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Moments of event shapes in electron-positron annihilation at next-to-next-to-leading order

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This article gives the perturbative next-to-next-to-leading order results for the moments of the most commonly used event shape variables associated to three-jet events in electron-positron annihilation: thrust, heavy jet mass, wide jet broadening, total jet broadening, C parameter and the Durham three-to-two-jet transition variable.

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I. INTRODUCTION

The electron-positron annihilation experiments at the colliders LEP (CERN), SLC (SLAC) and PETRA (DESY) have collected a wealth of data with hadronic final state over a wide range of energies. Of particular interest are three-jet events, which can be used to extract the value of the strong coupling. Three-jet events are well suited for this task because the leading term in a perturbative calculation of three-jet observables is already proportional to the strong coupling. In comparing experiments to theory it is important to restrict oneself to infrared safe observables. For three-jet events in electron-positron annihilation there is a well established set of infrared safe observables, which is widely used. These are the event shape variables consisting of thrust, heavy jet mass, wide jet broadening, total jet broadening, the C parameter and the Durham three-to-two-jet transition variable. All these observables can be calculated in perturbation theory. Next-to-next-to-leading-order (NNLO) results for the distributions of these observables have been presented in [1–4]. Apart from the distributions also the moments of these observables are of interest to the experimentalists. For an observable O which takes values between 0 and 1 the nth moment is defined by

$$\langle O^n \rangle = \frac{1}{\sigma_{\rm tot}} \int_0^1 O^n \frac{d\sigma}{dO} dO. \qquad (1)$$

All observables are chosen such that $O \to 0$ corresponds to the two-jet region. On the other hand, large values of O correspond to the multijet region. For all n the nth moment receives contributions from both regions, but the two regions are weighted differently for different n: For higher n more weight is given to the multijet region.

In this article I present the QCD NNLO results for the moments of the event shape observables. The present calculation is based on the numerical Monte Carlo program Mercutio2 [5–7]. Independent results for the moments have been published in [8].

The results of this paper rely heavily on research carried out in the past years related to differential NNLO calculations: integration techniques for two-loop amplitudes [9–16], the calculation of the relevant tree-, one- and two-loop-amplitudes [17–29], routines for the numerical evaluation of polylogarithms [30–32], methods to handle infrared singularities [33–54] and experience from the NNLO calculations of $e^+e^- \to 2$ jets and other processes [43,55–73].

This article reports the pure QCD perturbative results for the moments of the event shapes. Not included are soft-gluon resummations [74–77] or power corrections [78,79]. Perturbative electroweak corrections to three-jet observables have been reported recently in [80,81].

From a technical point of view the calculation of the $n = 1$ moment is particularly challenging. The first moment receives sizable contributions from the close-to-two-jet region. The calculation of the NNLO correction in the close-to-two-jet region involves the integration over three unresolved partons—one additional unresolved parton more as compared to a "standard" NNLO calculation. The integration is done numerically by Monte Carlo methods and highly nontrivial.

This paper is organized as follows: In the next section the set of event shape variables is introduced. In Sec. III the perturbative calculation is described. The numerical results for the moments are given in Sec. IV. Finally, Sec. V contains the conclusions.

II. DEFINITION OF THE OBSERVABLES

In this section I briefly recall the definitions of the relevant event shape observables. The event shape variable thrust [82,83] is defined by

$$T = \frac{\max\limits_{\vec{n}} \sum\limits_j |\vec{p}_j \cdot \vec{n}|}{\sum\limits_j |\vec{p}_j|}, \qquad (2)$$

where $\vec{p}_j$ denotes the three-momentum of particle j and the sum runs over all particles in the final state. The thrust variable maximizes the total longitudinal momentum along the unit vector $\vec{n}$. The value of $\vec{n}$, for which the maximum is attained, is called the thrust axis and denoted by $\vec{n}_T$. The value of thrust ranges between $1/2$ and 1, where $T = 1$ corresponds to an ideal collinear two-jet event and $T = 1/2$ corresponds to a perfectly spherical event. Usually one considers instead of thrust T the variable $(1 - T)$, such that the two-jet region corresponds to $(1 - T) \to 0$.

The plane orthogonal to the thrust axis divides the space into two hemispheres H_1 and H_2. These are used to define the following event shape variables: The hemisphere masses [84] are defined by

$$M_i^2 = \Big(\sum_{j\in H_i} p_j\Big)^2, \qquad i = 1, 2, \tag{3}$$

where p_j denotes the four-momentum of particle j. The heavy hemisphere mass M_H is then defined by

$$M_H^2 = \max(M_1^2, M_2^2). \tag{4}$$

It is convenient to introduce the dimensionless quantity

$$\rho = \frac{M_H^2}{Q^2}, \tag{5}$$

where Q is the center-of-mass energy. In leading-order the distribution of the heavy jet mass ρ is identical to the distribution of $(1 - T)$. Experimentalists also use the square root of ρ as observable:

$$\sqrt{\rho} = \frac{M_H}{Q}. \tag{6}$$

In this paper we provide in addition to the results for the moments of ρ also the results for the moments of $\sqrt{\rho}$. The odd moments of $\sqrt{\rho}$ give new information, while the even moments of $\sqrt{\rho}$ are related to the moments of ρ as follows: The $2n$th moment of $\sqrt{\rho}$ is identical to the nth moment of ρ.

The hemisphere broadenings [85,86] are defined by

$$B_i = \frac{\sum_{j\in H_i} |\vec{p}_j \times \vec{n}_T|}{2\sum_k |\vec{p}_k|}, \qquad i = 1, 2, \tag{7}$$

where the sum over j runs over all particles in one of the hemispheres, whereas the sum over k is over all particles in the final state. The wide jet broadening B_W and the total jet broadening B_T are defined by

$$B_W = \max(B_1, B_2), \qquad B_T = B_1 + B_2. \tag{8}$$

The C parameter [87,88] is obtained from the linearized momentum tensor

$$\theta^{ij} = \frac{1}{\sum_l |\vec{p}_l|} \sum_k \frac{p_k^i p_k^j}{|\vec{p}_k|}, \qquad i, j = 1, 2, 3, \tag{9}$$

where the sum runs over all final-state particles and p_k^i is the ith component of the three-momentum $\vec{p}_k$ of particle k in the c.m. system. The tensor θ is normalized to have unit trace. In terms of the eigenvalues of the θ tensor, λ_1, λ_2, λ_3, with $\lambda_1 + \lambda_2 + \lambda_3 = 1$, one defines

$$C = 3(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1). \tag{10}$$

The C parameter exhibits in perturbation theory a singularity at the three-parton boundary $C = 3/4$ [89].

For the three-to-two-jet transition variable y_{23} one first defines jets according to the Durham jet algorithm [90]. The clustering procedure of the jet algorithm is defined through the following steps:

(1) Define a resolution parameter $y_{\rm cut}$.
(2) For every pair (p_k, p_l) of final-state particles compute the corresponding resolution variable y_{kl}.
(3) If y_{ij} is the smallest value of y_{kl} computed above and $y_{ij} < y_{\rm cut}$, then combine (p_i, p_j) into a single jet ("pseudoparticle") with momentum p_{ij} according to a recombination prescription.
(4) Repeat until all pairs of objects (particles and/or pseudoparticles) have $y_{kl} > y_{\rm cut}$.

For the Durham algorithm the resolution variable is given by

$$y_{ij} = \frac{2\min(E_i^2, E_j^2)(1 - \cos\theta_{ij})}{Q^2}, \tag{11}$$

while the recombination prescription is given by

$$E_{ij} = E_i + E_j, \qquad \vec{p}_{ij} = \vec{p}_i + \vec{p}_j. \tag{12}$$

Here, E_i and E_j are the energies of particles i and j, and θ_{ij} is the angle between the three-momenta $\vec{p}_i$ and $\vec{p}_j$. Q is the center-of-mass energy. The jet transition variable y_{23} is the value of the jet resolution parameter $y_{\rm cut}$, for which the event changes from a three-jet to a two-jet configuration.

III. PERTURBATIVE EXPANSION

The perturbative expansion of a differential distribution weighted by the nth power of the observable O can be written for any infrared-safe observable for the process $e^+e^- \to 3$ jets up to NNLO as

$$\frac{O^n}{\sigma_{\rm tot}(\mu)}\frac{d\sigma(\mu)}{dO} = \frac{\alpha_s(\mu)}{2\pi}\frac{O^n d\bar{A}_O(\mu)}{dO} + \left(\frac{\alpha_s(\mu)}{2\pi}\right)^2 \times \frac{O^n d\bar{B}_O(\mu)}{dO} + \left(\frac{\alpha_s(\mu)}{2\pi}\right)^3 \frac{O^n d\bar{C}_O(\mu)}{dO}. \tag{13}$$

$\bar{A}_O$ gives the LO result, $\bar{B}_O$ the NLO correction and $\bar{C}_O$ the NNLO correction. $\sigma_{\rm tot}$ denotes the total hadronic cross section calculated up to the relevant order. The arbitrary renormalization scale is denoted by μ. All observables are chosen such that they take values between 0 and 1. The nth moment is given by

$$\langle O^n\rangle = \frac{1}{\sigma_{\rm tot}}\int_0^1 O^n \frac{d\sigma}{dO} dO. \tag{14}$$

In practice the numerical program computes the distribution

$$\frac{O^n}{\sigma_0(\mu)}\frac{d\sigma(\mu)}{dO} = \frac{\alpha_s(\mu)}{2\pi}\frac{O^n dA_O(\mu)}{dO} + \left(\frac{\alpha_s(\mu)}{2\pi}\right)^2 \times \frac{O^n dB_O(\mu)}{dO} + \left(\frac{\alpha_s(\mu)}{2\pi}\right)^3 \frac{O^n dC_O(\mu)}{dO}, \tag{15}$$

normalized to σ_0, which is the LO cross section for $e^+e^- \to$ hadrons, instead of the normalization to σ_{tot}. There is a simple relation between the two distributions: The functions A_O, B_O and C_O are related to the functions $\bar{A}_O$, $\bar{B}_O$ and $\bar{C}_O$ by

$$\begin{aligned} \bar{A}_O &= A_O, \qquad \bar{B}_O = B_O - A_{\text{tot}}A_O, \\ \bar{C}_O &= C_O - A_{\text{tot}}B_O - (B_{\text{tot}} - A_{\text{tot}}^2)A_O, \end{aligned} \tag{16}$$

where

$$\begin{aligned} A_{\text{tot}} &= \frac{3(N_c^2-1)}{4N_c}, \\ B_{\text{tot}} &= \frac{N_c^2-1}{8N_c}\left[\left(\frac{243}{4} - 44\zeta_3\right)N_c + \frac{3}{4N_c} + (8\zeta_3 - 11)N_f\right]. \end{aligned} \tag{17}$$

N_c denotes the number of colors and N_f the number of light quark flavors. A_{tot} and B_{tot} are obtained from the perturbative expansion of σ_{tot} [91–93]:

$$\sigma_{\text{tot}} = \sigma_0\left(1 + \frac{\alpha_s}{2\pi}A_{\text{tot}} + \left(\frac{\alpha_s}{2\pi}\right)^2 B_{\text{tot}} + O(\alpha_s^3)\right). \tag{18}$$

The perturbative calculation of the inclusive hadronic cross section $\langle\sigma\rangle^{(\text{tot})}$ is actually known to $O(\alpha_s^3)$ [94,95], although we need here only the coefficients up to $O(\alpha_s^2)$.

It is sufficient to calculate the functions $\bar{A}_O$, $\bar{B}_O$ and $\bar{C}_O$ for a fixed renormalization scale μ_0, which can be taken conveniently to be equal to the center-of-mass energy: $\mu_0 = Q$. The scale variation can be restored from the renormalization group equation

$$\begin{aligned} \mu^2\frac{d}{d\mu^2}\left(\frac{\alpha_S}{2\pi}\right) &= -\frac{1}{2}\beta_0\left(\frac{\alpha_S}{2\pi}\right)^2 - \frac{1}{4}\beta_1\left(\frac{\alpha_S}{2\pi}\right)^3 - \frac{1}{8}\beta_2\left(\frac{\alpha_S}{2\pi}\right)^4 + O(\alpha_s^5), \\ \beta_0 &= \frac{11}{3}C_A - \frac{4}{3}T_R N_f, \\ \beta_1 &= \frac{34}{3}C_A^2 - 4\left(\frac{5}{3}C_A + C_F\right)T_R N_f, \\ \beta_2 &= \frac{2857}{54}C_A^3 - \left(\frac{1415}{27}C_A^2 + \frac{205}{9}C_A C_F - 2C_F^2\right)T_R N_f + \left(\frac{158}{27}C_A + \frac{44}{9}C_F\right)T_R^2 N_f^2. \end{aligned} \tag{19}$$

The color factors are defined as usual by

$$C_A = N_c, \qquad C_F = \frac{N_c^2-1}{2N_c}, \qquad T_R = \frac{1}{2}. \tag{20}$$

The values of the functions $\bar{A}_O$, $\bar{B}_O$ and $\bar{C}_O$ at a scale μ are then obtained from the ones at the scale μ_0 by

$$\begin{aligned} \bar{A}_O(\mu) &= \bar{A}_O(\mu_0), \\ \bar{B}_O(\mu) &= \bar{B}_O(\mu_0) + \frac{1}{2}\beta_0 \ln\left(\frac{\mu^2}{\mu_0^2}\right)\bar{A}_O(\mu_0), \\ \bar{C}_O(\mu) &= \bar{C}_O(\mu_0) + \beta_0 \ln\left(\frac{\mu^2}{\mu_0^2}\right)\bar{B}_O(\mu_0) + \frac{1}{4}\left[\beta_1 + \beta_0^2 \ln\left(\frac{\mu^2}{\mu_0^2}\right)\right]\ln\left(\frac{\mu^2}{\mu_0^2}\right)\bar{A}_O(\mu_0). \end{aligned} \tag{21}$$

Finally, an approximate solution of Eq. (19) for α_s is given by

$$\frac{\alpha_s(\mu)}{2\pi} = \frac{2}{\beta_0 L}\left\{1 - \frac{\beta_1}{\beta_0^2}\frac{\ln L}{L} + \frac{\beta_1^2}{\beta_0^4 L^2}\left[\left(\frac{1}{2} - \ln L\right)^2 + \frac{\beta_0\beta_2}{\beta_1^2} - \frac{5}{4}\right]\right\}, \tag{22}$$

where $L = \ln(\mu^2/\Lambda^2)$.

The function C_O can be decomposed into six color pieces

$$C_O = \frac{(N_c^2-1)}{8N_c}\left[N_c^2 C_O^{lc} + C_O^{sc} + \frac{1}{N_c^2}C_O^{ssc} + N_f N_c C_O^{nf} + \frac{N_f}{N_c}C_O^{nfsc} + N_f^2 C_O^{nfnf}\right]. \tag{23}$$

In addition, there are singlet contributions, which arise from interference terms of amplitudes, where the electroweak boson couples to two different fermion lines. These singlet contributions are expected to be numerically small [28,96,97] and neglected in the present calculation. We define

$$\begin{aligned} C_O|_{lc} &= \frac{(N_c^2-1)}{8N_c}N_c^2 C_O^{lc}, \quad C_O|_{sc} = \frac{(N_c^2-1)}{8N_c}C_O^{sc}, \\ C_O|_{ssc} &= \frac{(N_c^2-1)}{8N_c}\frac{1}{N_c^2}C_O^{ssc}, \quad C_O|_{nf} = \frac{(N_c^2-1)}{8N_c}N_f N_c C_O^{nf}, \\ C_O|_{nfsc} &= \frac{(N_c^2-1)}{8N_c}\frac{N_f}{N_c}C_O^{nfsc}, \quad C_O|_{nfnf} = \frac{(N_c^2-1)}{8N_c}N_f^2 C_O^{nfnf}; \end{aligned} \tag{24}$$

e.g. the function $C_O|_{lc}$ includes the color factors.

The functions A_O, B_O and C_O are calculated for a fixed renormalization scale equal to the center-of-mass energy: $\mu_0 = Q$. They depend only on the value of the observable O. Since only QCD corrections with nonsinglet quark couplings are taken into account and singlet contributions to C_O are neglected, the functions A_O, B_O and C_O do not depend on electroweak couplings.

IV. NUMERICAL RESULTS

In this section I present the results for the perturbative coefficients $A_{O,n}$, $B_{O,n}$ and $C_{O,n}$. These are related by Eq. (16) to the coefficients $\bar{A}_{O,n}$, $\bar{B}_{O,n}$ and $\bar{C}_{O,n}$. The perturbative expansion of the moments in terms of the latter is given by

$$\langle O^n \rangle = \frac{\alpha_s}{2\pi}\bar{A}_{O,n} + \left(\frac{\alpha_s}{2\pi}\right)^2 \bar{B}_{O,n} + \left(\frac{\alpha_s}{2\pi}\right)^3 \bar{C}_{O,n} + O(\alpha_s^4). \tag{25}$$

The coefficients $A_{O,n}$, $B_{O,n}$ and $C_{O,n}$ are calculated at the renormalization scale equal to the center-of-mass energy $\mu_0 = Q$. The scale dependence is given by Eq. (21).

All results have been obtained by numerical Monte Carlo integration. The Monte-Carlo integration introduces a statistical error. For the infrared singularities a hybrid method between subtraction and slicing has been used. The dimensionless slicing parameter is denoted by

$$\eta = \frac{s_{\min}}{Q^2}. \tag{26}$$

The slicing procedure introduces in addition to the statistical error from the Monte Carlo integration a systematic error. The size of this error can be estimated by varying the slicing parameter η. However, lowering the slicing parameter will increase the statistical error. A practical criterion is to require that the variation due to the slicing parameter is smaller than the statistical error, with the possible exception for the boundaries of the distributions. Imposing this criterion $\eta = 10^{-5}$ turns out to be a good compromise between accuracy and efficiency in the region away from the two-jet region. However the first moment of the distributions receives non-negligible contributions from the close-to-two-jet region. In this region a more careful analysis has to be performed. The moments are calculated as follows: We introduce a second parameter $\kappa = 10^{-3}$ and split the integral into

$$\frac{1}{\sigma_{\text{tot}}}\int_0^1 O^n \frac{d\sigma}{dO} dO = \frac{1}{\sigma_{\text{tot}}}\int_0^\kappa O^n \frac{d\sigma}{dO} dO + \frac{1}{\sigma_{\text{tot}}} \times \int_\kappa^1 O^n \frac{d\sigma}{dO} dO. \tag{27}$$

In the region $[\kappa, 1]$ the integral is calculated with the slicing parameter $\eta = 10^{-5}$, whereas in the region $[0, \kappa]$ the slicing parameter $\eta = 10^{-9}$ is used for the computation of the integral. In order to estimate the systematical error due to the slicing procedure the moments are recalculated in the region $[0, \kappa]$ with the slicing parameters $\eta = 10^{-7}$ and $\eta = 10^{-5}$. Therefore we obtain the three results I_5, I_7 and I_9 for the integral over $[0, \kappa]$, corresponding to the values $\eta = 10^{-5}$, $\eta = 10^{-7}$ and $\eta = 10^{-9}$, respectively. The systematic error due to the slicing procedure is taken as

$$\max(|I_9 - I_7|, |I_9 - I_5|, |I_7 - I_5|) \tag{28}$$

and added linearly to the statistical error. This defines the total error, which is quoted in all results.

The results of this paper are the perturbative coefficients $A_{O,n}$, $B_{O,n}$ and $C_{O,n}$ for the first ten moments of the event shape variables thrust, heavy jet mass, wide jet broadening, total jet broadening, C parameter and the three-to-two-jet transition variable. Since the experimentalists use the heavy jet mass as well as the square root of the heavy jet mass, the results for the moments of both are given for convenience. Of course, the $2n$th moment of $\sqrt{\rho}$ is identical to the nth moment of ρ. The perturbative coefficients for the first ten moments are tabulated in Tables I, II, III, IV, V, VI, and VII. The total error is indicated in these tables.

The NNLO coefficient $C_{O,n}$ can be split up into the contributions from the individual color factors

$$C_O|_{lc}, \quad C_O|_{sc}, \quad C_O|_{ssc}, \quad C_O|_{nf}, \quad C_O|_{nfsc}, \quad C_O|_{nfnf} \tag{29}$$

defined in Eq. (24). For the moments the contributions

TABLE I. Coefficients of the leading-order ($A_{(1-T),n}$), next-to-leading-order ($B_{(1-T),n}$) and next-to-next-to-leading-order ($C_{(1-T),n}$) contributions to the nth moment of the thrust distribution.

n	$A_{(1-T),n}$	$B_{(1-T),n}$	$C_{(1-T),n}$
1	$2.10344(3) \times 10^{0}$	$4.499(5) \times 10^{1}$	$1.10(3) \times 10^{3}$
2	$1.90190(5) \times 10^{-1}$	$6.2568(4) \times 10^{0}$	$1.829(2) \times 10^{2}$
3	$2.9874(1) \times 10^{-2}$	$1.1278(1) \times 10^{0}$	$3.617(3) \times 10^{1}$
4	$5.8576(3) \times 10^{-3}$	$2.4619(5) \times 10^{-1}$	$8.447(8) \times 10^{0}$
5	$1.29460(9) \times 10^{-3}$	$6.003(2) \times 10^{-2}$	$2.184(2) \times 10^{0}$
6	$3.0839(3) \times 10^{-4}$	$1.5709(6) \times 10^{-2}$	$6.026(7) \times 10^{-1}$
7	$7.7332(8) \times 10^{-5}$	$4.318(2) \times 10^{-3}$	$1.739(2) \times 10^{-1}$
8	$2.0132(2) \times 10^{-5}$	$1.2307(8) \times 10^{-3}$	$5.184(8) \times 10^{-2}$
9	$5.3929(7) \times 10^{-6}$	$3.608(3) \times 10^{-4}$	$1.584(3) \times 10^{-2}$
10	$1.4775(2) \times 10^{-6}$	$1.0821(9) \times 10^{-4}$	$4.939(9) \times 10^{-3}$

TABLE II. Coefficients of the leading-order ($A_{\rho,n}$), next-to-leading-order ($B_{\rho,n}$) and next-to-next-to-leading-order ($C_{\rho,n}$) contributions to the nth moment of the heavy jet mass distribution.

n	$A_{\rho,n}$	$B_{\rho,n}$	$C_{\rho,n}$
1	$2.10344(3) \times 10^{0}$	$2.331(4) \times 10^{1}$	$4.2(3) \times 10^{2}$
2	$1.90190(5) \times 10^{-1}$	$3.087(1) \times 10^{0}$	$4.15(2) \times 10^{1}$
3	$2.9874(1) \times 10^{-2}$	$4.572(3) \times 10^{-1}$	$4.53(4) \times 10^{0}$
4	$5.8576(3) \times 10^{-3}$	$8.350(7) \times 10^{-2}$	$5.01(9) \times 10^{-1}$
5	$1.29460(9) \times 10^{-3}$	$1.755(2) \times 10^{-2}$	$4.3(3) \times 10^{-2}$
6	$3.0839(3) \times 10^{-4}$	$4.082(5) \times 10^{-3}$	$-1.8(8) \times 10^{-3}$
7	$7.7332(8) \times 10^{-5}$	$1.026(2) \times 10^{-3}$	$-2.6(2) \times 10^{-3}$
8	$2.0132(2) \times 10^{-5}$	$2.744(5) \times 10^{-4}$	$-1.06(8) \times 10^{-3}$
9	$5.3929(7) \times 10^{-6}$	$7.71(1) \times 10^{-5}$	$-3.4(3) \times 10^{-4}$
10	$1.4775(2) \times 10^{-6}$	$2.257(5) \times 10^{-5}$	$-1.00(8) \times 10^{-4}$

TABLE III. Coefficients of the leading-order ($A_{B_W,n}$), next-to-leading-order ($B_{B_W,n}$) and next-to-next-to-leading-order ($C_{B_W,n}$) contributions to the nth moment of the wide jet broadening distribution.

n	$A_{B_W,n}$	$B_{B_W,n}$	$C_{B_W,n}$
1	$4.067(4) \times 10^{0}$	$-1.01(9) \times 10^{1}$	$2(1) \times 10^{3}$
2	$3.36874(5) \times 10^{-1}$	$4.533(1) \times 10^{0}$	$5.09(4) \times 10^{1}$
3	$4.7552(1) \times 10^{-2}$	$6.667(2) \times 10^{-1}$	$6.24(3) \times 10^{0}$
4	$8.3108(3) \times 10^{-3}$	$1.0681(4) \times 10^{-1}$	$5.99(7) \times 10^{-1}$
5	$1.62954(7) \times 10^{-3}$	$1.863(1) \times 10^{-2}$	$1.2(2) \times 10^{-2}$
6	$3.4328(2) \times 10^{-4}$	$3.458(3) \times 10^{-3}$	$-1.82(4) \times 10^{-2}$
7	$7.5946(4) \times 10^{-5}$	$6.711(7) \times 10^{-4}$	$-8.2(1) \times 10^{-3}$
8	$1.7410(1) \times 10^{-5}$	$1.347(2) \times 10^{-4}$	$-2.74(3) \times 10^{-3}$
9	$4.1002(3) \times 10^{-6}$	$2.773(5) \times 10^{-5}$	$-8.28(8) \times 10^{-4}$
10	$9.8630(8) \times 10^{-7}$	$5.83(1) \times 10^{-6}$	$-2.39(2) \times 10^{-4}$

TABLE IV. Coefficients of the leading-order ($A_{B_T,n}$), next-to-leading-order ($B_{B_T,n}$) and next-to-next-to-leading-order ($C_{B_T,n}$) contributions to the nth moment of the total jet broadening distribution.

n	$A_{B_T,n}$	$B_{B_T,n}$	$C_{B_T,n}$
1	$4.067(4) \times 10^{0}$	$6.41(3) \times 10^{1}$	$2.3(9) \times 10^{3}$
2	$3.36874(5) \times 10^{-1}$	$1.473(2) \times 10^{1}$	$3.325(4) \times 10^{2}$
3	$4.7552(1) \times 10^{-2}$	$2.766(2) \times 10^{0}$	$7.202(4) \times 10^{1}$
4	$8.3108(3) \times 10^{-3}$	$6.070(5) \times 10^{-1}$	$1.676(1) \times 10^{1}$
5	$1.62954(7) \times 10^{-3}$	$1.472(1) \times 10^{-1}$	$4.232(3) \times 10^{0}$
6	$3.4328(2) \times 10^{-4}$	$3.810(3) \times 10^{-2}$	$1.135(1) \times 10^{0}$
7	$7.5946(4) \times 10^{-5}$	$1.0326(7) \times 10^{-2}$	$3.181(3) \times 10^{-1}$
8	$1.7410(1) \times 10^{-5}$	$2.897(2) \times 10^{-3}$	$9.21(1) \times 10^{-2}$
9	$4.1002(3) \times 10^{-6}$	$8.349(5) \times 10^{-4}$	$2.737(3) \times 10^{-2}$
10	$9.8630(8) \times 10^{-7}$	$2.461(1) \times 10^{-4}$	$8.30(1) \times 10^{-3}$

from the individual color factors to $C_{O,n}$ are given in Tables VIII, IX, X, XI, XII, XIII, and XIV. Again, the total error is indicated in these tables.

TABLE V. Coefficients of the leading-order ($A_{C,n}$), next-to-leading-order ($B_{C,n}$) and next-to-next-to-leading-order ($C_{C,n}$) contributions to the nth moment of the C parameter distribution.

n	$A_{C,n}$	$B_{C,n}$	$C_{C,n}$
1	$8.6378(1) \times 10^{0}$	$1.728(3) \times 10^{2}$	$4.2(1) \times 10^{3}$
2	$2.43160(4) \times 10^{0}$	$8.1160(5) \times 10^{1}$	$2.332(2) \times 10^{3}$
3	$1.07919(3) \times 10^{0}$	$4.2752(3) \times 10^{1}$	$1.3608(8) \times 10^{3}$
4	$5.6848(2) \times 10^{-1}$	$2.5804(3) \times 10^{1}$	$8.791(5) \times 10^{2}$
5	$3.2720(1) \times 10^{-1}$	$1.6865(3) \times 10^{1}$	$6.074(4) \times 10^{2}$
6	$1.98719(9) \times 10^{-1}$	$1.1595(2) \times 10^{1}$	$4.386(3) \times 10^{2}$
7	$1.25095(6) \times 10^{-1}$	$8.256(2) \times 10^{0}$	$3.265(2) \times 10^{2}$
8	$8.0789(4) \times 10^{-2}$	$6.033(2) \times 10^{0}$	$2.485(2) \times 10^{2}$
9	$5.3184(3) \times 10^{-2}$	$4.500(1) \times 10^{0}$	$1.925(2) \times 10^{2}$
10	$3.5534(2) \times 10^{-2}$	$3.413(1) \times 10^{0}$	$1.511(1) \times 10^{2}$

TABLE VI. Coefficients of the leading-order ($A_{y_{23},n}$), next-to-leading-order ($B_{y_{23},n}$) and next-to-next-to-leading-order ($C_{y_{23},n}$) contributions to the nth moment of the three-to-two-jet transition distribution.

n	$A_{y_{23},n}$	$B_{y_{23},n}$	$C_{y_{23},n}$
1	$8.9419(2) \times 10^{-1}$	$1.2648(2) \times 10^{1}$	$1.32(2) \times 10^{2}$
2	$8.1408(3) \times 10^{-2}$	$1.2912(3) \times 10^{0}$	$1.353(9) \times 10^{1}$
3	$1.28529(9) \times 10^{-2}$	$1.9873(9) \times 10^{-1}$	$1.86(2) \times 10^{0}$
4	$2.5226(2) \times 10^{-3}$	$3.770(3) \times 10^{-2}$	$3.08(6) \times 10^{-1}$
5	$5.5680(6) \times 10^{-4}$	$8.028(8) \times 10^{-3}$	$5.6(2) \times 10^{-2}$
6	$1.3232(2) \times 10^{-4}$	$1.837(3) \times 10^{-3}$	$1.03(5) \times 10^{-2}$
7	$3.3091(5) \times 10^{-5}$	$4.410(9) \times 10^{-4}$	$1.8(1) \times 10^{-3}$
8	$8.590(2) \times 10^{-6}$	$1.096(3) \times 10^{-4}$	$2.8(4) \times 10^{-4}$
9	$2.2945(5) \times 10^{-6}$	$2.79(1) \times 10^{-5}$	$2(1) \times 10^{-5}$
10	$6.269(1) \times 10^{-7}$	$7.26(3) \times 10^{-6}$	$-9(4) \times 10^{-6}$

TABLE VII. Coefficients of the leading-order ($A_{\sqrt{\rho},n}$), next-to-leading-order ($B_{\sqrt{\rho},n}$) and next-to-next-to-leading-order ($C_{\sqrt{\rho},n}$) contributions to the nth moment of the square root of the heavy jet mass distribution.

n	$A_{\sqrt{\rho},n}$	$B_{\sqrt{\rho},n}$	$C_{\sqrt{\rho},n}$
1	$1.372(5) \times 10^{1}$	$-3.4(1) \times 10^{2}$	$2(2) \times 10^{4}$
2	$2.10344(3) \times 10^{0}$	$2.336(3) \times 10^{1}$	$4.18(6) \times 10^{2}$
3	$5.6445(1) \times 10^{-1}$	$8.87(1) \times 10^{0}$	$1.289(4) \times 10^{2}$
4	$1.90190(5) \times 10^{-1}$	$3.090(5) \times 10^{0}$	$4.16(2) \times 10^{1}$
5	$7.2514(2) \times 10^{-2}$	$1.152(2) \times 10^{0}$	$1.359(7) \times 10^{1}$
6	$2.9874(1) \times 10^{-2}$	$4.58(1) \times 10^{-1}$	$4.53(4) \times 10^{0}$
7	$1.29764(6) \times 10^{-2}$	$1.915(5) \times 10^{-1}$	$1.52(2) \times 10^{0}$
8	$5.8576(3) \times 10^{-3}$	$8.36(3) \times 10^{-2}$	$5.01(9) \times 10^{-1}$
9	$2.7224(2) \times 10^{-3}$	$3.78(1) \times 10^{-2}$	$1.57(5) \times 10^{-1}$
10	$1.29460(9) \times 10^{-3}$	$1.758(8) \times 10^{-2}$	$4.3(3) \times 10^{-2}$

In Eq. (27) we have split up the integral defining the moments into two pieces. The integral over the region $[\kappa, 1]$

$$\langle O^n \rangle_{\text{incomplete}} = \frac{1}{\sigma_{\text{tot}}} \int_\kappa^1 O^n \frac{d\sigma}{dO} dO \tag{30}$$

is like the complete moment a measurable quantity. It has the advantage that the perturbative calculation of this quantity is much less affected by the numerical issues related to the close-to-two-jet region. For this reason the results for these "incomplete" moments are provided as well. The perturbative coefficients are given for $\kappa = 10^{-3}$ for the various event shape variables in Tables XV, XVI, XVII, XVIII, XIX, XX, and XXI. The indicated error corresponds to the statistical error.

In a recent calculation the logarithmic terms of the NNLO coefficient of the thrust distribution have been calculated based on soft-collinear effective theory [76]:

TABLE VIII. Individual contributions from the different color factors to the next-to-next-to-leading-order contribution to the nth moment of the thrust distribution.

n	N_c^2	N_c^0	N_c^{-2}	$N_f N_c$	N_f/N_c	N_f^2
1	$2.51(1) \times 10^3$	$3.7(5) \times 10^1$	$6(5) \times 10^{-1}$	$-1.69(3) \times 10^3$	$-1.2(5) \times 10^1$	$2.55(5) \times 10^2$
2	$3.460(1) \times 10^2$	$-2.003(2) \times 10^1$	$-2.293(5) \times 10^{-1}$	$-1.6275(8) \times 10^2$	$7.776(4) \times 10^0$	$1.2152(3) \times 10^1$
3	$6.368(3) \times 10^1$	$-4.697(4) \times 10^0$	$2(7) \times 10^{-5}$	$-2.568(1) \times 10^1$	$1.482(1) \times 10^0$	$1.3903(4) \times 10^0$
4	$1.4240(7) \times 10^1$	$-1.137(1) \times 10^0$	$5.20(2) \times 10^{-3}$	$-5.196(4) \times 10^0$	$3.221(3) \times 10^{-1}$	$2.1322(9) \times 10^{-1}$
5	$3.563(2) \times 10^0$	$-2.956(3) \times 10^{-1}$	$2.137(8) \times 10^{-3}$	$-1.200(1) \times 10^0$	$7.751(8) \times 10^{-2}$	$3.678(3) \times 10^{-2}$
6	$9.572(7) \times 10^{-1}$	$-8.13(1) \times 10^{-2}$	$7.39(3) \times 10^{-4}$	$-3.006(3) \times 10^{-1}$	$2.000(3) \times 10^{-2}$	$6.503(8) \times 10^{-3}$
7	$2.699(2) \times 10^{-1}$	$-2.330(3) \times 10^{-2}$	$2.461(9) \times 10^{-4}$	$-7.95(1) \times 10^{-2}$	$5.423(9) \times 10^{-3}$	$1.070(2) \times 10^{-3}$
8	$7.885(7) \times 10^{-2}$	$-6.90(1) \times 10^{-3}$	$8.12(3) \times 10^{-5}$	$-2.185(3) \times 10^{-2}$	$1.526(3) \times 10^{-3}$	$1.275(7) \times 10^{-4}$
9	$2.366(2) \times 10^{-2}$	$-2.093(4) \times 10^{-3}$	$2.68(1) \times 10^{-5}$	$-6.18(1) \times 10^{-3}$	$4.416(9) \times 10^{-4}$	$-8.8(2) \times 10^{-6}$
10	$7.253(8) \times 10^{-3}$	$-6.48(1) \times 10^{-4}$	$8.91(4) \times 10^{-6}$	$-1.791(3) \times 10^{-3}$	$1.307(3) \times 10^{-4}$	$-1.511(8) \times 10^{-5}$

TABLE IX. Individual contributions from the different color factors to the next-to-next-to-leading-order contribution to the nth moment of the heavy jet mass distribution.

n	N_c^2	N_c^0	N_c^{-2}	$N_f N_c$	N_f/N_c	N_f^2
1	$1.23(3) \times 10^3$	$9.0(3) \times 10^1$	$8(1) \times 10^0$	$-1.10(2) \times 10^3$	$-7.1(6) \times 10^1$	$2.72(5) \times 10^2$
2	$1.234(1) \times 10^2$	$-2.47(2) \times 10^0$	$-8.45(4) \times 10^{-2}$	$-9.575(7) \times 10^1$	$1.422(4) \times 10^0$	$1.5031(3) \times 10^1$
3	$1.574(3) \times 10^1$	$-6.32(4) \times 10^{-1}$	$-1.20(1) \times 10^{-2}$	$-1.290(1) \times 10^1$	$3.167(9) \times 10^{-1}$	$2.0142(4) \times 10^0$
4	$2.458(9) \times 10^0$	$-1.13(1) \times 10^{-1}$	$-2.09(3) \times 10^{-3}$	$-2.269(4) \times 10^0$	$6.12(2) \times 10^{-2}$	$3.6598(9) \times 10^{-1}$
5	$4.40(2) \times 10^{-1}$	$-2.04(3) \times 10^{-2}$	$-4.67(9) \times 10^{-4}$	$-4.66(1) \times 10^{-1}$	$1.299(7) \times 10^{-2}$	$7.700(3) \times 10^{-2}$
6	$8.77(7) \times 10^{-2}$	$-3.8(1) \times 10^{-3}$	$-1.22(3) \times 10^{-4}$	$-1.062(3) \times 10^{-1}$	$3.04(2) \times 10^{-3}$	$1.7616(7) \times 10^{-2}$
7	$1.92(2) \times 10^{-2}$	$-7.4(3) \times 10^{-4}$	$-3.50(9) \times 10^{-5}$	$-2.607(9) \times 10^{-2}$	$7.73(7) \times 10^{-4}$	$4.248(2) \times 10^{-3}$
8	$4.60(7) \times 10^{-3}$	$-1.5(1) \times 10^{-4}$	$-1.06(3) \times 10^{-5}$	$-6.77(3) \times 10^{-3}$	$2.11(2) \times 10^{-4}$	$1.0610(6) \times 10^{-3}$
9	$1.20(2) \times 10^{-3}$	$-3.0(3) \times 10^{-5}$	$-3.3(1) \times 10^{-6}$	$-1.839(9) \times 10^{-3}$	$6.11(7) \times 10^{-5}$	$2.713(2) \times 10^{-4}$
10	$3.37(8) \times 10^{-4}$	$-6(1) \times 10^{-6}$	$-1.08(3) \times 10^{-6}$	$-5.18(3) \times 10^{-4}$	$1.84(2) \times 10^{-5}$	$7.044(6) \times 10^{-5}$

TABLE X. Individual contributions from the different color factors to the next-to-next-to-leading-order contribution to the nth moment of the wide jet broadening distribution.

n	N_c^2	N_c^0	N_c^{-2}	$N_f N_c$	N_f/N_c	N_f^2
1	$2(1) \times 10^3$	$0(2) \times 10^1$	$8(3) \times 10^1$	$-7(3) \times 10^2$	$-4.5(7) \times 10^2$	$7.3(4) \times 10^2$
2	$1.846(3) \times 10^2$	$6.94(7) \times 10^0$	$2.13(8) \times 10^{-1}$	$-1.712(2) \times 10^2$	$-3.19(3) \times 10^0$	$3.364(2) \times 10^1$
3	$2.395(3) \times 10^1$	$-1.20(4) \times 10^{-1}$	$-1.054(8) \times 10^{-2}$	$-2.155(1) \times 10^1$	$1.330(7) \times 10^{-1}$	$3.8438(4) \times 10^0$
4	$3.287(7) \times 10^0$	$-6.55(9) \times 10^{-2}$	$-1.89(2) \times 10^{-3}$	$-3.271(3) \times 10^0$	$3.97(2) \times 10^{-2}$	$6.1019(9) \times 10^{-1}$
5	$4.65(2) \times 10^{-1}$	$-1.19(2) \times 10^{-2}$	$-2.92(4) \times 10^{-4}$	$-5.618(7) \times 10^{-1}$	$7.29(4) \times 10^{-3}$	$1.1390(2) \times 10^{-1}$
6	$6.35(4) \times 10^{-2}$	$-1.61(5) \times 10^{-3}$	$-4.8(1) \times 10^{-5}$	$-1.046(2) \times 10^{-1}$	$1.21(1) \times 10^{-3}$	$2.3348(5) \times 10^{-2}$
7	$7.3(1) \times 10^{-3}$	$-1.1(1) \times 10^{-4}$	$-8.5(3) \times 10^{-6}$	$-2.060(5) \times 10^{-2}$	$1.82(3) \times 10^{-4}$	$5.083(1) \times 10^{-3}$
8	$2.8(3) \times 10^{-4}$	$3.2(3) \times 10^{-5}$	$-1.59(9) \times 10^{-6}$	$-4.22(1) \times 10^{-3}$	$2.22(7) \times 10^{-5}$	$1.1533(3) \times 10^{-3}$
9	$-2.25(7) \times 10^{-4}$	$2.04(9) \times 10^{-5}$	$-3.1(2) \times 10^{-7}$	$-8.94(3) \times 10^{-4}$	$9(2) \times 10^{-7}$	$2.6970(9) \times 10^{-4}$
10	$-1.17(2) \times 10^{-4}$	$7.8(2) \times 10^{-6}$	$-6.5(7) \times 10^{-8}$	$-1.940(9) \times 10^{-4}$	$-7.3(5) \times 10^{-7}$	$6.452(2) \times 10^{-5}$

TABLE XI. Individual contributions from the different color factors to the next-to-next-to-leading-order contribution to the nth moment of the total jet broadening distribution.

n	N_c^2	N_c^0	N_c^{-2}	$N_f N_c$	N_f/N_c	N_f^2
1	$4.5(7) \times 10^3$	$2(1) \times 10^2$	$3(2) \times 10^1$	$-3.0(3) \times 10^3$	$-2.2(7) \times 10^2$	$7.0(4) \times 10^2$
2	$7.443(3) \times 10^2$	$-2.480(6) \times 10^1$	$-2.516(7) \times 10^0$	$-4.359(2) \times 10^2$	$2.356(3) \times 10^1$	$2.790(2) \times 10^1$
3	$1.4437(4) \times 10^2$	$-9.048(7) \times 10^0$	$-3.009(2) \times 10^{-1}$	$-7.031(2) \times 10^1$	$4.884(2) \times 10^0$	$2.4263(4) \times 10^0$
4	$3.198(1) \times 10^1$	$-2.354(2) \times 10^0$	$-4.384(5) \times 10^{-2}$	$-1.4127(5) \times 10^1$	$1.0712(7) \times 10^0$	$2.359(1) \times 10^{-1}$
5	$7.810(3) \times 10^0$	$-6.163(5) \times 10^{-1}$	$-7.22(1) \times 10^{-3}$	$-3.222(2) \times 10^0$	$2.560(2) \times 10^{-1}$	$1.100(3) \times 10^{-2}$
6	$2.0388(9) \times 10^0$	$-1.671(2) \times 10^{-1}$	$-1.225(4) \times 10^{-3}$	$-7.948(5) \times 10^{-1}$	$6.516(6) \times 10^{-2}$	$-5.754(8) \times 10^{-3}$
7	$5.579(3) \times 10^{-1}$	$-4.686(5) \times 10^{-2}$	$-1.92(1) \times 10^{-4}$	$-2.068(1) \times 10^{-1}$	$1.737(2) \times 10^{-2}$	$-3.332(3) \times 10^{-3}$
8	$1.5813(9) \times 10^{-1}$	$-1.352(2) \times 10^{-2}$	$-2.04(5) \times 10^{-5}$	$-5.592(5) \times 10^{-2}$	$4.797(6) \times 10^{-3}$	$-1.3251(8) \times 10^{-3}$
9	$4.605(3) \times 10^{-2}$	$-3.995(5) \times 10^{-3}$	$2.7(2) \times 10^{-6}$	$-1.558(2) \times 10^{-2}$	$1.363(2) \times 10^{-3}$	$-4.721(3) \times 10^{-4}$
10	$1.371(1) \times 10^{-2}$	$-1.204(2) \times 10^{-3}$	$3.06(6) \times 10^{-6}$	$-4.446(5) \times 10^{-3}$	$3.961(6) \times 10^{-4}$	$-1.607(1) \times 10^{-4}$

TABLE XII. Individual contributions from the different color factors to the next-to-next-to-leading-order contribution to the nth moment of the C parameter distribution.

n	N_c^2	N_c^0	N_c^{-2}	$N_f N_c$	N_f/N_c	N_f^2
1	$9.8(1) \times 10^3$	$2.9(2) \times 10^2$	$7(5) \times 10^0$	$-6.87(8) \times 10^3$	$-1.1(2) \times 10^2$	$1.12(2) \times 10^3$
2	$4.528(1) \times 10^3$	$-2.248(2) \times 10^2$	$-5.047(4) \times 10^0$	$-2.2420(7) \times 10^3$	$1.0303(5) \times 10^2$	$1.7314(2) \times 10^2$
3	$2.4430(7) \times 10^3$	$-1.672(1) \times 10^2$	$-9.47(2) \times 10^{-1}$	$-1.0292(4) \times 10^3$	$5.944(3) \times 10^1$	$5.5725(9) \times 10^1$
4	$1.5070(5) \times 10^3$	$-1.1508(8) \times 10^2$	$6.1(1) \times 10^{-2}$	$-5.713(2) \times 10^2$	$3.628(2) \times 10^1$	$2.2186(6) \times 10^1$
5	$1.0065(3) \times 10^3$	$-8.138(6) \times 10^1$	$3.29(1) \times 10^{-1}$	$-3.509(2) \times 10^2$	$2.362(2) \times 10^1$	$9.238(4) \times 10^0$
6	$7.069(3) \times 10^2$	$-5.934(4) \times 10^1$	$3.844(9) \times 10^{-1}$	$-2.290(1) \times 10^2$	$1.611(1) \times 10^1$	$3.512(3) \times 10^0$
7	$5.140(2) \times 10^2$	$-4.437(3) \times 10^1$	$3.709(8) \times 10^{-1}$	$-1.557(1) \times 10^2$	$1.136(1) \times 10^1$	$8.27(2) \times 10^{-1}$
8	$3.832(2) \times 10^2$	$-3.386(3) \times 10^1$	$3.357(7) \times 10^{-1}$	$-1.0894(8) \times 10^2$	$8.214(9) \times 10^0$	$-4.39(2) \times 10^{-1}$
9	$2.913(1) \times 10^2$	$-2.627(2) \times 10^1$	$2.956(6) \times 10^{-1}$	$-7.793(7) \times 10^1$	$6.059(8) \times 10^0$	$-1.002(2) \times 10^0$
10	$2.250(1) \times 10^2$	$-2.067(2) \times 10^1$	$2.570(5) \times 10^{-1}$	$-5.674(6) \times 10^1$	$4.544(6) \times 10^0$	$-1.209(1) \times 10^0$

$$\frac{dC_\tau}{d\tau} = \frac{1}{\tau}[a_5 \ln^5\tau + a_4 \ln^4\tau + a_3 \ln^3\tau + a_2 \ln^2\tau + a_1 \ln\tau + a_0 + O(\tau)],$$
$$\tau = 1 - T. \quad (31)$$

The values of the a_j's are for $N_f = 5$

$$a_5 = -18.96, \quad a_4 = -207.4, \quad a_3 = -122.3,$$
$$a_2 = 1488.3, \quad a_1 = -822.3, \quad a_0 = -683.4. \quad (32)$$

TABLE XIII. Individual contributions from the different color factors to the next-to-next-to-leading-order contribution to the nth moment of the three-to-two-jet transition distribution.

n	N_c^2	N_c^0	N_c^{-2}	$N_f N_c$	N_f/N_c	N_f^2
1	$4.87(1) \times 10^2$	$1.27(4) \times 10^1$	$5.63(6) \times 10^{-1}$	$-4.448(9) \times 10^2$	$-6.56(2) \times 10^0$	$8.34(2) \times 10^1$
2	$4.539(8) \times 10^1$	$-1.38(1) \times 10^0$	$-1.91(3) \times 10^{-2}$	$-3.664(4) \times 10^1$	$6.67(2) \times 10^{-1}$	$5.511(1) \times 10^0$
3	$6.60(2) \times 10^0$	$-2.83(3) \times 10^{-1}$	$-2.98(7) \times 10^{-3}$	$-5.383(8) \times 10^0$	$1.278(5) \times 10^{-1}$	$7.954(2) \times 10^{-1}$
4	$1.198(5) \times 10^0$	$-5.70(7) \times 10^{-2}$	$-5.6(2) \times 10^{-4}$	$-1.009(2) \times 10^0$	$2.54(1) \times 10^{-2}$	$1.5085(6) \times 10^{-1}$
5	$2.44(1) \times 10^{-1}$	$-1.21(2) \times 10^{-2}$	$-1.19(6) \times 10^{-4}$	$-2.147(6) \times 10^{-1}$	$5.43(4) \times 10^{-3}$	$3.285(2) \times 10^{-2}$
6	$5.34(4) \times 10^{-2}$	$-2.69(6) \times 10^{-3}$	$-2.7(2) \times 10^{-5}$	$-4.94(2) \times 10^{-2}$	$1.22(1) \times 10^{-3}$	$7.777(5) \times 10^{-3}$
7	$1.22(1) \times 10^{-2}$	$-6.1(2) \times 10^{-4}$	$-6.4(5) \times 10^{-6}$	$-1.196(5) \times 10^{-2}$	$2.84(3) \times 10^{-4}$	$1.948(1) \times 10^{-3}$
8	$2.85(4) \times 10^{-3}$	$-1.42(5) \times 10^{-4}$	$-1.5(2) \times 10^{-6}$	$-3.00(2) \times 10^{-3}$	$6.7(1) \times 10^{-5}$	$5.082(4) \times 10^{-4}$
9	$6.8(1) \times 10^{-4}$	$-3.3(2) \times 10^{-5}$	$-3.8(5) \times 10^{-7}$	$-7.75(5) \times 10^{-4}$	$1.62(3) \times 10^{-5}$	$1.367(1) \times 10^{-4}$
10	$1.61(4) \times 10^{-4}$	$-7.7(5) \times 10^{-6}$	$-9(2) \times 10^{-8}$	$-2.04(2) \times 10^{-4}$	$3.93(9) \times 10^{-6}$	$3.767(3) \times 10^{-5}$

TABLE XIV. Individual contributions from the different color factors to the next-to-next-to-leading-order contribution to the nth moment of the square root of the heavy jet mass distribution.

n	N_c^2	N_c^0	N_c^{-2}	$N_f N_c$	N_f/N_c	N_f^2
1	$2(1) \times 10^4$	$-1(1) \times 10^3$	$4.0(5) \times 10^2$	$5(2) \times 10^3$	$-2.3(2) \times 10^3$	$3.1(2) \times 10^3$
2	$1.209(6) \times 10^3$	$8.87(6) \times 10^1$	$6.81(4) \times 10^0$	$-1.088(2) \times 10^3$	$-6.57(1) \times 10^1$	$2.665(1) \times 10^2$
3	$3.816(4) \times 10^2$	$2.66(5) \times 10^0$	$-2.91(9) \times 10^{-2}$	$-3.076(2) \times 10^2$	$-3.73(8) \times 10^{-1}$	$5.2590(4) \times 10^1$
4	$1.234(1) \times 10^2$	$-2.48(2) \times 10^0$	$-8.44(4) \times 10^{-2}$	$-9.574(6) \times 10^1$	$1.423(4) \times 10^0$	$1.5030(2) \times 10^1$
5	$4.274(7) \times 10^1$	$-1.416(9) \times 10^0$	$-3.22(2) \times 10^{-2}$	$-3.362(3) \times 10^1$	$7.21(2) \times 10^{-1}$	$5.2026(7) \times 10^0$
6	$1.574(3) \times 10^1$	$-6.32(4) \times 10^{-1}$	$-1.20(1) \times 10^{-2}$	$-1.290(1) \times 10^1$	$3.167(9) \times 10^{-1}$	$2.0142(4) \times 10^0$
7	$6.10(2) \times 10^0$	$-2.68(2) \times 10^{-1}$	$-4.83(5) \times 10^{-3}$	$-5.281(7) \times 10^0$	$1.380(5) \times 10^{-1}$	$8.377(2) \times 10^{-1}$
8	$2.458(9) \times 10^0$	$-1.13(1) \times 10^{-1}$	$-2.09(3) \times 10^{-3}$	$-2.269(4) \times 10^0$	$6.12(2) \times 10^{-2}$	$3.6598(9) \times 10^{-1}$
9	$1.025(5) \times 10^0$	$-4.78(6) \times 10^{-2}$	$-9.6(2) \times 10^{-4}$	$-1.013(2) \times 10^0$	$2.79(1) \times 10^{-2}$	$1.6566(5) \times 10^{-1}$
10	$4.40(2) \times 10^{-1}$	$-2.04(3) \times 10^{-2}$	$-4.67(9) \times 10^{-4}$	$-4.66(1) \times 10^{-1}$	$1.299(7) \times 10^{-2}$	$7.700(3) \times 10^{-2}$

TABLE XV. Coefficients of the leading-order ($A_{(1-T),n}$), next-to-leading-order ($B_{(1-T),n}$) and next-to-next-to-leading-order ($C_{(1-T),n}$) contributions to the nth moment of the thrust distribution with a cut $(1-T) > \kappa$ for $\kappa = 10^{-3}$.

n	$A_{(1-T),n}$	$B_{(1-T),n}$	$C_{(1-T),n}$
1	$2.065\,27(3) \times 10^{0}$	$4.8055(3) \times 10^{1}$	$1.076(1) \times 10^{3}$
2	$1.901\,72(5) \times 10^{-1}$	$6.2578(4) \times 10^{0}$	$1.829(1) \times 10^{2}$
3	$2.9874(1) \times 10^{-2}$	$1.1278(1) \times 10^{0}$	$3.617(3) \times 10^{1}$
4	$5.8576(3) \times 10^{-3}$	$2.4619(3) \times 10^{-1}$	$8.447(8) \times 10^{0}$
5	$1.294\,60(8) \times 10^{-3}$	$6.0029(9) \times 10^{-2}$	$2.184(2) \times 10^{0}$
6	$3.0839(2) \times 10^{-4}$	$1.5710(3) \times 10^{-2}$	$6.026(7) \times 10^{-1}$
7	$7.7332(7) \times 10^{-5}$	$4.318(1) \times 10^{-3}$	$1.739(2) \times 10^{-1}$
8	$2.0132(2) \times 10^{-5}$	$1.2307(3) \times 10^{-3}$	$5.184(8) \times 10^{-2}$
9	$5.3929(6) \times 10^{-6}$	$3.608(1) \times 10^{-4}$	$1.584(3) \times 10^{-2}$
10	$1.4775(2) \times 10^{-6}$	$1.0821(4) \times 10^{-4}$	$4.939(9) \times 10^{-3}$

The logarithmic terms give a good description of the thrust distribution in the close-to-two-jet region. We may therefore use Eq. (31) to estimate the NNLO contribution to the integral

$$\frac{1}{\sigma_{\text{tot}}} \int_0^{\kappa} (1-T)^n \frac{d\sigma}{d(1-T)} d(1-T). \quad (33)$$

TABLE XVI. Coefficients of the leading-order ($A_{\rho,n}$), next-to-leading-order ($B_{\rho,n}$) and next-to-next-to-leading-order ($C_{\rho,n}$) contributions to the nth moment of the heavy jet mass distribution with a cut $\rho > \kappa$ for $\kappa = 10^{-3}$.

n	$A_{\rho,n}$	$B_{\rho,n}$	$C_{\rho,n}$
1	$2.065\,27(3) \times 10^{0}$	$2.6555(3) \times 10^{1}$	$3.55(1) \times 10^{2}$
2	$1.901\,72(5) \times 10^{-1}$	$3.0879(5) \times 10^{0}$	$4.16(2) \times 10^{1}$
3	$2.9874(1) \times 10^{-2}$	$4.572(1) \times 10^{-1}$	$4.53(4) \times 10^{0}$
4	$5.8576(3) \times 10^{-3}$	$8.350(3) \times 10^{-2}$	$5.01(9) \times 10^{-1}$
5	$1.294\,60(8) \times 10^{-3}$	$1.7550(9) \times 10^{-2}$	$4.3(3) \times 10^{-2}$
6	$3.0839(2) \times 10^{-4}$	$4.082(3) \times 10^{-3}$	$-1.8(8) \times 10^{-3}$
7	$7.7332(7) \times 10^{-5}$	$1.0262(9) \times 10^{-3}$	$-2.6(2) \times 10^{-3}$
8	$2.0132(2) \times 10^{-5}$	$2.744(3) \times 10^{-4}$	$-1.06(8) \times 10^{-3}$
9	$5.3929(6) \times 10^{-6}$	$7.711(9) \times 10^{-5}$	$-3.4(3) \times 10^{-4}$
10	$1.4775(2) \times 10^{-6}$	$2.257(3) \times 10^{-5}$	$-1.00(8) \times 10^{-4}$

TABLE XVII. Coefficients of the leading-order ($A_{B_W,n}$), next-to-leading-order ($B_{B_W,n}$) and next-to-next-to-leading-order ($C_{B_W,n}$) contributions to the nth moment of the wide jet broadening distribution with a cut $B_W > \kappa$ for $\kappa = 10^{-3}$.

n	$A_{B_W,n}$	$B_{B_W,n}$	$C_{B_W,n}$
1	$3.991\,53(5) \times 10^{0}$	$5.929(8) \times 10^{0}$	$3.56(4) \times 10^{2}$
2	$3.368\,39(5) \times 10^{-1}$	$4.5391(5) \times 10^{0}$	$5.06(2) \times 10^{1}$
3	$4.7552(1) \times 10^{-2}$	$6.667(1) \times 10^{-1}$	$6.24(3) \times 10^{0}$
4	$8.3108(3) \times 10^{-3}$	$1.0680(2) \times 10^{-1}$	$5.99(7) \times 10^{-1}$
5	$1.629\,54(7) \times 10^{-3}$	$1.8634(6) \times 10^{-2}$	$1.2(2) \times 10^{-2}$
6	$3.4328(2) \times 10^{-4}$	$3.458(2) \times 10^{-3}$	$-1.82(4) \times 10^{-2}$
7	$7.5946(4) \times 10^{-5}$	$6.711(4) \times 10^{-4}$	$-8.2(1) \times 10^{-3}$
8	$1.7410(1) \times 10^{-5}$	$1.347(1) \times 10^{-4}$	$-2.74(3) \times 10^{-3}$
9	$4.1002(3) \times 10^{-6}$	$2.773(3) \times 10^{-5}$	$-8.28(8) \times 10^{-4}$
10	$9.8630(7) \times 10^{-7}$	$5.831(8) \times 10^{-6}$	$-2.39(2) \times 10^{-4}$

TABLE XVIII. Coefficients of the leading-order ($A_{B_T,n}$), next-to-leading-order ($B_{B_T,n}$) and next-to-next-to-leading-order ($C_{B_T,n}$) contributions to the nth moment of the total jet broadening distribution with a cut $B_T > \kappa$ for $\kappa = 10^{-3}$.

n	$A_{B_T,n}$	$B_{B_T,n}$	$C_{B_T,n}$
1	$3.991\,53(5) \times 10^{0}$	$7.9498(8) \times 10^{1}$	$9.72(4) \times 10^{2}$
2	$3.368\,39(5) \times 10^{-1}$	$1.473\,65(7) \times 10^{1}$	$3.324(2) \times 10^{2}$
3	$4.7552(1) \times 10^{-2}$	$2.7659(2) \times 10^{0}$	$7.202(4) \times 10^{1}$
4	$8.3108(3) \times 10^{-3}$	$6.0697(5) \times 10^{-1}$	$1.676(1) \times 10^{1}$
5	$1.629\,54(7) \times 10^{-3}$	$1.4718(2) \times 10^{-1}$	$4.232(3) \times 10^{0}$
6	$3.4328(2) \times 10^{-4}$	$3.8099(5) \times 10^{-2}$	$1.135(1) \times 10^{0}$
7	$7.5946(4) \times 10^{-5}$	$1.0326(2) \times 10^{-2}$	$3.181(3) \times 10^{-1}$
8	$1.7410(1) \times 10^{-5}$	$2.8966(6) \times 10^{-3}$	$9.21(1) \times 10^{-2}$
9	$4.1002(3) \times 10^{-6}$	$8.349(2) \times 10^{-4}$	$2.737(3) \times 10^{-2}$
10	$9.8630(7) \times 10^{-7}$	$2.4608(6) \times 10^{-4}$	$8.30(1) \times 10^{-3}$

TABLE XIX. Coefficients of the leading-order ($A_{C,n}$), next-to-leading-order ($B_{C,n}$) and next-to-next-to-leading-order ($C_{C,n}$) contributions to the nth moment of the C parameter distribution with a cut $C > \kappa$ for $\kappa = 10^{-3}$.

n	$A_{C,n}$	$B_{C,n}$	$C_{C,n}$
1	$8.590\,06(9) \times 10^{0}$	$1.7964(1) \times 10^{2}$	$3.930(6) \times 10^{3}$
2	$2.431\,58(4) \times 10^{0}$	$8.1163(4) \times 10^{1}$	$2.332(1) \times 10^{3}$
3	$1.079\,19(3) \times 10^{0}$	$4.2752(3) \times 10^{1}$	$1.3608(8) \times 10^{3}$
4	$5.6848(2) \times 10^{-1}$	$2.5804(2) \times 10^{1}$	$8.791(5) \times 10^{2}$
5	$3.2720(1) \times 10^{-1}$	$1.6865(2) \times 10^{1}$	$6.074(4) \times 10^{2}$
6	$1.987\,19(9) \times 10^{-1}$	$1.1595(1) \times 10^{1}$	$4.386(3) \times 10^{2}$
7	$1.250\,95(6) \times 10^{-1}$	$8.256(1) \times 10^{0}$	$3.265(2) \times 10^{2}$
8	$8.0789(4) \times 10^{-2}$	$6.033(1) \times 10^{0}$	$2.485(2) \times 10^{2}$
9	$5.3184(3) \times 10^{-2}$	$4.4999(8) \times 10^{0}$	$1.925(2) \times 10^{2}$
10	$3.5534(2) \times 10^{-2}$	$3.4129(7) \times 10^{0}$	$1.511(1) \times 10^{2}$

TABLE XX. Coefficients of the leading-order ($A_{y_{23},n}$), next-to-leading-order ($B_{y_{23},n}$) and next-to-next-to-leading-order ($C_{y_{23},n}$) contributions to the nth moment of the three-to-two-jet transition distribution with a cut $y_{23} > \kappa$ for $\kappa = 10^{-3}$.

n	$A_{y_{23},n}$	$B_{y_{23},n}$	$C_{y_{23},n}$
1	$8.7701(2) \times 10^{-1}$	$1.3044(2) \times 10^{1}$	$1.367(6) \times 10^{2}$
2	$8.1400(3) \times 10^{-2}$	$1.2913(3) \times 10^{0}$	$1.353(9) \times 10^{1}$
3	$1.285\,29(8) \times 10^{-2}$	$1.9873(7) \times 10^{-1}$	$1.86(2) \times 10^{0}$
4	$2.5226(2) \times 10^{-3}$	$3.770(2) \times 10^{-2}$	$3.08(6) \times 10^{-1}$
5	$5.5680(6) \times 10^{-4}$	$8.028(6) \times 10^{-3}$	$5.6(2) \times 10^{-2}$
6	$1.3232(2) \times 10^{-4}$	$1.837(2) \times 10^{-3}$	$1.03(5) \times 10^{-2}$
7	$3.3091(4) \times 10^{-5}$	$4.410(6) \times 10^{-4}$	$1.8(1) \times 10^{-3}$
8	$8.590(1) \times 10^{-6}$	$1.096(2) \times 10^{-4}$	$2.8(4) \times 10^{-4}$
9	$2.2945(4) \times 10^{-6}$	$2.794(5) \times 10^{-5}$	$2(1) \times 10^{-5}$
10	$6.269(1) \times 10^{-7}$	$7.26(2) \times 10^{-6}$	$-9(4) \times 10^{-6}$

TABLE XXI. Coefficients of the leading-order ($A_{\sqrt{\rho},n}$), next-to-leading-order ($B_{\sqrt{\rho},n}$) and next-to-next-to-leading-order ($C_{\sqrt{\rho},n}$) contributions to the nth moment of the square root of the heavy jet mass distribution with a cut $\sqrt{\rho} > \kappa$ for $\kappa = 10^{-3}$.

n	$A_{\sqrt{\rho},n}$	$B_{\sqrt{\rho},n}$	$C_{\sqrt{\rho},n}$
1	$1.35675(6) \times 10^{1}$	$-2.714(2) \times 10^{2}$	$8.86(4) \times 10^{3}$
2	$2.10336(3) \times 10^{0}$	$2.3347(3) \times 10^{1}$	$4.12(1) \times 10^{2}$
3	$5.6445(1) \times 10^{-1}$	$8.857(1) \times 10^{0}$	$1.289(4) \times 10^{2}$
4	$1.90190(5) \times 10^{-1}$	$3.0867(5) \times 10^{0}$	$4.16(2) \times 10^{1}$
5	$7.2514(2) \times 10^{-2}$	$1.1506(2) \times 10^{0}$	$1.359(7) \times 10^{1}$
6	$2.9874(1) \times 10^{-2}$	$4.572(1) \times 10^{-1}$	$4.53(4) \times 10^{0}$
7	$1.29764(6) \times 10^{-2}$	$1.9132(6) \times 10^{-1}$	$1.52(2) \times 10^{0}$
8	$5.8576(3) \times 10^{-3}$	$8.350(3) \times 10^{-2}$	$5.01(9) \times 10^{-1}$
9	$2.7224(2) \times 10^{-3}$	$3.773(2) \times 10^{-2}$	$1.57(5) \times 10^{-1}$
10	$1.29460(8) \times 10^{-3}$	$1.7550(9) \times 10^{-2}$	$4.3(3) \times 10^{-2}$

TABLE XXII. Contribution to the next-to-next-to-leading-order coefficient $C_{(1-T),n}$ for the nth moment of the thrust distribution obtained from integrating the logarithmic terms from 0 to κ for various values of κ.

n	$\kappa = 10^{-3}$	$\kappa = 10^{-5}$	$\kappa = 10^{-7}$	$\kappa = 10^{-9}$
1	-2.8×10^{1}	1.4×10^{1}	1.2×10^{0}	4.9×10^{-2}
2	-3.0×10^{-2}	4.5×10^{-5}	4.7×10^{-8}	2.1×10^{-11}
3	-2.0×10^{-5}	2.6×10^{-10}	2.9×10^{-15}	1.3×10^{-20}
4	-1.5×10^{-8}	1.8×10^{-15}	2.1×10^{-22}	9.7×10^{-30}
5	-1.2×10^{-11}	1.4×10^{-20}	1.6×10^{-29}	7.6×10^{-39}
6	-1.0×10^{-14}	1.1×10^{-25}	1.3×10^{-36}	6.3×10^{-48}
7	-8.6×10^{-18}	9.4×10^{-31}	1.1×10^{-43}	5.4×10^{-57}
8	-7.5×10^{-21}	8.1×10^{-36}	9.9×10^{-51}	4.7×10^{-66}
9	-6.6×10^{-24}	7.1×10^{-41}	8.8×10^{-58}	4.1×10^{-75}
10	-5.9×10^{-27}	6.4×10^{-46}	7.9×10^{-65}	3.7×10^{-84}

These results are reported for $\kappa = 10^{-3}$, 10^{-5}, 10^{-7} and 10^{-9} in Table XXII.

The first five moments of the six event shape variables have also been calculated independently in Ref. [8]. The results of Tables I, II, III, IV, V, and VI for the perturbative coefficients $A_{O,n}$, $B_{O,n}$ and $C_{O,n}$ have their correspondence in Table 3 of Ref. [8]. The results of Tables VIII, IX, X, XI, XII, and XIII for the individual color factors to $C_{O,n}$ have their correspondence in Tables 1 and 2 of Ref. [8]. The compatibility of the two results is not perfect, but acceptable given the complexity of the calculation. The higher moments ($n \geq 2$) agree typically within 5%. The agreement in the individual color factors is in general better (at the level of 2% for the numerically dominant color factors N_c^2 and $N_f N_c$ for $n \geq 2$). There is a simple explanation for the fact that one observes a better agreement for the individual color factors as compared to the complete NNLO coefficient $C_{O,n}$: The complete NNLO coefficient involves large cancellations between different color factors (in particular between the numerically dominant color factors N_c^2 and $N_f N_c$). The numerically not so important color factors N_c^0 and N_c^{-2} show slightly worse agreement. As they give numerically small contributions they have been calculated with lower statistics. The agreement for the complete NNLO coefficients for the first moment ($n = 1$) is at the level of 20% for thrust, C parameter and the three-to-two-jet transition variable, 50% for the heavy jet mass and 70%–100% for the jet broadenings. It should be noted that the first moment receives sizeable contributions from the close-to-two-jet region, in particular, the jet broadenings. In comparing the two results one should take into account that Tables 1–3 of Ref. [8] do not include systematic errors. For the first moment the systematic error from the close-to-two-jet region dominates over the statistical error. As in the case for the higher moments one also observes for the first moment a better agreement for the individual color factors as compared to the complete NNLO results, with the same explanation of large cancellations between different color factors as above.

V. CONCLUSIONS

In this article I reported on the NNLO calculation of the moments of the event shape observables associated to three-jet events in electron-positron annihilation. I provided NNLO results for the moments of the event shape variables thrust, heavy jet mass, wide jet broadening, total jet broadening, C parameter and the Durham three-to-two-jet transition variable. The results of this paper will be useful for an extraction of α_s from three-jet quantities [77,98–102].

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