Hamiltonian of a spinning test particle in curved spacetime

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Using a Legendre transformation, we compute the unconstrained Hamiltonian of a spinning test particle in a curved spacetime at linear order in the particle spin. The equations of motion of this unconstrained Hamiltonian coincide with the Mathisson-Papapetrou-Pirani equations. We then use the formalism of Dirac brackets to derive the constrained Hamiltonian and the corresponding phase space algebra in the Newton-Wigner spin supplementary condition, suitably generalized to curved spacetime, and find that the phase space algebra $(\mathbf{q}, \mathbf{p}, \mathbf{S})$ is canonical at linear order in the particle spin. We provide explicit expressions for this Hamiltonian in a spherically symmetric spacetime, both in isotropic and spherical coordinates, and in the Kerr spacetime in Boyer-Lindquist coordinates. Furthermore, we find that our Hamiltonian, when expanded in post-Newtonian (PN) orders, agrees with the Arnowitt-Deser-Misner canonical Hamiltonian computed in PN theory in the test particle limit. Notably, we recover the known spin-orbit couplings through 2.5PN order and the spin-spin couplings of type $S_{\mathrm{Kerr}}S$ (and S^2_{Kerr}) through 3PN order, S_{Kerr} being the spin of the Kerr spacetime. Our method allows one to compute the PN Hamiltonian at any order, in the test particle limit and at linear order in the particle spin. As an application we compute it at 3.5PN order.

I. INTRODUCTION

The dynamics of spinning bodies in general relativity is a complicated problem which has been investigated in several papers during the last 70 years, starting from the pioneering work by Mathisson [1], Papapetrou [2–4], Pirani [5], Tulczyjew [6,7], and Dixon [8]. Spin effects on the free motion of a test particle were first derived in the form of a coupling to the spacetime curvature in Refs. [2–4]. The computation assumes that the test particle can be described by a pole-dipole energy-momentum tensor [6,7], thus neglecting the quadrupole moment (and higher multipole moments) and providing spin couplings only at linear order in the test particle's spin.

The two-body dynamics of spinning objects can also be computed in post-Newtonian (PN) theory [9], which is basically an expansion in powers of v/c and $GM/(c^2 r)$, where v is the characteristic velocity of the system and r is the binary's separation. Currently, spin couplings have been computed in the two-body equations of motion through 2.5PN order [10–15], and in the Arnowitt-Deser-Misner (ADM) canonical Hamiltonian through 3PN order [16–20] and partially at higher PN orders [21,22]. These coupling terms agree with those computed via effective-field-theory techniques at 1.5PN, 2PN, and 3PN order [23–26].

The main motivation for describing as accurately as possible the dynamics of a binary system of spinning compact bodies in general relativity comes from the forthcoming observation of gravitational waves with ground and space-based detectors. In particular, LIGO, Virgo, and GEO could observe signals emitted by stellar-mass black-hole and neutron-star binaries, and LISA could de-

tect signals from supermassive black-hole binaries and extreme-mass ratio binaries.

In this paper we compute the Hamiltonian of a test particle in a curved background spacetime, including all couplings linear in the test particle's spin. Starting from the Lagrangian given in Ref. [27], we apply a Legendre transformation to derive the *unconstrained* Hamiltonian. The Hamiltonian is unconstrained in the sense that the test particle's spin variables are given by an antisymmetric tensor $S^{\mu\nu}$, which *a priori* contains 6 degrees of freedom instead of 3. It is well known that in order to fix the unphysical degrees of freedom associated with the arbitrariness in the definition of $S^{\mu\nu}$, a choice must be made for the so-called spin supplementary condition (SSC). The arbitrariness can be interpreted, in the case of extended bodies,[1] as the freedom of choosing the point, internal to the body, whose motion is followed [29].

Building on the work by Hanson and Regge [30] and generalizing the Newton-Wigner (NW) SSC to curved spacetime, we then derive the *constrained* Hamiltonian and the corresponding Dirac brackets, which should replace the Poisson brackets when computing the equations of motion from that Hamiltonian. Quite interestingly, we find that the NW SSC leads, at least at linear order in the particle spin, to canonical Dirac brackets, *i.e.* the standard sympletic structure for a set of dynamical variables

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[1]It should be stressed that *any* spinning ''particle'' must actually have a small nonfinite size. An intuitive argument for this can be found in Ref. [28], Ex. 5.6, where it is shown that any spinning body must have a minimal size in order not to rotate at velocities larger than c. A more rigorous proof can be found in Ref. [29], Sec. 2.

(q, p, S). As a consistency check of our results we also compare our constrained Hamiltonian with the ADM canonical Hamiltonian for spinning bodies, as computed in PN theory through 3PN order. In addition we provide explicit expressions for the Hamiltonian of a spinning particle moving in a generic spherically symmetric spacetime (using both isotropic and spherical coordinates), as well as in the Kerr spacetime (in Boyer-Lindquist coordinates).

Another important application of this work will be developed in a subsequent paper, where we will use the Hamiltonian derived here to build a new effective-one-body Hamiltonian [31–34] for spinning objects. This application is crucial to take full advantage of the analytical and numerical treatment of the dynamics of spinning bodies throughout the inspiral, merger, and ringdown, and build accurate templates for the search of gravitational waves with ground-based and space-based detectors.

The paper is organized as follows. In Sec. II we briefly summarize our notations. In Sec. III we apply a Legendre transformation to compute the unconstrained Hamiltonian and show that the equations of motion that follow from it coincide with the well-known Mathisson-Papapetrou-Pirani (MPP) equations of motion. In Sec. IV, after reviewing the Dirac bracket formalism, we derive the constrained Hamiltonian and the corresponding Dirac brackets using the generalized NW SSC. In Sec. V, we specialize our results to spherically symmetric spacetimes and to the Kerr spacetime in Boyer-Lindquist coordinates. In Sec. VI we restrict ourselves to the Kerr spacetime in ADM coordinates, expand the Hamiltonian computed in the NW SSC in a PN series through 3.5PN order and find agreement with the ADM canonical Hamiltonian in the test particle limit through 3PN order. Section VII summarizes our main conclusions.

II. NOTATIONS

Throughout this paper, we will use the signature $(-, +, +, +)$ for the metric. Spacetime tensor indices (ranging from 0 to 3) will be denoted with Greek letters, while spatial tensor indices (ranging from 1 to 3) will be denoted with lowercase Latin letters. Also, we will often use t as alternate for the timelike index 0.

We define a tetrad field as a set consisting of a timelike future-oriented vector $\tilde{e}_T^\mu$ and three spacelike vectors $\tilde{e}_I^\mu$ ($I = 1, \ldots, 3$)—collectively denoted as $\tilde{e}_A^\mu$ ($A = 0, \ldots, 3$)—satisfying[2]

$$\tilde{e}_A^{\ \mu} \tilde{e}_B^{\ \nu} g_{\mu\nu} = \eta_{AB}, \qquad (2.1)$$

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[2]We use the notation $\tilde{e}_A^\mu$ to denote any choice of tetrad given a background spacetime. The tetrad without the tilde e_A^μ refers to a special tetrad, namely, the one carried by the test particle. The tetrad e_A^μ is special in the sense that it is a dynamical variable whose evolution along the worldline is prescribed by some Lagrangian.

where $\eta_{TT} = -1$, $\eta_{TI} = 0$, $\eta_{IJ} = \delta_{IJ}$ (δ_{IJ} being the Kronecker symbol). Thus the internal tetrad space is Lorentz-invariant, *i.e.* one can obtain any tetrad from an existing one by applying a Lorentz transformation $\tilde{e}_A' = \Lambda_A^{\ B} \tilde{e}_B$, where

$$\Lambda_A^{\ C} \Lambda_C^{\ B} = \Lambda^C_{\ A} \Lambda^B_{\ C} = \delta_A^B. \qquad (2.2)$$

Internal tetrad indices denoted with the uppercase Latin letters A, B, C, and D always run from 0 to 3, while internal tetrad indices with the uppercase Latin letters I, J, K, and L, associated with the spacelike tetrad vectors, run from 1 to 3 only. The timelike tetrad index is denoted by T.

Tetrad indices are raised and lowered with the metric η_{AB} [e.g., $\tilde{e}_A^\mu = \eta_{AB}(\tilde{e}^B)^\mu$]. With this convention the relation (2.1) can be easily shown to be equivalent to the completeness relation

$$\tilde{e}_A^{\ \mu} \tilde{e}_\nu^A = \delta_\nu^\mu. \qquad (2.3)$$

We will denote the projections of a vector V onto the tetrad with $V^A \equiv V^\mu \tilde{e}_\mu^A$, and similarly for tensors of higher rank, as well as Christoffel symbols. Partial derivatives will be denoted with a comma or with ∂, covariant derivatives with a semicolon, while total covariant derivatives with respect to a parameter σ will be denoted by $D/D\sigma$. Finally, we will denote the operation of antisymmetrization with respect to the indices μ and ν as $A^{\cdots[\mu} B^{\nu]\cdots} \equiv (A^{\cdots\mu} B^{\nu\cdots} - A^{\cdots\mu} B^{\nu\cdots})/2$.

We use geometric units $G = c = 1$ throughout the paper, except in Sec. VI where the factors of c are restored, playing the role of PN bookkeeping parameters.

III. UNCONSTRAINED HAMILTONIAN

In this section we derive the unconstrained Hamiltonian by applying a Legendre transformation to the Lagrangian describing the motion of a spinning particle in a generic curved spacetime.

A. The Lagrangian and the Mathisson-Papapetrou-Pirani equations

Building on the classic work of Hanson and Regge [30] which analyzes the dynamics of a relativistic top in a flat spacetime, Porto showed in Ref. [27] that the equations of motion of a spinning particle in curved spacetime can be obtained from the action

$$S = \int L(a_1, a_2, a_3, a_4) d\sigma, \qquad (3.1)$$

σ being a parameter along the representative worldline. The Langrangian L is a function of the four Lorentz-invariant scalars

$$a_1 = u_\mu u^\mu, \qquad (3.2)$$

$$a_2 = \Omega_{\mu\nu}\Omega^{\mu\nu}, \tag{3.3}$$

$$a_3 = u_\mu \Omega^{\mu\nu}\Omega_{\nu\rho}u^\rho, \tag{3.4}$$

$$a_4 = \Omega_{\lambda\mu}\Omega^{\mu\nu}\Omega_{\nu\rho}\Omega^{\rho\lambda}, \tag{3.5}$$

where $u^\mu \equiv dx^\mu/d\sigma$ is the tangent vector to the representative worldline, and where the antisymmetric tensor $\Omega^{\mu\nu}$ describes how the tetrad e_A^μ carried by the particle rotates along the worldline

$$\Omega^{\mu\nu} = \eta^{AB}e_A^\mu \frac{De_B^\nu}{D\sigma} = e^{A\mu}\frac{de_A^\nu}{d\sigma} + \Gamma^\mu_{\alpha\beta}g^{\alpha\nu}u^\beta. \tag{3.6}$$

Moreover, the action (3.1) is assumed to be reparametrization-invariant (i.e. its form must be independent of the particular parameter used to follow the particle's worldline), which translates in the requirement that the Lagrangian L be a homogeneous function of degree one in the "velocities" u^μ and $\Omega^{\mu\nu}$ [30]. Porto then shows that if one *defines* the four-momentum vector and the spin tensor of the particle as[3]

$$p_\mu \equiv \frac{\partial L}{\partial u^\mu}\bigg|_\Omega, \tag{3.7}$$

$$S_{\mu\nu} \equiv 2\frac{\partial L}{\partial \Omega^{\mu\nu}}\bigg|_u \tag{3.8}$$

[note that p_μ is *not* the momentum conjugate to the coordinates x^μ because $\Omega^{\mu\nu}$ depends on u^μ, as can be seen in Eq. (3.6)], then a variation of the action with respect to e_A^μ which preserves the defining property (2.1) of a tetrad gives the precession equation for the spin tensor

$$\frac{DS^{\mu\nu}}{D\sigma} = S^{\mu\lambda}\Omega^\nu_\lambda - \Omega^{\mu\lambda}S^\nu_\lambda = p^\mu u^\nu - p^\nu u^\mu. \tag{3.9}$$

The second equality in Eq. (3.9) follows from definitions (3.7) and (3.8), and from the fact that the Lagrangian in the action (3.1) depends only on a_1, a_2, a_3, and a_4 [30]. Moreover, a variation of the action with respect to the particle's position x^μ gives [27]

$$\frac{Dp^\mu}{D\sigma} = -\frac{1}{2}R^\mu_{\ \alpha\beta\gamma}u^\alpha S^{\beta\gamma}. \tag{3.10}$$

Thus one precisely recovers the well-known MPP equations from the action (3.1), which therefore encodes the dynamics of a spinning test particle in curved spacetime, at linear order in the particle's spin.

Notice however that the set of Eqs. (3.9) and (3.10) consists of ten equations and 13 independent variables (p^μ, u^μ, and $S^{\mu\nu}$, subject to the normalization constraint[4] of the tangent vector u^μ) and is therefore not closed. This underdetermination can be addressed by imposing an SSC, which is typically expressed as

$$S^{\mu\nu}\omega_\nu = 0, \tag{3.11}$$

where ω_ν is some suitably chosen timelike vector. Equation (3.11) contains three independent constraints, and is therefore expected to reduce the number of independent variables from 13 to 10, thus closing the system of Eqs. (3.9) and (3.10). This is indeed what happens, as the requirement that Eq. (3.11) be valid at all points along the worldline implies the following implicit relationship between p^μ and u^μ:

$$p^\mu = \frac{1}{\omega_\nu u^\nu}\left[(\omega_\nu p^\nu)u^\mu - S^{\mu\nu}\frac{D\omega_\nu}{D\sigma}\right]. \tag{3.12}$$

It should be stressed once again that it is the *underdetermination* of the unconstrained MPP system that allows one to impose *any* constraint of the form (3.11), and that the constraint will be *automatically* conserved by the time evolution of the system because of Eq. (3.12). Of course different constraints of the form (3.11) will produce different systems of equations describing the evolution of the particle's worldline. The physical reason for this is easy to understand: the SSC (3.11) binds the test particle described by the Lagrangian to a specific, SSC-dependent worldline lying inside the worldtube spanned by the spinning body, namely, the center of energy of the body as seen by an observer with four-velocity parallel to ω^μ (see e.g. Ref. [29] for a lucid discussion of the physical meaning of SSCs).

B. Deriving the Hamiltonian through a Legendre transformation

It is convenient to rewrite the action (3.1) as

$$S = \int L\left(x^\mu, u^\mu, \phi^a, \frac{d\phi^a}{d\sigma}\right)d\sigma, \tag{3.13}$$

where the Langrangian L can be now considered as a function of the coordinates x^μ, the four-vector $u^\mu = dx^\mu/d\sigma$, the six parameters ϕ^a, and their time derivatives. The set $\{\phi^a\}$ consists simply of the parameters of the internal Lorentz transformation describing the orientation of the tetrad field e_A^μ carried by the particle with respect to an arbitrary, but fixed, reference tetrad field $\tilde{e}_A^\mu(x)$ covering

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[3]Because of reparametrization invariance of the action (3.1), these definitions maintain the same form whatever parameter σ is chosen along the wordline, as appropriate for physical quantities like the four-momentum and the spin.

[4]For example one is free to select a parameter $\sigma = \tau$ such that $u_\mu u^\mu = -1$, since the action (3.1) is reparametrization-invariant. Any other choice of parameter simply yields a different normalization constraint $u_\mu u^\mu = -(d\sigma/d\tau)^{-2}$.

the whole spacetime.[5] Therefore, the tetrad carried by the particle is given by

$$e_A^\mu(\phi, x) = \Lambda_A{}^B(\phi)\tilde{e}_B^\mu(x), \qquad (3.14)$$

where $\Lambda_A{}^B$ is a Lorentz transformation. We also note that the parameters ϕ^a and their time derivatives enter the Lagrangian only through the antisymmetric tensor $\Omega^{\mu\nu}$, which we write explicitly as

$$\Omega^{\mu\nu} = \eta^{AB} e_A^\mu(\phi, x)\left[\frac{d\phi^a}{d\sigma}\frac{\partial e_B^\nu}{\partial\phi^a}(\phi, x) + u^\beta e_{B,\beta}^\nu(\phi, x)\right]$$
$$+ \Gamma_{\alpha\beta}^\nu g^{\mu\alpha} u^\beta. \qquad (3.15)$$

To construct the Hamiltonian we need to choose a particular $3 + 1$ decomposition of the background metric. We take $\sigma = t$, where t is the time coordinate of that particular decomposition. Using reparametrization invariance, we can write

$$S = \int L\left(x^\mu, u^\mu, \phi^a, \frac{d\phi^a}{d\sigma}\right)d\sigma,$$
$$= \int L(x^i, v^i, \phi^a, \dot\phi^a, t)dt, \qquad (3.16)$$

where $x^0 = t$, $u^0 = 1$, $u^i = v^i = dx^i/dt$, and $\dot\phi^a = d\phi^a/dt$. The configuration space of the spinning particle therefore consists of the set $\{x^i, \phi^a\}$. The total variation of the Lagrangian considered as function of x^i, v^i, ϕ^a, and $\dot\phi^a$ is

$$\delta L = \frac{\partial L}{\partial x^i}\delta x^i + \frac{\partial L}{\partial v^i}\delta v^i + \frac{\partial L}{\partial\phi^a}\delta\phi^a + \frac{\partial L}{\partial\dot\phi^a}\delta\dot\phi^a,$$
$$\equiv \frac{\partial L}{\partial x^i}\delta x^i + P_i\delta v^i + \frac{\partial L}{\partial\phi^a}\delta\phi^a + P_{\phi^a}\delta\dot\phi^a, \qquad (3.17)$$

where we denoted by P_i and P_{ϕ^a} the momenta conjugate to x^i and ϕ^a, respectively. The total variation of the Lagrangian considered as function of x^i, v^i, and $\Omega^{\mu\nu}$ is instead

$$\delta L = \frac{\partial L}{\partial x^i}\bigg|_\Omega \delta x^i + \frac{\partial L}{\partial v^i}\bigg|_\Omega \delta v^i + \frac{\partial L}{\partial\Omega^{\mu\nu}}\bigg|_{x,v}\delta\Omega^{\mu\nu}. \qquad (3.18)$$

Using Eq. (3.15), Eq. (3.18) can be rewritten as

$$\delta L = \left(\frac{\partial L}{\partial x^i}\bigg|_\Omega + \frac{\partial L}{\partial\Omega^{\mu\nu}}\bigg|_{x,v}\frac{\partial\Omega^{\mu\nu}}{\partial x^i}\bigg|_{v,\phi,\dot\phi}\right)\delta x^i$$
$$+ \left(\frac{\partial L}{\partial v^i}\bigg|_\Omega + \frac{\partial L}{\partial\Omega^{\mu\nu}}\bigg|_{x,v}\frac{\partial\Omega^{\mu\nu}}{\partial v^i}\bigg|_{x,\phi,\dot\phi}\right)\delta v^i$$
$$+ \frac{\partial L}{\partial\Omega^{\mu\nu}}\bigg|_{x,v}\frac{\partial\Omega^{\mu\nu}}{\partial\phi^a}\bigg|_{x,v,\dot\phi}\delta\phi^a$$
$$+ \frac{\partial L}{\partial\Omega^{\mu\nu}}\bigg|_{x,v}\frac{\partial\Omega^{\mu\nu}}{\partial\dot\phi^a}\bigg|_{x,v,\phi}\delta\dot\phi^a. \qquad (3.19)$$

Comparing Eq. (3.17) with Eq. (3.19), and using Eqs. (3.7), (3.8), and (3.15), we obtain the conjugate momenta

$$P_i = p_i + \tfrac{1}{2}\eta^{AB}S_{\mu\nu}e_A^\mu e_{B;i}^\nu,$$
$$= p_i + \tfrac{1}{2}\eta^{AB}S_{\mu\nu}\tilde{e}_A^\mu \tilde{e}_{B;i}^\nu,$$
$$\equiv p_i + E_{i\mu\nu}S^{\mu\nu}, \qquad (3.20)$$

and

$$P_{\phi^a} = \frac{1}{2}\eta^{AB}S_{\mu\nu}e_A^\mu \frac{\partial e_B^\nu}{\partial\phi^a},$$
$$= \frac{1}{2}S_{\mu\nu}\lambda_a^{AB}\tilde{e}_A^\mu \tilde{e}_B^\nu, \qquad (3.21)$$

where we have introduced the tensor

$$E_{\lambda\mu\nu} \equiv \tfrac{1}{2}\eta_{AB}\tilde{e}_\mu^A \tilde{e}_{\nu;\lambda}^B, \qquad (3.22)$$

which is antisymmetric in the last two indices, and the antisymmetric tensor [30]

$$\lambda_a^{AB}(\phi) \equiv \Lambda_C{}^A \frac{\partial\Lambda^{CB}}{\partial\phi^a}. \qquad (3.23)$$

A necessary condition to go from the Lagrangian formalism to the Hamiltonian one in the usual way (i.e. by means of a Legendre transformation) is that the Langrangian is *regular* [35], i.e. it satisfies[6]

$$\det\left(\frac{\partial^2 L}{\partial\dot q^i\partial\dot q^j}\right) \neq 0, \qquad (3.24)$$

where $\mathbf{q} = (x^i, \phi^a)$. Under this condition, we can perform the usual Legendre transformation to get the Hamiltonian

$$H = P_i v^i + P_{\phi^a}\dot\phi^a - L. \qquad (3.25)$$

Since L is homogeneous of degree one in the "velocities" [because of the reparametrization invariance of the action

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[5]In what follows we will prove that the equations of motion are independent of the choice of this tetrad field. This had to be expected, based on Refs. [27,30].

[6]While this condition is sufficiently generic to leave our Lagrangian essentially undetermined, it should be noticed that there are famous examples in physics where this regularity condition does not hold, such as the electromagnetic field (see for instance Ref. [36], chapter 5), the Dirac field (see for instance Ref. [37], problem 9.2d), the Schrodinger equation (see Ref. [38] and references therein) and general relativity (see for instance Ref. [36], chapter 9).

(3.1)], Euler's theorem implies that

$$u^\mu \frac{\partial L}{\partial u^\mu}\bigg|_\Omega + \Omega^{\mu\nu}\frac{\partial L}{\partial \Omega^{\mu\nu}}\bigg|_u = v^i p_i + p_t + \frac{1}{2}S_{\mu\nu}\Omega^{\mu\nu} = L,$$
(3.26)

where we have used the definitions (3.7) and (3.8), as well as the fact that with our time slicing $u^0 = 1$ and $u^i = v^i$. Using now Eqs. (3.26), (3.20), (3.21), and (3.15) (with $u^0 = 1$ and $u^i = v^i$) in Eq. (3.25), simple algebra allows one to write the Hamiltonian as

$$H = -p_t - \tfrac{1}{2}\eta^{AB}S_{\alpha\beta}e_A^\alpha e_{B;t}^\beta,$$
(3.27a)

$$= -p_t - \tfrac{1}{2}\eta^{AB}S_{\alpha\beta}\tilde{e}_A^\alpha \tilde{e}_{B;t}^\beta,$$
(3.27b)

$$\equiv -p_t - E_{t\mu\nu}S^{\mu\nu},$$
(3.27c)

where the covariant derivative with respect to t in the second term of Eq. (3.27a) above is a shorthand for covariant derivative with respect to $x^0 = t$, i.e. one can pull the Lorentz transformation $\Lambda_A{}^B(\phi)$ outside the covariant derivative as it is independent of x^0. It should be noted that using the tensor $E_{\lambda\mu\nu}$ defined in Eq. (3.22), one can combine H and P_i into a four-vector P_α such that

$$P_\alpha = (-H, P_i) = p_\alpha + E_{\alpha\mu\nu}S^{\mu\nu}.$$
(3.28)

The MPP equations of motion can be derived from the Hamiltonian (3.27c) as follows. On one hand we have

$$\frac{dP_{\phi^a}}{dt} = -\frac{\partial H}{\partial \phi^a} = \frac{\partial L}{\partial \phi^a}\bigg|_{x,u,\dot\phi} = \frac{\partial L}{\partial \Omega^{\mu\nu}}\bigg|_{x,u}\frac{\partial \Omega^{\mu\nu}}{\partial \phi^a},$$
(3.29)

where the second equality follows from the definition of the Hamiltonian (3.25) with the regularity condition (3.24). [One could also derive the second equality by comparing the Hamiltonian and Lagrange equations, but it should be stressed that these two sets of equations are equivalent only if the regularity condition (3.24) is satisfied [35].] Using then Eqs. (3.15) and (3.21), as well as the definition (3.8), a straightforward computation gives the precession equation

$$\frac{DS^{\mu\nu}}{Dt} = S^{\lambda\mu}\Omega^\nu{}_\lambda - \Omega^{\mu\lambda}S^\nu{}_\lambda.$$
(3.30)

The translational equations of motion can be obtained following a similar procedure. In the neighborhood of any event located on the particle's worldline we can choose Riemann normal coordinates and write

$$\frac{dP_i}{dt} = \frac{dp_i}{dt} + \frac{1}{2}\frac{d}{dt}(S_{\mu\nu}\eta^{AB}e_A^\mu e_{B;i}^\nu),$$

$$= -\frac{\partial H}{\partial x^i} = \frac{\partial L}{\partial x^i}\bigg|_{u,\phi,\dot\phi},$$

$$= \frac{\partial L}{\partial \Omega^{\mu\nu}}\bigg|_{x,u}\frac{\partial \Omega^{\mu\nu}}{\partial x^i},$$
(3.31)

where the last equality follows from the compatibility of

the metric with the connection, i.e. $g_{\mu\nu;i} = 0$, which becomes $g_{\mu\nu,i} = 0$ in Riemann normal coordinates.[7] Making use of Eq. (3.15) and using the fact that in Riemann normal coordinates $\Gamma^\lambda_{\mu\nu} = 0$, while their derivatives are nonzero, we get

$$\frac{dp_i}{dt} = -\frac{1}{2}R_{i\alpha\beta\gamma}u^\alpha S^{\beta\gamma},$$
(3.32)

where the Riemann tensor term arises from the derivatives of the Christoffel symbols appearing in Eq. (3.15). Rewriting Eq. (3.32) in a generic coordinates system, we immediately get the spatial part of the translational MPP equations

$$\frac{Dp_i}{Dt} = -\frac{1}{2}R_{i\alpha\beta\gamma}u^\alpha S^{\beta\gamma}.$$
(3.33)

The unconstrained equation of motion for p_t is obtained as follows. One starts from the formal expression

$$\frac{dp_t}{dt} = \{p_t, H\} + \frac{\partial p_t}{\partial t}.$$
(3.34)

In Riemann normal coordinates, the left-hand side is equal to Dp_t/Dt. To evaluate the right-hand side, one makes use of Eq. (3.27c) to eliminate p_t in favor of the Hamiltonian and other quantities whose explicit expressions in terms of the phase space variables $\{x^i, P_i, \phi^a, P_{\phi^a}\}$ are known. Straightforward algebra then yields

$$\frac{Dp_t}{Dt} = -\frac{1}{2}R_{t\alpha\beta\gamma}u^\alpha S^{\beta\gamma},$$
(3.35)

which can be combined with Eqs. (3.33) in the well-known equation translational MPP equations

$$\frac{Dp_\mu}{Dt} = -\frac{1}{2}R_{\mu\alpha\beta\gamma}u^\alpha S^{\beta\gamma}.$$
(3.36)

Before concluding this section, we provide explicit expressions for the Poisson brackets of the variables x^i, P_i, $S^{\mu\nu}$, and $\Lambda^{\mu\nu} \equiv \Lambda^{AB}\tilde{e}_A^\mu \tilde{e}_B^\nu$. Using the definition of Poisson bracket,

$$\{f, g\} \equiv \frac{\partial f}{\partial \mathbf{q}}\cdot\frac{\partial g}{\partial \boldsymbol{\pi}} - \frac{\partial g}{\partial \mathbf{q}}\cdot\frac{\partial f}{\partial \boldsymbol{\pi}},$$
(3.37)

where $\mathbf{q} = (x^i, \phi^a)$ and $\boldsymbol{\pi} = (P_i, P_{\phi^a})$, we trivially have

$$\{x^i, P_j\} = \delta^i_j,$$
(3.38a)

$$\{x^i, x^j\} = \{P_i, P_j\} = 0.$$
(3.38b)

To compute the Poisson brackets involving $S^{\mu\nu}$, let us first invert Eq. (3.21) [30]:

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[7]We stress that one is allowed to set $g_{\mu\nu,i} = 0$ in this equation as we do not need to take derivatives of it (in which case, of course, the terms containing $g_{\mu\nu,i}$ would give a contribution, as in general $g_{\mu\nu,ij} \neq 0$ even in Riemann normal coordinates).

$$S^{\alpha\beta} = \tilde{e}_A^\alpha \tilde{e}_B^\beta \rho_a^{AB} P_{\phi^a}, \tag{3.39}$$

where $\rho_a^{AB}(\phi)$ satisfies

$$\rho_a^{AB} \lambda_{bAB} = 2\delta_{ab}, \tag{3.40}$$

$$\lambda_a^{AB} \rho_a^{CD} = \eta^{AC}\eta^{BD} - \eta^{AD}\eta^{BC}. \tag{3.41}$$

Using these relations together with the identity

$$\frac{\partial \lambda_a^{AB}}{\partial \phi^b} - \frac{\partial \lambda_b^{AB}}{\partial \phi^a} = \lambda_a^{AC}\lambda_{bC}{}^B - \lambda_b^{AC}\lambda_{aC}{}^B, \tag{3.42}$$

which can be immediately derived [30] by taking the derivative of Eq. (3.23), it is straightforward to prove that ρ_a^{AB} is a realization of the Lie algebra of the Lorentz group:

$$\rho_b^{AB} \frac{\partial \rho_a^{CD}}{\partial \phi^b} - \rho_b^{CD} \frac{\partial \rho_a^{AB}}{\partial \phi^b} = -\rho_a^{AC}\eta^{BD} - \rho_a^{BD}\eta^{AC}$$
$$+ \rho_a^{AD}\eta^{BC} + \rho_a^{BC}\eta^{AD}. \tag{3.43}$$

Simple algebra then yields

$$\{S^{\mu\nu}, S^{\alpha\beta}\} = S^{\mu\alpha}g^{\nu\beta} + S^{\nu\beta}g^{\mu\alpha} - S^{\mu\beta}g^{\nu\alpha} - S^{\nu\alpha}g^{\mu\beta}, \tag{3.44a}$$

while using Eqs. (2.3) and (3.39) we easily obtain

$$\{S^{\mu\nu}, P_i\} = S^{\alpha\nu}\tilde{e}_\alpha^A \tilde{e}_{A,i}^\mu + S^{\mu\alpha}\tilde{e}_\alpha^A \tilde{e}_{A,i}^\nu, \tag{3.44b}$$

$$\{S^{\mu\nu}, x^i\} = 0. \tag{3.44c}$$

Finally, it is straightforward to show that Λ^{AB} satisfies [30]

$$\{\Lambda^{AB}, x^i\} = \{\Lambda^{AB}, P_i\} = \{\Lambda^{AB}, \Lambda^{CD}\} = 0, \tag{3.45}$$

$$\{\Lambda^{AB}, S^{CD}\} = \Lambda^{AC}\eta^{BD} - \Lambda^{AD}\eta^{BC}, \tag{3.46}$$

or, in terms of $\Lambda^{\mu\nu} \equiv \Lambda^{AB}\tilde{e}_A^\mu \tilde{e}_B^\nu$

$$\{\Lambda^{\mu\nu}, x^i\} = \{\Lambda^{\mu\nu}, \Lambda^{\alpha\beta}\} = 0, \tag{3.47}$$

$$\{\Lambda^{\mu\nu}, P_i\} = \Lambda^{\alpha\nu}\tilde{e}_\alpha^A \tilde{e}_{A,i}^\mu + \Lambda^{\mu\alpha}\tilde{e}_\alpha^A \tilde{e}_{A,i}^\nu, \tag{3.48}$$

$$\{\Lambda^{\mu\nu}, S^{\alpha\beta}\} = \Lambda^{\mu\alpha}g^{\nu\beta} - \Lambda^{\mu\beta}g^{\nu\alpha}. \tag{3.49}$$

IV. CONSTRAINED HAMILTONIAN

A. Imposing constraints in phase space: a Dirac bracket primer

Let us briefly recall how constraints are imposed in the Hamiltonian formalism (a very detailed review on the subject can be found in Ref. [39]). Let us consider a Hamiltonian $H(\mathbf{q}, \boldsymbol{\pi}, t)$ living in a $2n$-dimensional phase space and a binary "bracket" operation $\{\ldots, \ldots\}$ which is antisymmetric, bilinear, and which satisfies the Leibniz rule, as well as the Jacobi identity, i.e.

$$\{A, B\} = -\{B, A\}, \tag{4.1a}$$

$$\{aA + bB, C\} = a\{A, C\} + b\{B, C\}, \tag{4.1b}$$

$$\{AB, C\} = \{A, C\}B + A\{B, C\}, \tag{4.1c}$$

and

$$\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0. \tag{4.1d}$$

In Eqs. (4.1), A, B, and C are arbitrary phase space functions, while a and b are constants. Let us also assume that the bracket operation gives the equations of motion for a generic phase space function A through the Hamilton equations

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \{A, H\}. \tag{4.2}$$

If we consider now a set of constraints $\xi_i = 0$, $i = 1, \ldots, 2m$ (with $m < n$) such that the matrix

$$C_{ij} \equiv \{\xi_i, \xi_j\} \tag{4.3}$$

is not singular,[8] these constraints can be imposed simply by replacing the original brackets with the so-called *Dirac brackets*. The Dirac brackets are in essence the projection of the original symplectic structure onto the phase space surface defined by the constraints. For two arbitrary phase space functions A and B, the Dirac brackets are given by

$$\{A, B\}_{\text{DB}} = \{A, B\} + \{A, \xi_i\}\{B, \xi_j\}[C^{-1}]_{ij}. \tag{4.4}$$

It can be shown (see e.g. Secs. 1.3.2, 1.3.3, and Ex. 1.12 in Ref. [39]), that the Dirac brackets are bilinear, antisymmetric, that they satisfy the Leibniz rule and the Jacobi identity, and that they provide the correct equations of motion for the constrained system through the Hamilton equations

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \{A, \bar{H}\}_{\text{DB}}, \tag{4.5}$$

where A is an arbitrary phase space function, and where the new Hamiltonian $\bar{H}$ is obtained simply by inserting the constraints in the original Hamiltonian H.

In summary, given a Hamiltonian H and a bracket operation (e.g., the Poisson brackets in the case of an unconstrained Hamiltonian), in order to impose a set of constraints satisfying $\det(C_{ij}) \neq 0$ [with $\mathbf{C}$ given by Eq. (4.3)], we need to replace the original bracket operation with the Dirac bracket operation (4.4), and insert the constraints directly in the original Hamiltonian.

In Secs. IV B and IV C we start from the unconstrained Hamiltonian (3.27c) and the unconstrained algebra (3.38a), (3.38b), (3.44a)–(3.44c), (3.47), (3.48), and (3.49), and use the procedure outlined in this subsection to impose the generalized NW SSC. In particular, in Sec. IV B we com-

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[8]In the literature, constraints satisfying this condition are known as *second class constraints* [39].

pute the Dirac brackets in the NW SSC, showing that they are canonical (i.e., they reduce to the usual Poisson brackets) at linear order in the particle's spin, while in Sec. IV C we explicitly write the constrained Hamiltonian.

B. Dirac brackets in the generalized Newton-Wigner spin supplementary condition

In this section, we consider the NW SSC generalized to curved spacetime,

$$V^\mu \equiv S^{\mu\nu}\omega_\nu = 0, \tag{4.6}$$

with

$$\omega_\mu = p_\mu - m\tilde{e}_\mu^T, \tag{4.7}$$

where $m = \sqrt{-p_\mu p^\mu}$ is a function of phase space varia-bles that we define as the mass of the particle.[9] We stress that the vector $\boldsymbol{\omega}$ is the sum of two timelike future-oriented vectors and is therefore timelike itself, which implies that Eqs. (4.6) and (4.7) do indeed yield a legitimate SSC [29]. (We recall that with our notation one has $\tilde{e}^T = -\tilde{e}_T$, and that $\tilde{e}_T$ is future-oriented.)

While the NW SSC is well known to be the *only* SSC condition which yields canonical variables in flat space-time[10] [30,40,41], there is no *a priori* guarantee that this is the case in curved spacetime. In this section we show that the NW SSC *does* indeed yield canonical variables at linear order in the particle's spin.

Because $V^\mu\omega_\mu = 0$, only three of the four constraints (4.6) are independent. Since $\boldsymbol{\omega}$ is a timelike vector, it is natural to take the three independent constraints to be the spatial components V^i. The constraints V^i may be viewed as constraints on the momenta P_{ϕ^a}, as there is a one-to-one mapping between the spin tensor $S^{\mu\nu}$ and the six momenta conjugate to the ϕ^a's. This implies that by themselves, the constraints V^i do not form a consistent set of constraints on phase space: an additional set of three constraints must be imposed on the configuration coordinates ϕ^a themselves in order to retain a symplectic structure, i.e. that the constraint hypersurface contains the same number of configuration coordinates and conjugate momenta. The additional con-straints we choose to impose are given by [23,30]

$$\chi_\mu = (e_T)_\mu - \frac{p_\mu}{m} = \Lambda_T{}^A(\tilde{e}_A)_\mu - \frac{p_\mu}{m} = 0. \tag{4.8}$$

It is worth pointing out once again that the mass m is a function on phase space, and therefore its Poisson brackets with coordinates and momenta are nonvanishing. It will acquire a special status as a constant of motion (at linear

order in spin) only at the end of this subsection. Equa-tion (4.8) may be alternatively rewritten as

$$\Lambda_T{}^A = \frac{1}{m} p_\mu (\tilde{e}^A)^\mu = \frac{p^A}{m}, \tag{4.9}$$

which shows explicitly that it constrains the three velocity parameters, say $\phi^{4,5,6}$, of the Lorentz transformation that relates the tetrad carried by the particle to the background tetrad. Since $\Lambda_T{}^T$ is fully determined by $\Lambda_T{}^I$, only three of the four constraints given in Eqs. (4.8) or (4.9) are inde-pendent.[11] We will take the spatial components $\chi_i = 0$ as our three independent constraints on the coordinates ϕ^a.

In summary, for the generalized NW SSC, the vector of constraints is

$$\boldsymbol{\xi} \equiv (V^1, V^2, V^3, \chi_1, \chi_2, \chi_3). \tag{4.10}$$

In principle, the computation of the matrix $\boldsymbol{C}$ defined in Eq. (4.3) can be performed directly using the uncon-strained symplectic algebra (3.38a), (3.38b), (3.44a)–(3.44c), (3.47), (3.48), and (3.49). However, since the con-straints are formulated in terms of the momentum four-vector p_μ rather than the conjugate momenta P_i and the Hamiltonian H, it turns out to be quite useful to first compute Poisson brackets between p_μ and other phase space quantities, and then make use of these results to compute the matrix $\boldsymbol{C}$. The relevant Poisson brackets are

$$\{x^i, p_j\} = \delta^i_j, \tag{4.11a}$$

$$\{x^i, p_t\} = -v^i, \tag{4.11b}$$

$$\{x^i, m\} = -\frac{1}{m}(p^i - p^t v^i), \tag{4.11c}$$

$$\{p_i, p_j\} = -\frac{1}{2}R_{ij\mu\nu}S^{\mu\nu}, \tag{4.11d}$$

$$\{p_i, p_t\} = \frac{1}{2}R_{ik\mu\nu}v^k S^{\mu\nu} - \Gamma^\mu_{i\nu}p_\mu u^\nu, \tag{4.11e}$$

$$\{p_i, m\} = -\frac{1}{m}p_\mu\Gamma^\mu_{ik}(p^k - p^t v^k), \tag{4.11f}$$

$$\{p_t, m\} = \frac{1}{m}p_\mu\Gamma^\mu_{k\nu}(p^\nu v^k - p^k u^\nu), \tag{4.11g}$$

$$\{p_i, (e_T)_j\} = -\Gamma^\mu_{ij}(e_T)_\mu, \tag{4.11h}$$

$$\{p_t, (e_T)_j\} = \frac{1}{m}(-\Omega_{j\nu}p^\nu + \Gamma^\mu_{jk}p_\mu v^k), \tag{4.11i}$$

$$\{S^{\mu\nu}, p_i\} = 2S^{\lambda[\mu}\Gamma^{\nu]}_{i\lambda}, \tag{4.11j}$$

$$\{S^{\mu\nu}, p_t\} = -2p^{[\mu}u^{\nu]} - 2S^{\lambda[\mu}\Gamma^{\nu]}_{\lambda k}v^k, \tag{4.11k}$$

$$\{S^{\mu\nu}, m\} = 2p^{[\mu}u^{\nu]}, \tag{4.11l}$$

$$\{S^{\mu\nu}, (e_T)_j\} = 2\delta^{[\mu}_j e^{\nu]}_T, \tag{4.11m}$$

$$\{(e_T)_i, m\} = -\frac{1}{m^2}p^t[p^\nu\Omega_{i\nu} + p_\mu\Gamma^\mu_{ij}(p^j - p^t v^j)], \tag{4.11n}$$

$$\{(\tilde{e}_T)^\mu, m\} = -\frac{1}{m}(\tilde{e}_T)^\mu_{,k}(p^k - p^t v^k), \tag{4.11o}$$

where the Poisson bracket between an arbitrary phase

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[9]Note that at this stage there is no guarantee that this function on phase space is a constant of motion. We will show later that it is indeed the case, but we emphasize that this is a nontrivial result.

[10]We note that in quantum mechanics and flat spacetime the NW SSC holds a special place [40,41].

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[11]One can also see this from the fact that $\boldsymbol{\chi}$ is orthogonal to the timelike vector $\boldsymbol{e}_T + \boldsymbol{p}/(mc)$. Hence only its three spacelike components are independent.

space function A and the quantity p_t is obtained as follows:

$$\{A, p_t\} = \left\{A, -H - \frac{1}{2}\eta^{AB}(\tilde{e}_A)_\alpha(\tilde{e}_B)_{\beta;t}S^{\alpha\beta}\right\},$$

$$= \frac{\partial A}{\partial t} - \frac{dA}{dt} - \frac{1}{2}\eta^{AB}\{A, (\tilde{e}_A)_\alpha(\tilde{e}_B)_{\beta;t}S^{\alpha\beta}\}. \quad (4.12)$$

The total time derivative dA/dt is then evaluated with the help of the unconstrained equations of motion. The Poisson brackets (4.11) along with Eqs. (3.12) and (3.44a) yield

$$\{V^i, V^j\} = \omega_\mu\omega^\mu S^{ij} + \mathcal{O}(S^2), \quad (4.13)$$

$$\{V^i, \chi_j\} = \frac{\omega_\mu p^\mu}{m}\left(\delta^i_j - \frac{p^i\omega_j}{\omega_\mu p^\mu}\right) + S^{i\lambda}\tilde{e}^T_{\lambda;\nu}\left(\delta^\nu_j + \frac{p^\nu p_j}{m^2}\right)$$

$$+ \mathcal{O}(S^2), \quad (4.14)$$

$$\{\chi_i, \chi_j\} = \frac{1}{2m^4}(p_i R_{j\lambda\mu\nu} - p_j R_{i\lambda\mu\nu})p^\lambda S^{\mu\nu}$$

$$- \frac{1}{2m^2}R_{ij\mu\nu}S^{\mu\nu} + \mathcal{O}(S^2), \quad (4.15)$$

The inverse matrix $\boldsymbol{C}^{-1}$ can be easily computed at linear order in the spin, the result being

$$\boldsymbol{C}^{-1} = \boldsymbol{K}^{-1} - \boldsymbol{K}^{-1}\boldsymbol{\Sigma}\boldsymbol{K}^{-1} + \mathcal{O}(S^2), \quad (4.20)$$

where

$$\boldsymbol{K}^{-1} = \begin{pmatrix} \mathbb{0}_3 & -(\boldsymbol{Q}^{-1})^T \\ \boldsymbol{Q}^{-1} & \mathbb{0}_3 \end{pmatrix}, \quad (4.21)$$

with

The remainders scaling as the square of the particle's spin are dropped, since the pole-dipole particle model is valid only at linear order in the particle's spin. The matrix $\boldsymbol{C}$ defined in Eq. (4.3) is therefore given by

$$\boldsymbol{C} = \boldsymbol{K} + \boldsymbol{\Sigma} + \mathcal{O}(S^2) \quad (4.16)$$

where the matrices $\boldsymbol{K}$ and $\boldsymbol{\Sigma}$ are defined as

$$\boldsymbol{K} = \begin{pmatrix} \mathbb{0}_3 & \boldsymbol{Q} \\ -\boldsymbol{Q}^T & \mathbb{0}_3 \end{pmatrix}, \quad (4.17)$$

with

$$Q^i_j = \frac{\omega_\mu p^\mu}{m}\left(\delta^i_j - \frac{p^i\omega_j}{\omega_\mu p^\mu}\right), \quad (4.18)$$

and

$$\Sigma^{ij} \equiv \begin{pmatrix} \omega_\mu\omega^\mu S^{ij} & S^{i\mu}\tilde{e}^T_{\mu;\nu}(\delta^\nu_j + \frac{p^\nu p_j}{m^2}) \\ -S^{j\mu}\tilde{e}^T_{\mu;\nu}(\delta^\nu_i + \frac{p^\nu p_i}{m^2}) & -\frac{1}{2m^2}R_{k\lambda\mu\nu}S^{\mu\nu}[\delta^k_i\delta^\lambda_j + \frac{p^\lambda}{m^2}(\delta^k_i p_j - \delta^k_j p_i)] \end{pmatrix}. \quad (4.19)$$

$$[Q^{-1}]^i_j = \frac{m}{\omega_\mu p^\mu}\left(\delta^i_j + \frac{\omega_j p^i}{\omega_t p^t}\right). \quad (4.22)$$

To compute the Dirac brackets between two phase space functions, one also needs the Poisson brackets between those phase space functions and the constraints. For our purposes, the relevant brackets are given by

$$\{x^i, V^j\} = -S^{ij} - S^{jt}\frac{p^i}{p^t} + \mathcal{O}(S^2), \quad (4.23a)$$

$$\{x^i, \chi_j\} = -\frac{1}{m}\left[\delta^i_j + \frac{p_j}{m^2}(p^i - p^t u^i)\right] + \mathcal{O}(S^2), \quad (4.23b)$$

$$\{P_i, V^j\} = p_A S^{j\alpha}\tilde{e}^A_{\alpha,i} - S^{jt}\Gamma^\mu_{\nu i}\frac{p^\nu p_\mu}{p^t} + \mathcal{O}(S^2) = p_A\tilde{e}^A_{k,i}\left(S^{jk} - S^{jt}\frac{p^k}{p^t}\right) + \mathcal{O}(S^2), \quad (4.23c)$$

$$\{P_i, \chi_j\} = -\frac{1}{m}p_A\tilde{e}^A_{j,i} + \frac{2p_j}{m^3}p^{[\mu}u^{\nu]}(E_{i\mu\nu} - p_\lambda\Gamma^\lambda_{i\mu}\delta^t_\nu) + S^{\mu\nu}\left[\frac{1}{2m^2}R_{ij\mu\nu} - \frac{1}{m}(E_{i\mu\nu;j} + \Gamma^\lambda_{ij}E_{\lambda ij})\right] + \mathcal{O}(S^2), \quad (4.23d)$$

$$\{S^{AB}, V^i\} = S^{Ai}\omega^B - S^{Bi}\omega^A + \mathcal{O}(S^2), \quad (4.23e)$$

$$\{S^{AB}, \chi_i\} = -\frac{2}{m}p^{[A}\tilde{e}^{B]}_i + \frac{2p_i}{m^3}p^{[A}u^{B]} - \frac{2}{m}S^{\mu[A}\Gamma^{B]}_{i\mu} + \mathcal{O}(S^2). \quad (4.23f)$$

The matrix (4.20) and (4.23), together with the unconstrained algebra given by Eqs. (3.38) and (3.44), is all one needs to compute the Dirac brackets according to Eq. (4.4). Our results for the Dirac brackets involving x^i and P_j are given by

$$\{x^i, x^j\}_{\mathrm{DB}} = \left[\frac{\omega^\mu \omega_\mu - 2p^\nu \omega_\nu}{(p^\sigma \omega_\sigma)^2}\right]\left(S^{ij} - S^{it}\frac{p^j}{p^t} + S^{jt}\frac{p^i}{p^t}\right) + \mathcal{O}(S^2) = \mathcal{O}(S^2),\tag{4.24a}$$

$$\{x^i, P_j\}_{\mathrm{DB}} = \delta^i_j + \left(S^{ik} - S^{it}\frac{p^k}{p^t} + S^{kt}\frac{p^i}{p^t}\right)\left[\frac{\omega^\mu \omega_\mu - 2p^\nu \omega_\nu}{(p^\sigma \omega_\sigma)^2}\right]p_\alpha \tilde{e}^\alpha_A \tilde{e}^A_{k,j} + \mathcal{O}(S^2) = \delta^i_j + \mathcal{O}(S^2),\tag{4.24b}$$

$$\{P_i, P_j\}_{\mathrm{DB}} = \left(S^{kl} - S^{kt}\frac{p^l}{p^t} + S^{lt}\frac{p^k}{p^t}\right)\left[\frac{\omega^\mu \omega_\mu - 2p^\nu \omega_\nu}{(p^\sigma \omega_\sigma)^2}\right]p_\alpha \tilde{e}^\alpha_A \tilde{e}^A_{k,i}p_\beta \tilde{e}^\beta_B \tilde{e}^B_{l,j} + \mathcal{O}(S^2) = \mathcal{O}(S^2).\tag{4.24c}$$

The crucial point now is that Eq. (4.7) implies $\omega^\mu \omega_\mu = 2p^\mu \omega_\mu$, and therefore all terms linear in the particle's spin on the right-hand side of Eqs. (4.24) vanish. Hence the Dirac bracket algebra between x^i and P_j is canonical up to terms quadratic in the particle's spin.

The Dirac brackets involving the spin variables are most effectively computed by considering the projection of the spin tensor onto the spacelike background tetrad vectors, i.e. $S^{IJ} = S^{\mu\nu}\tilde{e}^I_\mu \tilde{e}^J_\nu$. We find

$$\{x^i, S^{KL}\}_{\mathrm{DB}} = \left[\frac{\omega^\mu \omega_\mu - 2p^\nu \omega_\nu}{(p^\sigma \omega_\sigma)^2}\right]\left(S^{i\alpha} + S^{\alpha t}\frac{p^i}{p^t}\right)p^\beta(\tilde{e}^K_\beta \tilde{e}^L_\alpha - \tilde{e}^L_\beta \tilde{e}^K_\alpha) + \mathcal{O}(S^2) = \mathcal{O}(S^2)\tag{4.25a}$$

$$\{P_i, S^{KL}\}_{\mathrm{DB}} = \left[\frac{\omega^\mu \omega_\mu - 2p^\nu \omega_\nu}{(p^\sigma \omega_\sigma)^2}\right]\left(S^{\gamma k} - S^{\gamma t}\frac{p^k}{p^t}\right)p^\alpha p_\beta \tilde{e}^\beta_C \tilde{e}^C_{k,i}(\tilde{e}^L_\alpha \tilde{e}^K_\gamma - \tilde{e}^L_\gamma \tilde{e}^K_\alpha) + \mathcal{O}(S^2) = \mathcal{O}(S^2),\tag{4.25b}$$

$$\{S^{IJ}, S^{KL}\}_{\mathrm{DB}} = \left[\frac{\omega^\mu \omega_\mu - 2p^\nu \omega_\nu}{(p^\sigma \omega_\sigma)^2}\right]S^{\gamma\delta}p^\alpha p^\beta(\tilde{e}^L_\alpha \tilde{e}^K_\delta - \tilde{e}^L_\delta \tilde{e}^K_\alpha)(\tilde{e}^J_\beta \tilde{e}^I_\gamma - \tilde{e}^J_\gamma \tilde{e}^I_\beta) + S^{IK}\delta^{JL} + S^{JL}\delta^{IK} - S^{IL}\delta^{JK}$$
$$- S^{JK}\delta^{IL} + \mathcal{O}(S^2) = S^{IK}\delta^{JL} + S^{JL}\delta^{IK} - S^{IL}\delta^{JK} - S^{JK}\delta^{IL} + \mathcal{O}(S^2),\tag{4.25c}$$

where we have used $\omega^I = p^I$, which follows directly from Eq. (4.7). Again the terms proportional to $\omega_\mu \omega^\mu - 2p^\nu \omega_\nu$ disappear. Defining a three-dimensional spin vector by

$$S^I = \tfrac{1}{2}\epsilon^{IJK}S^{JK}\tag{4.26}$$

one can immediately rewrite Eqs. (4.25) as

$$\{x^i, S^J\}_{\mathrm{DB}} = \mathcal{O}(S^2),\tag{4.27a}$$
$$\{P_i, S^J\}_{\mathrm{DB}} = \mathcal{O}(S^2),\tag{4.27b}$$
$$\{S^I, S^J\}_{\mathrm{DB}} = \epsilon_{IJK}S^K + \mathcal{O}(S^2).\tag{4.27c}$$

Equations (4.27) imply that the phase space variables $\{x^i, P_j, S^K\}$ provided by the generalized NW SSC are canonical at linear order in the particle's spin.

C. Hamiltonian in the generalized Newton-Wigner SSC

In this section, we provide an explicit expression for the Hamiltonian (3.27c) in the NW SSC, at linear order in the particle's spin. As explained in Sec. IVA, this is simply obtained by inserting the NW SSC directly into the unconstrained Hamiltonian. Also, we express this constrained Hamiltonian in terms of the variables x^i, P_j, S^K, which have been proven in Sec. IV B to be canonical at linear order in the particle spin.

We begin by rewriting the quantity p_t appearing in the unconstrained Hamiltonian (3.27c) in terms of the mass $m = \sqrt{-p_\mu p^\mu}$ and the spatial components p_i of the momentum four-vector. The result is

$$p_t = -\beta^i p_i - \alpha\sqrt{m^2 + \gamma^{ij}p_i p_j},\tag{4.28}$$

where

$$\alpha = \frac{1}{\sqrt{-g^{tt}}},\tag{4.29a}$$

$$\beta^i = \frac{g^{ti}}{g^{tt}},\tag{4.29b}$$

$$\gamma^{ij} = g^{ij} - \frac{g^{ti}g^{tj}}{g^{tt}}.\tag{4.29c}$$

The crucial usefulness of Eq. (4.28) resides in the fact that the canonical phase space variables $\{x^i, P_j, S^K\}$ have vanishing Dirac brackets with the mass at linear order in the particle spin. We have established this result by explicit computation. As an illustration, we provide the details of the computation of the Dirac bracket between x^i and the mass (the other brackets involving the mass are computed in a similar fashion). We start from

$$\{x^i, m\}_{\mathrm{DB}} = \{x^i, \sqrt{-p_\mu p^\mu}\}_{\mathrm{DB}},$$

$$= -\frac{1}{2m}\{x^i, g^{\mu\nu}p_\mu p_\nu\}_{\mathrm{DB}},$$

$$= -\frac{1}{m}p^\mu\{x^i, p_\mu\}_{\mathrm{DB}},\tag{4.30}$$

the last line following from $\{x^i, x^j\}_{\mathrm{DB}} = \mathcal{O}(S^2)$. Using Eq. (3.28) together with the fact that the Dirac bracket with the Hamiltonian gives the constrained equations of motion yields

$$\{x^i, m\}_{\text{DB}} = -\tfrac{1}{m} p^\mu \{x^i, P_\mu - E_{\mu\alpha\beta} S^{\alpha\beta}\}_{\text{DB}},$$
$$= -\tfrac{1}{m}(p^i - p^t v^i) + \tfrac{1}{m} p^\mu E_{\mu\alpha\beta} \{x^i, S^{\alpha\beta}\}_{\text{DB}},$$
$$(4.31)$$

where Eq. (3.12) must be employed to express the four-velocity components v^i in terms of canonical variables. Substituting Eq. (4.7) into Eq. (3.12), it is straightforward to show that

$$p^i - p^t v^i = -\frac{m}{\omega_\nu p^\nu} \tilde{e}^T_{\lambda;\sigma} p^\sigma \left(S^{i\lambda} - S^{t\lambda} \frac{p^i}{p^t}\right). \quad (4.32)$$

Next the Dirac bracket between x^i and $S^{\alpha\beta}$ at linear order in spin can be computed directly following the procedure outlined in Sec. IV B, the result being

$$\{x^i, S^{\alpha\beta}\}_{\text{DB}} = -\frac{2m(\tilde{e}^T)^{[\alpha}}{\omega_\nu p^\nu} \left(S^{\beta]i} + S^{\beta]k} \frac{\omega_k p^i}{\omega_t p^t}\right). \quad (4.33)$$

Hence, since $E_{\mu\alpha\beta}$ is antisymmetric in $\alpha \leftrightarrow \beta$, we get

$$E_{\mu\alpha\beta}\{x^i, S^{\alpha\beta}\}_{\text{DB}} = E_{\mu\alpha\beta} \frac{2m(\tilde{e}^T)^\alpha}{\omega_\nu p^\nu} \left(S^{\beta i} + S^{\beta k} \frac{\omega_k p^i}{\omega_t p^t}\right)$$
$$= \frac{m}{\omega_\nu p^\nu} (\tilde{e}^T)_{\beta;\mu} \left(S^{\beta i} + S^{\beta k} \frac{\omega_k p^i}{\omega_t p^t}\right), \quad (4.34)$$

the second line following from the definition $2E_{\mu\alpha\beta} = \eta^{AB}(\tilde{e}_A)_\alpha (\tilde{e}_B)_{\beta;\mu}$. Substituting Eqs. (4.32), (4.33), and (4.34) into Eq. (4.31) one obtains

$$\{x^i, m\}_{\text{DB}} = \frac{1}{\omega_\nu p^\nu} \left[(\tilde{e}^T)_{\lambda;\sigma} p^\sigma \left(S^{i\lambda} - S^{t\lambda} \frac{p^i}{p^t}\right) \right.$$
$$\left. + p^\mu (\tilde{e}^T)_{\beta;\mu} \left(S^{\beta i} + S^{\beta k} \frac{\omega_k p^i}{\omega_t p^t}\right) \right]. \quad (4.35)$$

Renaming dummy indices and making use of the NW SSC to rewrite $S^{t\lambda} = -S^{k\lambda} \omega_k / \omega_t$, one can see that all terms cancel, therefore showing that the mass commutes with x^i under the Dirac brackets.

Since the constrained Hamiltonian depends only on $\{x^i, P_j, S^K\}$ and the mass m, it follows that the mass may be treated as a constant when taking the Dirac bracket between an arbitrary function of constrained phase space variables and the Hamiltonian.

Our Hamiltonian (3.27c) now takes the form

$$\bar{H} = \beta^i p_i + \alpha\sqrt{m^2 + \gamma^{ij} p_i p_j} - E_{tAB} S^{AB}. \quad (4.36)$$

Equation (4.26) implies $S^{IJ} = \epsilon^{IJK} S^K$, while the NW SSC [Eqs. (4.6) and (4.7)] implies

$$S^{TI} = \frac{S^{IJ}\omega_J}{\omega_T}, \quad (4.37)$$

where

$$\omega_T = \omega_\mu \tilde{e}^\mu_T = p_\mu \tilde{e}^\mu_T - m, \quad (4.38a)$$
$$\omega_I = \omega_\mu \tilde{e}^\mu_I = p_\mu \tilde{e}^\mu_I. \quad (4.38b)$$

The canonical momenta P_i are related to the linear momenta p_i by Eq. (3.20), which may be rewritten in terms of the canonical spin variables as

$$P_i = p_i + E_{iAB} S^{AB},$$
$$= p_i + \left(2E_{iTJ} \frac{\omega_K}{\omega_T} + E_{iJK}\right) \epsilon^{JKL} S^L. \quad (4.39)$$

In principle, in order to express the Hamiltonian (4.36) in terms of the canonical momenta P_i, one must invert Eq. (4.39) to obtain p_i as function of canonical variables (recall that ω_μ depends on p_μ). However, because our Hamiltonian is valid only at linear order in the test particle's spin, it is sufficient to write

$$p_i = P_i - \left(2E_{iTJ} \frac{\bar{\omega}_K}{\bar{\omega}_T} + E_{iJK}\right) \epsilon^{JKL} S^L + \mathcal{O}(S^2), \quad (4.40)$$

where

$$\bar{\omega}_\mu = \bar{P}_\mu - m\tilde{e}^T_\mu \quad (4.41a)$$
$$\bar{P}_i = P_i, \quad (4.41b)$$
$$\bar{P}_t = -\beta^i P_i - \alpha\sqrt{m^2 + \gamma^{ij} P_i P_j}, \quad (4.41c)$$
$$\bar{\omega}_T = \bar{\omega}_\mu \tilde{e}^\mu_T = \bar{P}_\mu \tilde{e}^\mu_T - m, \quad (4.41d)$$
$$\bar{\omega}_I = \bar{\omega}_\mu \tilde{e}^\mu_I = \bar{P}_\mu \tilde{e}^\mu_I. \quad (4.41e)$$

We may now write the constrained Hamiltonian (4.36) as

$$\bar{H} = \beta^i p_i + \alpha\sqrt{m^2 + \gamma^{ij} p_i p_j} - F^K_t S_K + \mathcal{O}(S^2), \quad (4.42)$$

where

$$F^K_\mu = \left(2E_{\mu TI} \frac{\bar{\omega}_J}{\bar{\omega}_T} + E_{\mu IJ}\right) \epsilon^{IJK}. \quad (4.43)$$

By substituting expression (4.40) for p_i into Eq. (4.42) and expanding to linear order in spin, one arrives at last at the following Hamiltonian:

$$\bar{H} = \bar{H}_{\text{NS}} - \left(\beta^i F^K_i + F^K_t + \frac{\alpha\gamma^{ij} P_i F^K_j}{\sqrt{m^2 + \gamma^{ij} P_i P_j}}\right) S_K, \quad (4.44)$$

where $\bar{H}_{\text{NS}}$ is the Hamiltonian for a nonspinning particle, simply given by

$$\bar{H}_{\text{NS}} = \beta^i P_i + \alpha\sqrt{m^2 + \gamma^{ij} P_i P_j}. \quad (4.45)$$

V. EXPLICIT HAMILTONIAN FOR SPECIFIC BACKGROUND SPACETIMES

A. Spherically symmetric spacetime in isotropic coordinates

The line element for a generic spherically symmetric spacetime in isotropic coordinates is given by

$$ds^2 = -f(\rho)dt^2 + h(\rho)(dx^2 + dy^2 + dz^2), \quad (5.1)$$

where $\rho^2 = x^2 + y^2 + z^2$. The natural tetrad associated with this spacetime and coordinate system is

$$\tilde{e}_T^\mu = \frac{1}{\sqrt{f}}\delta_0^\mu, \quad (5.2a)$$

$$\tilde{e}_I^\mu = \frac{1}{\sqrt{h}}\delta_I^\mu, \quad (5.2b)$$

where the symbol δ_I^μ is equal to 0 when $\mu = 0$ and it is equal to 1 when $\mu = I$ numerically.[12] With a metric and a convenient tetrad in hand, one may now compute the quantity $E_{\mu AB}$ as follows:

$$E_{\mu AB} = -\frac{1}{2}[(\tilde{e}_A)_\lambda(\tilde{e}_B^\lambda)_{,\mu} + (\tilde{e}_A)_\lambda\Gamma_{\mu\gamma}^\lambda\tilde{e}_B^\gamma]. \quad (5.3)$$

The algebra is straightforward and the result is

$$E_{\mu TI} = \frac{f'}{4\sqrt{fh}}\delta_\mu^0 n_I, \quad (5.4a)$$

$$E_{\mu JK} = -\frac{h'}{2h}\delta_{\mu[J}n_{K]}, \quad (5.4b)$$

where the prime symbol denotes a derivative with respect to ρ, and where $n_I = (x/\rho, y/\rho, z/\rho)$. The last ingredients needed in order to obtain the explicit Hamiltonian are $\bar{\omega}_T$ and $\bar{\omega}_K$ defined in Eqs. (4.41d) and (4.41e). A quick computation yields

$$\bar{\omega}_T = -\sqrt{m^2 + \gamma^{ij}P_iP_j} - m,$$
$$\equiv -m(1 + \sqrt{Q}), \quad (5.5a)$$

$$\bar{\omega}_K = \frac{1}{\sqrt{h}}P_K, \quad (5.5b)$$

where $Q = 1 + \gamma^{ij}P_iP_j/m^2$ and $P_K = P_j\delta_K^j$. By substituting Eqs. (5.4) and (5.5) into Eq. (4.43), we obtain the following expression for the quantity F_μ^I:

$$F_0^I = -\frac{1}{m(1 + \sqrt{Q})}\frac{f'}{2\sqrt{f}h}\epsilon^{IJK}n_JP_K, \quad (5.6a)$$

$$F_j^I = -\frac{h'}{2h}\epsilon^{IJK}\delta_{Jj}n_K. \quad (5.6b)$$

Finally, by substituting Eq. (5.6) into the Hamiltonian

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[12]More precisely, even though the spacetime index μ and the internal tetrad index I are completely different in character, both indices may take on the same numerical value (1, 2, or 3) associated with x, y, and z, respectively.

(4.44) and performing simple algebra, we arrive at

$$\bar{H} = \bar{H}_{NS} + \left[\frac{\sqrt{Q}(f'h - fh') - fh'}{2M\rho\sqrt{f}h^2\sqrt{Q}(1 + \sqrt{Q})}\right](\boldsymbol{L} \cdot \boldsymbol{S}^*), \quad (5.7)$$

where $\bar{H}_{NS}$ is the Hamiltonian for a nonspinning particle, and where

$$Q = 1 + \frac{1}{h}\hat{\boldsymbol{P}}^2, \quad (5.8)$$

$$\hat{\boldsymbol{P}}^2 = \delta^{JK}\frac{P_JP_K}{m^2} = \delta^{jk}\frac{P_jP_k}{m^2}, \quad (5.9)$$

$$\boldsymbol{L} \cdot \boldsymbol{S}^* = \rho\epsilon^{IJK}n_IP_J\left(\frac{MS_K}{m}\right). \quad (5.10)$$

The quantity M in Eqs. (5.7) and (5.10) is introduced in anticipation of specialization to the Schwarzschild metric below. Since a spherically symmetric spacetime possesses an $SO(3)$ symmetry (associated with rotation of the x, y, z coordinates among themselves) that is shared by the internal tetrad space, one may accompany any coordinate rotation by the corresponding tetrad rotation, thereby preserving the functional form of the Hamiltonian (5.7), as well as the quantities (5.9) and (5.10). Thus one may meaningfully identify the vectors $L_I = \rho\epsilon^{IJK}n_JP_K$ and S_I (which really live in the tetrad internal space) with spacetime vectors L_i and S_i which transform accordingly under rotations of the coordinates x, y, z.

In the limit of flat spacetime, the Hamiltonian reduces to $\bar{H}_{NS}$ as expected, since the Cartesian components of the spin are all constants of motion. For the Schwarzschild spacetime in isotropic coordinates, we have

$$ds^2 = -\left[\frac{1 - M/(2\rho)}{1 + M/(2\rho)}\right]^2 dt^2$$
$$+ \left(1 + \frac{M}{2\rho}\right)^4(dx^2 + dy^2 + dz^2). \quad (5.11)$$

Substituting these explicit expressions for $f(\rho)$ and $h(\rho)$ in the Hamiltonian (5.7), one finds

$$\bar{H} = \bar{H}_{NS} + \frac{\psi^6}{\rho^3\sqrt{Q}(1 + \sqrt{Q})}$$
$$\times \left[1 - \frac{M}{2\rho} + 2\left(1 - \frac{M}{4\rho}\right)\sqrt{Q}\right](\boldsymbol{L} \cdot \boldsymbol{S}^*), \quad (5.12)$$

where $\psi = (1 + M/2\rho)^{-1}$.

B. Spherically symmetric spacetime in spherical coordinates

In this case, the metric takes the form

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2. \quad (5.13)$$

Note that the functions f and h appearing above are not the

same as in the isotropic case. However we follow here generally accepted notation conventions. The natural tetrad associated with this spacetime and coordinate system is

$$\tilde{e}_T^\mu = \frac{1}{\sqrt{f}} \delta_t^\mu, \tag{5.14a}$$

$$\tilde{e}_1^\mu = \frac{1}{\sqrt{h}} \delta_r^\mu, \tag{5.14b}$$

$$\tilde{e}_2^\mu = \frac{1}{r} \delta_\theta^\mu, \tag{5.14c}$$

$$\tilde{e}_3^\mu = \frac{1}{r\sin\theta} \delta_\phi^\mu. \tag{5.14d}$$

The metric (5.13) and the tetrad (5.14) then lead to the following result

$$E_{tAB} = \frac{f'}{2\sqrt{fh}} \delta_{[A}^T \delta_{B]}^1, \tag{5.15a}$$

$$E_{rAB} = 0, \tag{5.15b}$$

$$E_{\theta AB} = \frac{1}{\sqrt{h}} \delta_{[A}^1 \delta_{B]}^2, \tag{5.15c}$$

$$E_{\phi AB} = \frac{\sin\theta}{\sqrt{h}} \delta_{[A}^1 \delta_{B]}^3 + \cos\theta \delta_{[A}^2 \delta_{B]}^3. \tag{5.15d}$$

Next the computation of $\bar{\omega}_T$ and $\bar{\omega}_K$ yields

$$\bar{\omega}_T = -m(1 + \sqrt{Q}), \tag{5.16a}$$

$$\bar{\omega}_1 = \frac{1}{\sqrt{h}} P_r, \tag{5.16b}$$

$$\bar{\omega}_2 = \frac{1}{r} P_\theta, \tag{5.16c}$$

$$\bar{\omega}_3 = \frac{1}{r\sin\theta} P_\phi, \tag{5.16d}$$

Equations (5.15) and (5.16) then allow us to obtain F_μ^I. The result is

$$F_\mu^1 = \cos\theta \delta_\mu^\phi, \tag{5.17a}$$

$$F_\mu^2 = \left(\frac{f'}{2r\sin\theta\sqrt{fh}}\right)\left(\frac{\hat{P}_\phi}{1 + \sqrt{Q}}\right)\delta_\mu^t - \frac{\sin\theta}{\sqrt{h}}\delta_\mu^\phi, \tag{5.17b}$$

$$F_\mu^3 = -\left(\frac{f'}{2r\sqrt{fh}}\right)\left(\frac{\hat{P}_\theta}{1 + \sqrt{Q}}\right)\delta_\mu^t + \frac{1}{\sqrt{h}}\delta_\mu^\theta, \tag{5.17c}$$

where again $\hat{P}_i = P_i/m$. The Hamiltonian then follows immediately

$$\bar{H} = \bar{H}_{NS} + \frac{f'}{2(1 + \sqrt{Q})r\sqrt{fh}}\left(-\frac{1}{\sin\theta}\hat{P}_\phi S_2 + \hat{P}_\theta S_3\right)$$
$$- \sqrt{\frac{f}{Q}}\left(\frac{\cos\theta}{r^2\sin^2\theta}\hat{P}_\phi S_1 - \frac{\hat{P}_\phi S_2}{r^2\sqrt{h}\sin\theta} + \frac{\hat{P}_\theta S_3}{r^2\sqrt{h}}\right). \tag{5.18}$$

The spin terms in the first line of the Hamiltonian (5.18) are the spherical coordinate equivalent of the $\boldsymbol{L} \cdot \boldsymbol{S}^*$ terms of the isotropic Hamiltonian (5.7). The spin terms in the second line of Eq. (5.18) do not vanish in the flat space

limit $f = h = 1$, and therefore represent coordinate effects related to the fact that the components of the spin in spherical coordinates and its associated tetrad must evolve, even in the absence of spin-orbit coupling. Such spin terms in the Hamiltonian represent therefore a type of gauge terms.

Notice, however, that one could in principle eliminate these gauge terms in the Hamiltonian by picking a "Cartesian" tetrad, even though the coordinate system chosen is the spherical one. For example one could pick the "isotropic" tetrad (5.2), taking care of transforming the components of $\tilde{\boldsymbol{e}}_A$ from isotropic to spherical coordinates. In that case the spin degrees of freedom S_K, which live in the internal tetrad space, behave as the components of the spin in Cartesian coordinates, and in that case the flat space limit of the Hamiltonian should be free of gauge terms and should reduce to the nonspinning Hamiltonian.

For the Schwarzschild spacetime, $f = 1/h = 1 - 2M/r$, and we obtain

$$\bar{H} = \bar{H}_{NS} + \frac{M}{r^3(1 + \sqrt{Q})}\left(-\frac{1}{\sin\theta}\hat{P}_\phi S_2 + \hat{P}_\theta S_3\right)$$
$$- \frac{1 - 2M/r}{\sqrt{Q}}\left[\frac{\cos\theta}{r^2\sin^2\theta}\left(1 - \frac{2M}{r}\right)^{-1/2}\hat{P}_\phi S_1 \right.$$
$$\left. - \frac{\hat{P}_\phi S_2}{r^2\sin\theta} + \frac{\hat{P}_\theta S_3}{r^2}\right], \tag{5.19}$$

where

$$Q = 1 + \left(1 - \frac{2M}{r}\right)\hat{P}_r^2 + \frac{1}{r^2}\hat{P}_\theta^2 + \frac{1}{r^2\sin^2\theta}\hat{P}_\phi^2. \tag{5.20}$$

C. Kerr spacetime in Boyer-Lindquist coordinates

Not surprisingly the computation of the Hamiltonian is much more involved in Kerr spacetime, whose line element, in Boyer-Lindquist coordinates, is given by

$$ds^2 = \left(-1 + \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4aMr\sin^2\theta}{\Sigma}dtd\phi$$
$$+ \frac{\Lambda\sin^2\theta}{\Sigma}d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2, \tag{5.21}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta, \tag{5.22a}$$
$$\Delta = r^2 + a^2 - 2Mr, \tag{5.22b}$$
$$\varpi^2 = r^2 + a^2, \tag{5.22c}$$
$$\Lambda = \varpi^4 - a^2\Delta\sin^2\theta. \tag{5.22d}$$

For sake of shortening some further formulas, we also introduce the quantity

$$\rho^2 = r^2 - a^2\cos^2\theta. \tag{5.23}$$

Our choice for the reference tetrad is given by the "spheroidal" tetrad

$$\tilde{e}_\mu^T = \delta_\mu^t \sqrt{\frac{\Delta\Sigma}{\Lambda}}, \tag{5.24a}$$

$$\tilde{e}_\mu^1 = \delta_\mu^r \sqrt{\frac{\Sigma}{\Delta}}, \tag{5.24b}$$

$$\tilde{e}_\mu^2 = \delta_\mu^\theta \sqrt{\Sigma}, \tag{5.24c}$$

$$\tilde{e}_\mu^3 = -\frac{2aMr\sin\theta}{\sqrt{\Lambda\Sigma}} \delta_\mu^t + \delta_\mu^\phi \sin\theta \sqrt{\frac{\Lambda}{\Sigma}}, \tag{5.24d}$$

which reduces to the "spherical" tetrad (5.14) for $a = 0$. This tetrad then leads to the following components for the quantities $E_{\mu AB}$:

$$E_{tT1} = \frac{M\varpi^2\rho^2}{2\sqrt{\Lambda}\Sigma^2}, \tag{5.25a}$$

$$E_{tT2} = -\frac{a^2\sqrt{\Delta}Mr\cos\theta\sin\theta}{\sqrt{\Lambda}\Sigma^2}, \tag{5.25b}$$

$$E_{tT3} = 0, \tag{5.25c}$$

$$E_{t12} = 0, \tag{5.25d}$$

$$E_{t13} = \frac{a\sqrt{\Delta}M\rho^2\sin\theta}{2\sqrt{\Lambda}\Sigma^2}, \tag{5.25e}$$

$$E_{t23} = -\frac{aMr\varpi^2\cos\theta}{\sqrt{\Lambda}\Sigma^2}, \tag{5.25f}$$

$$E_{rT1} = 0, \tag{5.26a}$$

$$E_{rT2} = 0, \tag{5.26b}$$

$$E_{rT3} = -\frac{aM(2r^2\Sigma + \varpi^2\rho^2)\sin\theta}{2\sqrt{\Delta}\Lambda\Sigma}, \tag{5.26c}$$

$$E_{r12} = \frac{a^2\cos\theta\sin\theta}{2\sqrt{\Delta}\Sigma}, \tag{5.26d}$$

$$E_{r13} = 0, \tag{5.26e}$$

$$E_{r23} = 0, \tag{5.26f}$$

$$E_{\theta T1} = 0, \tag{5.27a}$$

$$E_{\theta T2} = 0, \tag{5.27b}$$

$$E_{\theta T3} = \frac{a^3\sqrt{\Delta}Mr\cos\theta\sin^2\theta}{\Lambda\Sigma}, \tag{5.27c}$$

$$E_{\theta 12} = \frac{\sqrt{\Delta}r}{2\Sigma}, \tag{5.27d}$$

$$E_{\theta 13} = 0, \tag{5.27e}$$

$$E_{\theta 23} = 0, \tag{5.27f}$$

$$E_{\phi T1} = -\frac{aM\sin^2\theta}{2\sqrt{\Lambda}\Sigma^2}(2r^2\Sigma + \varpi^2\rho^2), \tag{5.28a}$$

$$E_{\phi T2} = \frac{a^3\sqrt{\Delta}Mr\cos\theta\sin^3\theta}{\sqrt{\Lambda}\Sigma^2}, \tag{5.28b}$$

$$E_{\phi T3} = 0, \tag{5.28c}$$

$$E_{\phi 12} = 0, \tag{5.28d}$$

$$E_{\phi 13} = \frac{\sqrt{\Delta}\sin\theta}{2\sqrt{\Lambda}\Sigma^2}(r\Sigma^2 - a^2M\rho^2\sin^2\theta), \tag{5.28e}$$

$$E_{\phi 23} = \frac{(2Mr\varpi^4 + \Delta\Sigma^2)\cos\theta}{2\sqrt{\Lambda}\Sigma^2}, \tag{5.28f}$$

while $\bar{\omega}_T$ and $\bar{\omega}_K$ are easily found to be

$$\bar{\omega}_T = -m(1 + \sqrt{Q}), \tag{5.29a}$$

$$\bar{\omega}_1 = P_r\sqrt{\frac{\Delta}{\Sigma}}, \tag{5.29b}$$

$$\bar{\omega}_2 = P_\theta\sqrt{\frac{1}{\Sigma}}, \tag{5.29c}$$

$$\bar{\omega}_3 = P_\phi\frac{\sqrt{\Sigma}}{\sin\theta\sqrt{\Lambda}}, \tag{5.29d}$$

where

$$\begin{aligned} Q &= 1 + \gamma^{ij}\hat{P}_i\hat{P}_j \\ &= 1 + \frac{\Delta}{\Sigma}\hat{P}_r^2 + \frac{1}{\Sigma}\hat{P}_\theta^2 + \frac{\Sigma}{\Lambda\sin^2\theta}\hat{P}_\phi^2, \end{aligned} \tag{5.30}$$

with $\hat{P}_i \equiv P_i/m$. The coefficients F_μ^K are finally given by

$$F_t^1 = 2aMr\cos\theta\left[\frac{a\sqrt{\Delta}}{\Lambda(1 + \sqrt{Q})\Sigma^{3/2}}\hat{P}_\phi - \frac{\varpi^2}{\sqrt{\Lambda}\Sigma^2}\right], \tag{5.31a}$$

$$F_r^1 = -\frac{aM(2r^2\Sigma + \varpi^2\rho^2)\sin\theta}{\sqrt{\Delta}\Lambda(1 + \sqrt{Q})\Sigma^{3/2}}\hat{P}_\theta, \tag{5.31b}$$

$$F_\theta^1 = \frac{2a^3Mr\cos\theta\sin^2\theta}{\Lambda(1 + \sqrt{Q})}\sqrt{\frac{\Delta}{\Sigma^3}}\hat{P}_\theta, \tag{5.31c}$$

$$F_\phi^1 = \cos\theta\left[\frac{2Mr\varpi^4 + \Delta\Sigma^2}{\sqrt{\Lambda}\Sigma^2} - \frac{2a^3Mr\sin^2\theta}{\Lambda(1 + \sqrt{Q})}\sqrt{\frac{\Delta}{\Sigma^3}}\hat{P}_\phi\right], \tag{5.31d}$$

$$F_t^2 = \frac{M\rho^2}{\Sigma^2}\left[\frac{\varpi^2\sqrt{\Sigma}}{\Lambda(1+\sqrt{Q})\sin\theta}\hat{P}_\phi - a\sqrt{\frac{\Delta}{\Lambda}}\sin\theta\right], \quad (5.32\text{a})$$

$$F_r^2 = \frac{aM(2r^2\Sigma + \rho^2\varpi^2)\sin\theta}{\Lambda(1+\sqrt{Q})\Sigma^{3/2}}\hat{P}_r, \quad (5.32\text{b})$$

$$F_\theta^2 = -\frac{2a^3Mr\Delta\cos\theta\sin^2\theta}{\Lambda(1+\sqrt{Q})\Sigma^{3/2}}\hat{P}_r, \quad (5.32\text{c})$$

$$F_\phi^2 = -\sin\theta\left[\frac{aM(2r^2\Sigma + \varpi^2\rho^2)}{\Lambda(1+\sqrt{Q})\Sigma^{3/2}}\hat{P}_\phi \right.$$
$$\left. + \sqrt{\frac{\Delta}{\Lambda}}\left(\frac{r\Sigma^2 - a^2M\rho^2\sin^2\theta}{\Sigma^2}\right)\right], \quad (5.32\text{d})$$

$$F_t^3 = -\frac{M}{\sqrt{\Lambda}(1+\sqrt{Q})\Sigma^{5/2}}(\rho^2\varpi^2\hat{P}_\theta + 2a^2r\Delta\sin\theta\cos\theta\hat{P}_r),$$
$$(5.33\text{a})$$

$$F_r^3 = \frac{a^2\cos\theta\sin\theta}{\sqrt{\Delta}\Sigma}, \quad (5.33\text{b})$$

$$F_\theta^3 = \frac{\sqrt{\Delta}r}{\Sigma}, \quad (5.33\text{c})$$

$$F_\phi^3 = \frac{aM\sin^2\theta}{\sqrt{\Lambda}(1+\sqrt{Q})\Sigma^{5/2}}[2a^2r\Delta\cos\theta\sin\theta\hat{P}_r$$
$$+ (2r^2\Sigma + \rho^2\varpi^2)\hat{P}_\theta]. \quad (5.33\text{d})$$

Inserting these results into the Hamiltonian (4.44), a long but straightforward computation yields

$$\bar{H} = \bar{H}_{\text{NS}} + \bar{H}_I S^I, \quad (5.34)$$

where

$$\bar{H}_1 = -\left[\frac{\sqrt{\Delta}\cos\theta}{\Lambda^2\sqrt{\Sigma}\sqrt{Q}(1+\sqrt{Q})\sin^2\theta}\right][(1+\sqrt{Q})(\Delta\Sigma^2 + 2Mr\varpi^4) + 2a^2Mr\varpi^2\sqrt{Q}\sin^2\theta]\hat{P}_\phi$$
$$+ \left[\frac{aM\Delta(2r^2\Sigma + \varpi^2\rho^2)\sin\theta}{\Lambda^{3/2}\Sigma^2\sqrt{Q}(1+\sqrt{Q})}\right]\hat{P}_r\hat{P}_\theta + \left[\frac{2a^3Mr\Delta\cos\theta\sin^2\theta}{\Lambda^{3/2}\Sigma\sqrt{Q}(1+\sqrt{Q})}\right]\left(1 + \sqrt{Q} + \frac{2\Sigma}{\Lambda\sin^2\theta}\hat{P}_\phi^2 + \frac{\Delta}{\Sigma}\hat{P}_r^2\right), \quad (5.35)$$

$$\bar{H}_2 = \left[\frac{\Delta(1+\sqrt{Q})(r\Sigma^2 - a^2M\rho^2\sin^2\theta) - M\sqrt{Q}(\rho^2\varpi^4 - 4a^2Mr^3\sin^2\theta)}{\Lambda^2\sqrt{\Sigma}\sqrt{Q}(1+\sqrt{Q})\sin\theta}\right]\hat{P}_\phi + \left[\frac{2a^3Mr\Delta^{3/2}\cos\theta\sin^2\theta}{\Lambda^{3/2}\Sigma^2\sqrt{Q}(1+\sqrt{Q})}\right]\hat{P}_r\hat{P}_\theta$$
$$+ \left[\frac{aM\sqrt{\Delta}(2r^2\Sigma + \varpi^2\rho^2)\sin\theta}{\Lambda^{3/2}\Sigma\sqrt{Q}(1+\sqrt{Q})}\right]\left(1 + \sqrt{Q} + \frac{2\Sigma}{\Lambda\sin^2\theta}\hat{P}_\phi^2 + \frac{1}{\Sigma}\hat{P}_\theta^2\right), \quad (5.36)$$

$$\bar{H}_3 = -\left[\frac{a^2\Delta\cos\theta\sin\theta}{(\Lambda\Sigma)^{3/2}\sqrt{Q}(1+\sqrt{Q})}\right](\Lambda + \sqrt{Q}\Delta\Sigma)\hat{P}_r - \left[\frac{r\Lambda\Delta + \varpi^2\Sigma\sqrt{Q}(r\Delta - M(r^2 - a^2))}{(\Lambda\Sigma)^{3/2}\sqrt{Q}(1+\sqrt{Q})}\right]\hat{P}_\theta$$
$$- \left[\frac{aM\sqrt{\Delta}}{\Lambda^2\Sigma\sqrt{Q}(1+\sqrt{Q})}\right][2a^2r\Delta\cos\theta\sin\theta\hat{P}_r + (2r^2\Sigma + \varpi^2\rho^2)\hat{P}_\theta]\hat{P}_\phi. \quad (5.37)$$

Setting $a = 0$ in this result and noting that for $a = 0$ one has $\Lambda = r^4$, $\Sigma = r^2$, and $\Delta = r(r - 2M)$, it is easy to check that this Hamiltonian reduces to the Schwarzschild result (5.19) in the nonspinning case.

VI. COMPARING THE HAMILTONIAN IN THE GENERALIZED NEWTON-WIGNER SSC WITH THE ADM CANONICAL HAMILTONIAN OF PN THEORY

In this section we specialize our results to the case of the Kerr spacetime, but this time using ADM transverse traceless coordinates. By expanding our Hamiltonian (4.42) following the prescription of PN theory, we verify explicitly that we recover the known test particle limit results of the Arnowitt-Deser-Misner (ADM) canonical Hamiltonian computed within PN theory alone. The latter is currently known through 2.5PN order for the terms linear in the spin [17], and through 3PN order for the terms quadratic in the spin [18–20,23–25,42]. We cannot reproduce the PN couplings of the test particle's spin with itself because the MPP equations, as we have already stressed, are only valid to linear order in the particle's spin. In addition we also obtain the terms linear in the spins at 3.5PN order of the canonical ADM Hamiltonian in the test particle limit. Those contributions have never been computed before.

In order to make the PN expansion as clear as possible, we restore factors of c in this section. However these factors of c must be viewed purely as dimensionless PN bookkeeping parameters, and as such we are still formally employing geometric units.

First, let us introduce the Kerr metric in ADM-TT coordinates [21],

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_i\beta^i & -\beta_i \\ -\beta_j & \gamma_{ij} \end{pmatrix}, \tag{6.1}$$

$$g^{\mu\nu} = \begin{pmatrix} -1/\alpha^2 & -\beta^i/\alpha^2 \\ -\beta^j/\alpha^2 & \gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{pmatrix}, \tag{6.2}$$

where $\gamma^{ik}\gamma_{kj} = \delta^i_j$ and $\beta^i = \gamma^{ik}\beta_k$. Defining $n^i \equiv x^i/r$ and introducing a dimensionless three-vector $\boldsymbol{\chi}$ defined as

$$\boldsymbol{\chi} \equiv \frac{\boldsymbol{S}_{\text{Kerr}}}{M^2}, \tag{6.3}$$

where M is the mass of the Kerr black hole and $\boldsymbol{S}_{\text{Kerr}}$ its spin, the lapse function is given by [21]

$$\begin{aligned} \alpha = c &- \frac{M}{rc} + \frac{1}{2}\frac{M^2}{r^2c^3} - \frac{1}{4}\frac{M^3}{r^3c^5} + \frac{1}{8}\frac{M^4}{r^4c^7} \\ &+ \frac{1}{2}\frac{M^3[3(\boldsymbol{\chi}\cdot\boldsymbol{n})^2 - \chi^2]}{r^3c^5} + \frac{1}{2}\frac{M^4[5\chi^2 - 9(\boldsymbol{\chi}\cdot\boldsymbol{n})^2]}{r^4c^7} \\ &+ \mathcal{O}(9), \end{aligned} \tag{6.4}$$

the shift vector is given by

$$\begin{aligned} \beta^i = &\left\{ \frac{2M^2}{r^2c^3} - \frac{6M^3}{r^3c^5} + \frac{21}{2}\frac{M^4}{r^4c^7} \right. \\ &\left. - \frac{M^4[5(\boldsymbol{\chi}\cdot\boldsymbol{n})^2 - \chi^2]}{r^4c^7} \right\}\epsilon^{ijk}\chi_j n_k + \mathcal{O}(9), \end{aligned} \tag{6.5}$$

and the spatial metric γ^{ij} is given by

$$\gamma^{ij} = \tfrac{1}{A}\delta^i_j - \delta^{ik}\delta^{jl}h^{\text{TT}}_{kl} + \mathcal{O}(10), \tag{6.6}$$

where $\epsilon_{ijk} = \epsilon^{ijk}$ is the Levi-Civita symbol (with $\epsilon_{123} = \epsilon^{123} = 1$), and where the quantities A and h^{TT}_{kl} are defined as

$$\begin{aligned} A = &\left(1 + \frac{M}{2rc^2}\right)^4 + \frac{M^3[\chi^2 - 3(\boldsymbol{\chi}\cdot\boldsymbol{n})^2]}{r^3c^6} + \frac{1}{2}\frac{M^4\chi^2}{r^4c^8} \\ &- \frac{3M^4(\boldsymbol{\chi}\cdot\boldsymbol{n})^2}{r^4c^8}, \end{aligned} \tag{6.7}$$

$$\begin{aligned} h^{\text{TT}}_{ij} = &-\frac{7}{2}\frac{M^4\chi^2}{r^4c^8}\delta_{ij} + 7\frac{M^4(\boldsymbol{\chi}\cdot\boldsymbol{n})^2}{r^4c^8}\delta_{ij} + 7\frac{M^4\chi^2 n_i n_j}{r^4c^8} \\ &- 21\frac{M^4(\boldsymbol{\chi}\cdot\boldsymbol{n})^2 n_i n_j}{r^4c^8} + \frac{7}{2}\frac{M^4\chi_i\chi_j}{r^4c^8}. \end{aligned} \tag{6.8}$$

For the reference tetrad appearing in the Hamiltonian, we chose

$$\tilde{e}^T_\mu = \delta^t_\mu \alpha, \tag{6.9a}$$

$$\tilde{e}^\mu_I = \frac{\delta^\mu_I}{\sqrt{A}} + \mathcal{O}(8). \tag{6.9b}$$

It turns out, however, that we only need the spatial triad $\tilde{\boldsymbol{e}}_I$ through order $1/c^7$ for our purposes. (This makes the spatial triad very simple because the spatial metric is diagonal at that order).

The canonical spin S^I appearing in the Hamiltonian (4.42) scales as the physical spin of the test particle. To conform with standard power counting in PN theory, this spin variable carries a power of $1/c$. Therefore when restoring the factors of $1/c$ for the purpose of PN bookkeeping, we make the replacement[13]

$$S^I \to \frac{S^I}{c}. \tag{6.10}$$

Finally we define the orbital angular momentum as

$$L^i \equiv \epsilon^{ijk}x^j P_k, \tag{6.11}$$

and rescaled momentum and spin as

$$\hat{\boldsymbol{P}} = \frac{1}{m}\boldsymbol{P}, \tag{6.12a}$$

$$\boldsymbol{S}^* = \frac{M}{m}\boldsymbol{S}, \tag{6.12b}$$

which are useful to abbreviate formulas below. With these tools it is straightfroward to expand the Hamiltonian (4.42) in powers of $1/c$ as

$$\begin{aligned} \bar{H} = &mc^2 + \bar{H}_{\text{N}} + \frac{1}{c^2}\bar{H}_{\text{1PN}} + \frac{1}{c^3}\bar{H}_{\text{1.5PN}} + \frac{1}{c^4}\bar{H}_{\text{2PN}} \\ &+ \frac{1}{c^5}\bar{H}_{\text{2.5PN}} + \frac{1}{c^6}\bar{H}_{\text{3PN}} + \frac{1}{c^7}\bar{H}_{\text{3.5PN}} + \mathcal{O}(8) + \mathcal{O}(S^2), \end{aligned} \tag{6.13}$$

where

$$\bar{H}_{\text{N}} = m\left(\frac{\hat{P}^2}{2} - \frac{M}{r}\right), \tag{6.14}$$

$$\bar{H}_{\text{1PN}} = m\left(-\frac{\hat{P}^4}{8} - \frac{3M}{2r}\hat{P}^2 + \frac{M^2}{2r^2}\right), \tag{6.15}$$

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[13]This is appropriate if the particle is a black hole or a rapidly rotating compact star. In the black hole case, $S = am^2/c$, with a ranging from 0 to 1 [see Eq. (6.3)]. In the rapidly spinning star case one has $S = mv_{\text{rot}}R \sim mcR_s \sim m^2/c$ (where we have assumed that the rotational velocity v_{rot} is comparable to c and that the stellar radius R is of order of the Schwarzschild radius $R_s = m/c^2$).

$$\bar{H}_{1.5\text{PN}} = \frac{1}{r^3}\left(2\boldsymbol{S}_{\text{Kerr}} + \frac{3}{2}\boldsymbol{S}^*\right) \cdot \boldsymbol{L}, \qquad (6.16)$$

$$\bar{H}_{2\text{PN}} = m\left(\frac{\hat{\boldsymbol{P}}^6}{16} + \frac{5M}{8r}\hat{\boldsymbol{P}}^4 + \frac{5M^2}{2r^2}\hat{\boldsymbol{P}}^2 - \frac{M^3}{4r^3}\right)$$
$$+ \frac{m}{2Mr^3}(3n_{ij} - \delta_{ij})S^i_{\text{Kerr}}(S^j_{\text{Kerr}} + 2S^*_j), \qquad (6.17)$$

$$\bar{H}_{2.5\text{PN}} = \frac{1}{r^3}\left[-\frac{M}{r}(6\boldsymbol{S}_{\text{Kerr}} + 5\boldsymbol{S}^*) - \frac{5}{8}\hat{\boldsymbol{P}}^2\boldsymbol{S}^*\right] \cdot \boldsymbol{L}, \qquad (6.18)$$

$$\bar{H}_{3\text{PN}} = m\left(-\frac{5\hat{\boldsymbol{P}}^8}{128} - \frac{7M}{16r}\hat{\boldsymbol{P}}^6 - \frac{27M^2}{16r^2}\hat{\boldsymbol{P}}^4 - \frac{25M^3}{8r^3}\hat{\boldsymbol{P}}^2 + \frac{M^4}{8r^4}\right)$$
$$+ \frac{m}{2Mr^3}S^{ij}_{\text{Kerr}}\left[\frac{3}{2}\hat{\boldsymbol{P}}^2(3n_{ij} - \delta_{ij}) - \frac{M}{r}(9n_{ij} - 5\delta_{ij})\right]$$
$$+ \frac{3mn_{ij}}{2Mr^3}[2\hat{P}^i S^k_{\text{Kerr}}\hat{P}^{[j}S^{*k]} - (\hat{\boldsymbol{P}} \times \boldsymbol{S}^*)^i(\hat{\boldsymbol{P}} \times \boldsymbol{S}_{\text{Kerr}})^j]$$
$$+ \frac{6m}{r^4}S^{*i}S^j_{\text{Kerr}}(\delta_{ij} - 2n_{ij}), \qquad (6.19)$$

where $n_{ij} = n_i n_j$ and $S^{ij}_{\text{Kerr}} = S^i_{\text{Kerr}}S^j_{\text{Kerr}}$. The nonspinning terms in the Hamiltonian (6.13) coincide with the corresponding terms computed in PN theory in the test particle limit [32]; the linear terms in the spins at 1.5PN and 2.5PN order agree with the terms computed in the test particle limit in PN theory [16,17]; the terms quadratic in the spin of the larger body coincide with what derived in PN theory at 2PN [33] and 3PN order [19,20]. We find that the contributions at 3.5PN are given by

$$\bar{H}_{3.5\text{PN}} = \frac{9m}{2M^2r^4}(\boldsymbol{S}_{\text{Kerr}} \cdot \boldsymbol{n})(\boldsymbol{S}^* \times \boldsymbol{S}_{\text{Kerr}}) \cdot \hat{\boldsymbol{P}} - \frac{1}{4M^2r^5}$$
$$\times [5(\boldsymbol{S}_{\text{Kerr}} \cdot \boldsymbol{n})^2 - \boldsymbol{S}^2_{\text{Kerr}}](9\boldsymbol{S}^* + 4\boldsymbol{S}_{\text{Kerr}}) \cdot \boldsymbol{L}$$
$$+ \frac{21M^2}{2r^5}\boldsymbol{S}_{\text{Kerr}} \cdot \boldsymbol{L}$$
$$+ \left(\frac{7}{16r^3}\hat{\boldsymbol{P}}^4 + \frac{27M}{8r^4}\hat{\boldsymbol{P}}^2 + \frac{105}{8}\frac{M^2}{r^5}\right)(\boldsymbol{S}^* \cdot \boldsymbol{L}). \qquad (6.20)$$

While the terms of this expression which are cubic in the spins (S^3_{Kerr} and $S^2_{\text{Kerr}}S^*$) have already been calculated for generic mass ratios in Refs. [21,22], with which we agree in the test particle limit, the terms linear in the spins are, as far as we are aware, a new result. Of course, because our Hamiltonian is only valid at linear order in the particle's spin, this result is still incomplete as it does not include terms $(S^*)^3$ and $S_{\text{Kerr}}(S^*)^2$, which are still unknown.

Finally, we stress that at leading order our generalized NW SSC reduces to the so-called baryonic SSC of Refs. [11,16]. In fact, at leading order $p_i \approx m\upsilon^i$, $p_t \approx$

$-mc^2$, and $\tilde{e}^T_\mu \approx c\delta^t_\mu$, which yields $\omega_t \approx -2mc^2$ and $\omega_i \approx m\upsilon^i$. Therefore, our generalized NW SSC becomes

$$S^{it} \approx \frac{1}{2}S^{ij}\frac{\upsilon^j}{c^2}, \qquad (6.21)$$

in agreement with Refs. [11,16].

VII. CONCLUSIONS

In summary: starting from the Lagrangian put forward in Ref. [27] building on the classical work of Ref. [30] on the relativistic spherical top dynamics, we derived the unconstrained Hamiltonian for a spinning test particle in a curved spacetime, at linear order in the particle's spin. The equations of motion for this Hamiltonian coincide with the MPP equations of motion. The latter are well known to describe the motion and spin-precession of a test particle, but are expressed in terms of the spin tensor $S^{\mu\nu}$ carrying 6 degrees of freedom. In order to eliminate three of these degrees of freedom (which can be shown to correspond to the choice of the point internal to the spinning body whose worldline is followed [29]), we impose the so-called NW spin supplementary condition, suitably generalized to curved spacetime. Using the formalism of Dirac brackets [39] we computed the Hamiltonian and phase space algebra of the constrained system. In particular, we showed that, in a generic curved spacetime, the resulting phase space algebra is canonical, i.e. it has the standard sympletic structure for the set of dynamical variables $(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{S})$, at least at linear order in the particle's spin. As a consequence, the equations of motion can be derived from our constrained Hamiltonian by means of the usual well-known Hamilton equations.

As an application, making specific choices of the tetrad field, we computed explicitly the constrained Hamiltonian for a spherically symmetric spacetime, both in isotropic and in spherical coordinates, as well as for the Kerr spacetime in Boyer-Lindquist coordinates. We notice that different choices of the tetrad field would lead to different Hamiltonians connected by canonical transformations. Also, we expanded our Hamiltonian in PN orders and showed explicitly that it reduces to the test particle limit of the ADM canonical Hamiltonian computed in PN theory [16,17,19,20,33]. Notably, we recover the known spin-orbit couplings through 2.5PN order and the spin-spin couplings of type $S_{\text{Kerr}}S$ through 3PN order, S_{Kerr} being the spin of the Kerr spacetime. Our method allows one to compute the PN Hamiltonian, in the test particle limit and at linear order in the particle's spin, at *any* PN order, and as an application we computed it at 3.5PN order.

Another application of this work will be developed in a follow-up paper, where we will use our Hamiltonian to build a new effective-one-body Hamiltonian for spinning bodies [31–34]. Such work will be important to build templates for the search of gravitational waves with ground and space-based detectors, as it will permit taking full

advantage of the analytical and numerical treatment of the dynamics of spinning black hole binaries throughout the inspiral, merger, and ringdown phases.

ACKNOWLEDGMENTS

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