Extremal correlators and Hurwitz numbers in symmetric product orbifolds

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We study correlation functions of single-cycle chiral operators in $Sym^N T^4$, the symmetric product orbifold of N supersymmetric four-tori. Correlators of twist operators are evaluated on covering surfaces, generally of different genera, where fields are single valued. We compute some simple four-point functions and study how the sum over inequivalent branched covering maps splits under operator product expansions. We then discuss extremal n-point correlators, i.e. correlators of $n - 1$ chiral and one antichiral operators. They obey simple recursion relations involving numbers obtained from counting branched covering maps with particular properties. In most cases we are able to solve explicitly the recursion relations. Remarkably, extremal correlators turn out to be equal to Hurwitz numbers.

I. INTRODUCTION

In this paper we apply the technology developed in a companion article [1] to study correlation functions of single-cycle twist operators in $Sym^N T^4$, the symmetric product orbifold of N supersymmetric four-tori. We focus on *extremal* correlators, which by definition are correlators of $n - 1$ chiral and one antichiral operators, where the notion of chirality is that of a (2, 2) subalgebra of the full (4, 4) supersymmetry.

An important motivation for this work is the holographic duality between the symmetric product of T^4 and type IIB string theory on $AdS_3 \times S^3 \times T^4$ [2].[1] The early checks of this duality included comparison of the moduli spaces [7,8] and the spectra of both theories [9–12]. Recently, much progress was made in comparing correlation functions. The structure constants of single-cycle operators in the chiral ring of the symmetric product were computed early on in [13] and, for a subset of these operators they were extended in [14,15] to the full 1/2 Bogomol'nyi-Prasad-Sommerfield (BPS) $SU(2)$ multiplet. These three-point functions were exactly reproduced in the string theory/supergravity dual [16–19] (see also [20–22]), which also predicts some correlators not yet computed in the symmetric product [18].

This agreement between bulk and boundary correlators was at first surprising because the computations on the string and conformal field theory (CFT) FT sides are performed at very different points in the moduli space [7,8]. It can be explained by a nonrenormalization theorem proved in [23]. The nonrenormalization theorem also holds for extremal correlators (as first conjectured in [19]), so we

expect that they can be successfully compared on both sides of the AdS_3/CFT_2 correspondence as well.

Extremal correlators play a special role in the anti-de Sitter/conformal field theory (AdS/CFT) correspondence. Not only are they not renormalized: they offer a unique window into the bulk/boundary dictionary, as we review at the end of this introduction. The main purpose of this paper is to compute extremal correlators of (anti)chiral twist fields on the CFT side of the duality. Explicit computations in symmetric product orbifolds can be notoriously difficult (see e.g. [14,15,24]). As usual in orbifold theories, determining a correlator involves finding branched covering maps, with branching points fixed by the position and type of the different twist fields. The non-Abelian character of the permutation group implies that starting from three-point functions several branched coverings, generally of different genera, contribute to a given correlator.

In general finding the relevant covering maps is a challenging task to pursue analytically. When the covering surface is a sphere, which provides the leading contribution for large N, the covering maps are quotients of polynomials whose coefficients depend on the length of the cycles in the correlator (see [1] for a thorough discussion).

However, extremal correlators are special. First, their genus zero covering maps are among the simplest: either a polynomial or a polynomial divided by a monomial. Moreover, in the former case the *only* contribution to the correlators comes from the sphere. Second, extremal correlators have trivial spacetime dependence, and only the overall constant needs to be determined. One of our main results is that finding this constant does not require detailed knowledge of the branched covering maps. What is needed are the total number of maps and the numbers of maps contributing to the operator product expansions (OPEs) between two operators. As we will show, these numbers

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[1]See [3–6] for reviews.

enter in simple recursion relations that determine all the extremal correlators of single-cycle operators.

The problem of computing extremal correlators thus reduces to counting branched covering maps with given branching structure (and other additional restrictions). These enumeration questions have a long history [25] and are generally referred to as the "Hurwitz problem." The Hurwitz problem has an equivalent reformulation in terms of enumerating sets of elements of S_N that multiply to the identity. Symmetric product orbifolds provide a very intuitive picture of this equivalence [1].

We are able to compute most extremal correlators of single-cycle twist operators, by finding the solutions to the relevant enumeration problems when the covering surface is a sphere. The final expressions have a remarkable property: after a certain rescaling of the (anti)chiral operators, all correlators are equal to the relevant Hurwitz number, i.e. to the total number of maps from the covering surface! We conjecture that this result holds also for extremal correlators of multicycle twist operators. We emphasize that the simple final answer arises nontrivially after combining partial results. As always, miracles of this kind call for a deeper explanation. We suspect that a topologically twisted version of the symmetric orbifold CFT will be the natural framework to understand why extremal correlators compute Hurwitz numbers. We note in this respect that a topologically twisted version of the *world sheet* theory with $\text{AdS}_3 \times S^3$ target [26] computes correlators of spacetime chiral primary operators. It would be interesting to see if the methods of [26] can be used to simplify the calculation of extremal correlators on the string theory side.

We conclude this introduction with some general remarks about the role of extremal correlators in the AdS/CFT correspondence.[2] It is worth recalling the situation in the $\text{AdS}_5/\text{CFT}_4$ instance of the duality. Extremal correlators in $\mathcal{N} = 4$ SYM (and in the dual IIB string theory on $\text{AdS}_5 \times S^5$) are also believed to obey a nonrenormalization theorem. Evidence for their protection came first from their calculation on the supergravity side [31], which gives a result proportional to the free-field expression (whereas a nontrivial dependence on the cross ratios would be *a priori* expected), and it was confirmed by explicit perturbative calculations on the field theory side [32], as well as by formal arguments using $\mathcal{N} = 2$ harmonic superspace [33,34]. See also [35–37] for related work. What makes extremal correlators especially interesting is not just their protection, but the fact that they are uniquely sensitive to the color structure of operators. In $\mathcal{N} = 4$ super Yang-Mills (SYM) chiral primary operators of charge k (under a $U(1) \subset SU(4)$ subgroup the R symmetry) are linear combinations of the form

$$\text{Tr}Z^k + \frac{1}{N}\sum_\ell a_\ell \text{Tr}Z^{k-\ell}\text{Tr}Z^\ell + \frac{1}{N^2}\sum_{\ell_1,\ell_2} a_{\ell_1\ell_2}\text{Tr}Z^{k-\ell_1-\ell_2}$$
$$\times \text{Tr}Z^{\ell_1}\text{Tr}Z^{\ell_2} + \dots. \qquad (1.1)$$

If the coefficients $a_{\ell_1\dots\ell_m}$ are taken to be independent of N, each term in the sum has the same large N scaling (since each trace contributes a factor of N). However, when computing large N correlation functions, the usual factorization arguments imply that *generically* only the single-trace piece of the operator contributes. Extremal correlators are the exception to this rule: the multitrace admixtures contribute at the same order [31]. The dual statement on the gravity side is that extremal correlators are uniquely sensitive to boundary terms, which can instead be neglected in the nonextremal cases [31].

Finding the precise dictionary between bulk states and boundary operators requires knowing which precise admixture of single and multitrace operators map to single-particle states in the bulk. Extremal correlators offer a window to find this precise dictionary. It was found in [31] by a careful evaluation of supergravity boundary terms in extremal three-point functions in $\text{AdS}_5 \times S^5$ that at leading order in N one can match *single*-trace operators of $\mathcal{N} = 4$ SYM to bulk *single*-particle states: in other terms $a_\ell = O(1/N)$. By contrast in little string theory the holographic dictionary requires mapping single string states with admixtures of the form (1.1) with $a_{\ell_1\dots\ell_m}$ of order one [38].

In the symmetric product orbifold, twist operators are classified by conjugacy classes of the symmetric group S_N, i.e. by their cycle structure – with the single-cycle operators playing a somewhat similar role as the single-trace operators in a gauge theory. In $\text{AdS}_3/\text{CFT}_2$, the agreement between the string theory and the field theory calculations of three-point functions works by postulating the naive correspondence between single-strings and single-cycles [16–18] (in other terms the analogue of a_ℓ must be assumed to be zero at large N to find agreement). Most *supergravity* extremal three-point correlators also match assuming the naive dictionary [19]. The matching of some doubly exceptional supergravity three-point correlators (extremal three-point correlators containing states that saturate the Breitenlohner-Freedman bound) appears instead to require an order-one admixture of single and double-cycles [19].[3] We hope that the extremal n-point correlators computed in this paper will be useful in sharpening the bulk-to-boundary dictionary.

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[2]It is also interesting to note that in the context of Gopakumar's approach to string duals of free-field theories [27] extremal correlators play a (technically) preferred role due to the relative simplicity of obtaining them [28–30].

[3]Alternatively, this may just indicate a subtlety in the naive supergravity calculation, which was performed by analytic continuation away from extremality. A careful analysis of supergravity boundary terms is necessary to confirm the validity of the analytic continuation procedure, indeed we would expect boundary terms to be especially subtle for states sitting exactly at the Breitenlohner-Freedman bound.

The paper is organized as follows: In Sec. II, we review the basic observables and interactions in the single-cycle sector of the chiral ring. In Sec. III, we compute some nonextremal four-point functions and extract useful lessons about the structure of the OPE in symmetric product orbifolds. In Sec. IV, we compute extremal correlators of single-cycle operators, discover that they compute Hurwitz numbers, and present a general conjecture for arbitrary extremal correlators. Two appendices collect technical results used in the body of the paper.

II. SINGLE-CYCLE STATES IN THE CHIRAL RING

The symmetric product orbifold $Sym^N T^4$ is obtained by considering N copies of T^4 and identifying the coordinates under the action of the permutation group S_N. Each copy of T^4 has bosonic coordinates X_I^i with real fermionic partners χ_I^i, where $i = 1, 2, 3, 4$ and $I = 1, \ldots, N$.[4] The basic observables of a symmetric product orbifold are the twist fields $\sigma_{[g]}$, labeled by a conjugacy class $[g]$ of the permutation group. Conjugacy classes with one nontrivial cycle of length n will be denoted by $[n]$. Clearly the OPE of two single-cycle twist-fields gives both single-cycle and multicycle operators, a fact that will play a prominent role in the calculation of extremal correlators (see Sec. IV).

"Gauge-invariant" twist fields $\sigma_{[g]}$ can be constructed from "gauge-non-invariant" ones, σ_g, associated to a group element $g \in S_N$ and not to a conjugacy class. Single- and double-cycle group elements have the form

$$g_{single} = (12 \ldots n), \tag{2.1}$$

$$g_{double} = (12 \ldots n_1)(n_1 + 1 \ldots n_1 + n_2), \tag{2.2}$$

and similarly for higher-cycle operators. This representation indicates on which of the N elements the cyclical permutations act, and in which order. We refer to the set of values of I on which g acts as "colors" of an element g. The operator $\sigma_g(z, \bar{z})$ is defined as a "defect" imposing the following monodromies on the different copies of the fields:

$$X_I^i(e^{2\pi i} z)\sigma_g(0) = X_{g(I)}^i(z)\sigma_g(0), \tag{2.3}$$

and similarly for the fermionic fields. Single-cycle "gauge-invariant" operators are obtained by averaging over the group orbit,

$$\sigma_{[n]} \equiv \frac{1}{\sqrt{nN!(N-n)!}} \sum_{h \in S(N)} \sigma_{h^{-1}(12\ldots n)h}, \tag{2.4}$$

where the prefactor gives the normalization

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[4] For more details on symmetric product orbifolds we refer the reader to [1] and references therein. For more details on the chiral ring of the symmetric product of T^4 see [13].

$$\langle \sigma_{[m]}(0)\sigma_{[n]}(z)\rangle = \frac{\delta_{mn}}{|z|^{2\Delta_n}}. \tag{2.5}$$

There is a similar definition for double-cycle operators. The operators (2.4) have conformal dimension (see e.g. [24])

$$\Delta_n = \frac{6}{24}\left(n - \frac{1}{n}\right), \tag{2.6}$$

where the 6 in the numerator is the central charge of each copy of supersymmetric T^4.

The (anti)chiral operators are built by dressing the twist fields (2.4) with invariant contributions from the fermionic sector to satisfy the (anti)chiral relationship $\Delta = \pm Q$, where Q is the charge under the $U(1)$ of the $\mathcal{N} = 2$ subalgebra. There are three types of chiral operators: $O_n^{(0,0)}$, $O_n^{(a,\bar{a})}(a, \bar{a} = 1, 2)$, and $O_n^{(2,2)}$, corresponding to 0, 1, and 2-forms in T^4, respectively, with n being the length of the permutation cycle. We will consider only operators whose holomorphic and antiholomorphic quantum numbers are equal, but it is easy to extend our results to operators of mixed type, e.g. $O_n^{(0,2)}$.

More explicitly, the four real holomorphic fermions of T^4 can be combined, in each copy I, into two complex fermions ψ_I^1, ψ_I^2, and bosonized as

$$\psi_I^1 = e^{i\phi_I^1}, \tag{2.7}$$

$$\psi_I^2 = e^{i\phi_I^2}, \qquad I = 1, \ldots, N. \tag{2.8}$$

The $U(1)$ current of the $\mathcal{N} = 2$ algebra is

$$J = \frac{i}{2} \sum_{I=1}^{N} \partial\phi_I^1 + \partial\phi_I^2. \tag{2.9}$$

We define first the gauge-non-invariant chiral operators,

$$o_{(12\ldots n)}^{(0,0)} = e^{i((n-1)/2n)\sum_{I=1}^{n}(\phi_I^1 + \phi_I^2 + \bar{\phi}_I^1 + \bar{\phi}_I^2)}\sigma_{(12\ldots n)}, \tag{2.10}$$

$$o_{(12\ldots n)}^{(a=1,\bar{a}=1)} = e^{i((n+1)/2n)\sum_{I=1}^{n}(\phi_I^1 + \bar{\phi}_I^1) + i((n-1)/2n)\sum_{I=1}^{n}(\phi_I^2 + \bar{\phi}_I^2)}$$
$$\times \sigma_{(12\ldots n)}, \tag{2.11}$$

$$o_{(12\ldots n)}^{(a=2,\bar{a}=2)} = e^{i((n-1)/2n)\sum_{I=1}^{n}(\phi_I^1 + \bar{\phi}_I^1) + i((n+1)/2n)\sum_{I=1}^{n}(\phi_I^2 + \bar{\phi}_I^2)}$$
$$\times \sigma_{(12\ldots n)}, \tag{2.12}$$

$$o_{(12\ldots n)}^{(2,2)} = e^{i((n+1)/2n)\sum_{I=1}^{n}(\phi_I^1 + \phi_I^2 + \bar{\phi}_I^1 + \bar{\phi}_I^2)}\sigma_{(12\ldots n)}, \tag{2.13}$$

and the gauge-invariant operators are obtained by summing over the group orbit as in (2.4),

$$O_n^{(0,0)} = \frac{1}{\sqrt{nN!(N-n)!}} \sum_{h \in S(N)} o_{h^{-1}(12\ldots n)h}^{(0,0)}, \tag{2.14}$$

$$O_n^{(a,\bar{a})} = \frac{1}{\sqrt{nN!(N-n)!}} \sum_{h \in S(N)} o_{h^{-1}(12...n)h}^{(a,\bar{a})}, \qquad (2.15)$$

$$O_n^{(2,2)} = \frac{1}{\sqrt{nN!(N-n)!}} \sum_{h \in S(N)} o_{h^{-1}(12...n)h}^{(2,2)}. \qquad (2.16)$$

The conformal dimensions and charges are

$$\Delta_n^0 = Q_n^0 = \frac{n-1}{2}, \qquad (2.17)$$

$$\Delta_n^a = Q_n^a = \frac{n}{2}, \qquad (2.18)$$

$$\Delta_n^2 = Q_n^2 = \frac{n+1}{2}, \qquad (2.19)$$

and similarly for the antiholomorphic sector. The antichiral operators $O_n^{(0,0)\dagger}$, $O_n^{(a,\bar{a})\dagger}$, $O_n^{(2,2)\dagger}$ are obtained by reversing the sign in the exponents in (2.10), (2.11), (2.12), and (2.13). The theory has actually $\mathcal{N} = 4$ supersymmetry, and the (anti)chiral states are the (lowest)highest weights in an $SU(2)$ multiplet.

The fusion rules of the chiral ring are [13]

$$(0,0) \times (0,0) \rightarrow (0,0) + (2,2),$$
$$(0,0) \times (2,2) \rightarrow (2,2),$$
$$(0,0) \times (a,a) \rightarrow (a,a), \qquad (2.20)$$
$$(a,a) \times (a,a) \rightarrow (2,2).$$

These rules are easy to obtain by combining the composition law of the permutation group and the conservation of $U(1)$ charge. To leading order in $1/N$ the five structure constants corresponding to the above OPEs are [13]

$$\langle O_{n_3}^{(0,0)\dagger} O_{n_2}^{(0,0)} O_{n_1}^{(0,0)} \rangle = \left(\frac{1}{N}\right)^{1/2} \frac{(n_3)^{3/2}}{(n_2 n_1)^{1/2}}, \qquad (2.21)$$

$$\langle O_{n_3}^{(2,2)\dagger} O_{n_2}^{(2,2)} O_{n_1}^{(0,0)} \rangle = \left(\frac{1}{N}\right)^{1/2} \frac{(n_2)^{3/2}}{(n_3 n_1)^{1/2}}, \qquad (2.22)$$

$$\langle O_{n_3}^{(b,\bar{b})\dagger} O_{n_2}^{(a,\bar{a})} O_{n_1}^{(0,0)} \rangle = \delta^{ab} \delta^{\bar{a}\bar{b}} \left(\frac{1}{N}\right)^{1/2} \frac{(n_3 n_2)^{1/2}}{(n_1)^{1/2}}, \qquad (2.23)$$

$$\langle O_{n_3}^{(2,2)\dagger} O_{n_2}^{(a,\bar{a})} O_{n_1}^{(b,\bar{b})} \rangle = \epsilon^{ab} \epsilon^{\bar{a}\bar{b}} \left(\frac{1}{N}\right)^{1/2} \frac{(n_2 n_1)^{1/2}}{(n_3)^{1/2}}, \qquad (2.24)$$

$$\langle O_{n_3}^{(2,2)\dagger} O_{n_2}^{(0,0)} O_{n_1}^{(0,0)} \rangle = \left(\frac{1}{N}\right)^{1/2} \frac{1}{(n_3 n_2 n_1)^{1/2}}. \qquad (2.25)$$

Conservation of $U(1)$ charge imposes the relation $n_3 =$

$n_1 + n_2 - 1$ in all the cases, except in (2.25), where we have $n_3 = n_1 + n_2 - 3$.

Consider a correlator of gauge-invariant operators (ignoring the fermionic dressing and normalization factors),

$$\left\langle \prod_{j=1}^p \sigma_{[n_j]}(z_j, \bar{z}_j) \right\rangle \sim \sum_{h_j \in S(N)} \left\langle \prod_{j=1}^p \sigma_{h_j(12...n_j)h_j^{-1}}(z_j, \bar{z}_j) \right\rangle. \qquad (2.26)$$

A term in this expansion will be nonzero only if the product of its group elements *in a fixed order*[5] satisfies,

$$(n_p)(n_{p-1})\ldots(n_1) = 1, \qquad (2.27)$$

where $(n_j) \equiv h_j(12\ldots n_j)h_j^{-1}$.

Some terms of the sum (2.26) will be "disconnected," namely, the p-point function splits into two or more independent factors with no common colors, such as

$$\left\langle \prod_{j=1}^q \sigma_{h_j g_j h_j^{-1}}(z_j, \bar{z}_j) \right\rangle_{\text{conn}} \left\langle \prod_{j=q+1}^p \sigma_{h_j g_j h_j^{-1}}(z_j, \bar{z}_j) \right\rangle_{\text{conn}}. \qquad (2.28)$$

Each term in the expansion (2.26) has a certain number of active colors, which we denote by c. For example, in a three-point function, the term

$$\langle \sigma_{(4321)} \sigma_{(34)} \sigma_{(123)} \rangle \qquad (2.29)$$

has $c = 4$. It is convenient to organize the connected terms in the sum (2.26) into groups of terms with fixed number of colors

$$\left\langle \prod_{i=1}^p \sigma_{[n_i]}(z_i, \bar{z}_i) \right\rangle_{\text{conn.}} = \sum_c R_c(n_i), \qquad (2.30)$$

such that each R_c is a sum of terms

$$R_c(n_i) = F_c(n_i, N) \sum_{j=1}^{H_c} \langle \sigma_{n_p} \ldots \sigma_{n_1} \rangle_j, \qquad (2.31)$$

with a fixed c. H_c is the number of distinct ways of satisfying (2.27), up to a relabeling of colors reflected in the symmetry factors $F_c(n_i, N)$, which can be computed exactly (see [1] for details). These symmetry factors encode the dependence of the correlators on N.

The terms in (2.31) are computed, as usual in orbifolds, by going to the covering surface(s) where operators are single valued. The genus g of the covering surface is fixed by the Riemann-Hurwitz formula

$$\text{g} = \frac{1}{2} \sum_{j=1}^p (n-1) - c + 1. \qquad (2.32)$$

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[5]It is convenient to choose this order to coincide with the radial ordering of the operators in the correlator (see [1] for details).

Thus, we see that the sum over c in (2.30) is equivalently a sum over the genera of the covering surfaces. For every g the symmetry factor scales at large N as (see e.g. [1])

$$F_c(n_i, N) \sim N^{1-\mathsf{g}-(p/2)}, \qquad (2.33)$$

which shows that the leading terms in (2.30) come from the sphere ($\mathsf{g} = 0$).

An interesting property of the four structure constants (2.21), (2.22), (2.23), and (2.24) of the chiral ring, which will hold for their generalization to p-point extremal correlators as well, is that only genus zero covering surfaces contribute. Indeed, the Riemann-Hurwitz relation (2.32) and the relation $n_3 = n_1 + n_2 - 1$ give

$$\mathsf{g} = \frac{1}{2} \sum_{j=1}^{3} (n_j - 1) - c + 1 = n_3 - c. \qquad (2.34)$$

But the number of colors c is at least as big as the longest cycle, $c \geq n_3$, and therefore only the $c = n_3$, $\mathsf{g} = 0$ term in the genus expansion contributes. Therefore, using the proper $F_c(n_i)$ factors, one can write expressions valid for finite N [13],

$$\langle O_{n_3}^{(0,0)\dagger} O_{n_2}^{(0,0)} O_{n_1}^{(0,0)} \rangle = F(n_1, n_2) \frac{(n_3)^{3/2}}{(n_2 n_1)^{1/2}}, \qquad (2.35)$$

$$\langle O_{n_3}^{(2,2)\dagger} O_{n_2}^{(2,2)} O_{n_1}^{(0,0)} \rangle = F(n_1, n_2) \frac{(n_2)^{3/2}}{(n_3 n_1)^{1/2}}, \qquad (2.36)$$

$$\langle O_{n_3}^{(b,\bar{b})\dagger} O_{n_2}^{(a,\bar{a})} O_{n_1}^{(0,0)} \rangle = \delta^{ab} \delta^{\bar{a}\bar{b}} F(n_1, n_2) \frac{(n_3 n_2)^{1/2}}{(n_1)^{1/2}}, \qquad (2.37)$$

$$\langle O_{n_3}^{(2,2)\dagger} O_{n_2}^{(a,\bar{a})} O_{n_1}^{(b,\bar{b})} \rangle = \epsilon^{ab} \epsilon^{\bar{a}\bar{b}} F(n_1, n_2) \frac{(n_2 n_1)^{1/2}}{(n_3)^{1/2}}, \qquad (2.38)$$

where

$$F(n_1, n_2) = \left[\frac{(N - n_1)!(N - n_2)!}{(N - n_1 - n_2 + 1)!N!} \right]^{1/2}. \qquad (2.39)$$

In the large N limit, we can use

$$\lim_{N\to\infty} F(n_1, n_2) = \left(\frac{1}{N} \right)^{1/2}, \qquad (2.40)$$

and the structure constants (2.21), (2.22), (2.23), and (2.24) follow.

The case (2.25) is different, because we have $n_3 = n_1 + n_2 - 3$, which leads to

$$\mathsf{g} = \frac{1}{2} \sum_{j=1}^{3} (n_j - 1) - c + 1 = n_3 - c + 1. \qquad (2.41)$$

Here, we must distinguish between two cases. If $n_1 = 2$, $n_2 = n + 1$ then $n_3 = n$ and the only possible number of colors is $c = n + 1 = n_3 + 1$, so $\mathsf{g} = 0$. The finite N form of (2.25) is in this case [13]

$$\langle O_n^{(2,2)\dagger} O_2^{(0,0)} O_{n+1}^{(0,0)} \rangle = \left(\frac{N - n}{N(N - 1)} \right)^{1/2} \frac{1}{(2n(n + 1))^{1/2}}. \qquad (2.42)$$

When both $n_1 > 2$ and $n_2 > 2$, Eq. (2.41) allows $c = n_3$, $\mathsf{g} = 1$ or $c = n_3 + 1$, $\mathsf{g} = 0$. Here, there are contributions from covering surfaces with torus topology, which were not computed so far, and thus the finite N form of the correlator (2.25) is not known.

III. FOUR-POINT FUNCTIONS AND OPES

In this section we will compute planar contributions to some four-point functions involving chiral and antichiral operators, and we will explore their OPE limits.

A. The general form of the four-point functions

As we mentioned above, we are interested in mapping a covering sphere, S_{cover}^2, with coordinate t, to the physical sphere z, called S_{base}^2, such that the c fields $X_I(z)$, $I = 1, \ldots c$ at a generic location $z \in S_{\mathrm{base}}^2$ are traded for a single field $X(t_I(z))$, where $t_I(z) \in S_{\mathrm{cover}}^2$ are the c preimages of the point z. As z approaches a point z_i where a twist field with n_i colors is inserted, we require that n_i of the c preimages of z converge to the same point t_i on S_{cover}^2. This implies

$$t - t_i \sim (z - z_i)^{1/n_i}, \qquad (3.1)$$

and guarantees that the field $X(t)$ in S_{cover}^2 returns to its original position only after we make n_i full 2π rotations around point z_i on the base sphere S_{base}^2.

For a four-point function, we can use the $SL(2, \mathbb{C})$ invariance of S_{base} to fix the twist fields at

$$z_1 = 0, \qquad z_2 = u, \qquad z_3 = 1, \qquad z_4 = \infty, \qquad (3.2)$$

and the $SL(2, \mathbb{C})$ invariance of S_{cover} to fix the points where their preimages converge at

$$t_1 = 0, \qquad t_2 = x, \qquad t_3 = 1, \qquad t_4 = \infty. \qquad (3.3)$$

Thus, the map from S_{cover}^2 to S_{base}^2 is given by a c-sheeted map such that

$$\lim_{t\to 0} z(t) \sim b_1 t^{n_1}, \qquad (3.4)$$

$$\lim_{t\to x} z(t) \sim u + b_2 (t - x)^{n_2}, \qquad (3.5)$$

$$\lim_{t\to 1} z(t) \sim 1 + b_3 (t - 1)^{n_3}, \qquad (3.6)$$

$$\lim_{t\to\infty} z(t) \sim b_4 t^{n_4}. \qquad (3.7)$$

Now, once we obtained a function $z(t)$ satisfying the conditions (3.4), (3.5), (3.5), and (3.7),[6] we have not yet fixed the covering map. The reason is that x is a preimage of $z = u$ and thus has to be chosen to satisfy

$$u = z(t = x). \qquad (3.8)$$

This equation has in general several, say M, solutions x_j [which in general is different from the number c of preimages of $z(t)$ at a generic $t \neq x$, because x also appears as a parameter in $z(t)$ for $t \neq x$]. The number M is the number of different maps $z(t)$ with the required behavior (3.4), (3.5), (3.6), and (3.7). The problem of counting ramified coverings is referred to as the *Hurwitz problem* in the mathematical literature (see e.g. [39] for a review). We will denote each of the covering maps by $z_j(t)$ ($j = 1 \ldots M$), and the c inverse maps for each j as $t_{j,I}(z)$ ($I = 1 \ldots c$), and we will have

$$z_j(x_j) = u \qquad j = 1 \ldots M. \qquad (3.9)$$

The crucial observation now, is that M is precisely the number of terms H_c in (2.31) corresponding to $g = 0$. In other words, M is the number of solutions of the group theory condition (2.27) for $p = 4$, up to color relabeling, with the constraint

$$c = \tfrac{1}{2}(n_1 + n_2 + n_3 + n_4) - 1. \qquad (3.10)$$

For details of this correspondence see again [1]. Therefore, the correlator of gauge-invariant operators will be a sum over the M solutions of Eq. (3.8).

Let us define the operators

$$o_n^\alpha = e^{i\alpha \sum_{I=1}^n (\phi_I^1 + \phi_I^2 + \bar{\phi}_I^1 + \bar{\phi}_I^2)} \sigma_n, \qquad (3.11)$$

$$O_n^\alpha = \frac{1}{\sqrt{nN!(N-n)!}} \sum_{h \in S(N)} o_{h^{-1}(12\ldots n)h}^\alpha. \qquad (3.12)$$

Here, O_n^α is the normalized and gauge-invariant version of o_n^α. We will later specialize to $\alpha = \frac{n-1}{2n}$ or $\alpha = \frac{n+1}{2n}$ to get operators of type $O_n^{(0,0)}$ or $O_n^{(2,2)}$. One can easily generalize the discussion to include operators of type $O_n^{(a,\bar{a})}$. Consider the four-point function

$$\langle O_{n_4}^{\alpha_4}(z_4, \bar{z}_4) O_{n_3}^{\alpha_3}(z_3, \bar{z}_3) O_{n_2}^{\alpha_2}(z_2, \bar{z}_2) O_{n_1}^{\alpha_1}(z_1, \bar{z}_1) \rangle_{g=0}$$
$$= G(u, \bar{u}) z_{24}^{-2\Delta_2} z_{14}^{\Delta_2 + \Delta_3 - \Delta_1 - \Delta_4} z_{34}^{\Delta_1 + \Delta_2 - \Delta_3 - \Delta_4} z_{13}^{-\Delta_1 - \Delta_2 - \Delta_3 + \Delta_4}$$
$$\times \text{c.c.,} \qquad (3.13)$$

where

$$u = \frac{z_{12} z_{34}}{z_{13} z_{24}}. \qquad (3.14)$$

Expanding the sum over the S_N group in each operator $O_{n_i}^{\alpha_i}$ in (3.13), we will have a sum as (2.31):

$$\langle O_{n_4}^{\alpha_4}(\infty) O_{n_3}^{\alpha_3}(1) O_{n_2}^{\alpha_2}(u) O_{n_1}^{\alpha_1}(0) \rangle_{g=}$$
$$= G(u, \bar{u}) \sim \sum_{j=1}^M \langle o_{n_4}^{\alpha_4}(\infty) o_{n_3}^{\alpha_3}(1) o_{n_2}^{\alpha_2}(u) o_{n_1}^{\alpha_1}(0) \rangle_j$$
$$= \sum_{j=1}^M G_j(u, \bar{u}), \qquad (3.15)$$

where in order to evaluate each term we use a different map $z_j(t)$ to go to the covering surface. For this we will use the stress-tensor method of Dixon *et al.* [40]. We compute first the auxiliary function

$$g_j(z, u) = \frac{\langle T(z) o_{n_4}^{\alpha_4}(\infty) o_{n_3}^{\alpha_3}(1) o_{n_2}^{\alpha_2}(u, \bar{u}) o_{n_1}^{\alpha_1}(0) \rangle_j}{\langle o_{n_4}^{\alpha_4}(\infty) o_{n_3}^{\alpha_3}(1) o_{n_2}^{\alpha_2}(u) o_{n_1}^{\alpha_1}(0) \rangle_j}. \qquad (3.16)$$

Using now the OPE

$$T(z) o_{n_2}^{\alpha_2}(u) = \frac{\Delta_2}{(z-u)^2} o_{n_2}^{\alpha_2}(u) + \frac{1}{z-u} \partial o_{n_2}^{\alpha_2}(u) + \ldots, \qquad (3.17)$$

we deduce

$$\partial_u \ln G_j(u) = \{g_j(z, u)\}_{(1/(z-u))}, \qquad (3.18)$$

where $G_j(u)$ is the contribution to the holomorphic part of $G_j(u, \bar{u})$ and on the right-hand side we take the coefficient of $\frac{1}{z-u}$ in the expansion of $g_j(z, u)$. There is a similar antiholomorphic expression, so that $G_j(u, \bar{u}) = G_j(u) \bar{G}_j(\bar{u})$ up to an overall constant. The dependence on the index j comes from the fact that the computation of (3.16) is done by mapping, with $z_j(t)$, all the fields to the covering sphere, where the twist fields disappear. We need finally to sum over j as in (3.15). The relative coefficient is fixed so that the sum is single valued as a function of u. The overall factor is fixed by considering OPE limits.

Before proceeding, note that the dressing factors in (3.11) satisfy

$$e^{\alpha(\phi_1 + \ldots + \phi_n)} \sigma_{(12\ldots n)} = \left(\frac{dt}{dz}\right)^\Delta e^{n\alpha\phi}, \qquad (3.19)$$

where the left-hand side lives in the base sphere and the right-hand side lives in the covering surface.

To obtain the function (3.16) we can use that

$$T(z) = -\frac{1}{2} \sum_{i=1}^4 \partial X_I^i(z) \partial X_I^i(z) - \frac{1}{2} \sum_{i=1}^2 \partial \phi_I^i(z) \partial \phi_I^i(z), \qquad (3.20)$$

---

[6]In general there can be several functions satisfying these conditions. However, in the examples we will consider all the covering maps are obtained from a single $z(t)$, so this multiplicity will not play any role for us. For a discussion of these issues see [1].

$$= -\frac{1}{2}\lim_{w\to z}\left[\sum_{i=1}^{4}\partial X_I^i(z)\partial X_I^i(w) + \sum_{i=1}^{2}\partial\phi_I^i(z)\partial\phi_I^i(w)\right.$$
$$\left. + \frac{6N}{(z-w)^2}\right]. \tag{3.21}$$

Inserting this expression in (3.16), we must consider the two types of terms. The terms with X_I^i in (3.21) give

$$\frac{\langle\partial X_I^i(z)\partial X_I^i(w)o_{n_4}^{\alpha_4}(\infty)o_{n_3}^{\alpha_3}(1)o_{n_2}^{\alpha_2}(u)o_{n_1}^{\alpha_1}(0)\rangle_j}{\langle o_{n_4}^{\alpha_4}(\infty)o_{n_3}^{\alpha_3}(1)o_{n_2}^{\alpha_2}(u)o_{n_1}^{\alpha_1}(0)\rangle_j}$$
$$= -\frac{t_{j,I}'(z)t_{j,I}'(w)}{(t_{j,I}(z) - t_{j,I}(w))^2}, \tag{3.22}$$

and the terms with ϕ_I^i in (3.21) give

$$\frac{\langle\partial\phi_I^i(z)\partial\phi_I^i(w)o_{n_4}^{\alpha_4}(\infty)o_{n_3}^{\alpha_3}(1)o_{n_2}^{\alpha_2}(u)o_{n_1}^{\alpha_1}(0)\rangle_j}{\langle o_{n_4}^{\alpha_4}(\infty)o_{n_3}^{\alpha_3}(1)o_{n_2}^{\alpha_2}(u)o_{n_1}^{\alpha_1}(0)\rangle_j}$$
$$= -\frac{t_{j,I}'(z)t_{j,I}'(w)}{(t_{j,I}(z) - t_{j,I}(w))^2}$$
$$- (t_{j,I}'(z))^2\left(\frac{n_3\alpha_3}{t_{j,I}(z)-1} + \frac{n_2\alpha_2}{t_{j,I}(z)-x} + \frac{n_1\alpha_1}{t_{j,I}(z)}\right)^2, \tag{3.23}$$

where in the second term we took the limit $w\to z$ since there is no singularity.

In the sum over I in (3.21), we only need for $g_j(z,u)$ in (3.16) those n_2 terms whose index I is one of the colors appearing in the operator $o_{n_2}^{\alpha_2}$, since only those terms will contribute to the singularity as $z\to u$. Collecting all the terms, taking the $w\to z$ limit and subtracting the normal-order singularity as in (3.21), we get finally

$$g_j(z,u) = \frac{6}{12}\sum_{I=1}^{n_2}\{t_{j,I}, z\} + 2\sum_{I=1}^{n_2}\frac{(t_{j,I}'(z))^2}{2}$$
$$\times\left(\frac{n_3\alpha_3}{t_{j,I}(z)-1} + \frac{n_2\alpha_2}{t_{j,I}(z)-x} + \frac{n_1\alpha_1}{t_{j,I}(z)}\right)^2. \tag{3.24}$$

Here, $\{t, z\}$ is the Schwartzian derivative

$$\{t, z\} = \frac{t'''}{t'} - \frac{3}{2}\left(\frac{t''}{t'}\right)^2 = \left(\frac{t''}{t'}\right)' - \frac{1}{2}\left(\frac{t''}{t'}\right)^2, \tag{3.25}$$

and the factor of 2 in front of the second term comes from the two values $i = 1, 2$ in ϕ^i.

Now, the n_2 inverse maps in the sum of (3.24) behave as

$$t_{j,I} - x_j \sim e^{((2\pi Ii)/n_2)}(z-u)^{1/n_2} \qquad I = 1,\ldots,n_2. \tag{3.26}$$

Since the terms in (3.24) involve derivatives of $t_{j,I}(z)$, we see that all the n_2 terms contribute to the singularities of $g_j(z,u)$ as $z\to u$. Each of these n_2 terms has an expansion in powers of $(z-u)^{1/n_2}$, but since $g_j(z,u)$ has no monodromies as z goes around u, all the terms in (3.24) with fractional powers of $(z-u)$ cancel out. Thus, we can just take

$$g_j(z,u) = \frac{n_2}{2}\{t_{j,I}, z\} + n_2(t_{j,I}'(z))^2$$
$$\times\left(\frac{n_3\alpha_3}{t_{j,I}(z)-1} + \frac{n_2\alpha_2}{t_{j,I}(z)-x} + \frac{n_1\alpha_1}{t_{j,I}(z)}\right)^2, \tag{3.27}$$

where $t_{j,I}$ is any of the (3.26) maps, and we keep only the terms with integer powers of $(z-u)$ in the expansion of $g_j(z,u)$.

The residue in the $(z-u)^{-1}$ pole in (3.18) will be a function of x_j, so it is convenient to express the left-hand side of (3.18) as a function of x_j as well, using

$$\partial_u\log G_j(u) = u'(x_j)^{-1}\partial_{x_j}\log G_j(u). \tag{3.28}$$

The generic structure of the differential equation for $G_j(z)$ has then the form

$$u'(x)^{-1}\partial_x\log G = \left\{\mathrm{A}\left[\left(\frac{t''}{t'}\right)' - \frac{1}{2}\left(\frac{t''}{t'}\right)^2\right]\right.$$
$$\left. + (t')^2\left[\frac{\mathrm{B}}{t-x} + \frac{\mathrm{C}}{t} + \frac{\mathrm{D}}{t-1}\right]^2\right\}_{(1/(z-u))}, \tag{3.29}$$

where

$$\mathrm{A} = \frac{n_2}{2}, \tag{3.30}$$

$$\mathrm{B} = n_2^{3/2}\alpha_2, \tag{3.31}$$

$$\mathrm{C} = \sqrt{n_2}n_1\alpha_1, \tag{3.32}$$

$$\mathrm{D} = \sqrt{n_2}n_3\alpha_3, \tag{3.33}$$

and $t = t_{j,I}(z)$, $x = x_j$.

B. Polynomial maps: A simple example

To complete the computation we need to construct the explicit map $z(t)$ as a function of the integers n_i. For any p-point function, we expect this map to be a quotient of polynomials. For four-point functions, these polynomials are solutions of Heun's differential equation [1]. When the polynomial in the denominator of $z(t)$ has degree zero, $z(t)$ itself is a polynomial. If we choose the branching of order n_p at $z = \infty$ to correspond to $t = \infty$, this occurs, in a p-point function, whenever the branching numbers n_i satisfy

$$n_p = \sum_{i=1}^{p-1} n_i - p + 2. \tag{3.34}$$

This has been shown in detail in [1] for $p = 4$, and the generalization to arbitrary p is immediate. A correlator whose quantum numbers satisfy the above relation has *only* connected contributions [cf. (2.28)]. Moreover, in this case one can verify that $c = n_p$ and therefore the Riemman-

Hurwitz Eq. (2.32) implies that the only covering surface to contribute to the correlator will be a sphere. Note that the four structure constants (2.35), (2.36), (2.37), and (2.38) of the chiral ring satisfy the polynomial condition (3.34), but the fifth structure constant (2.25) does not.

In this section we will consider a simple polynomial map for a four-point function with branching numbers

$$n_1 = n, \tag{3.35}$$

$$n_2 = n_3 = 2, \tag{3.36}$$

$$n_4 = n + 2. \tag{3.37}$$

In this case we have $c = n + 2$ colors and as expected $g = 0$. In Appendix A we present the details of the $z(t)$ function as well as its inverse. The equation $u = z(x)$ is

$$u \equiv v(x) = x^{1+n} \frac{2 + n - nx}{(2 + n)x - n}, \tag{3.38}$$

and has $M = n + 2$ solutions. Let us choose a root x_j of (3.38). The Eq. (3.29) can be integrated and the result has the form

$$\log G_j(u) = \mathrm{A} f_A(x_j) + \mathrm{B}^2 f_{B^2}(x_j) + \mathrm{C}^2 f_{C^2}(x_j)$$
$$+ \mathrm{D}^2 f_{D^2}(x_j) + 2\mathrm{BC} f_{2BC}(x_j) + 2\mathrm{BD} f_{2BD}(x_j)$$
$$+ 2\mathrm{CD} f_{2CD}(x_j), \tag{3.39}$$

where

$$f_A = \frac{1}{8}\left(-2\log(-1+x) - \frac{-2+n+n^2}{n}\log x\right.$$
$$\left. + 2\frac{-2+2n+n^2}{n(2+n)}\log(-n+(2+n)x)\right),$$

$$f_{B^2} = -\frac{1}{4}\left(\log(x-1) + (n-1)\log x\right.$$
$$\left. - \frac{n\log((2+n)x - n)}{2+n}\right),$$

$$f_{C^2} = -\frac{(2+n)\log x - 2\log((2+n)x - n)}{2n(2+n)},$$

$$f_{2BC} = \frac{(2+n)\log x - \log((2+n)x - n)}{2(2+n)},$$

$$f_{D^2} = -\frac{1}{4}\log(x-1) + \frac{n\log((2+n)x - n)}{8+4n},$$

$$f_{2CD} = -\frac{\log 4(n-(2+n)x)}{2(2+n)},$$

$$f_{2BD} = \frac{(2+n)\log(x-1) - \log((2+n)x - n)}{2(2+n)}, \tag{3.40}$$

with $x = x_j$. Using the expressions in (3.39), we get $G_j(u)$ and similarly $\bar{G}_j(\bar{u})$. We can finally sum over j to get

$$G(u, \bar{u}) = C_4 \sum_{j=1}^{n+2} G_j(u)\bar{G}_j(\bar{u}), \tag{3.41}$$

$$= C_4 \sum_{j=1}^{n+2} |x_j(u) - 1|^{2\alpha}|x_j(u)|^{2\beta}\left|x_j(u) - \frac{n}{2+n}\right|^{2\gamma}, \tag{3.42}$$

where C_4 is a constant and the powers α, γ, and β are

$$\alpha = -\frac{1}{4}\left[\mathrm{A} + \mathrm{B}^2 - 4\mathrm{BD} + \mathrm{D}^2\right]$$
$$= -\frac{1}{2}\left[\frac{1}{2} + 4(\alpha_3 - \alpha_2)^2 - 8\alpha_3\alpha_2\right], \tag{3.43}$$

$$\beta = -\frac{1}{4}\left[\frac{-2+n+n^2}{2n}\mathrm{A} + (n-1)\mathrm{B}^2 - 4\mathrm{BC} + \frac{2}{n}\mathrm{C}^2\right]$$
$$= -\left[\frac{-2+n+n^2}{8n} + n(2\alpha_2^2 - 4\alpha_2\alpha_1 + \alpha_1^2) - 2\alpha_2^2\right], \tag{3.44}$$

$$\gamma = \frac{1}{4}\left[\frac{-2+2n+n^2}{n(2+n)}\mathrm{A} + \frac{n}{n+2}\mathrm{B}^2 - \frac{4}{n+2}\mathrm{B}(\mathrm{C}+\mathrm{D})\right.$$
$$\left. + \frac{4}{n+2}\left(\frac{\mathrm{C}}{\sqrt{n}} - \frac{\sqrt{n}}{2}\mathrm{D}\right)^2\right]$$
$$= \frac{1}{4n(n+2)}\left[n^2 + 2n - 2 + 8n^2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)\right.$$
$$\left. - 16(\alpha_1 n^2(\alpha_2 + \alpha_3) + 2n\alpha_2\alpha_3)\right]. \tag{3.45}$$

C. Some nonextremal four-point functions and their OPEs

We have now all the ingredients to compute some examples explicitly, using the polynomial map we introduced above. Consider

$$\langle O_{n+2}^{(0,0)\dagger}(\infty)O_2^{(0,0)}(1)O_2^{(0,0)\dagger}(u,\bar{u})O_n^{(2,2)}(0)\rangle = G(u,\bar{u}). \tag{3.46}$$

The quantum numbers are $\alpha_1 = \frac{n+1}{2n}$, $\alpha_2 = -\alpha_3 = -1/4$, from which we get $\alpha = -1$, $\beta = -n - 1$, $\gamma = 1$, and thus, from (3.42),

$$G(u, \bar{u}) = C_4 \sum_{i=1}^{n+2} |x_i(u) - 1|^{-2}|x_i(u)|^{-2n-2}$$
$$\times \left|x_i(u) - \frac{n}{2+n}\right|^2. \tag{3.47}$$

It only remains to find C_4, which can be done by considering OPE limits. Note that since the polynomial map leads to sphere contributions only, we expect to find the finite N expression for C_4.

Consider the limit $u \to 0$. The equation $u(x) = 0$ has $n + 1$ roots with $x = 0$ and one root with $x = \frac{2+n}{n}$. Taking

the former case, we obtain

$$u = -\frac{2+n}{n}x^{n+1} + O(x^{n+2}).$$
(3.48)

Inserting this into (3.47) we get that

$$G(u, \bar{u})|_{u\to 0} = C_4(n+1)|u|^{-2} + O(u^{-1}) + O(\bar{u}^{-1})$$
$$+ O(1),$$
(3.49)

where the factor $(n+1)$ comes from the number of terms in (3.47), which have the behavior (3.48). The terms of order $O(1)$ include the leading contribution to (3.47) from the root $x = \frac{2+n}{n}$, which is a constant.

Now, in the limit $u \to 0$ we have

$$O_2^{(0,0)\dagger}(u, \bar{u})O_n^{(2,2)}(0) \sim \frac{C_1 O_{n-1}^{(2,2)}(0)}{|u|^2} + \frac{C_2 O_{n+1}^{(0,0)}(0)}{|u|^2}.$$
(3.50)

The two operators appearing in this OPE happen to be chiral because (3.49) fixes their conformal dimension to be $\Delta = \Delta_2^0 + \Delta_n^2 - 1 = \frac{n}{2}$, and their charge is $Q = \frac{n}{2}$ from charge conservation, but in general the OPE of a chiral operator with an antichiral operator is neither chiral nor antichiral. Inserting this OPE in (3.46), we see that only the second term can survive due to the length of the cycles. Therefore, the OPE limit yields

$$C_4(n+1) = \langle O_{n+2}^{(0,0)\dagger} O_2^{(0,0)} O_{n+1}^{(0,0)}\rangle\langle O_n^{(2,2)} O_2^{(0,0)\dagger} O_{n+1}^{(0,0)\dagger}\rangle,$$
(3.51)

and using the structure constants (2.35) and (2.42) [the (a, a) ring has the same structure constants as the (c, c) ring], we get the finite N result

$$C_4 = \frac{(n+2)^{3/2}}{2(n+1)^2 n^{1/2}}\sqrt{\frac{(N-n)(N-n-1)}{N^2(N-1)^2}}.$$
(3.52)

We can verify this expression for C_4 by re-obtaining it in another limit. When $u \to \infty$, we get $n+1$ solutions with $x \to \infty$ and a single solution with $x \sim \frac{n}{2+n}$. In the former case one explicitly obtains that

$$u \sim -\frac{n}{n+2}x^{n+1}.$$
(3.53)

Inserting this into (3.47) gives

$$G(u, \bar{u})|_{u\to\infty} = C_4(n+1)\frac{n^2}{(n+2)^2}|u|^{-2} + O(1), \quad (3.54)$$

where the $(n+1)$ is again the number of terms in (3.47), which diverge in the OPE limit. The OPE leads again to a chiral intermediate state $O_{n+1}^{(2,2)}$, and thus we get

$$C_4\frac{n^2(n+1)}{(n+2)^2} = \langle O_2^{(0,0)\dagger} O_{n+2}^{(0,0)\dagger} O_{n+1}^{(2,2)}\rangle\langle O_{n+1}^{(2,2)\dagger} O_2^{(0,0)} O_n^{(2,2)}\rangle.$$
(3.55)

Inserting the structure constants (2.42) and (2.36), we get for C_4 exactly the same expression as in (3.52). One could try to further take the limit $u \to 1$, but in this case the intermediate state is neither chiral nor antichiral, so we cannot use the structure constants of the chiral ring.

Consider next the four-point function

$$\langle O_{n+2}^{(0,0)\dagger}(\infty)O_2^{(0,0)\dagger}(1)O_2^{(0,0)}(u)O_n^{(2,2)}(0)\rangle.$$
(3.56)

This correlator is obtained from (3.46) by interchanging the positions of $z_2 = u$ with $z_3 = 1$. Using (3.13), this implies that (3.56) is equal to

$$\langle O_{n+2}^{(0,0)\dagger}(\infty)O_2^{(0,0)}(1)O_2^{(0,0)\dagger}(1/u, 1/\bar{u})O_n^{(2,2)}(0)\rangle \times |u|^{-2}.$$
(3.57)

In general, $SL(2, \mathbb{C})$ transformations in the u sphere are not simple in the x sphere, but for our particular map (3.38), we have

$$1/v(x) = v(1/x),$$
(3.58)

and therefore

$$\langle O_{n+2}^{(0,0)\dagger}(\infty)O_2^{(0,0)\dagger}(1)O_2^{(0,0)}(u)O_n^{(2,2)}(0)\rangle$$
$$= C_4\sum_{j=1}^{n+2}|x_i(u)-1|^{-2}\left|x_i(u)-\frac{n}{2+n}\right|^2.$$
(3.59)

This result can be verified by computing the coefficients α, β, γ explicitly. Note that there is no singularity as $u \to 0$, since this corresponds to the OPE of two chiral operators.

IV. EXTREMAL CORRELATORS AND HURWITZ NUMBERS

Armed with this experience, we are ready to tackle the computation of extremal correlators.

A. Extremal four-point functions and double-cycle operators

Consider the following extremal four-point functions

$$\langle O_{n+2}^{(0,0)\dagger}(\infty)O_2^{(0,0)}(1)O_2^{(0,0)}(u, \bar{u})O_n^{(0,0)}(0)\rangle = G_1(u, \bar{u}),$$
(4.1)

$$\langle O_{n+2}^{(2,2)\dagger}(\infty)O_2^{(2,2)}(1)O_2^{(0,0)}(u, \bar{u})O_n^{(0,0)}(0)\rangle = G_2(u, \bar{u}),$$
(4.2)

which we can compute with the same map of the previous

section. The quantum numbers for G_1 are $\alpha_1 = \frac{n-1}{2n}$, $\alpha_2 = \alpha_3 = \frac{1}{4}$. Using these values in (3.43), (3.44), and (3.45) gives $\alpha = \beta = \gamma = 0$. Similarly, for G_2 we have $\alpha_1 = \frac{n-1}{2n}$, $\alpha_2 = \frac{1}{4}$, $\alpha_3 = \frac{3}{4}$, and we get again $\alpha = \beta = \gamma = 0$. From (3.42), this means that both G_1 and G_2 are constants. We get thus a nice check on our formulas for α, β, γ, since this is precisely what we expect: the limits $u \to 0$, 1 correspond to OPEs of chiral operators, so $G_1(u)$ and $G_2(u)$ should have no singularities at $u = 0$, 1. But these are the only singularities we expect in $G_1(u)$, $G_2(u)$. So $G_1(u)$, $G_2(u)$ are meromorphic functions with no singularities and therefore should be constants. This result is of course general, so we will have

$$\langle O_{n_4}^{(0,0)\dagger}(\infty) O_{n_3}^{(0,0)}(1) O_{n_2}^{(0,0)}(u, \bar{u}) O_{n_1}^{(0,0)}(0) \rangle = C_4, \quad (4.3)$$

with C_4 a constant and

$$n_4 = n_3 + n_2 + n_1 - 2, \quad (4.4)$$

from charge conservation, which is the familiar polynomial condition (3.34).

It only remains to determine C_4 from OPE limits. But here things are subtler than for nonextremal correlators, because the OPE of two chiral operators has no singularities. Taking for example $O_{n_2}^{(0,0)}$ and $O_{n_1}^{(0,0)}$, we get

$$O_{n_2}^{(0,0)} O_{n_1}^{(0,0)} = C_3 O_{\tilde{n}}^{(0,0)} + C_3' O_{\tilde{n}-2}^{(2,2)} + \sum_{i=2}^{\tilde{n}-1} (D_3(i) O_{(i,\tilde{n}-i+1)}^{(0,0),(0,0)} \\ + D_3'(i) O_{(i-1,\tilde{n}-i)}^{(2,2),(0,0)} + \cdots) + \cdots, \quad (4.5)$$

where

$$\tilde{n} = n_1 + n_2 - 1. \quad (4.6)$$

The operators in the parentheses in the second line of (4.5) are double-cycle operators. They are always present in the OPE of single-cycle operators and correspond, for instance, to a product of two single cycles without common colors. Since such a product has no singularities, we ignored these operators in the previous section. But in the OPE of two chiral operators they appear on par with the single-cycle operators.

The problem now is that we do not know the structure constants D_3, D_3' (and other structure constants including multicycle operators) in (4.5), so we cannot use this OPE to determine C_4 in (4.3). Of course, in principle these structure constants can be computed using, e.g. the techniques of [15]. But this has not been done so far, and as we will see below, it is not necessary.

The presence of the double-cycle terms cannot be avoided even in the large N limit. To see this, insert the OPE (4.5) into (4.3). This gives

$$C_4 = C_3 \langle O_{n_4}^{(0,0)\dagger} O_{n_3}^{(0,0)} O_{\tilde{n}}^{(0,0)} \rangle \\ + \sum_{i=2}^{\tilde{n}-1} D_3(i) \langle O_{n_4}^{(0,0)\dagger} O_{n_3}^{(0,0)} O_{(i,\tilde{n}-i+1)}^{(0,0),(0,0)} \rangle, \quad (4.7)$$

$$= \langle O_{\tilde{n}}^{(0,0)\dagger} O_{n_2}^{(0,0)} O_{n_1}^{(0,0)} \rangle \langle O_{n_4}^{(0,0)\dagger} O_{n_3}^{(0,0)} O_{\tilde{n}}^{(0,0)} \rangle \\ + \sum_{i=2}^{\tilde{n}-1} \langle O_{(i,\tilde{n}-i+1)}^{(0,0),(0,0)\dagger} O_{n_2}^{(0,0)} O_{n_1}^{(0,0)} \rangle \langle O_{n_4}^{(0,0)\dagger} O_{n_3}^{(0,0)} O_{(i,\tilde{n}-i+1)}^{(0,0),(0,0)} \rangle. \quad (4.8)$$

Remember from (2.33) that the three-point functions of single-cycle operators scale as $N^{-(1/2)}$, so the first term in (4.8) scales as N^{-1}. Using the same combinatorial arguments used to obtain (2.33) (see e.g. [1]), it is easy to show that at large N

$$\langle O_{(n_2,n_1)}^{(0,0),(0,0)\dagger} O_{n_2}^{(0,0)} O_{n_1}^{(0,0)} \rangle \sim 1, \quad (4.9)$$

$$\langle O_{n_4}^{(0,0)\dagger} O_{n_3}^{(0,0)} O_{(n_2,n_1)}^{(0,0),(0,0)} \rangle \sim \frac{1}{N}. \quad (4.10)$$

In particular, the correlator (4.9) only receives contributions from its disconnected terms (see (2.28)). It follows then that in the sum in (4.8) there is always a term that scales as N^{-1}, as the first term, and therefore cannot be discarded.

A similar situation occurs in a free gauge theory [31]. Consider the following correlator in free $U(N)$ gauge theory

$$I_4^{(\text{YM})} = \langle \text{Tr}\bar{Z}^J(x_0) \text{Tr}Z^{J_1}(x_1) \text{Tr}Z^{J_2}(x_2) \text{Tr}Z^{J_3}(x_3) \rangle, \quad (4.11)$$

where $J = J_1 + J_2 + J_3$. Let us take the OPE limit $x_1 \to x_0$. The leading contributions to this are

$$I_4^{(\text{YM})} = \frac{1}{|x_0 - x_1|^{2J_1}} [\langle \text{Tr}\bar{Z}^{J-J_1}(x_0) \text{Tr}Z^{J_2}(x_2) \text{Tr}Z^{J_3}(x_3) \rangle \\ \times \langle \text{Tr}Z^{J-J_1} \text{Tr}Z^{J_1} \text{Tr}\bar{Z}^J \rangle \\ + \langle (\text{Tr}\bar{Z}^{J_2} \text{Tr}\bar{Z}^{J_3})(x_0) \text{Tr}Z^{J_2}(x_2) \text{Tr}Z^{J_3}(x_3) \rangle \\ \times \langle (\text{Tr}Z^{J_2} \text{Tr}Z^{J_3}) \text{Tr}Z^{J_1} \text{Tr}\bar{Z}^J \rangle + \ldots]. \quad (4.12)$$

In general the second term coming from a double-trace state is subleading in $1/N$, but here the two terms are of the same order. To see this we normalize the operators so that the two-point functions will be $O(N^0)$ and thus the planar s-point functions of single traces behave as N^{2-s}. The first term in (4.12) is of order N^{-2} and the second naively is of order N^{-4}. However, the leading contribution to the three-point function

$$\langle (\text{Tr}\bar{Z}^{J_2} \text{Tr}\bar{Z}^{J_3})(x_0) \text{Tr}Z^{J_2}(x_2) \text{Tr}Z^{J_3}(x_3) \rangle$$

comes from the disconnected diagrams, which gives a scaling of N^{-2} to the second term in (4.12). Thus, the two terms have the same order in $1/N$, and both have to be counted.

B. The ϵ deformation

To bypass the problem of determining the structure constants involving two-cycle states we would like to deform slightly our extremal correlators, so as to split single-cycle from double-cycle terms in the OPE (4.5). We can achieve this using the following trick. Let us change infinitesimally the momentum of, say, $O_{n_2}^{(0,0)}$ and $O_{n_3}^{(0,0)}$, as $\alpha_2 \rightarrow \alpha_2 = \frac{n_2-1}{2n_2} - \epsilon$ and $\alpha_3 \rightarrow \alpha_3 = \frac{n_3-1}{2n_3} + \epsilon$. The OPE (4.5) is now[7]

$$O(\epsilon)_{n_2}^{(0,0)}(u, \bar{u})O_{n_1}^{(0,0)}(0)$$

$$= \frac{C_3(\epsilon)O(\epsilon)_{\tilde{n}}^{(0,0)}(0)}{|u|^{2\alpha_1\epsilon}} + \frac{C_3'(\epsilon)O(\epsilon)_{\tilde{n}-2}^{(2,2)}(0)}{|u|^{2\alpha_1\epsilon}}$$

$$+ \sum_{i=2}^{\tilde{n}-1}(D_3(i, \epsilon)O(\epsilon)_{(i,\tilde{n}-i)}^{(0,0),(0,0)}$$

$$+ D_3'(i, \epsilon)O(\epsilon)_{(i-1,\tilde{n}-i)}^{(2,2),(0,0)} + \cdots) + \cdots \qquad (4.13)$$

The four-point function (4.3) will now have a form similar to (3.42)

$$\langle O_{n_4}^{(0,0)\dagger}(\infty)O(-\epsilon)_{n_3}^{(0,0)}(1)O(\epsilon)_{n_2}^{(0,0)}(u, \bar{u})O_{n_1}^{(0,0)}(0)\rangle$$

$$= \frac{C_4}{H_4}\sum_{j=1}^{H_4}\tilde{C}_j(\epsilon)\prod_{i=1}^{t}|x_j(u) - q_t|^{2\beta_t(\epsilon)}, \qquad (4.14)$$

such that $\lim_{\epsilon\rightarrow 0}\beta_t(\epsilon) = 0$ and $\lim_{\epsilon\rightarrow 0}\tilde{C}_j(\epsilon) = 1$. The sum is over the number of maps from the covering surface, which for four-point functions satisfying (4.4) is $H_4 = n_4$ (see e.g. Section 3.3 in [1]).

We can now take the limit $u \rightarrow 0$. In [1] we proved that the number of terms in the sum over j in (4.14), which contribute to the leading singularity is precisely the number $\tilde{n}$ defined in (4.6). Thus, for $\tilde{n}$ terms in (4.14) we will have

$$\prod_{i=1}^{t}|x_j(u) - q_t|^{2\beta_t(\epsilon)} \rightarrow \frac{c_j^\epsilon}{|u|^{2\alpha_1\epsilon}}, \qquad (4.15)$$

with c_j some constant. Inserting now the OPE (4.13) in (4.14), and equating the terms with leading singularity $|u|^{-2\alpha_1\epsilon}$, we get

$$\frac{C_4}{n_4}\sum_{j=1}^{\tilde{n}}c_j^\epsilon = \langle O_{n_4}^{(0,0)\dagger}O(-\epsilon)_{n_3}^{(0,0)}O(\epsilon)_{\tilde{n}}^{(0,0)}\rangle$$

$$\times \langle O(\epsilon)_{\tilde{n}}^{(0,0)\dagger}O(\epsilon)_{n_2}^{(0,0)}O_{n_1}^{(0,0)}\rangle. \qquad (4.16)$$

We can now safely take the limit $\epsilon \rightarrow 0$, and we get,

$$C_4\frac{\tilde{n}}{n_4} = \langle O_{n_4}^{(0,0)\dagger}O_{n_3}^{(0,0)}O_{\tilde{n}}^{(0,0)}\rangle\langle O_{\tilde{n}}^{(0,0)\dagger}O_{n_2}^{(0,0)}O_{n_1}^{(0,0)}\rangle. \qquad (4.17)$$

Note that if we keep only the single-cycle terms in the OPE (4.5), we would get an expression similar to (4.17) but without the factor $\frac{\tilde{n}}{n_4}$. Thus, the combined effect of the double-cycle terms in (4.5) is precisely to add this factor.

Inserting now the three-point functions (2.35) into (4.17), gives finally

$$C_4 = F_4(n_i)\frac{n_4^{5/2}}{(n_1 n_2 n_3)^{1/2}}, \qquad (4.18)$$

where

$$F_4(n_i) = \left[\frac{(N-n_1)!(N-n_2)!(N-n_3)!}{(N-n_4)!(N!)^2}\right]^{1/2}. \qquad (4.19)$$

Note that the final expression (4.18) does not depend on the length $\tilde{n}$ of the intermediate state. In particular, we can repeat the derivation of C_4 using any other OPE limit and obtain the same result.

This computation can be easily extended to other extremal correlators. Using the chiral fusion rules (2.20), one can verify that all possible extremal correlators can be obtained by adding operators of type $(0, 0)$ to the structure constants of the chiral ring.

When adding one operator of type $(0, 0)$ to the structure constants (2.35), (2.36), (2.37), and (2.38) cases, we get again the polynomial relation (4.4). Thus, the counting of total and divergent terms is the same as in the above case, and we get

$$\langle O_{n_4}^{(0,0)\dagger}O_{n_3}^{(0,0)}O_{n_2}^{(0,0)}O_{n_1}^{(0,0)}\rangle = F_4(n_i)\frac{n_4^{5/2}}{(n_1 n_2 n_3)^{1/2}}, \qquad (4.20)$$

$$\langle O_{n_4}^{(2,2)\dagger}O_{n_3}^{(2,2)}O_{n_2}^{(0,0)}O_{n_1}^{(0,0)}\rangle = F_4(n_i)\frac{n_3^{3/2}n_4^{1/2}}{(n_2 n_1)^{1/2}}, \qquad (4.21)$$

$$\langle O_{n_4}^{(b,\bar{b})\dagger}O_{n_3}^{(a,\bar{a})}O_{n_2}^{(0,0)}O_{n_1}^{(0,0)}\rangle = \delta^{ab}\delta^{\bar{a}\bar{b}}F_4(n_i)\frac{n_4^{3/2}n_3^{1/2}}{(n_2 n_1)^{1/2}}, \qquad (4.22)$$

$$\langle O_{n_4}^{(2,2)\dagger}O_{n_3}^{(a,\bar{a})}O_{n_2}^{(b,\bar{b})}O_{n_1}^{(0,0)}\rangle = \epsilon^{ab}\epsilon^{\bar{a}\bar{b}}F_4(n_i)\frac{(n_4 n_3 n_2)^{1/2}}{n_1^{1/2}}. \qquad (4.23)$$

C. Extremal nonpolynomial four-point function

The correlator

$$\langle O_{n_4}^{(2,2)\dagger}O_{n_3}^{(0,0)}O_{n_2}^{(0,0)}O_{n_1}^{(0,0)}\rangle \qquad (4.24)$$

is different from the cases studied above because the conservation of charge here leads to

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[7]Note that since the radius of the bosons ϕ_I^i was fixed from bosonizing the fermions, a change of their momentum by ϵ is actually a change in the radius away from the fermionization point. This in turn implies that we should deform the momenta of all the operators in the correlator. But we can ignore this subtlety, since in the sphere the correlators are analytic functions of the external momenta, and there is no other dependence on the compactification radius.

$$n_4 = n_1 + n_2 + n_3 - 4. \tag{4.25}$$

The corresponding map is not polynomial [cf. (4.4)], but rather the quotient of a polynomial and a monomial. Of course one can still build the map following the prescription developed in [1],[8] but the ϵ-deformation technique that we introduced does not need the actual map. All we need is the total number of different maps and the number of terms in the sum over maps which diverge in the possible OPEs. But this information can be obtained from the symmetric-group theoretical version of the map counting, and then it reduces to a combinatorial problem.

Note that in the previous cases, when we took the OPE (4.5), the operator $O_{\tilde{n}-2}^{(2,2)}$ did not survive inside the correlation function. But in the four-point function (4.24) it does and therefore the ϵ-deformed four-point function is, to leading order,

$$\langle O_{n_4}^{(2,2)\dagger} O(-\epsilon)_{n_3}^{(0,0)} O(\epsilon)_{n_2}^{(0,0)} O_{n_1}^{(0,0)} \rangle$$
$$= |u|^{-2\epsilon\alpha_1} \times [\langle O_{n_4}^{(2,2)\dagger} O(-\epsilon)_{n_3}^{(0,0)} O(\epsilon)_{n_1+n_2-1}^{(0,0)} \rangle C_3(\epsilon)$$
$$+ \langle O_{n_4}^{(2,2)\dagger} O(-\epsilon)_{n_3}^{(0,0)} O(\epsilon)_{n_1+n_2-3}^{(2,2)} \rangle C_3'(\epsilon)], \tag{4.26}$$

$$= |u|^{-2\epsilon\alpha_1} \times [\langle O_{n_4}^{(2,2)\dagger} O(-\epsilon)_{n_3}^{(0,0)} O(\epsilon)_{n_1+n_2-1}^{(0,0)} \rangle$$
$$\times \langle O(\epsilon)_{n_1+n_2-1}^{(0,0)\dagger} O_{n_1}^{(0,0)} O(\epsilon)_{n_2}^{(0,0)} \rangle$$
$$+ \langle O_{n_4}^{(2,2)\dagger} O(-\epsilon)_{n_3}^{(0,0)} O(\epsilon)_{n_1+n_2-3}^{(2,2)} \rangle$$
$$\times \langle O(\epsilon)_{n_1+n_2-3}^{(2,2)\dagger} O_{n_1}^{(0,0)} O(\epsilon)_{n_2}^{(0,0)} \rangle]. \tag{4.27}$$

On the other hand, following the same logic as in the previous section, the above expression is equal to

$$|u|^{-2\epsilon\alpha_1} \frac{C_4}{H_4} \left[\sum_{j=1}^{\tilde{n}_b} c_j^\epsilon + \sum_{j=1}^{\tilde{n}_a} \tilde{c}_j^\epsilon \right], \tag{4.28}$$

where H_4 is the total number of mappings. We have separated the terms contributing to the OPE singularity into two sums. In the first $\tilde{n}_b$ terms, the operators $O_{n_2}^{(0,0)}$ and $O_{n_1}^{(0,0)}$ share one color, and in the other $\tilde{n}_a$ terms they share two colors. These two sums clearly correspond to the two terms in the right-hand side of (4.27). In Appendix B 3 we compute these numbers to be

$$\tilde{n}_a = n_1 + n_2 - 3, \qquad \tilde{n}_b = n_1 + n_2 - 1,$$
$$H_4 = 2n_4. \tag{4.29}$$

We can equate either of the two terms in (4.27) and (4.28). Taking $\epsilon \to 0$ gives the equations

$$C_4 \frac{\tilde{n}_a}{H_4} = \langle O_{n_4}^{(2,2)\dagger} O_{n_1+n_2-3}^{(2,2)} O_{n_3}^{(0,0)} \rangle \langle O_{n_1+n_2-3}^{(2,2)\dagger} O_{n_1}^{(0,0)} O_{n_2}^{(0,0)} \rangle, \tag{4.30}$$

$$C_4 \frac{\tilde{n}_b}{H_4} = \langle O_{n_4}^{(2,2)\dagger} O_{n_1+n_2-1}^{(0,0)} O_{n_3}^{(0,0)} \rangle \langle O_{n_1+n_2-1}^{(0,0)\dagger} O_{n_1}^{(0,0)} O_{n_2}^{(0,0)} \rangle, \tag{4.31}$$

and plunging the relevant structure constants we get, at large N,

$$C_4 = \frac{2}{N} \frac{(n_4)^{1/2}}{(n_1 n_2 n_3)^{1/2}}. \tag{4.32}$$

The fact that we get the same result by equating separately the first and second terms in (4.27) and (4.28) is a nontrivial check of the procedure. Note that to extend (4.32) to finite N we would have to compute torus contributions.

D. Extremal polynomial p-point functions

We can now generalize the above results to extremal p-point functions. As in the case of four-point functions, there are four polynomial correlators and one nonpolynomial. We will consider only the former for simplicity.

The coordinate dependence of extremal p-point correlators is

$$\langle O_{n_p}^{(0,0)\dagger}(z_p) O_{n_{p-1}}^{(0,0)}(z_{p-1}) \ldots O_{n_1}^{(0,0)}(z_1) \rangle$$
$$= C_p \prod_{i=1}^{p-1} |z_i - z_p|^{-4\Delta_i}, \tag{4.33}$$

where C_p is a constant. This is fixed by the absence of singularities when the chiral operators approach each other and by requiring invariance under global conformal transformations. We will put $O_{n_p}^{(0,0)\dagger}(z_p)$ at $z_p = \infty$, so this expression becomes just C_p.

To determine C_p from OPE limits, we proceed as above. In order to avoid the multicycle terms in the OPEs we deform again the momentum of $O_{n_2}^{(0,0)}$ by ϵ and the momentum of $O_{n_3}^{(0,0)}$ by $-\epsilon$. The correlator becomes the usual sum over all the maps from the covering surface

$$\langle O_{n_p}^{(0,0)\dagger} O_{n_{p-1}}^{(0,0)} \ldots O(\epsilon)_{n_2}^{(0,0)} O_{n_1}^{(0,0)} \rangle$$
$$= \frac{C_p}{H_p} \sum_{j=1}^{H_p} |k(x_j(z_1, z_2, \ldots))|^{2\epsilon}. \tag{4.34}$$

Note that charge conservation in (4.33) implies the polynomial condition

$$n_p = \sum_{i=1}^{p-1} n_i - p + 2. \tag{4.35}$$

Since $c = n_p$ for polynomial maps, the number H_p of terms in (4.34) is the number of n_p-sheeted covering maps from S_{cover}^2 to S_{base}^2 with p branching points, with branching numbers $n_1, n_2, \ldots n_p$. The problem of determining H_p is well known in the mathematical literature on branched coverings, and its solution when (4.35) holds is

[39,41]

$$H_p = n_p^{p-3}. \tag{4.36}$$

For $p = 4$ we proved this result in [1], and for $p = 5$ we present a proof in Appendix B 1 as an illustration of the diagrammatic description of symmetric products we introduced in [1].

The details of the functions $k(x_j(z_1, z_2, \ldots))$ in (4.34) have not been worked out for $p > 4$, but they are not important for us. We are only interested in the fact that as $z_2 \to z_1$, a certain number $\tilde{n}$ of terms in (4.34) will behave as

$$|k(x_j(z_1, z_2, \ldots))|^{2\epsilon} \to \frac{c_j^\epsilon}{|z_{12}|^{2\epsilon\alpha_1}}. \tag{4.37}$$

This number $\tilde{n}$ counts how many terms in (4.34) contribute to the singularity of the deformed OPE (4.13). In Appendix B 2 we prove that $\tilde{n} = (n_1 + n_2 - 1)n_p^{p-4}$. Therefore, in the limit $z_2 \to z_1$, the coefficients of the leading $|z_{12}|^{-2\epsilon\alpha_1}$ singularity at both sides of (4.34) satisfy

$$\frac{C_p}{n_p^{p-3}} \sum_{j=1}^{(n_1+n_2-1)n_p^{p-4}} c_j^\epsilon = \langle O(\epsilon)_{n_1+n_2-1}^{(0,0)\dagger} O(\epsilon)_{n_2}^{(0,0)} O_{n_1}^{(0,0)} \rangle C(\epsilon)_{p-1}, \tag{4.38}$$

where

$$C(\epsilon)_{p-1} = \langle O_{n_p}^{(0,0)\dagger} O_{n_{p-1}}^{(0,0)} \ldots O(-\epsilon)_{n_3}^{(0,0)} O(\epsilon)_{n_1+n_2-1}^{(0,0)} \rangle. \tag{4.39}$$

We can take now the limit $\epsilon \to 0$ in both sides of (4.38) to get

$$C_p \frac{(n_1 + n_2 - 1)}{n_p} = \langle O_{n_1+n_2-1}^{(0,0)\dagger} O_{n_2}^{(0,0)} O_{n_1}^{(0,0)} \rangle C_{p-1}, \tag{4.40}$$

$$= F(n_1, n_2) \frac{(n_1 + n_2 - 1)^{3/2}}{(n_1 n_2)^{1/2}} C_{p-1}, \tag{4.41}$$

or equivalently

$$C_p = F(n_1, n_2) \frac{n_p (n_1 + n_2 - 1)^{1/2}}{(n_1 n_2)^{1/2}} C_{p-1}. \tag{4.42}$$

Iterating this recursion relation $p - 3$ times, we get finally

$$\langle O_{n_p}^{(0,0)\dagger} O_{n_{p-1}}^{(0,0)} \ldots O_{n_2}^{(0,0)} O_{n_1}^{(0,0)} \rangle = F_p(n_i) \frac{(n_p)^{p-3/2}}{(n_1 n_2 \ldots n_{p-1})^{1/2}}, \tag{4.43}$$

where

$$F_p(n_i) = \left[\frac{\prod_{i=1}^{p-1}(N - n_i)!}{(N - n_p)!(N!)^{p-2}} \right]^{1/2}. \tag{4.44}$$

The same procedure can be applied to obtain the following extremal correlators:

$$\langle O_{n_p}^{(2,2)\dagger} O_{n_{p-1}}^{(2,2)} O_{n_{p-2}}^{(0,0)} \ldots O_{n_1}^{(0,0)} \rangle = F_p(n_i) \frac{(n_p)^{p-7/2}(n_{p-1})^{3/2}}{(n_{p-2} \cdots n_1)^{1/2}}, \tag{4.45}$$

$$\langle O_{n_p}^{(b,\bar{b})\dagger} O_{n_{p-1}}^{(a,\bar{a})} O_{n_{p-2}}^{(0,0)} \ldots O_{n_1}^{(0,0)} \rangle$$
$$= \delta^{ab} \delta^{\bar{a}\bar{b}} F_p(n_i) \frac{(n_p)^{p-5/2}(n_{p-1})^{1/2}}{(n_{p-2} \cdots n_1)^{1/2}}, \tag{4.46}$$

$$\langle O_{n_p}^{(2,2)\dagger} O_{n_{p-1}}^{(a,\bar{a})} O_{n_{p-2}}^{(b,\bar{b})} O_{n_{p-3}}^{(0,0)} \ldots O_{n_1}^{(0,0)} \rangle$$
$$= \epsilon^{ab} \epsilon^{\bar{a}\bar{b}} F_p(n_i) \frac{(n_p)^{p-7/2}(n_{p-1} n_{p-2})^{1/2}}{(n_{p-3} \cdots n_1)^{1/2}}. \tag{4.47}$$

The large N limit is obtained using

$$\lim_{N \to \infty} F_p(n_i) = \left(\frac{1}{N} \right)^{(p-2)/2}. \tag{4.48}$$

According to [23], extremal correlators are not renormalized under marginal deformations away from the orbifold point, so one expects the same expressions in the string/supergravity dual.

E. Extremal correlators compute Hurwitz numbers

If we now look back at our results, we notice that the five types of correlators (four polynomial and one nonpolynomial) can all be expressed in a uniform way by performing the rescaling

$$O_n^{(0,0)} \to \hat{O}_n^{(0,0)} = n^{1/2} O_n^{(0,0)}, \tag{4.49}$$

$$O_n^{(0,0)\dagger} \to \hat{O}_n^{(0,0)\dagger} = n^{-3/2} O_n^{(0,0)\dagger}, \tag{4.50}$$

$$O_n^{(a,\bar{a})} \to \hat{O}_n^{(a,\bar{a})} = n^{-1/2} O_n^{(a,\bar{a})}, \tag{4.51}$$

$$O_n^{(a,\bar{a})\dagger} \to \hat{O}_n^{(a,\bar{a})\dagger} = n^{-1/2} O_n^{(a,\bar{a})\dagger}, \tag{4.52}$$

$$O_n^{(2,2)} \to \hat{O}_n^{(2,2)} = n^{-3/2} O_n^{(2,2)}, \tag{4.53}$$

$$O_n^{(2,2)\dagger} \to \hat{O}_n^{(2,2)\dagger} = n^{1/2} O_n^{(2,2)\dagger}. \tag{4.54}$$

With this rescaling the two-point functions become

$$\langle \hat{O}_n^{(a,\bar{a})\dagger} \hat{O}_n^{(a,\bar{a})} \rangle = \langle \hat{O}_n^{(2,\bar{2})\dagger} \hat{O}_n^{(2,2)} \rangle = \langle \hat{O}_n^{(0,0)\dagger} \hat{O}_n^{(0,0)} \rangle = \frac{1}{n}, \tag{4.55}$$

and the five structure constants of the chiral ring become, at large N,[9]

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[9]As in (2.23) and (2.24), for the two types of extremal correlators with a, $\bar{a}$ indices we have to add appropriately δs and ϵs to (4.56), (4.57), and (4.58).

$$\langle \hat{O}_{n_3}^\dagger \hat{O}_{n_2} \hat{O}_{n_1} \rangle = \left(\frac{1}{N}\right)^{1/2}. \tag{4.56}$$

Remarkably, the five types of extremal correlators are now given by the simple expression

$$\hat{C}_p = F_p(n_i) H_p(\{n_i\}), \tag{4.57}$$

where $H_p(\{n_i\})$ is, as defined above (4.36) for the polynomial cases, the number of maps contributing to a given correlator. The relation (4.57) also holds, at large N, for the nonpolynomial four-point function (4.32), with $H_4(\{n_i\})$ given in (4.29).

The elegance of this result suggests that this relation might hold also for more general extremal correlators, which include (properly rescaled) multicycle states. In this case, the Hurwitz numbers count the number of maps with multicycle branching points. More precisely, we conjecture that in the large N limit the nonvanishing extremal correlators satisfy

$$\left\langle \hat{O}_{[g_0]}^{\mathcal{A}_0\dagger}(z_0, \bar{z}_0) \prod_{i=1}^{p-1} \hat{O}_{[g_i]}^{\mathcal{A}_i}(z_i, \bar{z}_i) \right\rangle = \frac{1}{N^{(p/2)-1}} H_p(\{[g_i]\}), \tag{4.58}$$

where $H_p(\{[g_i]\})$ is the number of maps from the base sphere to the covering sphere with p ramifications of type (g_i) at points z_i, and $\mathcal{A}_i$ denote the additional quantum numbers. Since the Hurwitz numbers are topological invariants it is likely that it will be possible to prove our conjecture from first principles by performing a topological twist of the symmetric product theory, and using localization techniques.[10] The topological A and B models, obtained in the standard fashion by twisting the (2, 2) supersymmetry, are equivalent in this case, because the model has (4, 4) supersymmetry. It is plausible that the full power of this bigger symmetry may play a role in proving our conjecture and extending our results, much as it did in

the proof of the nonrenormalization theorem [23]. The investigation of these ideas is left for future work.

APPENDIX A: DETAILS OF THE POLYNOMIAL MAP

In this appendix we derive the map relevant for the correlators discussed in Sec. III.

We discuss a map from a covering sphere, t, to a base sphere, z, with ramification points of order n at $z = 0$, order 2 at $z = 1$, order $n + 2$ at $z = \infty$ and order 2 at $z = u$. We take the images of the ramification points to be at $t = 0, 1, \infty$ and $t = x$, respectively. The relation between x and u will be derived shortly. The derivative of the map is given by

$$z'(y) = C(y + x)^{n-1}(y + x - 1)y, \tag{A1}$$

where $y = t - x$. Integrating the above we get

$$z(y) = C(x + y)^n \frac{n(n + 1)y^2 + (n^2(x - 1) - 2n)y + x(2 + n - nx)}{n(2 + 3n + n^2)} + v(x). \tag{A2}$$

We set C and $v(x)$ by demanding $z(y = 1 - x) = 1$ and $z(y = -x) = 0$,

$$v(x) = x^{1+n} \frac{2 + n - nx}{(n + 2)x - n}, \qquad z(t) = t^n \frac{n(n + 1)t^2 - n(n + 2)(1 + x)t + (n + 2)(n + 1)x}{(n + 2)x - n}. \tag{A3}$$

The relation between u and x is set by demanding x to satisfy $u = v(x)$.

Explicit construction of the differential equation satisfied by a four-point function requires computing several quantities built from the map in the limit of $t \to x$ (3.29), e.g. the Schwarzian derivative. Lets us find the expansion of $z - u$ in terms of y. Writing $z - u = y^2 \sum_k a_k y^k$ we obtain for the first coefficients,

$$a_0 = \frac{1}{\sum_{k=0}^{n-1} \sum_{l=0}^{1} \frac{2}{k+l+2}(-1)^{k+l+2}\binom{n-1}{k}[x^{-k}(x-1)^{2+k} - x^{2+l}(x-1)^{-l}]}, \tag{A4}$$

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[10]See e.g. [42] for a relation between Hurwitz numbers and topological strings.

$$a_1 = \frac{2}{3}a_0\left(\frac{n-1}{x} + \frac{1}{x-1}\right), \qquad a_2 = \frac{1}{2}a_0\left(\frac{n-1}{x(x-1)} + \frac{(n-1)(n-2)}{2x^2}\right). \tag{A5}$$

We can also compute the inverse expansion

$$y = \sum_{k=1}^{\infty} c_k (z-u)^{k/2}. \tag{A6}$$

The different expansion coefficients are related as

$$c_1 = a_0^{-1/2}, \qquad c_2 = -\frac{a_1 a_0^{-2}}{2},$$
$$c_3 = \frac{a_0^{-3/2}}{8}\left[5\frac{a_1^2}{a_0^2} - 4\frac{a_2}{a_0}\right]. \tag{A7}$$

The above results are needed to write down the differential Eq. (3.29). We get for the quantities appearing in this equation,

$$\left(\frac{t''}{t'}\right)' = \frac{1}{2}\frac{1}{(z-u)^2} - \frac{c_2}{c_1}\frac{1}{2}(z-u)^{-3/2} + \dots$$
$$\left(\frac{t''}{t'}\right)^2 = \frac{1}{4}\frac{1}{(z-u)^2} - \frac{c_2}{c_1}(z-u)^{-3/2} \tag{A8}$$
$$+ 3\left[\frac{c_2^2}{c_1^2} - \frac{c_3}{c_1}\right](z-u)^{-1} + \dots$$

From here we obtain

$$\left(\frac{t''}{t'}\right)' - \frac{1}{2}\left(\frac{t''}{t'}\right)^2 = \frac{3}{8}\frac{1}{(z-u)^2} - \frac{3}{2}\left[\frac{c_2^2}{c_1^2} - \frac{c_3}{c_1}\right](z-u)^{-1}$$
$$+ \dots. \tag{A9}$$

We will also need the following

$$(t')^2 = \frac{c_1^2}{4}(z-u)^{-1} + c_1 c_2 (z-u)^{-1/2}$$
$$+ \frac{1}{2}(2c_2^2 + 3c_1 c_3)\dots$$
$$\frac{1}{t-x} = \frac{1}{c_1}(z-u)^{-1/2} - \frac{c_2}{c_1^2} + \frac{c_2^2 - c_1 c_3}{c_1^3}(z-u)^{1/2}$$
$$+ \dots$$
$$\frac{1}{t-x+a} = \frac{1}{a} - \frac{c_1}{a^2}(z-u)^{1/2} - \left[\frac{c_2}{a^2} - \frac{c_1^2}{a^3}\right](z-u)$$
$$+ \frac{-c_1^3 + 2ac_1 c_2 - a^2 c_3}{a^4}(z-u)^{3/2} + \dots, \tag{A10}$$

where a is some complex number.

Plugging the above results into (3.29), and integrating the differential equation, expression (3.39) is obtained.

APPENDIX B: COUNTING MAPS AND OPE LIMITS

In this appendix we compute the total number of maps, and the number of maps contributing to the OPE limit in several cases relevant to the discussion in the bulk of the paper. It can be shown that counting covering maps is equivalent to counting certain types of graphs, and in fact there are many ways to define such graphs [39]. In [1] we developed a diagrammatic language suitable for symmetric product orbifolds. In what follows we use this language to solve the enumerative problems at hand. For notations and explanations of the diagrams we refer the reader to [1], where the case of four-point polynomial maps is treated in detail.

1. Diagrammatic counting of maps for five-point polynomial correlators

We want to compute the total number of maps, and the number of maps contributing to the OPE limit, in a five-point polynomial correlator. Consider a polynomial correlator with cycles $\tilde{n}_1 \le \tilde{n}_2 \le \tilde{n}_3 \le \tilde{n}_4$ inserted at finite points and cycle $\tilde{n}_5 = \tilde{n}_1 + \tilde{n}_2 + \tilde{n}_3 + \tilde{n}_4 - 3$ inserted at infinity. The different diagrams contributing to the polynomial correlator can be split into eight classes. These are depicted in Fig. 1. To count the number of diagrams we sum the contributions from each graph and each possible ordering, i.e. each assignment of the cycle lengths $\tilde{n}_i$ to the lengths n_i appearing in Fig. 1. Essentially we have to count all the possible ways to choose the numbers l and k in these diagrams. The results are summarized in the table below.

Class	Ordering	#
(I)	bcdea	$n_2 n_3$
(II)	ebcda	$n_3(n_4 - 1)$
(II)	becda	$n_3(n_4 - 1)$
(II)	bceda	$n_2(n_4 - 1)$
(III)	decba	$(n_1 - 1)n_3$
(III)	dceba	$(n_1 - 1)n_2$
(III)	dcbea	$(n_1 - 1)n_2$
(IV)	bdeca	$(n_2 - 1)(n_4 - 1)$
(IV)	bdcea	$(n_2 - 1)(n_3 - 1)$
(IV)	debca	$(n_1 - 1)(n_4 - 1)$
(IV)	dbeca	$(n_1 - 1)(n_4 - 1)$
(IV)	dbcea	$(n_1 - 1)(n_3 - 1)$
(V)	c e (d) b (d) a	$n_2(n_2 - 1)$
(VI)	(b) d (b) e c a	$n_3(n_3 - 1)$
(VII)	[b, c, d] e a	$(n_4 - 2)(n_4 - 1)$
(VIII)	e [b, c, d] a	$(n_1 - 2)(n_1 - 1)$

Summing all the contributions from the table above we get that the number of maps in the five-point extremal case is $\tilde{n}_5^2 = (\tilde{n}_1 + \tilde{n}_2 + \tilde{n}_3 + \tilde{n}_4 - 3)^2$. Let us count now the dia-
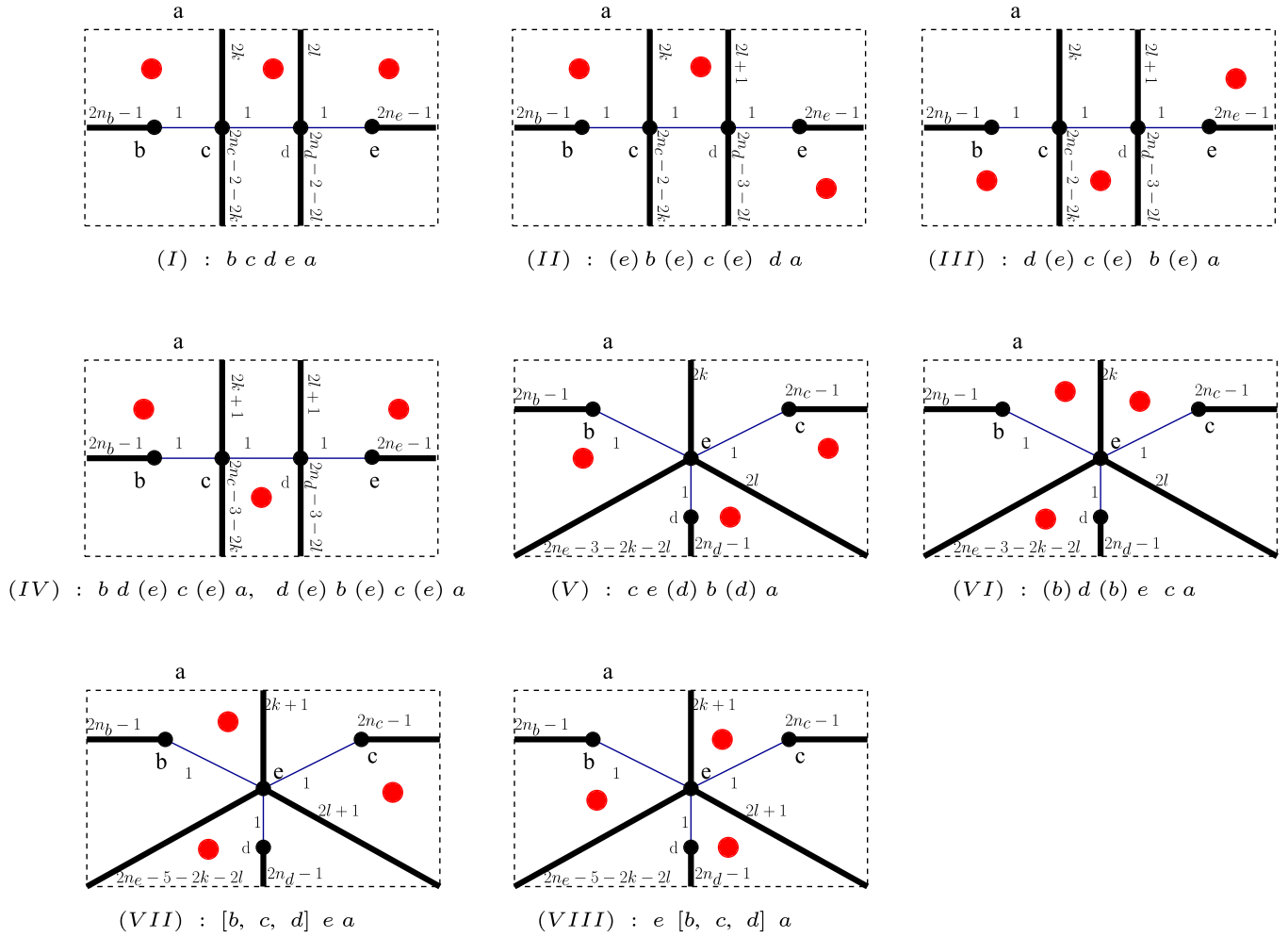
FIG. 1 (color online).   The eight classes of different diagrams contributing to a generic polynomial five-point function. The number over each line is the number of propagators joined in that line. The four vertices at finite positions are b, c, d, e, and a is the vertex at infinity. Below each diagram we indicate the ordering of the vertices. The commutator denotes that vertices commute, and parenthesis indicate the possible position of an operator.

grams that contribute in the OPE limit of $\tilde{n}_1$ cycle colliding with the $\tilde{n}_2$ cycle. The diagrams that contribute are the ones with a propagator stretched between $\tilde{n}_1$ and $\tilde{n}_2$. Thus, we count those diagram in the table above having a propagator between the two first cycles in column *Ordering*. We get that the number of such maps is $(\tilde{n}_1 + \tilde{n}_2 - 1)\tilde{n}_5$.

2. OPE counting

The number H_p of terms in (4.34) is the number of n_p-sheeted covering maps from S_{cover}^2 to S_{base}^2 with p branching points, with branching numbers $n_1, n_2, \ldots n_p$, and satisfying

$$n_p = \sum_{i=1}^{p-1} n_i - p + 2. \tag{B1}$$

The maps in this case are polynomial. This number can be shown [41] to be equal to

$$H_p = n_p^{p-3}. \tag{B2}$$

Let us call $\tilde{n}$ to the number of maps, out of H_p, such that the OPE of the operator with quantum number n_1 and to operator with quantum number n_2 gives a single cycle of size $n_1 + n_2 - 1$.

To determine $\tilde{n}$ consider an auxiliary $(p-1)$-point correlator with cycles of length $(n_1 + n_2 - 1), n_3, n_4, \ldots,$ n_{p-1}, n_p. From (B2) there are n_p^{p-4} such maps. To obtain a p-point correlator of form (B1) we have to split the $n_1 + n_2 - 1$ cycle of the auxiliary correlator into n_1 and n_2 cycle in all possible ways. Obviously, there are $n_1 + n_2 - 1$ ways to do so, so the total number of maps is

$$\tilde{n} = (n_1 + n_2 - 1)n_p^{s-4}. \tag{B3}$$

Note that in correlators satisfying (B1), two vertices can either have no common colors or just a single common color. And thus in the OPE between a pair of consecutive
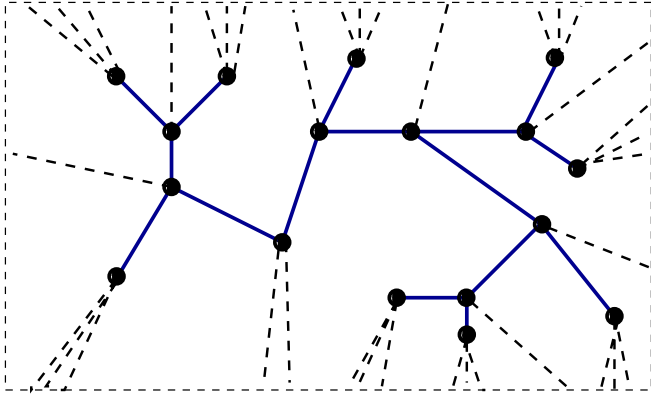
FIG. 2 (color online). An example of a generic graph of a polynomial map. The blue lines connect the first $p-1$ ramification points and the dashed black lines go to ramification point n_p. In this case we have a particular example of a diagram contributing to $\langle (\sigma_{[2]})^{17} \sigma_{[18]} \rangle$.

vertices, either the cycles n_i and n_{i+1} join to a single cycle of length $n_i + n_{i+1} - 1$ or to a double cycle of the form $(n_i)(n_{i+1})$. It is easy to see this fact diagrammatically. Any

diagram of such p-point correlator has all the propagators, except $p-2$ ones, going to the vertex n_p, and the remaining $p-2$ propagators connect the first $p-1$ vertices into some connected tree structure. This is exemplified in Fig. 2. It is easy to see that two vertices either do not have any common loop and thus no common color, or they have a single common color loop.

3. Planar maps in the extremal nonpolynomial four-point function

In this section we count diagrammatically all the planar maps contributing to the "near polynomial" case, $n_4 = n_1 + n_2 + n_3 - 4$. All the classes of diagrams are depicted in Fig. 3. We can count the number of diagrams in each class by just counting the number of values the parameter k can take in each class of diagrams in Fig. 3. The results are summarized in table below. The column labeled OPE denotes whether the diagrams contribute in the OPE limit (i) when two contracted cycles have two indices in common, or (ii) when a single index is common.
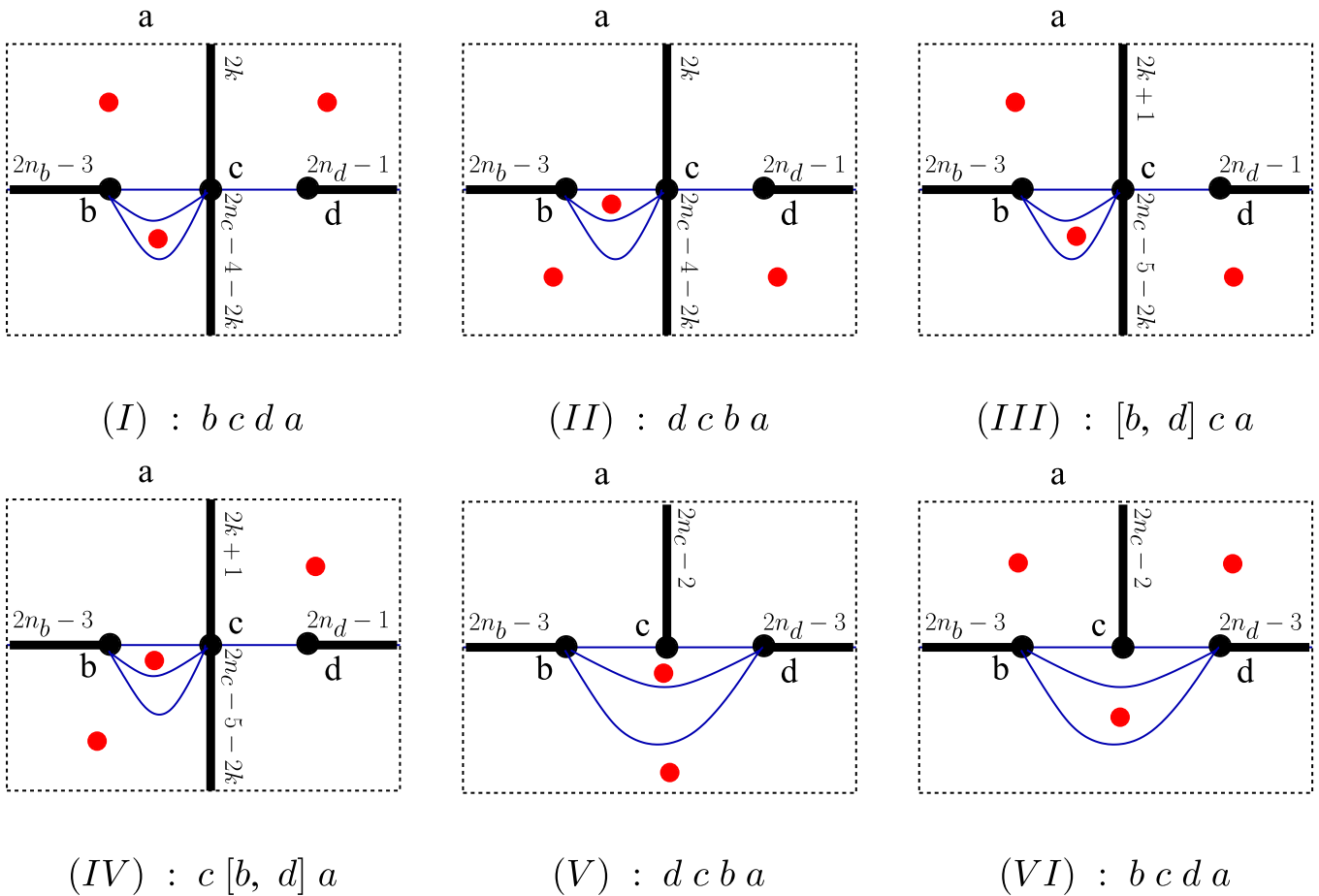

FIG. 3 (color online). The different classes of diagrams contributing to the "near polynomial" four-point function. Thin blue lines represent a single propagator of the graph.

Class	Ordering	#	OPE
(I)	bcda	$n_2 - 1$	i
(II)	dcba	$n_2 - 1$	ii
(III)	bdca	$n_3 - 2$	-
(III)	dbca	$n_3 - 2$	-
(IV)	cbda	$n_1 - 2$	i
(IV)	cdba	$n_1 - 2$	ii
(V)	dcba	1	ii
(VI)	bcda	1	ii

Summing all the contributions we get that the number of maps is $2(n_1 + n_2 + n_3 - 4) = 2n_4$. The number of maps contributing to (i) OPE limit is $n_1 + n_2 - 3$, and to the (ii) OPE limit is $n_1 + n_2 - 1$.

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