We study its thermodynamics and scalar response functions. The scalar wave equation turns out to be exactly solvable. Interestingly, the Green's functions do not exhibit the ultralocal behavior seen previously in the free Lifshitz scalar theory.

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**I. INTRODUCTION**

A great deal of progress has been made in the study of quantum field theories and their holographic duals. The possible scope of this enterprise is not yet clear; for example, the correspondence seems to extend to some systems without Lorentz invariance. Recently, attempts have been made to apply the holographic principle to study condensed matter systems near a critical point (for reviews, see [1,2]). There are many scale-invariant field theories that are not Lorentz invariant, which are of interest in studying such critical points. In such a theory, time and space can scale differently, i.e.,  $t \rightarrow \lambda^z t$ ,  $\vec{x} \rightarrow \lambda \vec{x}$  under dilatation. The relative scale dimension of time and space,  $z$ , is called the ‘‘dynamical exponent.’’ Such a scale invariance is exhibited by a Lifshitz theory, which we will take to mean an anisotropic scale-invariant theory that is not Galilean invariant. The following Gaussian action provides a simple example of a (free) Lifshitz theory in  $d$  space dimensions:

$$S[\chi] = \int d^d x dt [(\partial_t \chi)^2 - K(\nabla^2 \chi)^2]. \quad (1.1)$$

This action describes a fixed line parametrized by  $K$ , and the dynamical exponent is  $z = 2$ . This theory describes the critical behavior of, e.g., quantum dimer models [3]. In many ways, the  $d = 2$ ,  $z = 2$  version of the theory (1.1) is like a relativistic boson in  $1 + 1$  dimensions.<sup>1</sup> The scaling behavior of the ground-state entanglement entropy for this class of theories was studied recently in [5,6]. This analysis also supports the similarity with  $2d$  conformal field theory (CFT), in that a universal leading singular behavior is found.

<sup>1</sup>Similar statements apply whenever  $z = d$ . However, constructing a rotation-invariant, local spatial kinetic operator that scales like  $p^{2d}$  is tricky for  $d \neq 2^k$  for integer  $k$ . We note in passing that the existence of such theories seem to be suggested by the calculations of [4].

In the free theory, the boson has logarithmic correlators

$$\langle \chi(x) \chi(0) \rangle \sim \int d\omega d^2 k \frac{1}{\omega^2 - k^4} e^{i\vec{k}\cdot\vec{x} - i\omega t} \sim \ln x. \quad (1.2)$$

As in the familiar  $d = z = 1$  case, the operators of definite scaling dimension are not the canonical boson field itself, but rather its exponentials and derivatives. In connection with quantum dimer models, the boson field is a height variable constructed from the dimer configuration, and the exponentials of the boson field are order parameters for various dimer-solid orderings [3]. At zero temperature, the logarithmic behavior of the correlator of the boson fields implies that the two-point function of the order parameter decays as a power law. However, the equal-time correlators at finite temperature are *ultralocal* in the infinite-volume limit [7]: they vanish at any nonzero spatial separation. In [7], it was suggested that this might be a mechanism for the kind of local criticality (scaling in frequency, but not momentum) seen in the strange metal phase of the cuprates and in heavy fermion materials. One is led to wonder whether this property should be shared by interacting Lifshitz theories, and whether the Lifshitz scaling is sufficient to produce this behavior. In [7] the addition of perturbative interactions was shown to lead to a finite correlation length; these perturbations violate the Lifshitz scaling. Below we will show that interactions which preserve the Lifshitz scaling need not give ultralocal behavior.

Gravity solutions with Lifshitz-type scale invariance were found in [8]. They found that the following family of metrics, parametrized by  $z$ , provide a geometrical description of Lifshitz-like theories (with  $z$  as the dynamical exponent):

$$ds^2 = L^2 \left( -\frac{dt^2}{r^{2z}} + \frac{d\vec{x}^2 + dr^2}{r^2} \right), \quad (1.3)$$

where  $\vec{x}$  denotes a  $d$ -dimensional spatial vector.<sup>2</sup> For  $d = 2$ , this metric extremizes the following action:

<sup>2</sup>This metric appeared previously in [9].

$$\begin{aligned}
 S &= \frac{1}{2} \int d^4x (R - 2\Lambda) \\
 &\quad - \frac{1}{2} \int (F_{(2)} \wedge \star F_{(2)} + F_{(3)} \wedge \star F_{(3)}) \\
 &\quad - c \int B_{(2)} \wedge F_{(2)}, \tag{1.4}
 \end{aligned}$$

where  $F_{(2)} = dA_{(1)}$ ,  $F_{(3)} = dB_{(2)}$ , and  $\Lambda$  is the four-dimensional cosmological constant. They computed the two-point function for the case when  $z = 2$  and showed that it exhibits power law decay. They also studied the holographic renormalization group flow for this case and found that  $\text{AdS}_4$  is the only other fixed point of the flow. Lifshitz vacuum solutions were shown to be stable under perturbations of the bulk action in [10].

In this paper we shall study a black hole solution, which asymptotes to the Lifshitz spacetime with  $d = 2$ ,  $z = 2$ . In Sec. II, an analytical solution for a black hole that asymptotes to the planar Lifshitz spacetime is written down. We present several actions whose equations of motion it solves; they all involve some matter sector additional to (1.4). Section III presents an analysis of the thermodynamics of this black hole. In Sec. IV, we solve the wave equation for a massive scalar field in this background; surprisingly, this equation is exactly solvable. We use this solution to calculate the two-point functions of boundary operators in Sec. V.

Since we found the solution described in this paper, some related work has appeared. Reference [11] constructs a black hole solution in a related background with slightly different asymptotics. Danielsson and Thorlacius [12] found numerical solutions of black holes in global Lifshitz spacetime. Interestingly, these are solutions to precisely the system studied by [8], with no additional fields. Related solutions were found by [13,14]. Reference [15] found solutions of type IIB supergravity that are dual to Lifshitz-like theories with spatial anisotropy and  $z = 3/2$ ; these solutions have a scalar field that breaks the scaling symmetry. To our knowledge, a string embedding of  $z = 2$  Lifshitz spacetime is still not known; obstacles to finding such an embedding are described in [16].

## II. BLACK HOLE SOLUTION

### A. Vacuum solution

The tensor fields in [8] can be rewritten as one massive gauge field. The Chern-Simons-like coupling is responsible for the mass. A familiar example is that of a 2-form field strength  $F$  and a 3-form field strength  $H$  in five dimensions with  $L = F \wedge \star F + H \wedge \star H + F \wedge H$ : this gives the same equation of motion as  $L = F \wedge \star F + A^2$ . In the four-dimensional case studied in [8], the dual of the 3-form field strength in four dimensions is a scalar field  $\varphi$ . Then

$$\begin{aligned}
 B_2 \wedge F_2 &= -F_3 \wedge A_1 + \text{bdy terms} \\
 &= -\star d\varphi \wedge A_1 = -\sqrt{g} \partial^\mu \varphi A_\mu. \tag{2.1}
 \end{aligned}$$

The action then reduces to

$$F_2 \wedge \star F_2 + (\partial\varphi + A)^2, \tag{2.2}$$

and  $\varphi$  shifts under the  $A$  gauge symmetry, and we can fix it to zero, and this is just a massive gauge field.<sup>3</sup> Hence, the zero-temperature Lifshitz metric

$$ds^2 = -\frac{dt^2}{r^{2z}} + \frac{d\vec{x}^2 + dr^2}{r^2} \tag{2.3}$$

is a solution of gravity in the presence of cosmological constant and a massive gauge field, and the gauge field mass is  $m^2 = dz$ . The bulk curvature radius has been set to one here and throughout the paper; in these units, the cosmological constant is  $\Lambda = -\frac{z^2+(d-1)z+d^2}{2}$ . The gauge field profile is  $A = \Omega r^{-z} dt$  (in the  $r$  coordinate with the boundary at  $r = 0$ ), and the strength of the gauge field is (for  $d = 2$ )

$$\Omega^2 = 8 \frac{z^2 + z - 2}{z(z + 2)}.$$

We note in passing that the Schrödinger spacetime is a solution of the same action with a different mass for the gauge field and a different cosmological constant [17,18]. Therefore, we find the perhaps-unfamiliar situation where the same gravitational action has solutions with very different asymptopia. Another recent example where this happens is ‘‘chiral gravity’’ in three dimensions, which has asymptotically anti-de Sitter (AdS) solutions as well as various squashed and smushed and wipfed solutions [19].

Given this fact, one might expect that the Lifshitz spacetime can be embedded into the same type IIB truncations as the Schrödinger spacetime (see [20,21] and especially [22]). However, the scalar equation of motion is not satisfied by the Lifshitz background since  $F^2$  is nonzero.

### B. Black hole solution

We shall now study a black hole in four dimensions that asymptotically approaches the Lifshitz spacetime with  $z = 2$ . We first observe that there is such a black hole in a system with a strongly-coupled scalar (i.e., a scalar without kinetic terms). The action is

$$\begin{aligned}
 S_1 &= \frac{1}{2} \int d^4x (R - 2\Lambda) \\
 &\quad - \int d^4x \left( \frac{e^{-2\Phi}}{4} F^2 + \frac{m^2}{2} A^2 + (e^{-2\Phi} - 1) \right). \tag{2.4}
 \end{aligned}$$

A solution of this system is

<sup>3</sup>This was also observed in [11].

$$\begin{aligned}\Phi &= -\frac{1}{2} \log(1 + r^2/r_H^2), & A &= f/r^2 dt \\ ds^2 &= -f \frac{dt^2}{r^{2z}} + \frac{d\vec{x}^2}{r^2} + \frac{dr^2}{fr^2},\end{aligned}\quad (2.5)$$

with

$$f = 1 - \frac{r^2}{r_H^2}.$$

Note that the metric has the same simple form as in the RG flow solution [Eq. (4.1) of [8]].

We can get the same contributions to the stress tensor as from the scalar without kinetic terms from several more reasonable systems. One such system is obtained by adding a second massive gauge field  $B$ , which will provide the same stress energy as the scalar. It has a slightly unfamiliar action:

$$\begin{aligned}S_2 &= \frac{1}{2} \int d^4x (R - 2\Lambda) \\ &\quad - \int d^4x \left( \frac{1}{4} B^2 dA^2 + \frac{m_A^2}{2} A^2 + \frac{1}{4} dB^2 - \frac{m_B^2}{2} (1 - B^2) \right),\end{aligned}\quad (2.6)$$

where  $A, B$  are 1-forms, and  $m_A^2 = 4$  and  $m_B^2 = 2$ . The solution looks like  $B = B(r)dr, A = A(r)dt$  and the metric is same as (2.5). In the solution, the scalar functions take the form

$$B(r) = \sqrt{g_{rr} \left( 1 + \frac{r^2}{r_H^2} \right)}, \quad A(r) = \Omega f r^{-z} dt.$$

Note that  $B(r)$  is not gauge trivial (even though its field strength vanishes) because of the mass term. Since  $B(r)$  asymptotes to 1, the effective gauge coupling of the field  $A$  is not large at the boundary.

The system with a strongly-coupled scalar in (2.4) is not equivalent to the system (2.6) with two gauge fields. For example, there are solutions of (2.4) where the scalar has a profile that depends both on  $r$  and  $x$ ; such configurations do not correspond to solutions of (2.6).

It is not clear whether the solution written above is stable. We leave the analysis of the stability of such solutions to small perturbations to future work. As weak evidence for this stability, we show in the next section that these black holes are *thermodynamically* stable.

Another action with this Lifshitz black hole (2.5) as a solution is

$$\begin{aligned}S_3 &= \frac{1}{2} \int d^4x \left( R - 2\Lambda - \frac{1}{2} dB^2 - (\partial\Phi - B)^2 - m_A A^2 \right. \\ &\quad \left. - \frac{1}{2} e^{-2\Phi} F^2 - V(\Phi) \right),\end{aligned}\quad (2.7)$$

where  $V(\Phi) = 2e^{-2\Phi} - 2$ . In the solution, the metric and gauge field  $A$  take the same form as in (2.5). The other

fields are

$$e^{-2\Phi} = 1 + \frac{r^2}{r_H^2}, \quad B = d\Phi.$$

Note that the action (2.7) is *not* invariant under the would-be gauge transformation

$$B \rightarrow B + d\Lambda, \quad \Phi \rightarrow \Phi + \Lambda,$$

because of the coupling to  $F^2 - 4$  (the sum of the gauge kinetic term and the potential term).<sup>4</sup> We are not bothered by this: it means that in quantizing the model, mass terms for the fluctuations  $B$  will be generated; however, such a mass term is already present.

We would also like to point out that in the three systems  $S_{1,2,3}$  described above, the stress-energy tensor of the fields with local propagating degrees of freedom satisfy the dominant energy condition, i.e.,  $T_{\mu\nu}^{(\Phi, A, B)} (= R_{\mu\nu} - (\frac{1}{2}R + \Lambda)g_{\mu\nu})$  satisfies the following:

$$\frac{T_{tt}}{T_{xx}} = \frac{T_{tt}}{T_{yy}} > -1 \quad \text{and} \quad \frac{T_{tt}}{T_{rr}} > -1.$$

Hence, there are no superluminal effects in the bulk. This is basically a consequence of the fact that the squared masses of the gauge fields are positive.

### III. LIFSHITZ BLACK HOLE THERMODYNAMICS

The Hawking temperature and entropy can be calculated using the near-horizon geometry. The Hawking temperature is the periodicity of the Euclidean time direction in the near-horizon metric (proportional to the surface gravity), i.e.,  $T = \frac{\kappa}{2\pi} |_{r=r_H}$ , with

$$\kappa^2 = -\frac{1}{2} \nabla^a v^b \nabla_b v^a,$$

where  $v = \partial_t$ . Hence,

$$T = \frac{1}{2\pi r_H^2}. \quad (3.1)$$

The entropy of the black hole is

$$\mathcal{S} = \frac{\text{Area of Horizon}}{4G_N} = \frac{L_x L_y}{4G_4 r_H^2}. \quad (3.2)$$

We shall now evaluate the free energy, internal energy, and pressure by calculating the on-shell action and boundary stress tensor. In order to renormalize the action, it is essential to add counterterms that are intrinsic invariants of the boundary (see [23]).

<sup>4</sup>We note that this quantity does vanish on the solution of interest.

Consider the following gravitational action:

$$\begin{aligned}
 S = & \frac{1}{2} \int_M d^4x \sqrt{g} \left( R - 2\Lambda - \frac{e^{-2\Phi}}{4} F^2 - \frac{m_A^2}{2} A^2 - V(\Phi) \right) \\
 & - \int_{\partial M} d^3x \sqrt{\gamma} (K + c_N e^{-2\Phi} n^\mu A^\nu F_{\mu\nu}) \\
 & + \frac{1}{2} \int_{\partial M} d^3x \sqrt{\gamma} (2c_0 - c_1 \Phi - c_2 \Phi^2) \\
 & + \frac{1}{2} \int_{\partial M} d^3x \sqrt{\gamma} ((c_3 + c_4 \Phi) A^2 + c_5 A^4). \quad (3.3)
 \end{aligned}$$

The second line of (3.3) contains extrinsic boundary terms: the Gibbons-Hawking term, and a ‘‘Neumannizing term,’’ which changes the boundary conditions on the gauge field. The last line of (3.3) describes the intrinsic boundary counterterms.<sup>5</sup> In the above expression, we have set  $8\pi G = 1$ . We have written the analysis in terms of  $S_1$  (2.4); the analysis can be adapted for  $S_2$  (2.6) by simply replacing  $\Phi$  in (3.3) by  $-\frac{1}{2} \log B^2$ . If Neumann boundary conditions are imposed on the gauge field, then  $c_N = 1$  and  $c_i = 0$  for  $i \geq 3$ . Similarly,  $c_N = 0$ , if the Dirichlet boundary condition is imposed on the gauge field.

The boundary stress tensor resulting from (3.3) is

$$\begin{aligned}
 T_{\mu\nu} = & K_{\mu\nu} - \left( K - c_0 + \frac{1}{2} c_1 \Phi + \frac{1}{2} c_2 \Phi^2 \right) \gamma_{\mu\nu} \\
 & + \frac{e^{-2\Phi}}{2} (n^r A_\mu \partial_r A_\nu + n^r A_\nu \partial_r A_\mu - n^r A_\alpha \partial_r A^\alpha \gamma_{\mu\nu}) \\
 & + (c_3 + c_4 \Phi + 2c_5 A^2) A_\mu A_\nu \\
 & - \frac{1}{2} (c_3 + c_4 \Phi + c_5 A^2) A^2 \gamma_{\mu\nu}. \quad (34)
 \end{aligned}$$

The values of  $c_i$  are determined by demanding that the action is ‘‘well behaved.’’. The action is well behaved if the variation of the action vanishes on shell and if the residual gauge symmetries of the metric are not broken. The values of  $c_i$ , which make the action well defined, also render finite the action and boundary stress tensor (please see the Appendix). Implementing this procedure, we find for the energy density, pressure and free energy

$$\mathcal{E} = \mathcal{P} = -\mathcal{F} = \frac{1}{2} T S = \frac{L_x L_y}{2 r_H^4}. \quad (3.5)$$

Satisfying the first law of thermodynamics (in the Gibbs-Duhem form  $\mathcal{E} + \mathcal{P} = T S$ ) is a nice check on the sensi-

<sup>5</sup>The most general combination of counterterms, which do not vanish at the boundary, is

$$\begin{aligned}
 & \frac{1}{2} \int_{\partial M} d^3x \sqrt{\gamma} (2c'_0 + c'_1 \Phi + c'_2 \Phi^2) \\
 & + \frac{1}{2} \int_{\partial M} d^3x \sqrt{\gamma} ((c'_3 + c'_4 \Phi)(A^2 - 1) + c'_5 (A^2 - 1)^2),
 \end{aligned}$$

which has the same form as (3.3).

bility of our solution, since it is a relation between near-horizon ( $T$ ,  $S$ ) and near-boundary ( $\mathcal{E}$ ,  $\mathcal{P}$ ) quantities.

Recently, [24] described an alternative set of boundary terms for asymptotically Lifshitz theories. They do not include the Neumannizing term, but instead include an intrinsic but nonanalytic  $\sqrt{A^\mu A_\mu}$  term.

#### IV. SCALAR RESPONSE

In this section, we study a probe scalar in the black hole background (2.5). The scalar can be considered a proxy for the mode of the metric coupling to  $T^x_y$ .

##### A. Exact solution of the scalar wave equation

Consider a scalar field  $\phi$  of mass  $m$  in the black hole background (2.5).<sup>6</sup>

Let  $u \equiv \frac{r^2}{r_H^2}$ . Fourier expand

$$\phi = \sum_k \phi_k(u) e^{-i\omega t + i\vec{k} \cdot \vec{x}}.$$

The wave equation takes the form

$$0 = \frac{u(-fk^2 + u\omega^2) + m^2 f}{4f^2 u^2} \phi_k(u) - \frac{1}{fu} \phi'_k(u) + \phi''_k(u),$$

where  $k^2 \equiv \vec{k}^2$ . Near the horizon, the incoming (−) and outgoing (+) waves are

$$\phi_k \sim (1 - u)^{\pm i\omega/2}.$$

The solutions near the boundary at  $u = 0$  are

$$\phi_k \sim u^{1 \pm (1/2)\sqrt{4+m^2}}.$$

The *exact* solution to the wave equation is  $\phi_k(u) = f^{-i\omega/2} u^{1-(1/2)\sqrt{m^2+4}} G_k(u)$  with

$$\begin{aligned}
 G_k(u) = & A_{12} F_1(a_+, b_+; c_+, u) u^{\sqrt{m^2+4}} \\
 & + A_{22} F_1(a_-, b_-; c_-, u) \quad (4.1)
 \end{aligned}$$

and

<sup>6</sup>In the following we have set both the bulk radius of curvature and the horizon radius to one. This means that frequencies and momenta are ‘‘gothic’’ [25], i.e., measured in units of  $r_H$ . Note that since  $z = 2$ ,  $\omega$  needs two factors of  $r_H$  to make a dimensionless quantity.

$$(a_{\pm}, b_{\pm}; c_{\pm}) \equiv \left( -\frac{i\omega}{2} \pm \frac{\sqrt{m^2+4}}{2} - \frac{1}{2}\sqrt{-k^2 - \omega^2 + 1} \right. \\ \left. + \frac{1}{2}, -\frac{i\omega}{2} \pm \frac{\sqrt{m^2+4}}{2} \right. \\ \left. + \frac{1}{2}\sqrt{-k^2 - \omega^2 + 1} + \frac{1}{2}; 1 \pm \sqrt{m^2+4}; u \right).$$

We emphasize that this is the exact solution to the scalar wave equation in this black hole; such a solution is unavailable for the AdS<sub>d>3</sub> black hole. The difference is that the equation here has only three regular singular points, whereas the AdS<sub>5</sub> black hole wave equation has four. This is because in the AdS<sub>5</sub> black hole, the emblackening factor is  $f = 1 - u^2$ , which has two roots, whereas ours is just  $f = 1 - u$ .

The other example of a black hole with a solvable scalar wave equation is the Banados, Teitelboim, Zanelli (BTZ) black hole in AdS<sub>3</sub> [26].<sup>7</sup> The origin of the solvability in that case is the fact that BTZ is an orbifold of the zero-temperature solution. This is *not* the origin of the solvabil-

ity in our case—this black hole is not an orbifold of the zero-temperature solution. This may be seen by comparing curvature invariants: they are not locally diffeomorphic. More simply, if the black hole were an orbifold, it would solve the same equations of motion as the vacuum solution. The fact that we were forced to add an additional matter sector (such as  $\Phi$  or  $B_{\mu}$ ) to find the black hole solution immediately shows that they are not locally diffeomorphic.

Now we ask for the linear combination of (4.1), which is ingoing at the horizon. In terms of  $\nu \equiv \sqrt{4 + m^2}$ ,  $\gamma \equiv \sqrt{1 - \omega^2 - k^2}$ , this is the combination with

$$\frac{A_1}{A_2} = -(-1)^{\nu} \frac{\Gamma(\nu)}{\Gamma(-\nu)} \frac{\Gamma(\frac{1}{2}(1 - i\omega - \nu - \gamma))}{\Gamma(\frac{1}{2}(1 - i\omega + \nu - \gamma))} \\ \times \frac{\Gamma(\frac{1}{2}(1 - i\omega - \nu + \gamma))}{\Gamma(\frac{1}{2}(1 - i\omega + \nu + \gamma))}. \quad (4.2)$$

In the massless case, one of the hypergeometric functions in (4.1) specializes to a Meijer  $G$  function, and the solution is  $\phi_k = u^2 f^{-i\omega/2} G_k(u)$  with

$$G_k(u) = c_{22} F_1 \left( -\frac{i\omega}{2} - \frac{1}{2}\sqrt{-k^2 - \omega^2 + 1} + \frac{3}{2}, -\frac{i\omega}{2} + \frac{1}{2}\sqrt{-k^2 - \omega^2 + 1} + \frac{3}{2}; 3; u \right) \\ + c_1 G_{2,2}^{2,0} \left( u \left| \begin{matrix} \frac{1}{2}(i\omega - \sqrt{-k^2 - \omega^2 + 1} - 1), \frac{1}{2}(i\omega + \sqrt{-k^2 - \omega^2 + 1} - 1) \\ -2, 0 \end{matrix} \right. \right).$$

In this solution, the coefficient of  $c_1$  (the Meier- $G$  function) is purely ingoing at the horizon.

### B. Correlators of scalar operators

In the previous section we wrote the solution for the wave equation in this black hole for a scalar field with an arbitrary mass. As mentioned earlier, the BTZ black hole also shares this property of having a scalar wave equation whose solutions are hypergeometric. Hence, one might expect that the two-point function of scalar operators in a Lifshitz-like theory to have a form that is similar to that of 2D CFTs.

The momentum space correlator for a scalar operator of dimension  $\Delta = \Delta_-$  is determined from the ratio of the non-normalizable and normalizable parts of the solution. The asymptotic behavior of the solution in (4.1) is

$$\phi \sim u^{\Delta+/2} (A_1 + \mathcal{O}(u)) + u^{\Delta-/2} (A_2 + \mathcal{O}(u)). \quad (4.3)$$

Hence, the retarded Green's function (two-point function) is

$$G_{\text{ret}}(\omega, \vec{k}) = -\frac{A_1}{A_2} \\ = (-1)^{\nu} \frac{\Gamma(\nu)}{\Gamma(-\nu)} \frac{\Gamma(\frac{1}{2}(1 - i\omega - \nu - \gamma))}{\Gamma(\frac{1}{2}(1 - i\omega + \nu - \gamma))} \\ \times \frac{\Gamma(\frac{1}{2}(1 - i\omega - \nu + \gamma))}{\Gamma(\frac{1}{2}(1 - i\omega + \nu + \gamma))}, \quad (4.4)$$

with  $\nu$  and  $\gamma$  defined above Eq. (4.2). Note that the correlator has a form very similar to that of a 2D CFT. It would be nice to know the precise connection between  $z = 2$  Lifshitz-like theories in  $2 + 1$  D with 2D CFTs that is responsible for this similarity. Note that the poles of the retarded Green's function do not lie on a straight line in the complex frequency plane, as they do for 2D CFTs.

Next, we would like to see whether the correlators exhibit ultralocal behavior at finite temperature as observed in the free scalar Lifshitz theory [7]. We find that the Green's function is not ultralocal—this removes the possibility that Lifshitz-symmetric interactions require ultralocal behavior.

We will now calculate the two-point function of a scalar operator of dimension  $\Delta = 4$  at finite temperature. In this case, the correlator is given by the coefficient of  $r^4$  in the asymptotic expansion of the solution near  $r = 0$ . Kachru

<sup>7</sup>Another example, in two dimensions, is [27].

*et al.* [8] showed that the correlator exhibits a power law decay at zero temperature.

We can evaluate this correlator by extracting the coefficient of the  $u^2$  term (note that  $u \propto r^2$ ) in the asymptotic

expansion of the solution of the massless scalar wave equation. The behavior of the solution near  $u = 0$  is

$$\begin{aligned} \phi(u, \vec{k}, \omega) = & 1 - \frac{u}{4}(\vec{k}^2 + 2i\omega) - \frac{u^2}{64}((\vec{k}^2)^2 + 4\omega^2) \left[ -3 + 2\psi\left(\frac{1}{2}(-1 + i\omega - \sqrt{1 - \vec{k}^2 - \omega^2})\right) \right. \\ & \left. + 2\psi\left(\frac{1}{2}(-1 + i\omega + \sqrt{1 - \vec{k}^2 - \omega^2})\right) + 2\gamma_E + 2\ln u \right] + \mathcal{O}(u^3), \end{aligned} \quad (4.5)$$

where  $\gamma_E$  is Euler's constant,  $\psi$  is the digamma function. The behavior of the solution in the Euclidean black hole can be obtained by replacing  $\omega$  by  $-i|\omega|$ . The choice of the negative sign gives the solution, which is ingoing at the horizon, as appropriate to the retarded correlator [25]. Henceforth, we shall work with the solution for the Euclidean case. The correlator is the sum of the two digamma functions. All other terms in the coefficient of  $u^2$  are contact terms. Hence, the correlator in momentum space is

$$\langle \mathcal{O}(-\omega, -\vec{k}) \mathcal{O}(\omega, \vec{k}) \rangle \propto ((\vec{k}^2)^2 - 4\omega^2) \left[ \psi\left(\frac{1}{2}(-1 + |\omega| - \sqrt{1 - \vec{k}^2 + \omega^2})\right) + \psi\left(\frac{1}{2}(-1 + |\omega| + \sqrt{1 - \vec{k}^2 + \omega^2})\right) \right]. \quad (4.6)$$

After dropping the contact terms, the above expression can be written as follows:

$$\langle \mathcal{O}(-\omega, -\vec{k}) \mathcal{O}(\omega, \vec{k}) \rangle \propto \sum_{n=1}^{\infty} \mathcal{A}_n,$$

where

$$\begin{aligned} \mathcal{A}_n &= \frac{(2n-3) + |\omega|}{(2n-3)^2 + 2|\omega|(2n-3) + k^2 - 1} \\ &= \frac{a_n + |\omega|}{a_n^2 + 2a_n|\omega| + k^2 - 1}. \end{aligned}$$

We can now calculate the correlators in coordinate space by performing the Fourier transform of the above expression. This is given by

$$D(|\vec{x}|, t) = [(4\partial_t^2 - (\nabla^2)^2)] \sum_n \mathcal{F}_n, \quad (4.7)$$

where  $D(|\vec{x}|, t)$  is the two-point function and

$$\begin{aligned} \sum_n \mathcal{F}_n &= \sum_n \int k dk d\omega d\theta \frac{a_n + |\omega|}{a_n^2 + 2a_n|\omega| + k^2 - 1/4} \\ &\quad \times e^{ik|\vec{x}|\cos\theta + i\omega t}. \end{aligned} \quad (4.8)$$

The short distance ( $r \ll r_H$ ) behavior of the equal-time correlator is

$$D(|\vec{x}| \ll r_H, 0) = [(4\partial_t^2 - (\nabla^2)^2) \mathcal{F}]_{t=0, |\vec{x}| \rightarrow 0} \propto \frac{1}{|\vec{x}|^8}. \quad (4.9)$$

As a check, we note that, the short distance behavior of this expression reproduces the zero-temperature answer  $|\vec{x}|^{-8}$  found in [8].

The long distance ( $|\vec{x}| \gg r_H$ ) behavior is

$$\begin{aligned} D(|\vec{x}| \gg r_H, 0) &= [(4\partial_t^2 - (\nabla^2)^2) \mathcal{F}]_{t=0, |\vec{x}| \rightarrow \infty} \\ &\propto \frac{e^{-\sqrt{2}|\vec{x}|/r_H}}{|\vec{x}|^{3/2}}. \end{aligned} \quad (4.10)$$

The correlator is not ultralocal, unlike the thermal correlator in free scalar Lifshitz theory.<sup>8</sup>

## V. OUTLOOK

An important defect of our work that cannot have avoided the reader's attention is the fact that the matter content that produces the stress-energy tensor for this black hole is unfamiliar and contrived. There is no physical reason why terms such as  $A^2 B^2$  should not be added. In our defense, a perturbation analysis in the coefficient of such terms indicates that a corrected solution can be constructed.

It is not clear how to embed such solutions in a UV-complete gravity theory; a stringy description is not known yet even for the zero-temperature case. Such a description would help in finding specific Lifshitz-like field theories with gravity duals. It would be nice to understand the connection (if it exists) between the Lifshitz spacetime and non-Abelian Lifshitz-like gauge theories [28,29].

<sup>8</sup>The dependence on  $|\omega|$  does lead to remnants of ultralocal behavior (extended in time, but localized in space).

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## APPENDIX: REGULARIZING THE ACTION AND BOUNDARY STRESS TENSOR

In this Appendix, we will show that the on-shell action and boundary stress tensor can be rendered finite by making the action well behaved, i.e., the action is stationary on shell under an arbitrary normalizable variation of the bulk fields, and the boundary terms in the action must not break the residual gauge symmetries of the metric.

We will first find the constraints imposed by finiteness of the free energy, internal energy and pressure on  $c_i$ .

The free energy of the boundary theory is

$$\begin{aligned}
 -\mathcal{F} &= \frac{S_{\text{onshell}}}{\beta} \\
 &= \frac{1}{2} L_x L_y \left[ \frac{64c_N - 8c_0 + 16c_1 + 8c_2 + 6c_3 + 16c_4 - 15c_5}{32r_H^4} - \frac{32 + 4c_1 + 8c_0 + 6c_3 + 2c_4 - 5c_5}{16\epsilon^2 r_H^2} \right. \\
 &\quad \left. + \frac{24 + 2c_3 - c_5 - 8c_N + 8c_0}{8\epsilon^4} \right], \tag{A1}
 \end{aligned}$$

where  $\beta$  is inverse temperature. We must set  $-c_5 + 24 - 8c_N + 8c_0 + 2c_3 = 0$  and  $-c_1 - 8 - 2c_0 - 3/2c_3 - 1/2c_3 + 5/4c_5 = 0$  to get rid of the divergences in the on-shell action. Further, finiteness of the boundary stress tensor and conformal ward identities impose more constraints on the counterterms.

The internal energy of the boundary theory is

$$\begin{aligned}
 \mathcal{E} &= -L_x L_y \sqrt{\gamma} T_i^i \\
 &= -L_x L_y \left( \frac{16 + 8c_0 - 2c_3 + 3c_5 + 8c_N}{8\epsilon^4} - \frac{32 + 8c_0 + 4c_1 - 6c_3 - 2c_4 + 15c_5}{16r_H^2 \epsilon^2} \right. \\
 &\quad \left. - \frac{8c_0 - 16c_1 - 8c_2 + 6c_3 + 16c_4 - 45c_5 + 64c_N}{64r_H^4} \right). \tag{A2}
 \end{aligned}$$

Similarly, the expression for pressure is

$$\begin{aligned}
 \mathcal{P} &= \frac{1}{2} L_x L_y \sqrt{\gamma} T_i^i = L_x L_y \sqrt{\gamma} T_x^x \\
 &= \frac{1}{2} L_x L_y \left[ \frac{64c_N - 8c_0 + 16c_1 + 8c_2 + 6c_3 + 16c_4 - 15c_5}{32r_H^4} - \frac{32 + 4c_1 + 8c_0 + 6c_3 + 2c_4 - 5c_5}{16\epsilon^2 r_H^2} \right. \\
 &\quad \left. + \frac{24 + 2c_3 - c_5 - 8c_N + 8c_0}{8\epsilon^4} \right]. \tag{A3}
 \end{aligned}$$

Note that  $\mathcal{F} = -\mathcal{P}$ , as expected in the grand canonical ensemble. Hence, the condition for the divergences in pressure to cancel is the same as the condition for divergences in the on-shell action to cancel. However, finiteness of energy imposes additional constraints on the counterterms. In the case of the Schrödinger black hole, it is not possible to get rid of the divergence in the energy without the Neumannizing term [21].

The conformal Ward identity for conservation of the dilatation current requires  $z\mathcal{E} = d\mathcal{P}$ , and in our discussion  $d = z = 2$ . The residual gauge freedom of the metric is broken if this condition is not satisfied (see [17]). Note that making the boundary stress tensor finite does not ensure this condition. We must set  $c_2 = 7/2$  for the conformal

Ward identity to hold. After imposing these conditions, we find

$$\mathcal{E} = \mathcal{P} = -\mathcal{F} = L_x L_y \frac{15 - 2c_1 - 26c_N}{16r_H^4}. \tag{A4}$$

In order to have a well-defined variational principle, we must ensure that  $\delta S = 0$  on shell. We shall now determine the value of  $c_1$  using this condition.<sup>9</sup> The variation of the action is

<sup>9</sup>We have determined the value of  $c_1$  for the case where Dirichlet boundary condition is imposed on the gauge field. However, the method is general and can be used for other boundary conditions as well.

$$\begin{aligned} \delta S = & \int_{\text{bulk}} \text{EOM} + \frac{1}{2} \int_{\text{bdy}} d^3x \sqrt{\gamma} [T_\nu^\mu \delta \gamma_\mu^\nu \\ & + ((c_N - 1)e^{-2\Phi} n_\nu F^{\nu\mu} + (c_3 + c_4 \Phi + 2c_5 A^2) A^\mu) \\ & \times \delta A_\mu + c_N A^\mu \delta(n^\nu e^{-2\Phi} F_{\nu\mu}) \\ & - \frac{1}{2} (c_1 + 2c_2 \Phi - c_4 A^2 - 4c_N A^\mu n^\nu F_{\nu\mu} e^{-2\Phi}) \delta \Phi]. \end{aligned} \quad (\text{A5})$$

The first term vanishes on shell. Therefore, the boundary terms must also vanish on shell. Let us assume, for convenience that Dirichlet boundary condition is imposed on the gauge field ( $c_N = 0$ ). Prescribing boundary conditions is equivalent to prescribing the coefficient of the non-normalizable mode of the solution. The allowed variations at the boundary fall faster than the non-normalizable part of the solution, i.e.,

$$\begin{aligned} \delta \gamma_\nu^\mu &= \delta \gamma_{\nu(1)}^\mu r^2 + \delta \gamma_{\nu(2)}^\mu r^4 + \dots \\ \delta A_\mu &= r^{-2} (\delta A_{\mu(1)} r^2 + \delta A_{\mu(2)} r^4 + \dots) \\ \delta \Phi &= r^2 \delta \Phi_{(1)} + r^4 \delta \Phi_{(2)} + \dots \end{aligned} \quad (\text{A6})$$

Substituting these expressions in (A5) and using the con-

ditions on  $c_i$  for energy and pressure to be finite,<sup>10</sup> we find

$$\begin{aligned} \delta S = & \int d^3x \left( \sqrt{\gamma} T_\nu^\mu r^2 \delta \gamma_{\mu(1)}^\nu + \mathcal{O}(r^2) \delta A_{\mu(1)} \right. \\ & \left. + \left( \frac{c_2 - c_1}{r_H^2} \right) (\delta \Phi_1 + \mathcal{O}(r^2)) \right). \end{aligned} \quad (\text{A7})$$

Since  $\mathcal{E}$  and  $\mathcal{P}$  are finite, the first term in the integrand vanishes at the boundary. Hence,  $c_1 = c_2 = 7/2$  for the variation of the action to vanish on shell. Using the values of  $c_i$  found above in (A4) we get

$$\mathcal{E} = \mathcal{P} = -\mathcal{F} = \frac{L_x L_y}{2r_H^4}.$$

After restoring factors of  $8\pi G$ ,

$$\mathcal{E} = \mathcal{P} = -\mathcal{F} = \frac{L_x L_y}{16\pi G r_H^4} = -\frac{1}{2} T \frac{\partial \mathcal{F}}{\partial T} = \frac{1}{2} T S.$$

We have shown that the stress tensor and on-shell action can be regularized by making the action well behaved, i.e.,  $\delta S$  must vanish on shell, and the counterterms should not break any residual gauge symmetry.

<sup>10</sup> $c_0 = -(17 - c_1)/8$ ,  $c_3 = -5 - c_1$ ,  $c_4 = -2c_1$  and  $c_5 = -3 - c_1$ , when  $c_N = 0$ .

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