One-loop corrections to the photon propagator in curved-space QED

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We calculate and discuss the one-loop corrections to the photon sector of QED interacting to a background gravitational field. At high energies the fermion field can be taken as massless and the quantum terms can be obtained by integrating conformal anomaly. We present a covariant local expression for the corresponding effective action, similar to the one obtained earlier for the gravitational sector. At the moderate energies the quantum terms can be obtained through the heat-kernel method. In this way we derive the exact one-loop β function for the electric charge in the momentum subtraction scheme and explore both massless and large-mass limits. The relation between the two approaches is shown and the difference discussed in view of the possible applications to cosmology and astrophysics.

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I. INTRODUCTION

The interaction of electromagnetic field with gravity is an important subject due to various astrophysical and cosmological applications. There are many publications on the subject and, in particular, many interesting works devoted to the quantum corrections in the electromagnetic sector of the theory. In the early publications [1,2], direct calculations were performed through the Feynman diagrams and Schwinger source method and later on, starting from [3], by using different modifications of the heat-kernel approach [4–7]. The diagrams were also used in the traditional style calculations [8,9] in flat space-time and using a modified approach admitting a generalization to the curved space-time [10,11]. A general consideration, including one- and two-loop calculations and taking into account the temperature effects, has been given in [13].

One of the most important features of quantum corrections to the action of electromagnetic field is that they break the conformal invariance which the theory possesses at the classical level. Let us start by presenting a short list of the cosmological and astrophysical situations where the violation of conformal symmetry can be relevant. (1) The breaking of conformal symmetry changes the equation of state of the radiation, that can be relevant in the radiation-dominated epoch in the early Universe. The modified equation of state for the radiation enables one to construct interesting cosmological models (see, e.g., [14] and references therein), including the ones [15] based on the quantum corrections of [3]. The modified equation of state can produce the change in the law of expansion of the Universe, entropy production, and other phenomena. Also, this effect can slightly affect the redshift dependence of both energy density of radiation and cosmic microwave background (CMB) temperature. Because of the growing precision of astrophysical experiments, at some point this feature of quantum corrections can be relevant. (2) As a particular aspect of the previous point, due to the broken conformal symmetry the rate of creation of the photons in the reheating period after inflation may be affected by quantum corrections, leading to the potentially observable consequences. (3) The violation of conformal symmetry is needed for the creation of initial seeds of magnetic field of the galaxies, at the epoch of structure formation [16]. There is a well-known attempt to explain this violation by quantum effects similar to conformal anomaly [17], during the inflationary epoch (see also [18]). It would be rather interesting to have a more complete understanding of the field theoretical mechanisms behind such violation, including the ones which can occur at much later epochs. The problem was further discussed, e.g., in [19,20], and one can also see [21,22] for the recent reviews of the possible origin of the initial seeds of cosmic magnetic fields and related subjects. (4) Furthermore, the quantum corrections to the photon propagator can modify the position of the pole and hence produce the situation when the electromagnetic wave propagates with the velocity which is slightly different from the one in the purely classical case. A similar effect can take place in curved space and is sometimes characterized as a superluminal motion [23,24] (see [25] for the recent review). In particular, this effect may have a significant impact on the behavior of light in the vicinity of the black hole [26–28]. Indeed, a similar effect may also take place due to the presence of the boundaries and, in general, because of the macroscopic conditions [29]. Let us remark that the light propagation can be affected also by the nonlinear terms [30] which can result from the quantum corrections and by the classical and/or quantum interac-

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tions to external gravity [31] or to other fields such as k-essence [32]. It is obvious that the conformal symmetry forbids most of the possible changes in the wave equations and therefore the detailed study of the conformal symmetry breaking is relevant for this issue too.

If thinking about conformal symmetry, the first remarkable observation is that in the realistic theories such as QED or the standard model (SM), and also in its generalizations such as supersymmetric standard model or grand unified theories, the massless and conformal invariant electromagnetic field couples to the other fields, which are all massive and therefore conformally noninvariant. In the present paper we shall concentrate on the simplest case of QED and try to present the most general view of the violation of conformal symmetry within this theory. However, there is no qualitative difference from other mentioned theories, which can be also considered elsewhere.

One can formulate the two main questions:

(1) What is the mechanism of violation of the local conformal symmetry in the intermediate energies and how one can link its violations at the ultraviolet (UV) and infrared (IR) limit. Is it possible to derive such relations? This is not a purely technical question, because during its evolution the Universe is passing different phases and it is desirable to know how the conformal symmetry is violated not only at the inflation epoch or at the present epoch, but also around the recombination epoch or at the time period when the cosmic structure starts to form. The main purpose of the present paper is to explore this issue.

(2) To which extent the finite quantum corrections and, in particular, the violation of local conformal symmetry are universal? In other words, do we have some ambiguity in the quantum terms? Answering the last question is of course very significant, because the effective action, in general, is not a uniquely defined object. Usually, it depends on the choice of parametrization of quantum fields, in particular, on the gauge fixing choice, on the calculational schemes, regularization, renormalization, etc.

Our purpose is to derive the most general expression for the one-loop quantum correction to the electromagnetic sector of QED in curved space-time. As we have already mentioned above, the physical situations for such quantum corrections in the early Universe and at the later period are very different. In the first case the fermions can be treated as almost massless. In this case the mechanism for the violation of conformal symmetry is the well-known conformal anomaly. The advantage of conformal anomaly as a method of deriving quantum corrections to the classical effects is its simplicity, direct relation to the UV divergences, and consequent universality. At the opposite end of the energy scale the masses of the virtual fermions are much greater than the energies of the real photons or, equivalently, the energies of the external tails in the loops of massive fermions. In this case we observe the decoupling of the quantum contributions of the massive fields, according to the Appelquist and Carazzone theorem [33]. The violation of conformal symmetry still exists, but it is related to the remnant higher derivative terms in the effective action, which are quadratically suppressed by the fermion masses. Also, in curved space-time, there may be curvature-dependent terms which can break down the local conformal symmetry. Since we intend to study the curved-space theory, we will always concentrate our attention on the local conformal symmetry.

The paper is organized as follows. In the next section we consider the simplified version of QED with a massless fermion and obtain the noncovariant and also nonlocal covariant forms of the anomaly-induced effective action. Some part of this section was previously known, but we present these results for the sake of completeness, and also to establish the most general local representation for the covariant expression. In Sec. III we start to consider a massive case and calculate the first three coefficients of the Schwinger-DeWitt expansion in an arbitrary space-time dimension D. Even though the focus of our interest is mainly on the $4D$ case, it proves useful to have a more general D-dimensional result. The calculations are performed in two different schemes, such that we can try the limits of universality of the quantum corrections. The scheme dependence of the one-loop effective action is explored in a parallel paper [34]. In Sec. IV we use the heat-kernel method to calculate the complete one-loop form factors for the electromagnetic field sector. In the same section we consider the UV limit and establish the relation of the massless limit for the one-loop corrections for the massive case with the ones derived via conformal anomaly. In Sec. V we consider the renormalization group equations for the massive case and derive different forms of the low-energy decoupling law corresponding to the distinct calculational schemes. In Sec. VI we analyze the corresponding running of the effective charge and compare it to the one in the minimal-subtraction renormalization scheme. Finally, in the last section we present some discussions and draw our conclusions.

II. ANOMALY-INDUCED ACTION FOR THE METRIC AND ELECTROMAGNETIC FIELDS BACKGROUND

Let us start from a brief survey of massless conformal QED. The theory should be formulated in curved space-time and hence the action depends on the electromagnetic potential A_μ, Dirac spinor field ψ, and on the external metric $g_{\mu\nu}$. As far as we are interested in the local conformal symmetry, one of the useful parametrizations of the metric is

$$g_{\mu\nu} = \bar{g}_{\mu\nu} \cdot e^{2\sigma}, \qquad \sigma = \sigma(x), \tag{1}$$

where $\bar{g}_{\mu\nu}$ is the fiducial metric with fixed determinant. For example, in the case of the cosmological metric, using spherical coordinates, we have

$$\bar{g}_{\mu\nu} = \mathrm{diag}\left(1, -\frac{1}{1-kr^2}, -r^2\sin^2\theta, -r^2\right).$$

Separating $\sigma(x)$ in (1) proves to be a useful tool, especially because of the relation

$$\frac{2}{\sqrt{g}} g_{\mu\nu} \frac{\delta A[g_{\mu\nu}]}{\delta g_{\mu\nu}} = \frac{e^{-4\sigma}}{\sqrt{\bar{g}}} \frac{\delta A[\bar{g}_{\mu\nu} e^{2\sigma}]}{\delta\sigma}\Bigg|_{\bar{g}_{\mu\nu}\to g_{\mu\nu}, \sigma\to 0}, \tag{2}$$

which is valid for any functional $A[g_{\mu\nu}]$ of the metric and maybe other fields. If we replace the action of some theory in curved space at the place of $A[g_{\mu\nu}]$, then the left-hand side of the above relation is nothing else but the trace of the corresponding energy-momentum tensor, T^μ_μ. In order to remove the effect of other field variables, it is sufficient to use the corresponding equations of motion.

The vanishing trace of the energy-momentum tensor implies that the conformal factor of the metric decouples from the matter. However, the situation changes dramatically if we take quantum effects onto account. In the last case the corresponding theoretical phenomenon is called the trace anomaly [35] (see also [36,37] for review and many further references).

The classical action of electromagnetic field is

$$S_{\mathrm{em}} = -\frac{1}{4}\int d^4x\sqrt{g} F_{\mu\nu}F^{\mu\nu} \tag{3}$$

and possesses local conformal invariance, which means that this action does not change under simultaneous transformation of the metric and of the vector A_μ, namely

$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu}e^{2\sigma}, \qquad A_\mu \to A'_\mu = A_\mu. \tag{4}$$

Let us note that the difference between conformal weight and dimension for the vector field is due to the vector field definition in curved space-time,

$$A_\mu = A_b e^b_\mu, \qquad e^b_\mu e^a_\nu \eta_{ab} = g_{\mu\nu}, \qquad e^b_\mu e^a_\nu g^{\mu\nu} = \eta^{ab}. \tag{5}$$

We are interested in the corrections to the action (3) due to quantum effects of the fermion

$$S_f = i\int d^4x\sqrt{g}\{\bar{\psi}\gamma^\mu(\nabla_\mu - ieA_\mu)\psi - im\bar{\psi}\psi\}. \tag{6}$$

The conformal transformation rule for spinors is

$$\psi \to \psi' = \psi e^{-3\sigma/2}, \qquad \bar{\psi} \to \bar{\psi}' = \bar{\psi}e^{-3\sigma/2}.$$

The metric is always transformed like in (4). Indeed the action (6) is conformal invariant only when the spinor mass is zero, $m = 0$. All those fermions which couple to the electromagnetic field as in (6) are massive, however, the relevance of the mass terms depends on the energy scale. For instance, if we are interested in the quantum effects of fermions close to the inflationary epoch, the kinetic energy of the real fermions and (more important) of photons is much greater than the mass of the fermions. In this situation taking a massless spinor field is a legitimate approximation, so let us start from this case and take $m = 0$.

The derivation of conformal anomaly in the presence of background metric and electromagnetic field has been discussed before [35,36], and we can use the known result. The conformal anomaly can be used to construct the equation for the finite part of the one-loop correction to the effective action of the background metric and electromagnetic potential,

$$T^\mu_\mu = \frac{2}{\sqrt{g}} g_{\mu\nu}\frac{\delta\bar{\Gamma}_{\mathrm{ind}}}{\delta g_{\mu\nu}} = \frac{1}{(4\pi)^2}(wC^2 + bE + c\Box R + \tilde{\beta}F^2_{\mu\nu}), \tag{7}$$

where

$$C^2 = C^2_{\mu\nu\alpha\beta} = R^2_{\mu\nu\alpha\beta} - 2R^2_{\alpha\beta} + (1/3)R^2$$

is the square of the Weyl tensor and

$$E = R^2_{\mu\nu\alpha\beta} - 4R^2_{\alpha\beta} + R^2$$

is the integrand of the Gauss-Bonnet topological term. The coefficients ω, b, c depend on the number of scalar N_s, fermion N_f, and massless vector N_v fields as

$$\begin{aligned} \omega &= \frac{1}{120}N_s + \frac{1}{20}N_f + \frac{1}{10}N_v, \\ b &= -\frac{1}{360}N_s - \frac{11}{360}N_f - \frac{31}{180}N_v, \\ c &= \frac{1}{180}N_s + \frac{1}{30}N_f - \frac{1}{10}N_v. \end{aligned} \tag{8}$$

Also, $\tilde{\beta}$ depends on the number of charged scalars (in case of scalar QED) and spinors. As far as the massless approximation can be applied in the very early Universe, the number of fields which contribute to these coefficients is not necessarily restricted by the QED framework.

The solution of Eq. (7) is straightforward [38] (see also generalizations for the theory with torsion [39] and with a scalar field [40]). The simplest possibility is to parametrize the metric as in (1), separating the conformal factor $\sigma(x)$, and to rewrite Eq. (7) using (2). The solution for the effective action is

$$\begin{aligned} \bar{\Gamma} = S_c[\bar{g}_{\mu\nu}, A_\mu] + \frac{1}{(4\pi)^2}\int d^4x\sqrt{\bar{g}}\Big\{&\omega\sigma\bar{C}^2 + \tilde{\beta}\sigma\bar{F}^2_{\mu\nu} \\ &+ b\sigma\left(\bar{E} - \frac{2}{3}\Box\bar{R}\right) + 2b\sigma\bar{\Delta}_4\sigma - \frac{1}{12}\left(c + \frac{2}{3}b\right) \\ &\times[\bar{R} - 6(\bar{\nabla}\sigma)^2 - (\Box\sigma)]^2\Big\}, \end{aligned} \tag{9}$$

where $S_c[\bar{g}_{\mu\nu}, A_\mu] = S_c[g_{\mu\nu}]$ is an unknown conformal invariant functional of the metric and A_μ, which serves as an integration constant for the Eq. (7). All quantities with bars are constructed using the metric $\bar{g}_{\mu\nu}$, in particular

$$\bar{F}^2_{\mu\nu} = \bar{F}_{\mu\nu}\bar{F}_{\alpha\beta}\bar{g}^{\mu\alpha}\bar{g}^{\beta\nu}.$$

Furthermore, Δ_4 is the fourth derivative conformally covariant operator acting on dimensionless scalar

$$\Delta_4 = \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}R\Box + \frac{1}{3}R_{;\mu}\nabla^\mu. \tag{10}$$

The solution (9) has the merit of being simple, but an important disadvantage is that it is not covariant or, in other words, it is not expressed in terms of an original metric $g_{\mu\nu}$. In order to obtain the nonlocal covariant solution and after represent it in the local form using auxiliary fields, we shall follow [38,41]. The presence of the $\bar{F}^2_{\mu\nu}$ terms does not require any essential changes compared to the consideration presented in [37], in particular, this term can be always taken together with the $\bar{C}^2$ one. So, we present just the final result in the nonlocal form, which is expressed in terms of the Green function $G(x, y)$ of the operator (10),

$$\Delta_{4,x}G(x, y) = \delta(x, y).$$

Using the last formulas and (2) we find, for any $A(g_{\mu\nu}) = A(\bar{g}_{\mu\nu}e^{2\sigma})$, the relation

$$\frac{\delta}{\delta\sigma(y)}\int d^4x\sqrt{g(x)}A\Big(E - \frac{2}{3}\Box R\Big)\Big|_{g_{\mu\nu}=\bar{g}_{\mu\nu}} = 4\sqrt{\bar{g}}\bar{\Delta}_4 A = 4\sqrt{g}\Delta_4 A. \tag{11}$$

In particular, we obtain

$$\Gamma_{\text{induced}} = \Gamma_\omega + \Gamma_b + \Gamma_c, \tag{12}$$

where

$$\Gamma_\omega = \frac{1}{4}\int d^4x\sqrt{g(x)}\int d^4y\sqrt{-g(y)}(\omega C^2 + \tilde{\beta}F^2_{\mu\nu})_x G(x, y) \times \Big(E - \frac{2}{3}\Box R\Big)_y, \tag{13}$$

$$\Gamma_b = \frac{b}{8}\int d^4x\sqrt{g(x)}\int d^4y\sqrt{g(y)}\Big(E - \frac{2}{3}\Box R\Big)_x G(x, y) \times \Big(E - \frac{2}{3}\Box R\Big)_y \tag{14}$$

and

$$\Gamma_c = -\frac{c + \frac{2}{3}b}{12(4\pi)^2}\int d^4x\sqrt{g(x)}R^2(x). \tag{15}$$

The nonlocal expressions for the anomaly-induced effective action can be presented in a local form using two auxiliary scalar fields φ and ψ [41]. Let us give just a final result which has an extra electromagnetic terms compared to the one described in [37]

$$\Gamma = S_c[g_{\mu\nu}, A_\lambda] - \frac{3c + 2b}{36(4\pi)^2}\int d^4x\sqrt{g(x)}R^2(x) + \int d^4x\sqrt{g(x)}\Big\{\frac{1}{2}\varphi\Delta_4\varphi - \frac{1}{2}\psi\Delta_4\psi + \varphi\Big[\frac{\sqrt{-b}}{8\pi}\Big(E - \frac{2}{3}\Box R\Big) - \frac{1}{8\pi\sqrt{-b}}(aC^2 + \tilde{\beta}F^2_{\mu\nu})\Big] + \frac{1}{8\pi\sqrt{-b}}\psi(aC^2 + \tilde{\beta}F^2_{\mu\nu})\Big\}. \tag{16}$$

The local covariant form (16) is dynamically equivalent to the nonlocal covariant form (12). The complete definition of the Cauchy problem in the theory with the nonlocal action requires defining the boundary conditions for the Green functions $G(x, y)$, which show up independently in the two terms (13) and (14). The same can be achieved, in the local version, by imposing the boundary conditions on the two auxiliary fields φ and ψ.

Let us separate the part of the effective action (16), which has direct relation to the electromagnetic terms and therefore represents a one-loop correction for the classical action (3),

$$\Gamma = S_c[g_{\mu\nu}, A_\lambda] + \int d^4x\sqrt{g(x)}\frac{\sqrt{-b}}{8\pi}\varphi\Big(E - \frac{2}{3}\Box R\Big) + \frac{1}{2}\int d^4x\sqrt{g(x)}\Big[\varphi\Delta_4\varphi - \psi\Delta_4\psi + \frac{1}{4\pi\sqrt{-b}}(\psi - \varphi)\tilde{\beta}F^2_{\mu\nu}\Big]. \tag{17}$$

Let us note that the presence of the A_μ-independent term $\varphi[E - (2/3)\Box R]$ is relevant, because only this term provides violation of local conformal symmetry in the whole expression. For instance, the connection with the noncovariant presentation (9) is through the relations

$$\sqrt{\bar{g}}\bar{\Delta}_4 = \sqrt{g}\Delta_4 \quad \text{and} \quad \sqrt{g}\Big(E - \frac{2}{3}\Box R\Big) = \sqrt{\bar{g}}\Big(\bar{E} - \frac{2}{3}\Box\bar{R} + 4\bar{\Delta}_4\sigma\Big). \tag{18}$$

Another important observation is that the anomalous metric dependence of the $F^2_{\mu\nu}$ in Eq. (17) appears due to the coupling of $F^2_{\mu\nu}$ with the auxiliary scalars ψ and φ. This dependence is nontrivial because these two fields have different space-time behavior due to their distinct dynamical equations and independent initial and boundary conditions.

As an example of the use of the auxiliary fields ψ and φ, we mention that the different choices of their boundary conditions enable one to classify the vacuum states for the semiclassical black hole [42]. It would be interesting to explore the initial and boundary conditions for ψ and φ such that it could be applied, for example, for calculating the creation of the seeds of a magnetic field during inflation. However this issue requires (and deserves) a special detailed investigation which goes beyond the scope of the present paper.

III. MASSIVE FERMION CASE: SCHWINGER-DEWITT EXPANSION

The results of the previous section can be seen as follows: in the case of massless and conformal fields we can calculate an important (sometimes the most important) part of the one-loop effective action by using conformal anomaly. The qualitative explanation of this possibility is that the conformal anomaly is essentially controlled by the logarithmic divergences of the theory. And, on the other hand, the logarithmic divergences are intimately related to the UV behavior of the theory or, better to say, to its reaction to the scaling in the high-energy region. In the massless theory there is no natural scale and, therefore, all kind of quantum processes can be actually seen as "high energy". For this reason the UV divergences provide much more information on the finite part of effective action in the massless conformal case than they do for massive fields. The question is what we can do in this case.

The theory of a massive quantum field has the natural scale established by the mass. In this situation to notions of "high-energy region" (UV) and "low-energy region" (IR) assume some relations between the energy of the field and the mass of the quantum field. The quantum effects of massive fields are supposed to be close to the ones of the massless fields in the UV and follow the decoupling theorem [33] in the IR. In both cases the most relevant finite part of effective action is given by nonlocal expressions, but the structure of nonlocalities are rather different in the two cases. As far as we are interested in the photon propagation, all that we need is a form factor of the electromagnetic term. We shall present this result, at the one-loop level, in the next section. In the present section we consider the local terms which correspond to the Schwinger-DeWitt expansion of the effective action in the massive case. The analysis of the coefficients of this expansion will prove useful for better understanding of the structure of the form factors of our interest; moreover deriving these coefficients provides an efficient check of correctness of the consequent general calculations.

The one-loop effective action (EA) in the metric and electromagnetic sectors can be defined via the path integral

$$e^{i\Gamma[g_{\mu\nu},A_\mu]} = \int D\psi D\bar{\psi} e^{iS_{\rm QED}}, \tag{19}$$

where

$$S_{\rm QED} = S_f + S_{\rm em} \tag{20}$$

and the actions S_f, $S_{\rm em}$ are defined in (3) and (6). Since the action $S_{\rm QED}$ is bilinear in the spinor fields, we find (see, e.g., [43])

$$\bar{\Gamma}^{(1)} = -\frac{1}{2}{\rm Ln\,Det}\hat{H}, \tag{21}$$

where

$$\hat{H} = i(\gamma^\mu\nabla_\mu - im - ie\gamma^\mu A_\mu) \tag{22}$$

is the bilinear form of the action (6). Here and below we assume but usually do not write explicitly the identity matrix $\hat{1}$ in the space of Dirac spinors.

The derivation of (21) can be performed by several methods. Here we intend to use the heat-kernel approach and the Schwinger-DeWitt technique. For this end we need to reduce the problem to the derivation of ${\rm Ln\,Det}\hat{\mathcal{O}}$, where the operator $\mathcal{O}$ should have the form

$$\hat{\mathcal{O}} = \hat{\Box} + 2\hat{h}^\mu\nabla_\mu + \hat{\Pi}. \tag{23}$$

An obvious way to achieve the desired form of the operator is to multiply $\hat{H}$ by an appropriate conjugate operator $\hat{H}^*$,

$$\hat{\mathcal{O}} = \hat{H}\cdot\hat{H}^* \tag{24}$$

and use the relation

$${\rm Ln\ Det}\hat{H} = {\rm Ln\,Det}\hat{\mathcal{O}} - {\rm Ln\,Det}\hat{H}^*. \tag{25}$$

It is clear that one can obtain the desirable form of the product by simply choosing $\hat{H}^* = \hat{H}$. In this case we can obtain the result (21) by taking

$${\rm Ln\ Det}\hat{H} = \frac{1}{2}{\rm Ln\,Det}\hat{H}^2. \tag{26}$$

At the same time there are many other possible choices. For instance, the calculations can be performed in the most simple and economic way if we choose the conjugated operator with the opposite sign of the mass term,

$$\hat{H}_1^* = -i(\gamma^\mu\nabla_\mu + im - ie\gamma^\mu A_\mu). \tag{27}$$

According to [44], the contributions of the two operators are identical,

$${\rm Ln\ Det}\hat{H} = {\rm Ln\,Det}\hat{H}_1^*, \tag{28}$$

such that we can still use the formula similar to (26),

$${\rm Ln\ Det}\hat{H} = \frac{1}{2}{\rm Ln\,Det}(\hat{H}\hat{H}_1^*). \tag{29}$$

The expressions for the elements of the operator (23) in this case are rather simple, namely

$$\begin{aligned}\hat{h}_1^\mu &= -ieA^\mu \\ \hat{\Pi}_1 &= -\frac{1}{4}R + m^2 - \frac{ie}{2}\gamma^\mu\gamma^\nu F_{\mu\nu} - ie(\nabla_\mu A^\mu) \\ &\quad - ie^2A^\mu A_\mu.\end{aligned} \tag{30}$$

Another possible choice of $\hat{H}^*$ is

$$\hat{H}_2^* = -i(\gamma^\mu\nabla_\mu - im). \tag{31}$$

In this case we have to use the relation

$$\mathrm{Ln\ Det}\hat{H} = \mathrm{Ln\,Det}(\hat{H}\hat{H}_2^*) - \mathrm{Ln\,Det}\hat{H}_2^*. \qquad (32)$$

It is obvious that the contribution of $\hat{H}_1^*$ does not depend on A_μ, just because this operator does not depend on A_μ. Therefore, for evaluating the contribution of $\mathrm{Ln\,Det}\hat{H}$ we can take the A_μ-dependent terms from the contribution of the product $(\hat{H}\hat{H}_1^*)$ with the coefficient one, while for the A_μ-independent terms the coefficient must be one-half.

The UV divergences derived within the two schemes, the first one with $\hat{H}_1^*$ and the second one with $\hat{H}_2^*$, are equal to each other. At the same time, there is an essential difference between the finite parts of the corresponding effective actions derived within the two schemes. The general discussion of this difference (called multiplicative anomaly, [45,46]), is discussed in the parallel paper [34]. Let us present the technical parts of the consideration, which are also relevant for our main targets, that is: violation of conformal symmetry and, in general, quantum corrections.

The elements of the operator $\hat{\mathcal{O}}$ for the case of the operator $\hat{H}_2^*$ have the form

$$\hat{h}_2^{\mu} = -\frac{ie}{2}\gamma^\nu\gamma^\mu A_\nu \qquad \hat{\Pi}_2 = -\frac{1}{4}R + m^2 + eM\gamma^\mu A_\mu. \qquad (33)$$

It proves useful to calculate the first Schwinger-DeWitt coefficients in an arbitrary space-time dimension d. For this end we will need the expressions for the operators

$$\hat{P} = \hat{\Pi} + \frac{\hat{1}}{6}R - \nabla_\mu\hat{h}^\mu - \hat{h}_\mu\hat{h}^\mu \qquad (34)$$

and

$$\hat{S}_{\mu\nu} = \hat{R}_{\nu\mu} - \nabla_\mu\hat{h}_\nu + \nabla_\nu\hat{h}_\mu - \hat{h}_\mu\hat{h}_\nu + \hat{h}_\nu\hat{h}_\mu, \qquad (35)$$

where $\hat{R}_{\mu\nu} = \hat{1}[\nabla_\mu, \nabla_\nu]$ is the commutator of the two covariant derivatives acting on the corresponding fields. In our case of Dirac spinors space (see, e.g., [43]),

$$\hat{R}_{\mu\nu}\psi = [\nabla_\mu, \nabla_\nu]\psi = \frac{1}{4}R_{\mu\nu\lambda\tau}\gamma^\lambda\gamma^\tau\psi,$$
$$\gamma^\mu\gamma^\nu\nabla_\mu\nabla_\nu = \hat{\Box} - \frac{\hat{1}}{4}R. \qquad (36)$$

It is important to remember that formulas (34) and (35) do not depend on the dimension d, and the *general* expressions for the coincidence limits of the Schwinger-DeWitt coefficients do not depend on d either.

The expressions for $\hat{P}$ and $\hat{S}_{\mu\nu}$ for the two calculational schemes (with $\hat{H}_1^*$ and $\hat{H}_2^*$, correspondingly) are as follows:

$$\hat{P}_1 = m^2 - \frac{1}{12}R - \frac{ie}{2}\gamma^\mu\gamma^\nu F_{\mu\nu},$$
$$\hat{S}_{1,\mu\nu} = -\frac{1}{4}\gamma^\alpha\gamma^\beta R_{\alpha\beta\mu\nu} + ieF_{\mu\nu}; \qquad (37)$$

versus

$$\hat{P}_2 = m^2 - \frac{1}{12}R - \frac{ie}{4}\gamma^\mu\gamma^\nu F_{\mu\nu} + em\gamma^\mu A_\mu + \frac{ie}{2}(\nabla^\mu A_\mu) - \frac{e^2(d-2)}{4}A^\nu A_\nu,$$
$$\hat{S}_{2,\mu\nu} = -\frac{1}{4}\gamma^\alpha\gamma^\beta R_{\alpha\beta\mu\nu} + \frac{ie}{2}\gamma^\alpha(\gamma_\nu\nabla_\mu A_\alpha - \gamma_\mu\nabla_\nu A_\alpha) + \frac{e^2}{4}\gamma_\alpha(\gamma_\mu\gamma_\beta\gamma_\nu - \gamma_\nu\gamma_\beta\gamma_\mu)A^\alpha A^\beta. \qquad (38)$$

The operators (37) and (38) enable one to calculate the coincidence limits of the first coefficients of the Schwinger-DeWitt expansion, by just using the known general expressions (see, e.g., [47,48]).

Let us start from the first nontrivial coefficient. For the sake of simplicity we shall use the notation

$$a_k = \mathrm{Tr}\lim_{x'\to x} a_k(x, x')$$

for the functional traces (including space-time integrations) of the coincidence limits. We know [47,49] that for the operators of the form (23), the first coefficient is given by the expression

$$a_1 = \mathrm{sTr}\hat{P} = -\int d^d x\sqrt{g}\,\mathrm{tr}\hat{P},$$

where the operators $\hat{P}$ are given by the expressions (37) and (38). The symbol sTr means the functional trace, which is taken according to Grassmann parity of the corresponding functional matrices. Some pedagogical examples, including the operators with both bosonic and fermionic blocks can be found in [43]. For the general d-dimensional case we obtain the expressions

$$a_{1,1} = -\int d^d x\sqrt{g}\Big(4m^2 - \frac{1}{3}R\Big) \qquad (39)$$

and

$$a_{1,2} = -\int d^d x\sqrt{g}\Big\{4m^2 - \frac{1}{3}R + 2ie(\nabla^\mu A_\mu) - (d-2)e^2 A^\mu A_\mu\Big\}. \qquad (40)$$

The first expression (39) is independent of the electromagnetic potential and is, therefore, gauge invariant. However, the second expression is obviously different in both respects, in particular, it becomes gauge invariant only in the $d = 2$ case (in what follows we shall call it $2d$, and exactly the same way for other dimensions, e.g. $4d$ means $d = 4$, etc.).

How can we understand the difference between the results coming from the two calculational schemes, based on $\hat{H}_1^*$ and $\hat{H}_2^*$ conjugate operators? Let us note that the operation of introducing the conjugate operator (24) can be seen in two different ways.

First, we can see it as a change of the quantum variables in the path integral (19), that is taking $\psi = \hat{H}^*\chi$, where χ is a new quantum variable. In this case we have to take into account the contribution of the functional Jacobian of such a transformation. This Jacobian is in fact related to another path integral, similar to (19), namely, for the second case,

$$\int D\chi D\bar{\chi} \exp\left\{i\int d^dx\sqrt{g}\,\bar{\chi}\hat{H}_2^*\chi\right\}. \qquad (41)$$

This integral does not depend on the electromagnetic potential, but only on the metric, and hence it is gauge invariant. However, the result of the change of variables is quite different, for one meets another functional integral,

$$\int D\chi D\bar{\psi} \exp\left\{i\int d^dx\sqrt{g}\,\bar{\psi}(\hat{H}\cdot\hat{H}_2^*)\chi\right\}. \qquad (42)$$

This integral does not possess gauge invariance and hence it is not a surprise that the result of the change of variables also does not possess it.

Another possible understanding of the operation (24) is through the well-known relation

$$\text{Ln Det}(\hat{H}^*\cdot\hat{H}_2^*) = \text{Ln Det}\hat{H}^* + \text{Ln Det}\hat{H}_2^*. \qquad (43)$$

The two terms in the right-hand side are gauge invariant due to the reasons explained above, namely, the first term is a result of the gauge-invariant functional integration and the second term simply does not transform, just because it does not depend neither on A_μ neither on spinor field. Hence if we find the violation of gauge symmetry in the left-hand side (as we actually did), this indicates the violation of the ''identity'' (43).

The relation similar to (43),

$$\det(A\cdot B) = \det A\cdot\det B$$

can be easily proved for the finite-size square matrices A and B. However, the proof cannot be generalized for the differential operators, which have an infinite-size matrix representation. In fact, some mathematicians and physicists were looking, for a long time, for an example when this relation does not hold [45,46], by using the ζ-regularization technique [50]. The phenomenon of such possible violation has been called multiplicative anomaly. However, the results of the mentioned works met justified criticism [51–53] because it is in fact difficult to distinguish the effect from the usual renormalization ambiguity.

From the perspective of multiplicative anomaly we can observe that the key relation (43) holds in $2d$ and only in $2d$. However, this dimension is very special for the a_1, because in this particular dimension the coefficient a_1 defines the logarithmic divergence of the theory. At this point we conclude that, in the case under consideration, the relation (43) does hold for the logarithmically divergent part, but may be violated in the other sectors of the effective action. In a moment we shall see that this statement is also valid for the next two coefficients.

The general expression for the second Schwinger-DeWitt term (called also the ''magic'' coefficient) is

$$a_2 = \text{sTr}\hat{a}_2(x,x) = \text{sTr}\left\{\frac{\hat{1}}{180}(R^2_{\mu\nu\alpha\beta} - R^2_{\alpha\beta} + \Box R) + \frac{1}{2}\hat{P}^2 + \frac{1}{6}(\Box\hat{P}) + \frac{1}{12}\hat{S}^2_{\mu\nu}\right\}. \qquad (44)$$

Direct calculations using formula (44) and the expressions (37) and (38) yield the following results:

$$a_{2,1} = \frac{d}{288}\int d^dx\sqrt{g}(48e^2F_{\mu\nu}F^{\mu\nu} + R^2 - 24Rm^2 - 3R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + 144m^4), \qquad (45)$$

for the first calculational scheme based on $\hat{H}_1^*$ and

$$a_{2,2} = \frac{d}{288}\int d^dx\sqrt{g}\{24de^2(\nabla^\mu A^\nu)(\nabla_\mu A_\nu) - 96e^2(\nabla^\mu A^\nu)(\nabla_\nu A_\mu) + R^2 + 144m^4 - 24Rm^2 - 3R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + 12(d-4)(R-12m^2)A^\mu A_\mu e^2 + 6(d-2)(d-4)A_\mu A^\mu A_\nu A^\nu e^4\} \qquad (46)$$

for an alternative calculational scheme based on $\hat{H}_2^*$. It is easy to see that the two expressions for a_2 presented above follow the same pattern as the ones for a_1 which we considered earlier. Namely, the expression for $a_{2,1}$ is gauge invariant independent of the space-time dimension d and, in fact, its dependence on d is rather simple. On the contrary, $a_{2,2}$ manifest much more complicated dependence on d. This expression is gauge invariant only in the $4d$ case, when it also coincides with $a_{2,1}$. We know that this coincidence is because the a_2 defines the logarithmic divergences in $4d$ and only in $4d$. So, we observe that the logarithmic divergences of the theory are scheme independent and that for the divergent sector of the theory, in dimensional regularization, the relation (43) actually takes place. At the same time, for any other dimension $d \neq 4$, the two expressions are dramatically different. We can indicate two aspects of this difference. First, the $a_{2,2}(d\neq 4)$ expression is not gauge invariant. This means the quantum contributions depend not only on the transverse part of A_μ, but also on the longitudinal part of this vector. Let us note that the $a_{2,1}(d\neq 4)$ is perfectly gauge invariant and no longitudinal propagation takes place in this case.

Finally, as a final test of the difference between the two schemes of deriving the photon propagator, let us calculate the $a_{3,1}$ and $a_{3,2}$ coefficients. For this end we will need the general expression for the a_3 coefficient, which has been derived by Gilkey [54], and checked by Avramidi [55] who also derived a_4 coefficient. This general expression is rather bulky and we simply refer an interested reader, e.g., to the eq. (2.160) of [55]. One observation is in order. The result of [54,55] corresponds to the operator of the more simple form

$$\hat{O} = \hat{\Box} + \hat{\Pi} \tag{47}$$

if compared to our (23). However the way to generalize from (47) to (23) is already known for a long time from [48], where it was applied for the a_2 coefficient. One has to start by constructing the generalized covariant derivative acting in the fermionic space

$$\hat{D}_\mu = \hat{\nabla}_\mu + \hat{h}_\mu.$$

It is easy to see that the operator (23) can be presented in the form (47) with the new derivative and, also, $\hat{\Pi}$ replaced by $\hat{P}$. The last thing to do is to replace the commutator $\hat{R}_{\mu\nu}$ by $\hat{S}_{\mu\nu}$, defined in (35). Then the Schwinger-DeWitt technique can be developed with the covariant derivative D_μ instead of ∇_μ, and eventually gives the result in terms of $\hat{P}$, $\hat{S}_{\mu\nu}$, and curvature tensor.

Consider the most simple $\hat{H}_1^*$-based calculational scheme. Using the original approach for the operator (47) we arrive at the already known result of Ref. [3],

$$a_{3,1} = \frac{de^2}{360}(2R_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta} - 26R_{\alpha\nu}F^{\mu\nu}F_\mu{}^\alpha + 24\nabla_\nu F^{\mu\nu}\nabla_\alpha F_\mu{}^\alpha + 5RF^{\mu\nu}F_{\mu\nu}). \tag{48}$$

It is easy to see that this expression is gauge invariant independent of the space-time dimension d. Let us now see what is the situation with the second choice of the calculational scheme, based on the operator $\hat{H}_2^*$. In this case one has to use the approach described above, that is, to deal with the operator (23) and use generalized covariant derivative D_μ. After an involved algebra we arrive at the expression

$$\begin{aligned} a_{3,2}|_{AA} = \frac{de^2}{2880}\{&120(\nabla A)\Box(\nabla A) - 60F_{\mu\nu}\Box F^{\mu\nu} - 24\nabla_\nu F^{\mu\nu}\nabla^\alpha F_{\mu\alpha} + 24(\Box A^\alpha)[(d-3)(\Box A_\alpha) + 2\nabla_\alpha(\nabla A)] \\ &- 24(\nabla_\alpha\nabla_\mu A_\beta)[(\nabla^\beta\nabla^\mu A^\alpha) - (\nabla^\alpha\nabla^\mu A^\beta)] + A^\mu A_\mu[(18-7d)R^2_{\mu\nu\alpha\beta} - 8(9-d)R^2_{\mu\nu} - 6(5-d)R^2] \\ &+ 8R_{\mu\nu\alpha\beta}[4(\nabla^\alpha A^\nu)(\nabla^\mu A^\beta) - 8F^{\mu\nu}F^{\alpha\beta} - 3(d-4)(\nabla^\mu A^\alpha)(\nabla^\nu A^\beta) - R^{\lambda\nu\alpha\beta}A^\mu A_\lambda + 10R^{\mu\beta}A^\alpha A^\nu] \\ &+ 16R_{\mu\nu}[10(\nabla A)(\nabla^\mu A^\nu) + (\nabla^\alpha A^\mu)(5\nabla_\alpha A^\nu - 2\nabla^\nu A_\alpha) - (d-5)(\nabla^\mu A^\alpha)(\nabla^\nu A_\alpha) - 2R^\mu{}_\alpha A^\alpha A^\nu] \\ &+ 10R[2(d-5)(\nabla_\mu A_\nu)(\nabla^\mu A^\nu) - 2(\nabla A)^2 + 3F_{\mu\nu}F^{\mu\nu} + 2R_{\mu\nu}A^\mu A^\nu] - 24(\nabla^\nu R)[(\nabla_\nu A^\alpha A_\alpha) - (\nabla_\alpha A^\alpha A_\nu)] \\ &- 12(d-2)A^\alpha A_\alpha\Box R - 48(\nabla^\alpha R_{\mu\nu\alpha\beta})(\nabla^\nu A^\beta A^\mu)\}. \end{aligned} \tag{49}$$

The last expression is rather cumbersome and difficult to deal with. Apparently it is not gauge invariant and is different from (48). In order to complete the analysis, we need to answer two questions: (i) How we can prove that the two expressions are distinct or identical in one or another particular dimension? (ii) Is it true that the two expressions (48) and (49) are distinct, in one or another particular dimension? In the case of a_3 we are mainly interested in the 6d case, where one can hope to find the gauge invariance of (49) and also the equivalence between (48) and (49).

Let us start by making the most simple test. Consider the propagation of the purely transverse part of A_μ, that means to collect the terms which contain the terms $A_\mu\Box^2A^\mu$ in both expressions. It is important that the transverse terms cannot be affected by the possible violation of gauge invariance in the expression (49), so this particular test is independent of the rest. The relatively simple calculation shows that the two propagators are identical in 6d and only in 6d. This solves one-half of the item (ii), namely, we learn that out of 6d the two expressions (48) and (49) are different.

We could not precisely prove that the two expressions are identical in 6d for an arbitrary metric background. It is likely, however, that this is the case. The arguments in favor of this conclusion come from the analysis using a particular simple metric. Let us consider the case of de Sitter space-time, where one can actually check the equality of Eqs. (48) and (49). Even in this case it is not easy to work with (49), but one can perform some qualitative analysis of the expression for $a_{3,2}$. First of all, by direct inspection we can verify that the terms without scalar curvature are the same in both expressions.

As a next step, consider the terms proportional to the scalar curvature, which can be symbolically identified as RFF. In order to analyze these terms, one has to take into account also the terms proportional to the derivatives of the curvature tensor components in (49). For definiteness, we call them DR terms. If one integrates these terms by parts, it turns out that some of them do contribute to the R^2A^2 structures. Since these terms can be written as total derivative terms in de Sitter space-time, we can introduce them in $a_3^{(2)}$ with an arbitrary coefficient. To make the correct choice of this coefficient, one has to additionally evaluate those terms which are proportional to R^2A^2 and choose the coefficient of the DR terms such that they do cancel the R^2A^2 terms in (49). After that, the terms proportional to RFF, generated by integration by parts of the DR terms, sum with the ones which come from (49) and finally turn out to be the same as (48). We do not include the corresponding formulas here, because they are rather boring.

Finally, we have a strong argument to believe that qualitatively the situation with a_3 is the same as with the a_1 and a_2 coefficients. All these coefficients are universal (scheme independent) in the dimensions 2, 4, and 6, correspondingly. And, at the same time, they are essentially scheme dependent in other dimensions. It is easy to see that this fact implies that, in any dimension, the logarithmic divergences are scheme independent and gauge invariant. At the same time, quartic and quadratic divergences and the finite contributions are scheme dependent and, in case of the $\hat{H}_2^*$-based calculational scheme, they are not gauge invariant. Of course, this feature does not contradict the gauge-invariant renormalizability of the theory, but it makes the finite results scheme dependent. One has to remember that the effective action, in any particular dimension, is a sum of the series of the terms with $\hat{a}_0$, $\hat{a}_1$, $\hat{a}_2$, $\hat{a}_3$, etc. Therefore, we can conclude that the effective action is scheme dependent in any particular dimension.

The scheme dependence which we have explored (multiplicative anomaly), should be seen as a particular case of the nontrivial reaction of the effective action of quantum fields to the change of variables. The fact that such dependence can be relevant has been known for a long time (see, e.g., [56]). It is interesting to see what is the consequence of this fact for the sum of the Schwinger-DeWitt series. We perform the corresponding calculation in the next section.

IV. DERIVATION OF THE ONE-LOOP FORM FACTORS

The effective action (21) can be expressed by means of the proper time integral, involving the heat kernel $K(s)$ of the operator $\mathcal{O}$,

$$\bar{\Gamma}^1 = -\frac{1}{2}\int_0^\infty \frac{ds}{s}\mathrm{sTr}K(s). \qquad (50)$$

Let us derive this expression using the method developed previously in [57,58] (see also [59] for the general review). According to Ref. [57], we can perform calculations either by using Feynman diagrams or through the heat-kernel solution, which was obtained in [60,61]. The result of the two methods are going to be the same, but the covariant heat-kernel solution is much more simple, so we will use this approach.

The expression (50) can be expanded into the powers of the field strengths (curvatures), $R_{\mu\nu\alpha\beta}$, $\hat{S}_{\mu\nu}$, and $\hat{P}$. Up to the second order in the curvatures the expansion has the form [60]

$$\begin{aligned}\mathrm{Tr}\,K(s) = \frac{\mu^{4-2\omega}}{(4\pi s)^\omega}\int d^{2\omega-4}x\sqrt{g}e^{-sM^2}\mathrm{tr}\{\hat{1} + s\hat{P} \\ + s^2[R_{\mu\nu}f_1(-s\Box)R^{\mu\nu} + Rf_2(-s\Box)R \\ + \hat{P}f_3(-s\Box)R + \hat{P}f_4(-s\Box)\hat{P} \\ + \hat{S}_{\mu\nu}f_5(-s\Box)\hat{S}^{\mu\nu}]\},\end{aligned} \qquad (51)$$

where the expressions for $\hat{P}$ and $\hat{S}_{\mu\nu}$ are defined in (34) and (35), ω is the dimensional regularization parameter, μ is an arbitrary renormalization parameter with the dimension of mass and the functions f_i are given by

$$\begin{aligned} f_1(\tau) &= \frac{f(\tau) - 1 + \tau/6}{\tau^2}, \\ f_2(\tau) &= \frac{f(\tau)}{288} + \frac{f(\tau) - 1}{24\tau} - \frac{f(\tau) - 1 + \tau/6}{8\tau^2}, \\ f_3(\tau) &= \frac{f(\tau)}{12} + \frac{f(\tau) - 1}{2\tau}, \\ f_4(\tau) &= \frac{f(\tau)}{2}, \\ f_5(\tau) &= \frac{1 - f(\tau)}{2\tau}, \end{aligned} \qquad (52)$$

where

$$f(\tau) = \int_0^1 d\alpha e^{\alpha(1-\alpha)\tau}, \qquad \tau = -s\Box.$$

After some algebra, we arrive at the integral representation for the effective action up to the second order in $\hat{P}$, $\hat{S}_{\mu\nu}$, and $R_{\mu\nu}$, in dimensional regularization,

$$\begin{aligned}\bar{\Gamma}^1 = -\frac{1}{2(4\pi)^2}\int d^4x\sqrt{g}\left(\frac{m^2}{4\pi\mu^2}\right)^{\omega-2}\int_0^\infty dte^{-t}\sum_{k=1}^5\{l_k^{FF}e^2F_{\mu\nu}M_kF^{\mu\nu} + l_k^{*DA}e^2\nabla_\mu A^\mu M_k\nabla_\nu A^\nu + l_k^{DA}e^2\nabla_\mu A^\nu M_k\nabla_\nu A^\mu \\ + \lambda_k e^2 R_{\mu\nu}M_kA^\nu A^\mu + \lambda_k^* e^2A^\nu A^\mu M_kR_{\mu\nu} + l_k^{AR}e^2A_\alpha A^\alpha M_kR + l_k^{RA}e^2RM_kA_\alpha A^\alpha + l_k^*R_{\mu\nu}M_kR^{\mu\nu} + l_kRM_kR\},\end{aligned} \qquad (53)$$

where

$$M_1 = \frac{f(tu)}{t^{\omega-1}}, \qquad M_2 = \frac{f(tu)}{t^\omega u}, \qquad M_3 = \frac{f(tu)}{t^{\omega+1}u^2}, \qquad M_4 = \frac{1}{t^\omega u}, \qquad M_5 = \frac{1}{t^{\omega+1}u^2},$$

and the nonzero coefficients for $\hat{H}_1^*$ have the form

$$l_2 = -\frac{1}{16} \qquad l_3 = -\frac{1}{8}, \qquad l_4 = \frac{1}{24} \qquad l_5 = \frac{1}{8}, \qquad l_2^* = \frac{1}{4}, \qquad l_3^* = 1,$$
$$l_4^* = -\frac{1}{12}, \qquad l_5^* = -1, \qquad l_1^{FF_{(1)}} = 1, \qquad l_2^{FF_{(1)}} = 2, \qquad l_4^{FF_{(1)}} = -2.$$

Indeed, the coefficients l_k and l_k^* are the same as for the free fermion field [58], while the last set of coefficients is due to the interaction with A_μ and do not have analogues in the free-field case. For $\hat{H}_2^*$, we have

$$l_1^{FF_{(2)}} = \frac{1}{4}, \qquad l_2^{FF_{(2)}} = -\frac{1}{2}, \qquad l_4^{FF_{(2)}} = \frac{1}{2}, \qquad l_1^{*DA} = -\frac{1}{2}, \qquad l_2^{*DA} = -1, \qquad l_4^{*DA} = 1,$$
$$l_2^{DA} = -2, \qquad l_4^{DA} = 2, \qquad \lambda_2 = -1, \qquad \lambda_4 = 1, \qquad \lambda_2^* = -1, \qquad \lambda_4^* = 1, \tag{54}$$
$$l_1^{AR} = -\frac{1}{6}, \qquad l_2^{AR} = -1, \qquad l_4^{RA} = 1, \qquad l_1^{RA} = \frac{1}{6}, \qquad l_2^{RA} = 1, \qquad l_4^{RA} = -1.$$

As far as the purely gravitational terms were calculated in [58], here we are mainly interested in the one-loop corrections to the term $F_{\mu\nu}^2$. Let us first perform all the calculations for the operator $\hat{H}_1^*$. Substitute (38) into (51), then the result into (50), and take into account only the terms with f_4 and f_5. Using the notations [57],

$$t = sM^2, \qquad u = \frac{\tau}{t} = -\frac{\Box}{M^2}, \qquad Y = 1 - \frac{1}{a}\ln\left(\frac{2+a}{2-a}\right), \qquad a^2 = \frac{4\Box}{\Box - 4M^2} \tag{55}$$

and the known results for the integrals [58],

$$\left(\frac{M^2}{4\pi\mu^2}\right)^{\omega-2}\int_0^\infty dt e^{-t}\frac{f(tu)}{t^\omega u} = \left[\left(\frac{1}{12} - \frac{1}{a^2}\right)\left(\frac{1}{\epsilon} + 1\right) - \frac{4A}{3a^2} + \frac{1}{18}\right] + \mathcal{O}(2-\omega),$$
$$\left(\frac{M^2}{4\pi\mu^2}\right)^{\omega-2}\int_0^\infty dt e^{-t}\frac{1}{t^\omega u} = \left[\frac{a^2-4}{4a^2}\left(\frac{1}{\epsilon} + 1\right)\right] + \mathcal{O}(2-\omega),$$
$$\left(\frac{M^2}{4\pi\mu^2}\right)^{\omega-2}\int_0^\infty dt e^{-t}t^{1-\omega}f(ut) = \left(\frac{1}{\epsilon} + 2A\right) + \mathcal{O}(2-\omega),$$

where we also denoted

$$\frac{1}{\epsilon} = \frac{1}{2-\omega} + \ln\left(\frac{4\pi\mu^2}{M^2}\right).$$

After some algebra, we arrive at the explicit expression for the one-loop correction to the classical $F_{\mu\nu}^2$ term,

$$\bar{\Gamma}^{(1)}_{\sim F_{\mu\nu}^2} = -\frac{e^2}{2(4\pi)^2}\int d^4x\sqrt{g}F_{\mu\nu}\left[\frac{2}{3\epsilon} + k_1^{FF}(a)\right]F^{\mu\nu}, \tag{56}$$

where the electromagnetic form factor has the form [62]

$$k_1^{FF}(a) = Y\left(2 - \frac{8}{3a^2}\right) - \frac{2}{9}. \tag{57}$$

A simple way to check our result is to compare it to the expressions for the Schwinger-DeWitt coefficients derived in the previous section. Let us remember that the calculation of the form factor presented above has been performed in $4d$. There is nothing that can be compared to a_1, just because the quadratic divergences vanish when the dimensional regularization is used. The comparison to a_2 is absolutely successful and complete, because here we meet two perfect correspondences. First, it is easy to see that the coefficient of the pole term of (56) is exactly the a_2 from (45) at $4d$. Second, it is easy to check that the UV limit $a \to 0$ of the form factor (57) perfectly corresponds to the coefficient of the pole term of (56).

Furthermore, we can partially compare our form factor (57) to the expression for the $a_{3,1}$ in (48). Unfortunately, the complete verification is not possible because a certain part of (48) correspond to the next, third, order in the heat-kernel solution, which we do not use here. So, we can compare only the $F\nabla\nabla F$-type terms. If we expand the corresponding part of $k_1^{FF}(a)$ into power series of the parameter a, in the IR limit, where

$$a^2 \sim -4\frac{\Box}{M^2}, \tag{58}$$

use Maxwell equation

$$\nabla_\varrho F_{\mu\nu} + \nabla_\mu F_{\nu\varrho} + \nabla_\nu F_{\varrho\mu} = 0, \tag{59}$$

and integrate by parts, we obtain the corresponding term in the effective action (48),

$$-\frac{1}{120\pi^2}\nabla_\mu F^{\mu\nu}\nabla_\varrho F^\varrho{}_\nu, \tag{60}$$

which is actually well-known from [3].

Consider the second calculational scheme, based on the operator $\hat{H}_2^*$, defined in Eq. (31). The calculations are a bit more involved, but are in general analogous to the previous case. Finally, we arrive at the following result:

$$\bar{\Gamma}^{(1)}_{\sim A^2} = -\frac{e^2}{2(4\pi)^2}\int d^4x\sqrt{g}\Big\{F_{\mu\nu}\Big[\frac{2}{3\epsilon}+k_2^{FF}(a)\Big]F^{\mu\nu}+\nabla_\mu A^\mu\Big[Y\Big(\frac{8}{3a^2}-2\Big)+\frac{2}{3}\Big]\nabla_\nu A^\nu+\nabla_\mu A^\nu\Big(\frac{16Y}{3a^2}\Big)\nabla_\nu A^\mu$$
$$+R_{\mu\nu}\Big(\frac{8Y}{3a^2}\Big)A^\nu A^\mu+A^\nu A^\mu\Big(\frac{8Y}{3a^2}\Big)R_{\mu\nu}+A_\alpha A^\alpha\Big[Y\Big(\frac{1}{3}-\frac{4}{3a^2}\Big)\Big]R+R\Big[Y\Big(\frac{4}{3a^2}-\frac{1}{3}\Big)\Big]A_\alpha A^\alpha\Big\}, \tag{61}$$

$$k_2^{FF}(a) = Y\Big(1+\frac{4}{3a^2}\Big)+\frac{1}{9}. \tag{62}$$

Once again, the UV test and the comparison with the $a_{3,2}$ are perfectly successful. However, the low-energy expression is not (60) anymore in this case; instead we have

$$-\frac{1}{160\pi^2}\nabla_\mu F^{\mu\nu}\nabla_\varrho F^\varrho{}_\nu. \tag{63}$$

As we will see in Sec. VI, this divergence between (60) and (63) results also in the ambiguity in the decoupling theorem.

It is possible to make an interesting verification of the results for the two form factors $k_1^{FF}(a)$ and $k_2^{FF}(a)$ using the coefficients $\hat{a}_{3,1}$ and $\hat{a}_{3,2}$. If we expand (57) to the first order in operator $\Box$, we have

$$k_1^{FF}(a) = Y\Big(2-\frac{8}{3a^2}\Big)-\frac{2}{9}\simeq -\frac{2}{15}\frac{\Box}{M^2}, \tag{64}$$

that gives us a term in $\bar{\Gamma}^{(1)}$ which has the form

$$\bar{\Gamma}^{(1)}\sim\frac{e^2}{15(4\pi)^2}\int d^4x\sqrt{g}F^{\mu\nu}\frac{\Box}{M^2}F_{\mu\nu}. \tag{65}$$

Using (59) in (65), we can see that

$$\bar{\Gamma}^{(1)}\sim -\frac{2e^2}{15(4\pi)^2}\int d^4x\sqrt{g}\nabla_\nu F^{\mu\nu}\nabla_\alpha F_\mu{}^\alpha$$
$$= -\frac{24e^2}{180(4\pi)^2M^2}\int d^4x\sqrt{g}\nabla_\nu F^{\mu\nu}\nabla_\alpha F_\mu{}^\alpha \tag{66}$$

and that is the same contribution given to $\bar{\Gamma}^{(1)}$ in $4d$ by the third term of Eq. (48), as it should be. To the form factor $k_2^{FF}(a)$, we find

$$\bar{\Gamma}^{(1)}\sim -\frac{e^2}{10(4\pi)^2}\int d^4x\sqrt{g}A^\mu\frac{\Box^2}{M^2}A_\mu. \tag{67}$$

Now if we take the contribution of (49) to $\bar{\Gamma}^{(1)}$ in $4d$ the coefficient proportional to $A^\mu\Box^2A_\mu$ is the same as (67). It is important to note that the coefficients of the terms $A^\mu\Box^2A_\mu$ in Eqs. (66) and (67) are different in $4d$. They are equal only in $6d$ as it was explained in Sec. III.

V. MASSLESS LIMIT AND TRACE ANOMALY

In order to better understand the physical sense of the result (56) and also its relation with the minimal subtraction scheme-based anomaly-induced action (17), let us consider the high-energy limit. As was already explained in the previous sections, this can be done by taking the vanishing mass limit. On the other hand, this limit helps to establish the relation between the effective action (56) and the conformal anomaly (7).

Consider the high-energy limit, when the mass of the quantum field (i.e. of electron) is negligible. Taking the limit $a\to 2$ in the expression (56) with either one of the two available form factors, we arrive at the following leading-log behavior of the electromagnetic sector:

$$\tilde{\beta}F^{\mu\nu}\ln\Big(\frac{\Box}{\mu^2}\Big)F_{\mu\nu}. \tag{68}$$

Similar asymptotic behavior takes place also in the gravitational sector of the theory. For instance, the Weyl term has similar form factor [57,58],

$$\beta_1 C^{\mu\nu\alpha\beta}\ln\Big(\frac{\Box}{\mu^2}\Big)C_{\mu\nu\alpha\beta}. \tag{69}$$

Let us note that the same asymptotic behavior can be recovered in the minimal subtraction-based scheme of renormalization [43,63] (see also [64] for an alternative consideration), which is completely reliable in the massless case.

The expressions (68) and (69) are sufficient to derive the corresponding parts of the conformal anomaly, even in case of a local conformal symmetry. For this end, let us apply the conformal parametrization of the metric (4) and the differential relation (2). Consider the case of (68) as an example; (69) is completely analogous. If we replace the parametrization (4) into (68), the only place where the σ field shows up is the $\Box$. This operator becomes

$$\Box = e^{-2\sigma}[\bar{\Box}+\mathcal{O}(\partial\sigma)], \tag{70}$$

where we denote $\bar{\Box}$ the d'Alembertian operator constructed with the metric $\bar{g}_{\mu\nu}$,

$$\bar{\Box} = \bar{g}^{\mu\nu}\bar{\nabla}_\mu\bar{\nabla}_\mu\nu.$$

The explicit form of the terms $\mathcal{O}(\partial\sigma)$ is in fact irrelevant for us. When we apply (2), only the first term in the bracket (70) is important, because the other terms vanish after we set $\sigma\to 0$. Of course, the logarithmic dependence makes

$$\ln\frac{\Box}{\mu^2} = -2\sigma+\ln\frac{\bar{\Box}+\mathcal{O}(\partial\sigma)}{\mu^2}. \tag{71}$$

Finally, after applying (2) we arrive at

$$\langle T^{\mu}_{\mu}\rangle_{\rm em} = \tilde{\beta} F^2_{\mu\nu} \qquad (72)$$

in the electromagnetic sector. In a similar way one can obtain the ωC^2 term in the general formula for anomaly, Eq. (7). One has to note that the $\Box R$ term and the Gauss-Bonnet term can be also derived from the local and non-local finite parts of effective action [60,65]. At the same time, the $\Box R$ part is a subject of an important ambiguity. The origin and mechanism for this ambiguity has been explained recently in [66] (see also [37] for the review).

VI. RENORMALIZATION GROUP, LOW-ENERGY LIMIT, AND DECOUPLING

The two form factors (57) and (62) contain all necessary information about the scale dependence of the coupling parameter e at the one-loop level, within the corresponding calculational schemes. In the present section we shall mainly restrict our attention to the flat-space case and discuss this dependence in detail. Namely, our task here is to calculate the "physical" beta functions in the momentum-subtraction renormalization scheme and look at their UV and IR limits.

In the $\overline{\rm MS}$ scheme the β function of the effective charge e is defined as

$$\beta_e(\overline{\rm MS}) = \lim_{n\to 4} \mu \frac{de}{d\mu} = \frac{4e^3}{3(4\pi)^2}. \qquad (73)$$

The derivation of the β functions in the mass-dependent scheme has been described, e.g. in [67,68]. Starting from the polarization operator, one has to subtract the counter-term at the momentum $p^2 = M^2$, where M is the renormalization point. Then, the momentum-subtraction β function is defined as

$$\beta_e = \lim_{n\to 4} M \frac{de}{dM}. \qquad (74)$$

Mathematically, this is equivalent to taking the derivative (we also write the same operation in terms of a)

$$-ep\frac{d}{dp} = e(4-a^2)\frac{a}{4}\frac{d}{da} \qquad (75)$$

of the form factors in the polarization operator. If we apply this procedure to the form factor $k_1^{FF}(a)$ of the $F^2_{\mu\nu}$ term, the expression for the β function in a mass-dependent scheme is

$$\beta^1_e = \frac{e^3}{6a^3(4\pi)^2}\left\{20a^3 - 48a + 3(a^2-4)^2 \ln\left(\frac{2+a}{2-a}\right)\right\}, \qquad (76)$$

that is the general result for the one-loop β function valid at any scale [69].

As the special cases we meet the UV limit $p^2 \gg m^2$, or $a \to 2$,

$$\beta_e^{1\,{\rm UV}} = \frac{4e^3}{3(4\pi)^2} + \mathcal{O}\left(\frac{m^2}{p^2}\right), \qquad (77)$$

that is nothing else but the $\overline{\rm MS}$ scheme result (73) plus a small correction. In the IR regime, however, when $p^2 \ll m^2$, the result is quite different, and moreover depends on the calculational scheme. For the first case $\hat{H}_1^*$, we have

$$\beta_e^{1\,{\rm IR}} = \frac{e^3}{(4\pi)^2}\cdot\frac{4M^2}{15m^2} + \mathcal{O}\left(\frac{M^4}{m^4}\right). \qquad (78)$$

This is exactly the standard form of the decoupling theorem [33].

Similar calculations starting from the form factor $k_2^{FF}(a)$ give

$$\beta^2_e = \frac{e^3}{12a^3(4\pi)^2}\left\{4a(12+a^2) - 3(a^4-16)\ln\left(\frac{2-a}{2+a}\right)\right\}. \qquad (79)$$

In the UV limit $p^2 \gg m^2$, the above β function is in agreement with the standard result (77), while in the IR limit $p^2 \ll m^2$ we obtain

$$\beta_e^{2\,{\rm IR}} = \frac{e^3}{5(4\pi)^2}\cdot\frac{M^2}{m^2} + \mathcal{O}\left(\frac{M^4}{m^4}\right). \qquad (80)$$

As we can see from the expressions (78) and (80), there is a slight difference in how β functions go to zero in the IR limit. To discuss the physical sense of this fact, let us take the difference between the two form factors (57) and (62)

$$\Delta F^{FF} = k_1^{FF}(a) - k_2^{FF}(a) = A\left(1 - \frac{4}{a^2}\right) - \frac{1}{3} \qquad (81)$$

and expand ΔF^{FF} in power series of a. In the IR limit we have $a^2 \sim p^2/M^2 = -\Box/M^2$, so

$$\Delta F^{FF} = \frac{1}{30}\cdot\frac{\Box}{M^2} + \mathcal{O}(p^3/M^3). \qquad (82)$$

This is exactly the difference in the form factors which caused the ambiguity (calculational scheme dependence) in the decoupling theorem. In order to evaluate the source of this difference in the effective action, we should consider what would be the new terms in the equation of motion, generated by the term

$$F^{\mu\nu}\left(\frac{1}{30}\cdot\frac{\Box}{M^2}\right)F_{\mu\nu} \qquad (83)$$

As this term is proportional to the operator $\Box$, if we are working in flat spaces, we use Eq. (59) to obtain a term proportional to $(\nabla_\mu F^{\mu\nu})^2$. This term will not influence the equations of motion in flat space in the $\mathcal{O}(e^2)$ approximation [3]. However, as we will discuss later on, the situation can be different in curved space.

VII. RUNNING CHARGE

Let us now consider the running of the electromagnetic charge. The difference in the expressions for the form factors $k_1^{FF}(a)$ and $k_2^{FF}(a)$ and eventually in the expressions for the β functions (76) and (79), means that the effective charge (i.e., the running charge) may have different behavior for the different calculational schemes (that means, for the ones based on $\hat{H}_1^*$ and $\hat{H}_2^*$ operators). Also, we expect that the running within the momentum sSubtraction scheme will be distinct from the one within the minimal subtraction scheme, especially in the low-energy region. Let us check out what the real situation is.

To investigate the renormalization of the corresponding quantities, in the physical scheme, let us apply the operator $-epd/dp$ to $k_{1,2}^{FF}(a)$ with $\Box$ traded for $-p^2$. In doing so, we find the expressions for the β functions, which can be conveniently presented as

$$-\frac{(4\pi)^2}{e^3}\beta_1 = p\frac{d}{dp}k_1^{FF}(a) = \frac{d}{dt}k_1^{FF}(a) \quad \text{and}$$
$$-\frac{(4\pi)^2}{e^3}\beta_2 = p\frac{d}{dp}k_2^{FF}(a) = \frac{d}{dt}k_2^{FF}(a). \tag{84}$$

Here t is a dimensionless parameter defined by $t = \ln(p/m)$.

The UV limit is achieved for $p \gg m$, or equivalently $t \gg 1$, while in the IR limit we meet inverse relations $p \ll m$ and $t \ll 1$. With the help of the MATHEMATICA computer software, one can calculate explicitly the above beta functions and integrate them. For a clear illustration, we plot the beta functions for both cases as functions of the parameter a (see Fig. 1).

The integration of the renormalization equation corresponding to β_1 can be performed by using the MATHEMATICA software such that the integral curve describes the running coupling constant in the physical scheme. For comparison, we plot in Fig. 2 this curve

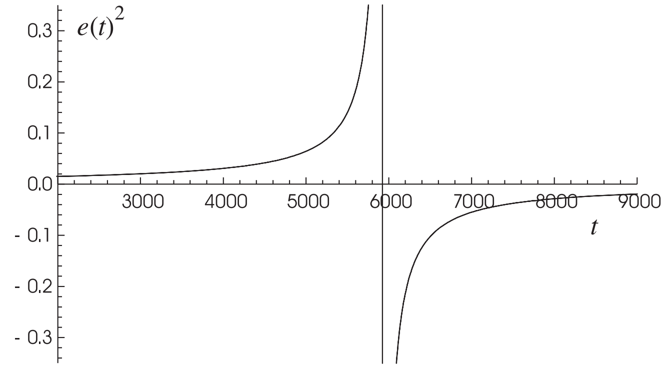

FIG. 2. The curves for the running parameter $e(t)^2$ seem to coincide for both MS and physical scheme (in the last one we use the prescription corresponding to the form factor k_1). Actually, both curves are slightly different as well as the Landau pole (see Fig. 3). We have used $e(0) = 0.1$.

together with the curve for the running parameter for the minimal subtraction scheme, for large value of t, where a Landau pole shows up. It is not easy to visualize a difference between both cases, which look almost identical. Actually, the curve for the running parameter in the physical scheme is shifted a little bit to the right. This difference can be made clear if we increase the plot scale in the region around $t \sim 5900$, as illustrated in Fig. 3. Here, one can see actually two vertical lines [the dashed one for the minimal subtraction (MS) scheme], indicating two different Landau poles. They are, however, very close, since the corresponding values for t differ by less than 0.02%. The situation is very similar to the one described earlier for the scalar field theory [71].

The running of the effective electromagnetic charge is shown in Fig. 4, where we plot e^{-2} versus $t = \ln(p/m)$. The plot for the $\overline{\text{MS}}$ case is a straight line, as it has to be. The plot for the momentum subtraction scheme represents two straight lines in the asymptotic (IR and UV) limits with

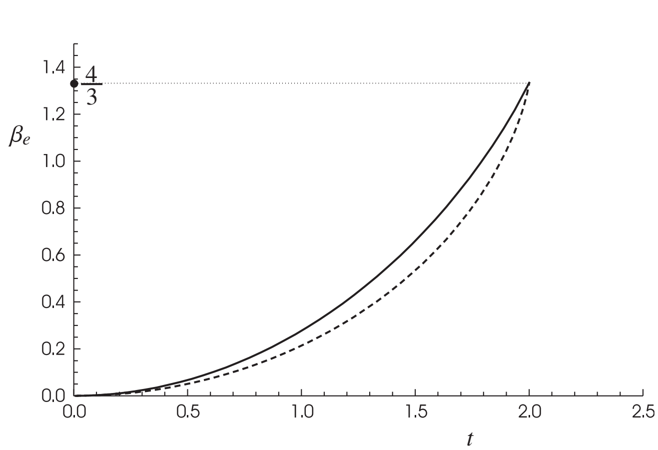

FIG. 1. The beta functions corresponding to form factors k_1 (solid line) and k_2 (dashed line). The vertical axis is drawn in units of $e^3/(4\pi)^2$.

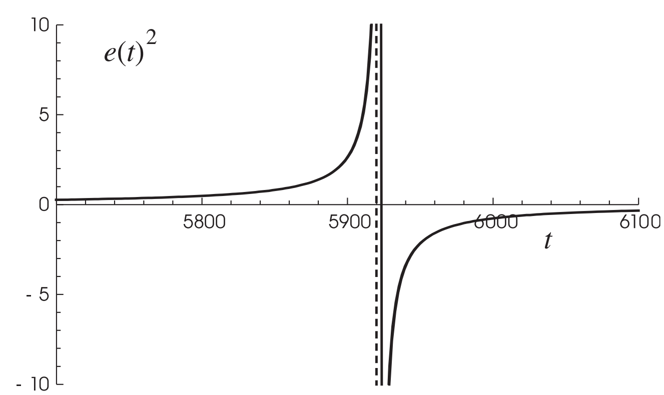

FIG. 3. The same plot as in Fig. 2, viewed with more detail around $t \sim 5900$. Here the two vertical lines indicate different Landau poles. The dashed line corresponds to the MS scheme. The value for t corresponding to the poles differs from each other by less than 0.02%.

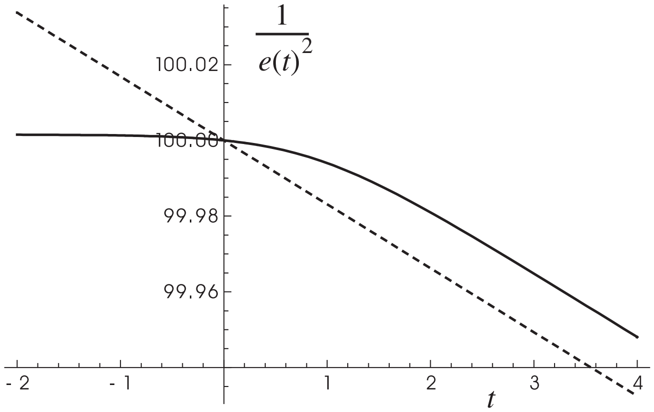

FIG. 4. Curves for $e(t)^{-2}$. The dashed line corresponds to the MS scheme and the solid line for the momentum subtraction scheme. For higher t, both curves are parallel straight lines. Substantial difference between the two plots takes place for $t < 1$.

the smooth transition between them in the intermediate region. It is easy to note that the curves are very similar to each other in the UV region, that is for $t \gg 1$ (in fact this feature holds already for $t \geq 2$). At the same time, even in the UV the two plots do not coincide and are represented by two parallel straight lines with slightly different initial points (at $t = 0$, that means $p = \mu_0$ and $\mu = \mu_0$). The effect of the effective UV shift of initial point will take place not only in QED, but also in QCD and electroweak sectors of the SM and beyond. This effect, despite being very small, may have interesting applications. For example, one can take into account the effective shift of the initial values of the couplings when calculating the UV effects in gauge theories via the renormalization group (see, e.g., [72]). The calculations of this sort are quite relevant, in particular for the physics related to future experiments on LHC. Taking the effect of the masses of the quantum fields into account may, in principle, improve the precision of the results. As a particular case, the effect of the initial data shift can be also observed in the supersymmetric models such as the minimal supersymmetric standard model and its extensions. As a result, there may be a slight violation of the exact convergence of the running coupling constants g, g' and g_s which will likely form a small triangle rather than meet in a single point. This feature of the massive theories has been already discussed in [73], but the effect can be done more explicitly when using analytic expressions for the β functions.

VIII. ON THE CONFORMAL VIOLATION IN THE MASSIVE CASE

In the previous sections we have considered the existing interface between the quantum contributions coming from the massive and massless field loops. In particular, we have seen that, in the high-energy limit, the form factors and the β functions tend to the ones in the massless case; that means they become close to the ones in the minimal subtraction scheme of renormalization ($\overline{\text{MS}}$, in our case). As we already learned in Sec. II, in this situation the violation of local conformal symmetry occurs due to anomaly and the corresponding EA can be presented in a closed form (17).

On the contrary, the low-energy quantum effects are characterized by the phenomenon of decoupling [33]. As a result, in the IR limit the quantum effects are quadratically suppressed and one can rely on the classical Maxwell equations. An interesting question is what is the difference between the violation of local conformal symmetry in the "far UV" limit and in the "far IR" limit, where we have a remnant quantum effects? Let us emphasize that there is no contradiction in having two different forms for the violation of conformal symmetry, because they correspond to the two different physical situations.

Let us first compare the dependences on the conformal factor in the two symmetry-violating terms. For the sake of simplicity we can take $\sigma = \text{const}$, that means the global conformal symmetry. The situation for the local case will not be very different. In the UV limit, exactly as in the purely massless case we have, according to the Eq. (71), the linear dependence on σ. Of course, the same result follows from the complete expression (17) derived for the precisely massless case. On the contrary, in the situation of IR decoupling, we can take the lowest order terms directly from the a_3 coefficient. In this case one meets the $F\Box F$-type expression (83) and also RFF-type terms. In both cases the σ dependence is exponential; that means the symmetry is violated by the terms which have a scaling law $e^{-2\sigma}$. So, it looks like the scaling of the symmetry-violating terms is even stronger for the low-energy sector. However, this is nothing but a wrong impression.

One can view the situation from a different position, if analyzing this question using the calculated expression (56). Then the physical sense of decoupling in the low-energy limit becomes much more explicit. If we compare the $F\Box F$-type expression (83) and the classical term $F^2_{\mu\nu}$, it is clear that the former has an extra factor of p^2/m^2, where p^2 is the square of the momentum of the photon and the m^2 is the square of the mass of the electron. In the low-energy limit $p^2 \ll m^2$, hence we meet the simplest (and very clear) form of the Appelquist and Carazzone [33] quadratic decoupling law. In the case of RFF-type terms, for most of the physical situations, the decoupling is even much stronger. The reason is that the scalar curvature is proportional to the square of the typical energy of the gravitons and this energy is much smaller than the one of the photon. For instance, in the cosmological setting we have $R \propto H^2$, where H is the Hubble parameter. In the present-day Universe the corresponding values are $H_0 \approx 10^{-42}$ GeV for the Hubble parameter and $\epsilon = \sqrt{p^2} \approx 10^{-12}$ GeV for the energy of the CMB photon. Similar relation between the two quantities holds during most of the evolution of the Universe. In this situation, the most important low-energy contribution to the violation of conformal symmetry comes from the (83) term, which has a nontrivial flat limit.

IX. DISCUSSIONS AND SUMMARY

The two distinct approaches to the derivation of one-loop quantum corrections to the photon sector of the curved-space QED have been explored. First, we derived the anomaly-induced action, coming from the integration of conformal anomaly. The effective action which we gain in this way has an enormous advantage of being exact, in the sense it is not related to some particular form of series expansion, except the one into the loops (taken in the first order). The representations obtained here can serve, in principle, as a consistent background for the investigation of quantum processes in the early Universe, when the masses of quantum matter (fermionic) fields are irrelevant. The anomaly-induced action has yet another advantage of being scheme independent, because it is based on the minimal subtraction renormalization scheme and is, after all, controlled by the one-loop divergences.

On the other hand, in the massive case there is the violation of conformal symmetry coming from the form factor in the electromagnetic sector of QED in curved space-time. This calculation is based on the physical renormalization scheme and hence it is supposed to be more adequate in the later Universe, when the masses of the fields play an important role. The price one has to pay for a more consistent physical approach is related to the restricted power of the available calculational methods, which are equivalent to the use of common Feynman diagrams for the linearized metric perturbations on the flat space-time background. The corresponding result which we obtain here includes terms which are quadratic in $F_{\mu\nu}$ and may also depend on the curvature tensor. It is complete in the case of a flat space-time background, but it is not supposed to be a complete one for the curved space-time, where we can expect many higher orders in curvature corrections which cannot be calculated exactly. An interesting point is that we have found that the quantum correction depends on the choice of the calculational scheme [34]. Thus we have proven the existence of the nonlocal and renormalization independent multiplicative anomaly in quantum field theory. One of the consequences of this anomaly is the ambiguity in the prediction of the Appelquist and Carazzone theorem [33], which provides two different coefficients of the quadratic decoupling law at low energies.

Since the quantum corrections in the electromagnetic sector include some ambiguity in the IR region, one should ask which one of the two schemes gives a correct result. In our opinion the advantage should be given to the one derived through the operator $\hat{H}_1^*$, because it is more natural and preserves gauge invariance. However, it is worthwhile to be aware of the ambiguity which is a manifestation of a typical property for the off-shell effective action in quantum field theory. In the present case this ambiguity becomes essential due to the presence of an external gravitational field. One can note that the simpler form of quantum correction (17), derived via conformal anomaly, is also ambiguous due to the presence of an arbitrary functional S_c. Of course, the two kinds of ambiguities are unrelated, but they can be seen as manifestations of a general feature of effective action.

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Note added.—After this paper was resubmitted, we learned about the recent paper [74] (see also [75]) where the same problem was considered using technically different approaches. We would like to cite this paper, especially because the parts of conformal anomaly-induced action are different only due to the choice of parametrization for the auxiliary fields. Also we would like to point out that, despite the mentioned differences, the results are qualitatively similar. In particular, in both approaches the conformal anomaly can be partially restored from the calculations of massive loop in the massless limit.

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