PHYSICAL REVIEW D **80,** 094501 (2009)

Gribov's horizon and the ghost dressing function

Ph. Boucaud,[1] J. P. Leroy,[1] A. Le Yaouanc,[1] J. Micheli,[1] O. Pène,[1] and J. Rodríguez-Quintero[2]

[1]*Laboratoire de Physique Théorique*[*]*1 Université de Paris XI, Bâtiment 210, 91405 Orsay Cedex, France*

[2]*Dpto. Física Aplicada, Fac. Ciencias Experimentales, Universidad de Huelva, 21071 Huelva, Spain*

(Received 15 September 2009; published 9 November 2009)

We study a relation recently derived by K. Kondo at zero momentum between the Zwanziger's horizon function, the ghost dressing function and Kugo's functions u and w. We agree with this result as far as bare quantities are considered. However, assuming the validity of the horizon gap equation, we argue that the solution $w(0) = 0$ is not acceptable since it would lead to a vanishing renormalized ghost dressing function. On the contrary, when the cutoff goes to infinity, $u(0) \to \infty$, $w(0) \to -\infty$ such that $u(0) + w(0) \to -1$. Furthermore w and u are not multiplicatively renormalizable. Relaxing the gap equation allows $w(0) = 0$ with $u(0) \to -1$. In both cases the bare ghost dressing function, $F(0, \Lambda)$, goes logarithmically to infinity at infinite cutoff. We show that, although the lattice results provide bare results not so different from the $F(0, \Lambda) = 3$ solution, this is an accident due to the fact that the lattice cutoffs lie in the range 1–3 GeV^{-1}. We show that the renormalized ghost dressing function should be finite and nonzero at zero momentum and can be reliably estimated on the lattice up to powers of the lattice spacing; from published data on a 80^4 lattice at $\beta = 5.7$ we obtain $F_R(0, \mu = 1.5\ \mathrm{GeV}) \simeq 2.2$.

DOI: 10.1103/PhysRevD.80.094501 PACS numbers: 11.15.Ha, 12.38.Aw, 12.38.Gc, 11.10.Gh

I. INTRODUCTION

Kondo has derived in recent papers [1,2] a relation between the $k = 0$ values of the ghost dressing function $F(k)$, Zwanziger's horizon function $h(k)$, Kugo's function $u(k)$, and an additional function $w(k)$. Applying to this relation Zwanziger's horizon gap equation and assuming that $w(0) = 0$ he derives the surprising result that $u(0) = -2/3$. This is surprising as so simple constraints on bare quantities are rare. We know only the case of the electric charge which benefits of the Ward identity. This surprising result deserves some closer investigation, even more so as lattice results are not so far from it as we shall see. Indeed it has given rise to several publications and there is far from a consensus on this matter [3–6].

To understand better the issue we try in this note to reconsider every point of the discussion from first principles. Our starting point is a set of relations between the functions we have just mentioned. They concern bare quantities, which imposes to use a finite ultraviolet cutoff, else we would have to deal with divergent quantities. In Sec. II we propose a faster derivation of these relations. If one assumes the validity of the Zwanziger horizon gap equation this boils down to very simple relations giving $u(0)$ and $w(0)$ as functions of $F(0)$. We then discuss whether $u(0)$ can be $-2/3$ or any finite quantity. We argue that it is not possible if we assume F to be multiplicatively renormalizable, which nobody would deny. In Sec. III we use lattice QCD and ghost-propagator Dyson-Swinger equation (GPDSeq) to get numbers. From the GPDSeq at small momentum we find that ratio of the bare (resp. renormalized) ghost dressing functions at small and zero momentum, assuming the latter to be finite, is essentially cutoff and renormalization point independent. We extract an estimate of the renormalized $F_R(0)$. Finally we discuss the status of the Zwanziger horizon gap equation on the lattice. Convinced that it has no reason to be valid, we generalize the result of Sec. II for a more general case.

II. GHOST DRESSING FUNCTION, HORIZON FUNCTION, *u* AND *w*

The discussion which follows deals with bare quantities. These are singular and need a regulator, or cutoff, which we will call Λ (in the lattice case, this regulator is a^{-1}, a being the lattice spacing). The dependence in Λ will often be kept implicit, to avoid heavy notations, but is always understood speaking of bare quantities. Renormalized quantities will be marked by the index R. There is no need to specify the renormalization scheme being used, since our results do not depend on a particular choice; however, regarding lattice results, we shall refer as usual to the MOM scheme.

A. Gribov-Zwanziger action

In [1,2], it has been claimed that three-point and four-point functions for gluon and ghost fields can be related in such a manner that the Zwanziger horizon condition strongly constrains the ghost propagator and the ghost-gluon vertex.

It is well-known, since Gribov's famous paper [7], that the gauge-fixing procedure in QCD using the standard Faddeev-Popov procedure is not unambiguous. It leads to a discrete set of solutions, named "Gribov copies". One solution, proposed by Zwanziger [8], *which aims at re-*

[*]Unité Mixte de Recherche 8627 du Centre National de la Recherche Scientifique

stricting the Gribov copies within the Gribov Horizon, consists in using the Gribov-Zwanziger partition function in Landau gauge,

$$Z_\gamma = \int [DA]\delta(\partial A)\det(M)e^{-S_{\rm YM}+\gamma\int d^Dx h(x)}, \qquad (1)$$

for the D-dimensional Euclidean Yang-Mills theory, where $S_{\rm YM}$ stands for the Yang-Mills action, M is the Faddeev-Popov operator,

$$M^{ab} = -\partial_\mu D_\mu^{ab} = -\partial_\mu(\partial_\mu\delta^{ab} + gf^{abc}A_\mu^c) \qquad (2)$$

and $h(x)$ is the Zwanziger horizon function,

$$h(x) = \int d^D y g f^{abc} A_\mu^b(x)(M^{-1})^{ce}(x,y) g f^{afe} A_\mu^f(y); \quad (3)$$

that restricts the integration over the gauge group to the first Gribov region, provided that the Gribov parameter, γ, is a positive number that is to be determined by solving the so-called gap equation:

$$\langle h(x)\rangle_\gamma = (N^2-1)D. \qquad (4)$$

The horizon function is a bare quantity depending on the cutoff parameter, as γ also does through the implementation of the horizon condition that requires that the gap equation be solved for every cutoff value. We will postpone the discussion of this gap equation to a later section. In any case, the horizon function is a well defined bare quantity and it is relevant to first derive, *independently of eq.* (4) the relation its v.e.v. has with other quantities.

B. Relating $h(0)$, $u(0)$, $w(0)$ and $F(0)$

In this subsection we propose a simplified derivation of Kondo's relation (cf [1,2]) which relates the v.e.v. of the horizon function $h(0)$ to the ghost dressing function at vanishing momentum and the Kugo-Ojima parameters. Then, contrarily to Kondo, we will add no assumption about the Kugo-Ojima parameters but simply combine Kondo's relation with the one discovered by Kugo between the ghost dressing in Landau gauge with these Kugo-Ojima parameters and discuss about their general implications.

$$\begin{aligned}\langle h(0)\rangle_{k=0} &= \lim_{k^2\to 0}\frac{1}{V_D}\int d^Dx\int d^Dy \langle gf^{abc}A_\mu^b(x)(M^{-1})^{ce} \\ &\quad\times (x,y) g f^{afe}A_\mu^f(y)\rangle e^{ik\cdot(x-y)} \\ &= \lim_{k\to 0}\frac{1}{V_D}\int d^Dx\int d^Dy\langle (gf^{abc}A_\mu^b c^c)_x \\ &\quad\times (gf^{afe}A_\mu^f\bar{c}^e)_y\rangle e^{ik\cdot(x-y)} \\ &= \lim_{k^2\to 0}\int d^D(x-y)\langle (gf^{abc}A_\mu^b c^c)_x \\ &\quad\times (gf^{afe}A_\mu^f\bar{c}^e)_y\rangle e^{ik\cdot(x-y)} \\ &= \langle (gf^{abc}A_\mu^b c^c)(gf^{afe}A_\mu^f\bar{c}^e)\rangle_{k^2\to 0}\end{aligned} \qquad (5)$$

where we use the simplified notation:

$$\langle(\ldots)(\ldots)\rangle_k \equiv \int d^D(x-y)\langle(\ldots)_x(\ldots)_y\rangle e^{ik\cdot(x-y)} \qquad (6)$$

that was introduced in Refs. [1,2] and that will be followed from now on. To establish Eq. (5), nothing is needed but the relation between the inverse Faddeev-Popov operator and the ghost and antighost fields and the translational invariance. Define then the function $u(k^2)$, the value of which at vanishing momentum gives the Kugo-Ojima parameter, as

$$\langle (D_\mu^{ab}c^b)(gf^{cde}A_\nu^d\bar{c}^e)\rangle_k = -\delta_{\mu\nu}^T\delta^{ac}u(k^2); \qquad (7)$$

where $k^2\delta_{\mu\nu}^T(k) \equiv k^2\delta_{\mu\nu} - k_\mu k_\nu$ and the transversality is guaranteed by the well-known identity:

$$\begin{aligned}\langle(\partial_\mu D_\mu^{ab}c^b)(gf^{cde}A_\nu^d\bar{c}^e)\rangle_k &= -ik_\mu\langle(D_\mu^{ab}c^b)(gf^{cde}A_\nu^d\bar{c}^e)\rangle_k \\ &= 0.\end{aligned} \qquad (8)$$

Now, by merely invoking the definitions of u (Eq. (7)) and of the covariant derivative,

$$D_\mu^{ab} \equiv \delta^{ab}\partial_\mu + gf^{acb}A_\mu^c, \qquad (9)$$

acting on the ghost and antighost fields, one obtains

$$\begin{aligned}\langle(gf^{abc}A_\mu^b c^c)(gf^{def}A_\nu^e\bar{c}^f)\rangle_k &= \underbrace{\langle(D_\mu^{ac}c^c)(gf^{def}A_\nu^e\bar{c}^f)\rangle_k}_{-\delta_{\mu\nu}^T(k)\delta^{ad}u(k^2)} \\ &\quad - \underbrace{\langle\partial_\mu c^a(gf^{def}A_\nu^e\bar{c}^f)\rangle_k}_{-ik_\mu\langle c^a(gf^{def}A_\nu^e\bar{c}^f)\rangle_k}\end{aligned} \qquad (10)$$

which proves that the transversal part of the left-hand side (lhs) of Eq. (10) is given by Eq. (7),

$$\delta_{\mu\mu'}^T\langle(gf^{abc}A_{\mu'}^b c^c)(gf^{def}A_\nu^e\bar{c}^f)\rangle_k = -\delta_{\mu\nu}^T(k)\delta^{ad}u(k^2). \qquad (11)$$

while the longitudinal part can be written as follows:

$$\begin{aligned}&k_\mu\langle(gf^{abc}A_\mu^b c^c)(gf^{def}A_\nu^e\bar{c}^f)\rangle_k \\ &\quad = ik^2\langle c^a(gf^{def}A_\nu^e\bar{c}^f)\rangle_k \\ &\quad = i\underbrace{k^2\langle c^a\bar{c}^{a'}\rangle}_{\delta^{aa'}F(k^2)}\langle c^{a'}(gf^{def}A_\nu^e\bar{c}^f)\rangle_k^{\rm 1PI};\end{aligned} \qquad (12)$$

where $F(k^2)$ is the bare ghost-propagator dressing function and 1PI notes the one-particle irreducible contribution to the v.e.v. obtained through the amputation of the external ghost leg. Let us then define the function $w(k^2)$ in order to parametrize this longitudinal contribution to Eq. (10) through

$$\langle c^a(gf^{def}A_\nu^e\bar{c}^f)\rangle_k^{\rm 1PI} = i\delta^{ad}k_\nu(u(k^2)+w(k^2)). \qquad (13)$$

This definition is equivalent to the one given in terms of diagrams in Refs. [1,2]. Note also that $w(0)$ was taken to be 0 in the seminal work by Kugo and Ojima.

Taking together Eqs. (10) and (13) one gets:

$$\langle (gf^{abc}A_\mu^b c^c)(gf^{def}A_\nu^e \bar{c}^f)\rangle_k = -\delta^{ad}\Big(\delta_{\mu\nu}^T u(k^2) + F(k^2) \times \frac{k_\mu k_\nu}{k^2}(u(k^2)+w(k^2))\Big), \tag{14}$$

and for the v.e.v of the horizon function [1,2]:

$$\begin{aligned}\frac{\langle h(0)\rangle_{k=0}}{D(N^2-1)} &= -\frac{1}{D(N^2-1)}\langle (gf^{abc}A_\mu^b c^c) \times (gf^{afe}A_\mu^f \bar{c}^e)\rangle_{k^2\to 0} \\ &= -\frac{1}{D}[(D-1)u(0) + F(0)(u(0)+w(0))].\end{aligned} \tag{15}$$

From now on we make explicit the dependence on the cutoff, Λ, of all the bare quantities, which generally will diverge in the infinite cutoff limit.[1]

Kugo has shown in Refs. [9,10] that the Landau gauge condition, $\partial_\mu A_\mu = 0$, can be exploited to give:

$$(1 + u(0,\Lambda) + w(0,\Lambda))F(0,\Lambda) = 1. \tag{16}$$

This result can also be easily derived from the ghost-propagator Dyson-Schwinger equation which, in Landau gauge, can be written as:

$$\begin{aligned}\frac{1}{F(k^2)} &= \frac{\delta^{ab}}{k^2(N^2-1)}\langle c^a \bar{c}^b\rangle^{-1} \\ &= 1 - ik_\mu \langle c^a(gA_\mu^e f^{def}\bar{c}^f)\rangle^{1PI}\frac{\delta^{ab}}{k^2(N^2-1)} \\ &= 1 + u(k^2,\Lambda) + w(k^2,\Lambda)\end{aligned} \tag{17}$$

Then, the two Eqs. (15) and (16) can be combined to obtain, *without any hypothesis about uand w*,

$$\begin{aligned}u(0,\Lambda) &= \frac{F(0,\Lambda)-1}{D-1} - \frac{D}{D-1}\left[\frac{\langle h(0)\rangle_{k=0}}{D(N^2-1)}\right] \\ w(0,\Lambda) &= -1 - u(0,\Lambda) + \frac{1}{F(0,\Lambda)} \\ &= -\frac{F(0,\Lambda)+(D-2)}{D-1} + \frac{1}{F(0,\Lambda)} + \frac{D}{D-1}\left[\frac{\langle h(0)\rangle_{k=0}}{D(N^2-1)}\right]\end{aligned} \tag{18}$$

as general solutions for the Kugo-Ojima parameters, $u(0,\Lambda)$, and $w(0,\Lambda)$ in terms of the bare ghost dressing function at vanishing momentum. *If, in addition, we assume that the gap equation eq.* (4) is satisfied the square bracket in Eq. (18) is equal to 1, independently of the cutoff and we get

[1]It is well-known that the ghost dressing function diverges logarithmically at infinite cutoff in the UV momentum domain.

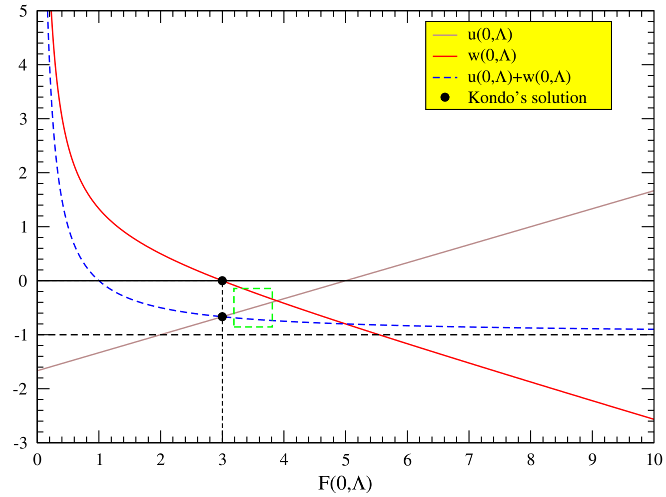

FIG. 1 (color online). The solutions for $u(0,\Lambda)$ and $w(0,\Lambda)$ given by Eq. (19) plotted as a function of $F(0,\Lambda)$. This plot can be understood as a function of $\tilde{Z}_3$ for a given nonzero value of $F_R(0,\mu^2)$. Then the infinite cutoff limit is the limit at infinity of the horizontal axis. The particular solution proposed in Refs. [1,2] (black circles), obtained by imposing $w(0,\Lambda)=0$, corresponds to the intersection of $u+w$ and u. It cannot hold when $\tilde{Z}_3 \to \infty$. The current lattice solutions for the bare ghost dressing functions at vanishing momentum lie inside the green dotted square (see Fig. 3). This apparent approximate agreement is misleading and due to the moderate cutoff value on the lattices.

$$\begin{aligned}u(0,\Lambda) &= \frac{F(0,\Lambda)-1-D}{D-1} \\ w(0,\Lambda) &= -1 - u(0,\Lambda) + \frac{1}{F(0,\Lambda)} \\ &= -\frac{F(0,\Lambda)-2}{D-1} + \frac{1}{F(0,\Lambda)}\end{aligned} \tag{19}$$

In Fig. 1, the solutions of Eq. (19) are plotted as functions of $F(0,\Lambda)$. It is obvious from Eq. (19) (see also Fig. 1) that, had we required $w(0,\Lambda)=0$, the solution proposed in Refs. [1,2] would emerge: $u(0,\Lambda) = -2/3$ and $F(0,\Lambda)=3$, for $D=4$. However we shall present in the next subsection the arguments which lead us to believe that no solution implying a cutoff independent and finite value[2] for $F(0)$ can be accepted.

C. Constraints from renormalizability

In this section we assume the validity of the relation (4) Let us start from the basic equation

$$F(0,\Lambda) = \tilde{Z}_3(\mu^2,\Lambda)\Big(F_R(0,\mu^2) + \mathcal{O}\Big(\frac{1}{\Lambda^n}\Big)\Big), \tag{20}$$

[2]This finite value is independent of the number of colors and, provided that $w(0,\Lambda)=0$, should be the same whichever regularized Yang-Mills action we use (any lattice action, for instance) or even including any nonzero number of quark flavors for the action.

where F_R is the renormalized dressing function in the infinite cutoff limit and n some positive number. For any quantity which is *multiplicatively* renormalizable, a field-theory nonperturbative renormalization scheme (in particular, those applied in lattice field theory) implies a relation of this kind where the crucial point is that *the cutoff dependence is an inverse power of the cutoff and cannot behave like some power of the cutoff's logarithm* [11,12]. Now, we know from perturbation theory how $\tilde{Z}_3(\mu^2, \Lambda)$ depends on the cutoff; choosing a fixed value Λ_0 and sending $\Lambda \to \infty$, $\tilde{Z}_3$ diverges logarithmically in the infinite cutoff limit, regardless of the renormalization procedure:

$$\frac{\tilde{Z}_3(\mu^2, \Lambda)}{\tilde{Z}_3(\mu^2, \Lambda_0)} = \left(\frac{\log(\Lambda/\Lambda_{\rm QCD})}{\log(\Lambda_0/\Lambda_{\rm QCD})}\right)^{9/44}(1 + \mathcal{O}(\alpha)), \quad (21)$$

Although this behavior is quite general, the specific value $\frac{9}{44}$ of the exponent is valid only in the case $N = 3$, $N_f = 0$. This provides us with two main objections for a finite bare ghost dressing function:

(i) A finite value of $F(0, \Lambda)$ requires, through Eq. (20), that $F_R(0, \mu^2)$ be zero; taking into account the fact that the subdominant terms, which vanish as $\Lambda \to \infty$, are supposed not to be logarithmic contributions but, at least, of the order of $1/\Lambda$ we are forced to conclude that zero is the *only* allowed finite value that the bare ghost dressing function can hit for Eq. (20) to be consistently satisfied.

(ii) We can apply Eq. (29a) which will be discussed later and implies at small q^2 a cutoff independent factor, decreasing with q^2, which multiplies $F(0, \Lambda)$ and as well $F_R(0, \mu^2)$. Recalling that the path integration has been limited by the Zwanziger procedure to a region in which the Faddeev-Popov operator is positive, we see that if $F_R(0, \mu^2) = 0$, F_R can only assume the value 0 throughout some range of q^2, which sounds weird. Even more, there are numerical evidences that the ghost dressing function $F(q^2, \Lambda)$ decreases for all q^2. Then $F_R(q^2, \mu^2) = 0$ should hold for any q^2.

Therefore, the only way out we see is that the bare ghost dressing function diverges logarithmically in the infinite cutoff limit and that multiplication by $\tilde{Z}_3^{(-1)}$ provides a strictly positive renormalized value. Then, Eq. (19) can be rewritten as:

$$\begin{aligned} F_R(0, \mu^2) &= \tilde{Z}_3^{-1}(\mu^2, \Lambda)((D-1)u(0, \Lambda) + D + 1) \\ &= (D-1)\left[\tilde{Z}_3^{-1}(\mu^2, \Lambda)u(0, \Lambda) \right. \\ &\quad \left. + \mathcal{O}\left(\frac{1}{\log\Lambda}\right)\right]u(0, \Lambda) + w(0, \Lambda) \\ &= -1 + \mathcal{O}\left(\frac{1}{\log\Lambda}\right). \end{aligned} \quad (22)$$

An important consequence of the first of Eqs. (22) is that the function $u(0, \Lambda)$ cannot be multiplicatively renormalized. The fact that, when multiplied by Z_3^{-1}, it gives a finite result in the infinite cutoff limit does not suffice. As we have already recalled, it has to differ from its asymptotic value by inverse powers of the cutoff, which is obviously wrong in Eqs. (22) where $\tilde{Z}_3^{-1}u(0, \Lambda)$ only converges up to inverse powers of the *logarithm of the cutoff*. The same is true for $w(0)$. This is not surprising since they are defined by the insertion of the composite operators shown in Eqs. (7) and (13).

Only the combination $1 + u(0, \Lambda) + w(0, \Lambda)$ vanishes logarithmically as $\Lambda \to \infty$ so that,

$$\begin{aligned} \tilde{Z}_3(\mu^2, \Lambda)(1 + u(0, \Lambda) + w(0, \Lambda)) \\ = \frac{1}{F_R(0, \mu^2)} + \mathcal{O}\left(\frac{1}{\Lambda}\right), \end{aligned} \quad (23)$$

while both $u(0, \Lambda)$ and $w(0, \Lambda)$ diverge but their divergences cancel in Eq. (23). Thus, as done in [2,13], one can consider $1 + u + w$ to be renormalized[3] by $\tilde{Z}_3$. However, let us repeat, $1 + u$ and w cannot be separately renormalized and $w(0, \Lambda) = 0$ cannot be accepted since in conjunction with Eqs. (4) and (18) it provides a finite $u(0, \Lambda) = -2/3$ and a finite $F(0, \Lambda) = 3$ which we have shown above to be forbidden.

III. COLLECTING AND EXTRAPOLATING BARE GHOST LATTICE DATA

Lattice simulations first provide us with estimates for bare quantities (correlation functions) in the lattice regularization scheme, where the role of the regularization cutoff is played by the inverse of the lattice spacing, a^{-1}. In present simulations a^{-1} is moderate, ranging from $\sim 1\ \text{GeV}^{-1}$ for $\beta = 5.7$ up to $\sim 3.5\ \text{GeV}^{-1}$ for $\beta = 6.4$. Those bare quantities should be further renormalized by applying MOM-like schemes. That this multiplicative renormalization procedure works has been proven by Reisz in Ref. [11]; the remaining corrections due to finite spacing (vanishing in the continuum limit) are considered to behave as powers of the lattice spacing. Those renormalized quantities are usually the reliable result of the simulations and the ones directly connected with physical predictions. On the contrary, the recent work [1,2] we discussed above supplies a prediction for a bare quantity: the bare ghost-propagator dressing function. Therefore, the nonrenormalized lattice estimates for this dressing function deserve by themselves a particular interest, as far as they could allow us to test that prediction.

In the last few years, many works have been devoted, at least partially, to the lattice computation of the ghost

[3]Renormalized in the sense that the (logarithmically vanishing) cutoff dependence can be killed at a given renormalization point up to vanishing powers of the cutoff

propagator. They mainly follow Ref. [14] in writing the Faddev-Popov operator as a lattice divergence:

$$M(U) = -\frac{1}{N}\nabla \cdot \tilde{D}(U), \tag{24}$$

where the operator $\tilde{D}$ reads

$$\tilde{D}_\mu(U)\eta(x) = \frac{1}{2}(U_\mu(x)\eta(x+\hat{\mu}) - \eta(x)U_\mu(x) + \eta(x+\hat{\mu})U_\mu^\dagger(x) - U_\mu^\dagger(x)\eta(x)). \tag{25}$$

Those definitions, complemented with conversion routines between the Lie algebra and the Lie group, allow for a very efficient lattice implementation. Some details about the procedure for the inversion of the Faddeev-Popov operator and some results will be found in [15].

The gauge fixing, in particular, for Landau gauge, is a more delicate issue. A minimization of the functional

$$F_U[g] = \mathrm{Re}\sum_x \sum_\mu \left(1 - \frac{1}{N} g(x)U_\mu(x)g^\dagger(x+\hat{\mu})\right) \tag{26}$$

can be achieved by the use of some algorithm driving the gauge configuration to a local minimum of $F_U[g]$. The gauge configurations obtained in this way will lie in the first Gribov region but, in general, they do not reach the fundamental modular region defined as the set of *absolute* minima of $F_U[g]$ on all gauge orbits. A "best-copy" algorithm (basically consisting in choosing the gauge configuration providing the lowest minimum after several minimizations) has also been used as well as a procedure that essentially consists in a simulated annealing technique and is claimed to reach gauge-functional values closer to the global minima than the standard approach (see for instance [16,17] and references therein). Figure 3 presents together results collected from Ref. [16] (for very big lattice-volume simulations with the simulated annealing gauge-fixing) and data from Refs. [15,18,19] (obtained using the standard gauge-fixing); it shows only a weak dependence in the cutoff a^{-1}. This is not surprising since one knows from Eq. (21) that it should behave at leading log as $\beta^{9/44}$ which gives no larger effect than 2.5% on the whole range of β's.

In order to compare the lattice data with the previous results we need to extrapolate them to zero momentum. To carry this task out we derive now from the bare ghost-propagator Dyson-Schwinger equation (GPDSeq) a small-momentum formula the coefficients of which are fitted against the lattice data. The bare GPDSeq can be regularized and evaluated with the help of a subtraction procedure at two different momenta p and k,

$$\frac{1}{F(p^2,\Lambda)} - \frac{1}{F(k^2,\Lambda)} = Ng^2(\Lambda)H_1(\Lambda)\int^{q^2<\Lambda}\frac{d^4q}{(2\pi)^4}\frac{F(q^2,\Lambda)}{q^2} \times \left(\frac{(k\cdot q)^2}{k^2} - q^2\right)\left(\frac{G((q-k)^2,\Lambda)}{(q-k)^4} - \frac{G((q-p)^2,\Lambda)}{(q-p)^4}\right) \tag{27}$$

as explained in Refs. [19,20].[4] In this equation N is the number of colors, $g(\Lambda)$ the bare, cutoff dependent, coupling constant and G stands for the gluon propagator dressing function and H_1 is one of the form factors for the bare ghost-gluon vertex,

$$\tilde{\Gamma}_\nu^{abc}(-q,k;q-k) = -ig_0 f^{abc}(q_\nu H_1(q,k) + (q-k)_\nu H_2(q,k)), \tag{28}$$

that should be finite and only weakly dependent on the momenta by virtue of Taylor's nonrenormalisation theorem [21] and that, consequently, is usually assumed to be constant with respect to the momenta. Such a bare (and cutoff dependent) GPDSeq can be numerically solved, as was done in Ref. [19], with the help of the lattice gluon propagator estimate to be inserted in the integral in Eq. (27). It is known (cf. [19]) that the solutions can belong to 2 different types: while the generic solution goes to a finite nonzero limit in the infrared there exists also an exceptional one which diverges as $1/\sqrt{k^2}$ in this same limit. The solutions appear to be dialed by the size of the coupling $g(\Lambda)$ in front of the integral in Eq. (27), the exceptional one resulting for a given critical value of the coupling. The same is found by the authors of Ref. [22] but depending on whether $F(0)$, taken as a boundary condition, is ∞ (exceptional solution) or not (regular). The lattice simulations clearly favor the first type and it is implied in the coming discussion that we are in this situation. The only unknown ingredient is the (constant) value for the ghost-gluon vertex form factor, $H_1(\Lambda)$. In addition, following Ref. [20], one can derive a small-momentum expansion for its solution

$$F(q^2,\Lambda) = F(0,\Lambda)\left[1 + \frac{NH_1(\Lambda)R(\Lambda)}{16\pi}q^2\log(q^2) + \mathcal{O}(q^2)\right], \tag{29a}$$

$$\text{with: } R(\Lambda) = \frac{g^2(\Lambda)}{4\pi}F(0,\Lambda)^2\left(\lim_{k\to0}\frac{G(k^2,\Lambda)}{k^2}\right) = \lim_{k\to0}\frac{\alpha_T(k^2)}{k^2} + \mathcal{O}\left(\frac{1}{\Lambda}\right); \tag{29b}$$

where α_T is the coupling constant defined in the Taylor

[4]In these references we dealt with renormalized quantities but everything applies straightforwardly for bare ones.

scheme [23]. Before turning to exploit this expansion we shall comment briefly on the various parameters it involves.

Insofar as the gluon propagator reaches a finite nonzero value at vanishing momentum (as it appears to be true on the lattice), $R(\Lambda)$ takes a finite value which depends on the cutoff by inverse powers. Thus, the dominant (in relative terms) q^2-dependent part of F in the vicinity of 0 [i.e. the second term in the bracket of Eq. (29a)] *does not require any renormalization*,

$$\lim_{\Lambda\to\infty} \frac{F(q^2,\Lambda)}{F(0,\Lambda)} = 1 + NH_1 \frac{\alpha_T(q^2)}{16\pi} \log(q^2) + O(q^2). \tag{30}$$

This (quasi-)independence of the slope with respect to the cutoff is of course important to ensure multiplicative renormalizability, since the latter demands that the q^2-dependence of the bare and renormalized Green's functions be the same up to negative powers of the cutoff. On the contrary the global multiplicative factor $F(0, \Lambda)$ is known to be logarithmically divergent with Λ, which implies that its variation could be appreciable.

Since $R(\Lambda)$ is to be evaluated from lattice estimates of zero-momentum gluon and ghost propagators, one expects it to be sensitive to finite-volume artefacts, to which much attention should therefore be paid. In Fig. 2 one notices the strong dependence of the zero-momentum gluon propagator on the lattice size, which implies an equally strong dependence for $R(\Lambda)$. When evaluated from the zero-momentum estimates for a 80^4-lattice at $\beta = 5.7$ in Ref. [16], $R(\beta = 5.7)$ takes on the value of 10.3 (while, for instance, from the data for a 32^4-lattice at $\beta = 5.8$ in Ref. [19], one would obtain $R(\beta = 5.8) \simeq 19$). The bare vertex form factor (supposed to be constant) was indirectly estimated in ref. [19] for a 32^4-lattice at $\beta = 5.8$ and appeared to be $H_1(\beta = 5.8) \simeq 1.2$.

Provided that finite-size and lattice-spacing artefacts can be neglected for the bare ghost-gluon vertex, the values of R and H_1 we have just determined can be used to attempt to describe the ghost dressing function at small momenta estimated from the 80^4 lattice at $\beta = 5.7$ in Ref. [16]. Then the only parameter in Eq. (29a) which remains to be determined by the best fit is the zero-momentum ghost dressing function. Actually, since a value for $F(0, \Lambda)$ is required in computing $R(\Lambda)$ one has to proceed by iterations: the known value of $G^{(2)}(0, \Lambda)$ and an initial guess of $F(0, \Lambda)$ are inserted in Eq. (29b) to produce a first estimate of $R(\Lambda)$. The latter is then used in Eq. (29a) to perform a new fit of the 80^4-lattice data deprived of the few momenta with $p < 4\pi/L$. It appears actually that, for all the lattice data sets plotted in Fig. 3, only the momenta satisfying his condition are affected by sizeable lattice-volume artefacts. This produces a new estimate of $F(0)$ and the process is iterated until it eventually converges to

$$R(\beta = 5.7, 80) = 10.3, \qquad F(\beta = 5.7, 80) = 3.50. \tag{31}$$

The fit is presented as a solid line in Fig. 3.

The impact of the finite-volume effect due to the lattice determination of $R(\Lambda)$ can be approximatively estimated in the following way. First, the zero-momentum gluon propagator data for the three different lattice volumes at $\beta = 5.7$ in Fig. 2 can be extrapolated up to infinite volume (we work only with data for the same β, in order avoid any mixing

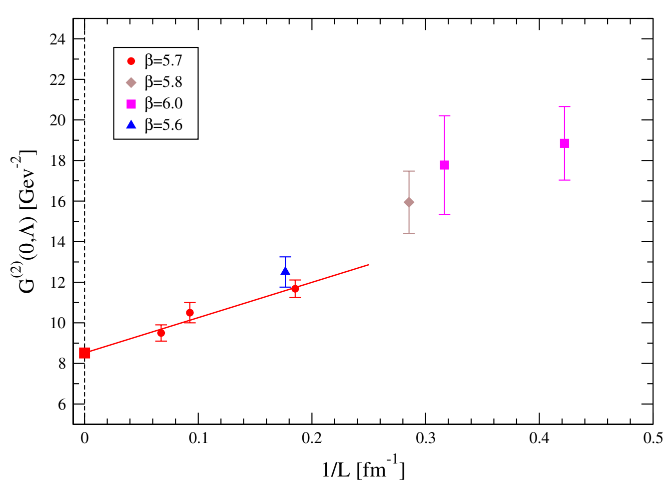

FIG. 2 (color online). Bare zero-momentum gluon propagator estimated from different lattice data sets plotted in terms of the inverse of the lattice size in physical units. The data for the two bigger lattice volumes are taken from Ref. [16], the smaller volume at $\beta = 5.7$ corresponds to Ref. [27] and the others to Refs. [15,19]. A linear fit for the three data at $\beta = 5.7$ to extrapolate at infinite volume is presented as a solid red line.

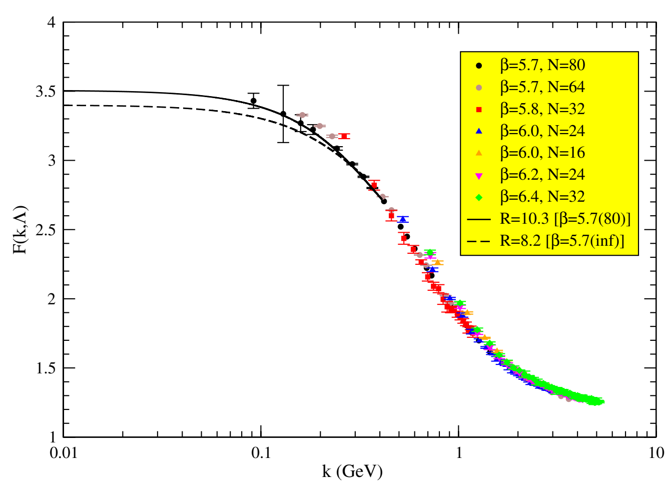

FIG. 3 (color online). Bare ghost dressing function estimated from different lattice data sets. The data for the two larger lattice volumes are taken from Ref. [16] and the others from Refs. [15,19]. the solid line is for the best fit with the small-momentum expansion in Eq. (29a) with $R(\beta = 5.7(80^4))$ and the dashed one stands for the best fit with $R(\beta = 5.7, \infty)$, both computed as explained in the text.

between lattice-spacing and volume effects). This is the starting point to repeat the iterative procedure explained above and one gets $R(\beta = 5.7, \infty) = 8.2$ and $F(\beta = 5.7, \infty) = 3.40$. The extrapolation is also shown (with a dashed line) in Fig. 3.

As for the bare Kugo-Ojima parameter $u(0, \Lambda)$, it is estimated to be of the order of $-(0.6 - 0.8)$ in Refs. [24,25] and very recently in Ref. [13] by using a mixed approach, analogous to the one previously applied to solve Eq. (27), in which DS equations are solved with the input of a lattice estimate of the gluon propagator.

A. Horizon gap equation and lattice QCD

A few words are in order to compare the two approaches we have considered in this note.

The Gribov-Zwanziger (GZ) approach Eq. (1) proposes a modification of the standard QCD action in order to limit the domain of the path integration to the domain within the Gribov horizon, in which the Faddeev-Popov (FP) operator is positive, i.e. all its eigenvalues are positive.

Lattice QCD uses the genuine QCD action. Some algorithm minimizes the functional which discretizes the functional $\int d^4 x A_a^\mu A_a^\mu$. These algorithms differ, but they all stop at a local minimum. Local minimum means that all the second order derivatives are positive, i.e. that the FP operator is positive.

Therefore the two approaches share the property that they limit themselves to the interior of the Gribov horizon. None of them manages to find the absolute minimum of that functional, i.e. to stay within the fundamental domain. There is also a "thermodynamic" argument claiming that one stays close to Gribov's horizon which means in a domain where the eigenvalues of the FP operator should be small. This also seems to be valid for both approaches.

Now come the differences. The GZ approach gives some weight to the different Gribov copies within the Gribov horizon. The lattice algorithms give another one, which, moreover, presumably depends on the specific features of each algorithm: it has been argued that stimulated annealing [16,17] leads to smaller values of the minimized functional.

On the lattice it is possible in principle to compute the v.e.v. of the horizon function $\langle h(0)\rangle$. Nothing imposes that it should be independent on the cutoff and we do not see any reason why it should verify the horizon gap equation Eq. (4). This is precisely a place where the differences we just mentioned could be visible.

In order to discuss this situation let us define a factor $\kappa(\Lambda)$ such that

$$\langle h(0)\rangle_{k=0} = \lim_{k\to 0} \frac{1}{V_D} \int d^D x \langle h(x)\rangle e^{ik\cdot x} = \kappa(\Lambda)(N^2 - 1)D. \quad (32)$$

The value $\kappa(\Lambda) = 1$ corresponds to Eq. (4). Equation (18) now reads

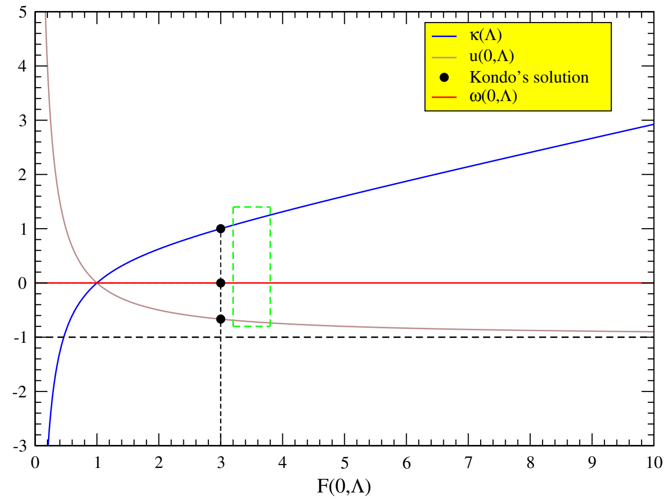

FIG. 4 (color online). The same plot shown in Fig. 1 but the gap equation is given here by Eq. (32), the solid blue line being for the new factor $\kappa(\Lambda)$ in that equation, and $w(0, \Lambda)$ is required to be zero, as explained in the text. Again, current lattice estimates lie inside the green dotted square.

$$\begin{aligned} u(0,\Lambda) &= \frac{F(0,\Lambda) - 1}{D-1} - \frac{D\kappa(\Lambda)}{D-1} \\ w(0,\Lambda) &= -1 - u(0,\Lambda) + \frac{1}{F(0,\Lambda)} \\ &= -\frac{F(0,\Lambda) + D - 2}{D-1} + \frac{1}{F(0,\Lambda)} + \frac{D\kappa(\Lambda)}{D-1} \end{aligned} \quad (33)$$

If the gap equation Eq. (4) is relaxed in this way it becomes possible to have $w(0) = 0$ as shown in [13,26] in the Landau background gauge and assumed in [1,2]. The solution becomes

$$\begin{aligned} \kappa(\Lambda) &= \frac{F(0,\Lambda)}{D} + \frac{D-2}{D} - \frac{D-1}{DF(0,\Lambda)} \\ u(0,\Lambda) &= \frac{1}{F(0,\Lambda)} - 1. \end{aligned} \quad (34)$$

implying that $\kappa(\Lambda) \to \infty$ and $u(0, \Lambda) \to -1$ when $\Lambda \to \infty$ (see Fig. 4). This does not change our major conclusion that no finite value of $F(0, \Lambda)$ independent of Λ is acceptable. In particular it happens that $\mathbf{u}(\mathbf{0}, \boldsymbol{\Lambda} \to \infty)$ *converges to* -1, *nevertheless Kugo-Ojima's condition only emerges at the infinite cutoff limit and thus* $\mathbf{0} < \mathbf{F_R}(\mathbf{0}, \boldsymbol{\mu}^2) < \infty$.

Lattice measurements, which correspond to $\Lambda \sim$ 1–4 GeV, give an estimate of $\kappa(\Lambda)$ which, with $u \sim (0.6–0.8)$ and $F(0) \sim 3.5$, leads to $\kappa(\Lambda) \simeq 1.2–1.3$. However this does not tell how $\kappa(\Lambda)$ depends on Λ.

IV. CONCLUSIONS

We have generalized the solution recently proposed by Kondo for the zero-momentum ghost dressing function and the Kugo-Ojima parameter, $u(0, \Lambda)$, by deriving Eqs. (18) where both Kugo-Ojima parameters, $u(0, \Lambda)$ and $w(0, \Lambda)$ appear written in terms of $F(0, \Lambda)$ and the horizon function, $\langle h(0,\Lambda)\rangle$, at any finite cutoff. In particular, we have shown that the one relating $u(0,\Lambda)$ and $F(0,\Lambda)$, after applying the gap equation, Eq. (4), is close to be verified by lattice estimates for $\Lambda = a^{-1} \sim 1$ GeV, but that this is a pure coincidence due to the small cutoff in lattice calculations. We have argued that neither u nor w can be multiplicatively renormalized. Indeed, from the anomalous dimension of the ghost-propagator renormalization constant we conclude that no solution with a cutoff independent bare $F(0,\Lambda)$ is possible. If the gap equation is valid, for $\Lambda \to \infty$, then $u(0,\Lambda) \to \infty$ and $w(0,\Lambda) \to -\infty$ such that $u(0,\Lambda) + w(0,\Lambda) \to -1$. If one relaxes the gap equation Eq. (32), one can satisfy $w(0,\Lambda) = 0$ with $\kappa(\Lambda) \to \infty$ and $u(0,\Lambda) \to -1$. The Kugo-Ojima condition $u(0) = -1$ is asymptotically fulfilled for the bare u while the renormalized ghost dressing function is finite and nonzero. We have argued that lattice QCD, notwithstanding some similarity with the Gribov-Zwanziger approach, has no reason to fulfill Zwanziger's horizon gap equation. We have shown, however, that this fact will not change much about our conclusions concerning the ghost propagator.

Our major conclusion about the ghost propagator is obtained from the joint use of lattice data and of a result stemming from the ghost-propagator Dyson-Schwinger equation: this result consists in a simple and cutoff independent formula for the ghost-propagator dependence at small momentum. If we choose 1.5 GeV as the renormalization scale, we get from lattice

$$F(1.5\ \text{GeV}) \equiv \tilde{Z}_3 \simeq 1.6 \quad \text{whence } F_R(0, 1.5\ \text{GeV}) \simeq 2.2. \tag{35}$$

This has of course to be refined particularly regarding finite-volume effects which should be considered with more care. The major point in this note, from the point of view of the renormalizability of the theory, is that lattice artefacts behave as powers of a (in this case $O(a^2)$). Of course at $\beta = 5.7$ lattice spacing is not yet small and this leads to a significant uncertainty which deserves further study. Any statement from lattice concerning bare quantities has to be taken with great caution since the very slow logarithmic dependence has chances to escape numerical observation.

ACKNOWLEDGMENTS

It is a pleasure for the authors to acknowledge fruitful discussions with K-I. Kondo, D. Dudal, A. C. Aguilar, D. Binosi and J. Papavassiliou. This work has been partially funded by the Spanish research project FPA2006-13825.

Note added in proof.—After the submission of this manuscript, a new paper [28] from K-I. Kondo appeared and dealt with the matter by implementing the horizon condition in the ghost-propagator DSE in a different way. The author discussed two different horizon-condition definitions and also did for both the same sort of analysis we proposed in this paper. The main source of discrepancy between his conclusions and ours in this paper comes from the proposal he made of a solution with a finite ghost dressing function, $F(k^2)$ at the infinite cutoff limit with $u(k^2,\Lambda) + w(k^2,\Lambda)$ being divergent. F, u and w being bare objects computed with bare fields and couplings. But this is in contradiction with the ghost-propagator DSE written in Eq. (17) [eq. (2.7) of [28]) because, as was discussed in Sec. II C, if $u(k^2,\Lambda) + w(k^2,\Lambda)$ diverges at the infinite cutoff limit, then $F(k^2, \Lambda \to \infty)$ should vanish for the ghost-propagator DSE Eq. (17) to be obeyed.

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