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# Bagger-Lambert-Gustavsson-motivated Lagrangian formulation for the chiral two-form gauge field in $D = 6$ and M5-branes

Paolo Pasti,[1,2] Igor Samsonov,[2,3] Dmitri Sorokin,[2] and Mario Tonin[1,2]

[1]*Dipartimento di Fisica "Galileo Galilei", Universitá degli Studi di Padova*
[2]*Istituto Nazionale di Fisica Nucleare, Sezione di Padova, via F. Marzolo 8, 35131 Padova, Italia*
[3]*Laboratory of Mathematical Physics, Tomsk Polytechnic University, 634050 Tomsk, Russia*

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We reveal nonmanifest gauge and $SO(1,5)$ Lorentz symmetries in the Lagrangian description of a six-dimensional free chiral field derived from the Bagger-Lambert-Gustavsson model in [P.-M. Ho and Y. Matsuo, J. High Energy Phys. 06 (2008) 105.] and make this formulation covariant with the use of a triplet of auxiliary scalar fields. We consider the coupling of this self-dual construction to gravity and its supersymmetrization. In the case of the nonlinear model of [P.-M. Ho, Y. Imamura, Y. Matsuo, and S. Shiba, J. High Energy Phys. 08 (2008) 014.] we solve the equations of motion of the gauge field, prove that its nonlinear field strength is self-dual and find a gauge-covariant form of the nonlinear action. Issues of the relation of this model to the known formulations of the M5-brane worldvolume theory are discussed.

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## I. INTRODUCTION

The problem of the Lagrangian formulation of the theory of self-dual or in general duality-symmetric fields, *i.e.* fields whose strengths are subject to a duality condition, has attracted a great deal of attention for decades. A classical physical example, the duality symmetry between electric and magnetic fields of free Maxwell equations, inspired Dirac to promote it to the gauge theory of electrically and magnetically charged particles by introducing the magnetic monopoles [1]. Since then duality-symmetric fields appeared and have played an important role in many field theories, in particular, in String Theory and M theory. The gauge fields whose field strength is self-dual are often called chiral ($p$-form) fields. In space-times of Lorentz signature such fields exist if $p = 2k$ ($k = 0, 1, \dots$) and the space-time dimension is $D = 2(p + 1)$.

Main problems of the Lagrangian formulation of the duality-symmetric and, in particular, the chiral fields are i) to construct an action whose variation would produce the *first-order* duality condition on the field strengths as a consequence of dynamical equations of motion; ii) to find a manifestly Lorentz-covariant form of such an action, which is of a great help for studying a (nonlinear) coupling of duality-symmetric fields to gravity and other fields in the theory; iii) to quantize such a theory.

The first two (classical) problems have been solved in a number of papers using different (classically equivalent) approaches. It has been realized that it is not possible to construct manifestly duality-symmetric and Lorentz-covariant actions without using auxiliary fields. In various space-time dimensions nonmanifestly Lorentz-covariant duality-symmetric actions were constructed and studied in [2–7], see also [8] for more recent developments based on a holographic formulation of self-dual theory. It is known that in these models the Lorentz-invariance gets restored at the level of the equations of motion (*i.e.* when the duality relation holds) and is actually a somewhat modified nonmanifest symmetry of the action (see e.g. [7]).

To make the Lorentz invariance of the duality-symmetric action manifest, in particular, that of the chiral-field action, one should introduce auxiliary fields. In different formulations their amount vary from infinity [9–12] to a few [13,14] or even one [15,16]. The relation between different noncovariant and covariant formulations was studied e.g. in [16–18]. The quantization of duality-symmetric and chiral gauge fields (which is a subtle and highly nontrivial problem, especially in topologically nontrivial backgrounds) has also been intensively studied, see e.g. [2,8–12,19–27] and references therein.

One more, noncovariant, Lagrangian formulation of a chiral 2-form gauge field in six space-time dimensions was derived in [28,29] from a Bagger-Lambert-Gustavsson (BLG) model of interacting Chern-Simons and matter fields in $D = 3$ [30,31]. This has been achieved by promoting the non-Abelian gauge symmetry of the BLG model to the infinite-dimensional local symmetry of volume preserving diffeomorphisms in an internal three-dimensional space, see also [32,33]. It was argued in [28,29] that when the initial $D = 3$ space-time and the three-dimensional internal space are treated as six-dimensional space-time, such a model describes a nonlinear effective field theory on the worldvolume of a 5-brane of M theory in a strong $C_3$ gauge field background. Other aspects of the relation of the M5-brane to the BLG model based on the 3-algebra associated with volume preserving diffeomorphisms were considered e.g. in [33–35]. In particular, the authors of [33] found a relation of the M5-brane action [36–38], in the limit of infinite M5-brane tension, to a Carrollian limit of the BLG model in which the speed of light is zero (which amounts to suppressing all spacial derivatives along the M2-brane).

The aim of this paper is to discuss and clarify some issues of the $D = 6$ chiral-field model of [28,29] regarding its space-time and gauge symmetries, and self-duality properties. We shall first consider the free chiral-field formulation of [28] and then its nonlinear generalization constructed in [29]. We shall also compare this model with the original actions for the $D = 6$ chiral 2-form gauge field [5,7,16], as well as with the M5-brane action [36,37] and equations of motion [39–43].

In the free-field case, we show that, like the actions of [5,7], the quadratic chiral-field action of [28] possesses a nonmanifest six-dimensional (modified) Lorentz symmetry and can be covariantized, coupled to gravity and supersymmetrized in a way similar to the approach of [15,16]. However it differs from the original PST formulation in the number of auxiliary fields required for making the $D = 6$ chiral-field action of [28] manifestly covariant. We show that the latter requires three scalar fields, taking values in the three-dimensional representation of a $GL(3)$ group, while the formulation of [15,16] makes use of a single auxiliary scalar field. This is expected, since in the model of [28,29] the six space-time directions are subject to $3 + 3$ splitting, instead of the $1 + 5$ splitting of [5,7,15,16,36,37].

We then consider the nonlinear chiral-field model of [28,29] neglecting its couplings to scalar and spinor matter fields. By solving the nonlinear field equations derived in [29] we find an explicit form of gauge field strength components that were missing in the formulation of [29] and show that the complete $D = 6$ field strength transforms as a scalar field under volume preserving diffeomorphisms and satisfies the complete set of Bianchi relations. We prove that the general solution of the nonlinear field equations results in the Hodge self-duality of the $D = 6$ nonlinear gauge field strength, thus confirming the assumption of [29]. We also find that the action of the nonlinear model can be rewritten in a form that involves solely the components of the chiral-field strength and hence is covariant under the volume preserving diffeomorphisms.

The paper is organized as follows. In Sec. II we recall the basic properties of a free 2-form chiral field in six-dimensional space-time (Sec. II A), consider the structure of a noncovariant action for the $D = 6$ chiral-gauge field a la [5,7] (Sec. II B) and overview the covariant Lagrangian description of the chiral fields proposed and developed in [15,16] (Sec. II C). In Secs. III A and III B we consider the alternative noncovariant formulation of [28,29] at the free-field level and reveal its hidden gauge and Lorentz symmetries. In Secs. III C, III D, and III E we propose its covariantization, coupling to gravity and supersymmetrization along the lines of the approach of [15,16]. In Sec. IV we consider the nonlinear generalization of the alternative chiral-field formulation and study its symmetry and self-duality properties. In Sec. V we briefly discuss issues of the relation of the model of [28,29] to the worldvolume theory of the M5-brane.

## II. ACTIONS FOR THE $D = 6$ CHIRAL FIELD

### A. The antisymmetric 2-rank gauge field in $D = 6$

Let $R^{1,5}$ be a six-dimensional Minkowski space having the metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1, 1, 1)$ and parametrized by coordinates $x^\mu$ ($\mu = 0, 1, \ldots, 5$). Let $A_{\mu\nu}$ be a two-rank antisymmetric tensor field with the field strength

$$F_{\mu\nu\rho} = \partial_\mu A_{\nu\rho} + \partial_\nu A_{\rho\mu} + \partial_\rho A_{\mu\nu}. \tag{2.1}$$

The field strength (2.1) is invariant under the gauge transformations[1]

$$\delta A_{\mu\nu} = 2\partial_{[\mu}\lambda_{\nu]}(x) = \partial_\mu \lambda_\nu - \partial_\nu \lambda_\mu. \tag{2.2}$$

These gauge transformations are reducible because of the residual gauge invariance of the gauge parameter,

$$\delta\lambda_\mu = \partial_\mu \lambda(x). \tag{2.3}$$

The classical action for this field is

$$S = -\frac{1}{4!} g^2 \int d^6x F_{\mu\nu\rho} F^{\mu\nu\rho}, \tag{2.4}$$

where $g$ is a coupling constant of mass dimensionality, which we shall put equal to one in what follows. The corresponding equation of motion is

$$\frac{\delta S}{\delta A_{\mu\nu}} = \partial_\rho F^{\mu\nu\rho} = 0. \tag{2.5}$$

By definition, the field (2.1) satisfies the Bianchi identity

$$\varepsilon^{\alpha\beta\gamma\delta\rho\sigma}\partial_\gamma F_{\delta\rho\sigma} = 0. \tag{2.6}$$

On the mass shell, such an antisymmetric tensor field $A_{\mu\nu}$ describes 6 degrees of freedom. This number can be reduced to three if one imposes an additional, self-duality, condition

$$F_{\mu\nu\rho} = \tilde{F}_{\mu\nu\rho}, \tag{2.7}$$

where

$$\tilde{F}_{\mu\nu\rho} := \frac{1}{6}\varepsilon_{\mu\nu\rho\alpha\beta\gamma}F^{\alpha\beta\gamma}. \tag{2.8}$$

The field $A_{\mu\nu}$ satisfying Eq. (2.7) is called the chiral field.

A natural question is whether one can derive the first-order self-duality condition (2.7) from an action principle as an equation of motion of $A_{\mu\nu}$. The answer is positive,

[1] We use the symmetrization and antisymmetrization of indices with "strength one", i.e. with the normalization factor $\frac{1}{n!}$.

though the construction is nontrivial, and the resulting action possess peculiar properties to be reviewed in the next section.

### B. Noncovariant action

Usually the actions for free bosonic fields are of a quadratic order in their field strengths, like Eq. (2.4). So if, in order to get a chiral-field action, one tries to modify the action (2.4) with some other terms depending solely on components of $F_{\mu\nu\rho}$, one gets the equations of motion that are of the second order in derivatives. Thus, the chiral-field action should have a structure and symmetries which would allow one to reduce the second order differential equations to the first-order self-duality condition. Such actions have been found for various types of chiral fields [2–7] but they turn out to be nonmanifestly space-time invariant. In the $D = 6$ case the self-dual action can be written in the following form

$$S = -\frac{1}{4!}\int d^6x[F_{\lambda\mu\nu}F^{\lambda\mu\nu} + 3(F-\tilde{F})_{0ij}(F-\tilde{F})_0{}^{ij}], \quad (i,j = 1,\ldots,5). \tag{2.9}$$

It contains the ordinary kinetic term for $A_{\mu\nu}$, and the second term which breaks manifest Lorentz-invariance down to its spatial subgroup $SO(5)$, since only the time components of $(F - \tilde{F})$ enter the action.[2] However, it turns out that Eq. (2.9) is (nonmanifestly) invariant under modified space-time transformations [4,7] which [in the gauge $A_{0i} = 0$ for the local symmetry (2.2)] look as follows

$$\delta A_{ij} = x^0 v^k \partial_k A_{ij} + x^k v^k \partial_0 A_{ij} - x^k v^k (F-\tilde{F})_{0ij}. \tag{2.10}$$

The first two terms in (2.10) are standard Lorentz boosts with a velocity $v_i$ which extend $SO(5)$ to $SO(1,5)$. The last term is a nonconventional one, it vanishes when (2.7) is satisfied, so that the transformations (2.10) reduce to the conventional Lorentz boosts on the mass shell.

From (2.9) one gets the $A_{\mu\nu}$ field equations, which have the form of Bianchi identities

$$\varepsilon^{ijklm}\partial_k(F-\tilde{F})_{lm0} = 0. \tag{2.11}$$

Their general (topologically trivial) solution is

$$(F-\tilde{F})_{ij0} = 2\partial_{[i}\phi_{j]}(x). \tag{2.12}$$

If the right hand side of (2.12) were zero, then

$$F_{ij0} = \tilde{F}_{ij0} = \frac{1}{6}\varepsilon_{ijklm}F^{klm} \tag{2.13}$$

and, hence, as one can easily check, the full covariant self-duality condition is satisfied. And this is what we would like to get. One could put the right-hand side (rhs) of (2.12) to zero if there is an additional local symmetry of (2.9) for which $\partial_{[i}\phi_{j]} = 0$ is a gauge fixing condition. And there is indeed such a symmetry [7] which acts on the components of $A_{\mu\nu}$ as follows

$$\delta A_{0i} = \Phi_i(x), \qquad \delta A_{ij} = 0, \qquad \delta(F-\tilde{F})_{ij0} = 2\partial_{[i}\Phi_{j]}. \tag{2.14}$$

The existence of this symmetry is the reason why the quadratic action describes the dynamics of the self-dual field $A_{\mu\nu}$ with twice less physical degrees of freedom than that of a non-self-dual one. It also implies that the components $A_{0i}$ are pure gauge and enter the action only under a total derivative. $A_{0i}$ can be thus put to zero directly in the action, which fixes the gauge symmetry (2.14). The action (2.9) then reduces to

$$S = -\frac{1}{4!}\int d^6x[2F_{ijk}F^{ijk} + \varepsilon^{ijklm}F_{klm}\partial_0 A_{ij}], \quad (i,j = 1,\ldots,5). \tag{2.15}$$

Equation (2.15) does not contain the $A_{0j}$ component of the six-dimensional chiral field. Thus, on the mass shell, the role of this component is taken by the ''integration'' function $\phi_j(x)$ of (2.12), which appears upon solving the second order field Eq. (2.11). We shall encounter the same feature in the alternative formulation of [28], but before describing the construction of [28] let us first review a covariant Lagrangian description of the chiral field proposed in [15,16].

### C. Lorentz-covariant formulation

The covariant formulation of [15,16] is constructed with the use of a single auxiliary scalar field $a(x)$. The covariant generalization of the action (2.9) for the $D = 6$ self-dual field looks as follows

$$S = -\frac{1}{4!}\int d^6x\Big[F_{\lambda\mu\nu}F^{\lambda\mu\nu} - \frac{3}{(\partial_\rho a\partial^\rho a)}\partial^\mu a(x)(F-\tilde{F})_{\mu\lambda\sigma}(F-\tilde{F})^{\lambda\sigma\nu}\partial_\nu a(x)\Big]. \tag{2.16}$$

In addition to standard gauge symmetry (2.3) of $A_{\mu\nu}(x)$ the covariant action (2.16) is invariant under two different local transformations:

$$\delta A_{\mu\nu} = 2\partial_{[\mu}a\Phi_{\nu]}(x), \qquad \delta a = 0; \tag{2.17}$$

$$\delta a = \varphi(x), \qquad \delta A_{\mu\nu} = \frac{\varphi(x)}{(\partial a)^2}(F-\tilde{F})_{\mu\nu\rho}\partial^\rho a. \tag{2.18}$$

The transformations (2.17) are a covariant counterpart of

[2]Alternatively, but equivalently, one might separate one spacial component from other five and construct an $SO(1,4)$ invariant action similar to (2.9) but in which the sign of the second term is changed and the time index 0 is replaced with a space index, e.g. 5. This choice is convenient when performing the dimensional reduction of the $D = 6$ theory to $D = 5$.

(2.14) and play the same role as the latter in deriving the self-duality condition (2.7).

Local symmetry (2.18) ensures the auxiliary nature of the field $a(x)$ required for keeping the space-time covariance of the action manifest [15]. An admissible gauge fixing condition for this symmetry is

$$\frac{\partial_\mu a(x)}{\sqrt{-\partial_\nu a\partial^\nu a}} = \delta^0_\mu. \tag{2.19}$$

In this gauge the action (2.16) reduces to (2.9). The modified space-time transformations (2.10), which preserve the gauge (2.19) arise as a combination of the Lorentz boost and the transformation (2.18) with $\varphi = -v^i x^i$, ($i = 1, 2, 3, 4, 5$).

One may wonder whether by using the gauge transformation (2.18) one can put the field $a(x)$ to zero. This is indeed possible if one takes into account the subtlety that by imposing such a gauge fixing one should handle a singularity in the action (2.16) in such a way that the ratio $\partial_\mu a\partial^\nu a/\partial_\rho a\partial^\rho a$ remains finite. This can be achieved by first imposing the gauge fixing condition $a(x) = \epsilon x^\mu n_\mu$, where $n_\mu$ is a constant timelike vector $n^2 = -1$ and then sending the constant parameter $\epsilon$ to zero. As one can see, such a limit is compatible with the gauge choice (2.19) with $n_\mu = \delta^0_\mu$.

For further analysis it is useful to note that the auxiliary field $a(x)$ enters the action (2.16) only through the combination which forms a projector matrix of rank one

$$P_\mu{}^\nu = \frac{1}{\partial_\rho a\partial^\rho a}\partial_\mu a\partial^\nu a, \qquad P_\mu{}^\rho P_\rho{}^\nu = P_\mu{}^\nu. \tag{2.20}$$

Then the action (2.16) takes the form

$$S = -\frac{1}{4!}\int d^6x[F_{\lambda\mu\nu}F^{\lambda\mu\nu} - 3(F-\tilde{F})^{\lambda\sigma\nu}P_\nu{}^\mu(F-\tilde{F})_{\mu\lambda\sigma}]. \tag{2.21}$$

It produces the following Lorentz-covariant counterpart of the self-duality condition (2.13)

$$\frac{1}{\sqrt{-(\partial a)^2}}F_{\mu\nu\rho}\partial^\rho a = \frac{1}{6\sqrt{-(\partial a)^2}}\partial^\rho a\varepsilon_{\rho\mu\nu\lambda\sigma\tau}F^{\lambda\sigma\tau} \equiv \tilde{F}_{\mu\nu}. \tag{2.22}$$

As one can easily see, Eq. (2.22) is equivalent to the self-duality condition (2.7).

## III. FREE $D = 6$ CHIRAL-GAUGE FIELD FROM THE BLG MODEL

### A. Noncovariant formulation

A different noncovariant Lagrangian description of the $D = 6$ chiral field was obtained in [28,29] from a Bagger-Lambert-Gustavsson (BLG) model [30,31] of interacting Chern-Simons and matter fields in $D = 3$ by promoting the gauge symmetry of the BLG model to the infinite-dimensional local symmetry of volume preserving diffeomorphisms of an internal three-dimensional space. The original three-dimensional space-time (supposed to be a worldvolume of coincident M2-branes) was assumed in [28,29] to combine with the three-dimensional internal space and to form the six-dimensional worldvolume of a 5-brane carrying a 2-form chiral field. So in the formulation of [28,29] the $D = 6$ Lorentz symmetry $SO(1,5)$ is (naturally) broken by the presence of membranes to $SO(1,2)\times SO(3)$. In particular, the action for the free chiral field is constructed with the use of components of $A_{\mu\nu}$ which are split into $SO(1,2)\times SO(3)$ tensors and is thus an $SO(1,2)\times SO(3)$ invariant counterpart of the $SO(5)$ (or $SO(1,4)$) covariant chiral-field Lagrangian of Sec. II B.

We shall now briefly review this formulation for the case of the free gauge field. The nonlinear chiral-field model of [28,29] will be discussed in Sec. IV.

With respect to the subgroup $SO(1,2)\times SO(3)$, the $SO(1,5)$ components of $A_{\mu\nu}$ split as follows

$$A_{\mu\nu} = (A_{ab}, A_{a\dot{b}}, A_{\dot{a}\dot{b}}), \tag{3.1}$$

where the indices $a = (0, 1, 2)$ and $\dot{a} = (1, 2, 3)$, correspond, respectively, to the $SO(1,2)$ and $SO(3)$ subgroup of the full $D = 6$ Lorentz group. Each of the antisymmetric fields $A_{ab}$ and $A_{\dot{a}\dot{b}}$ has three components, while $A_{a\dot{b}}$ has nine components. The $D = 6$ coordinates $x^\mu$ split into $x^a$ and $x^{\dot{a}}$.

Only the components $A_{a\dot{b}}$ and $A_{\dot{a}\dot{b}}$ were used in the construction of the chiral-field Lagrangian of [28], which has the form

$$L = -\frac{1}{4}F_{a\dot{b}\dot{c}}(F-\tilde{f})^{a\dot{b}\dot{c}} - \frac{1}{12}F_{\dot{a}\dot{b}\dot{c}}F^{\dot{a}\dot{b}\dot{c}}, \tag{3.2}$$

where

$$F_{a\dot{b}\dot{c}} = \partial_a A_{\dot{b}\dot{c}} - \partial_{\dot{b}}A_{a\dot{c}} + \partial_{\dot{c}}A_{a\dot{b}}, \tag{3.3}$$

$$F_{\dot{a}\dot{b}\dot{c}} = \partial_{\dot{a}} A_{\dot{b}\dot{c}} - \partial_{\dot{b}}A_{\dot{a}\dot{c}} + \partial_{\dot{c}}A_{\dot{a}\dot{b}}, \tag{3.4}$$

$$\tilde{f}_{a\dot{b}\dot{c}} = \frac{1}{2}\varepsilon_{abc}\varepsilon_{\dot{b}\dot{c}\dot{a}}f^{bc\dot{a}} \tag{3.5}$$

and

$$f_{ab\dot{c}} = \partial_a A_{b\dot{c}} - \partial_b A_{a\dot{c}}. \tag{3.6}$$

Here $\varepsilon_{abc}$ and $\varepsilon_{\dot{a}\dot{b}\dot{c}}$ are the antisymmetric unit tensors invariant under $SO(1,2)$ and $SO(3)$, respectively.

Note that the tensor (3.6) as well as the Lagrangian (3.2) do not contain the components $A_{ab}$ of the gauge potential. Because of this the Lagrangian (3.2) is invariant under the gauge transformations

$$\delta A_{a\dot{b}} = \partial_a\lambda_{\dot{b}} - \partial_{\dot{b}}\lambda_a \tag{3.7}$$

only modulo a total derivative.

As in the case of the formulation of Sec. II B, Eqs. (2.12) and (2.15), the $A_{ab}$ component of the chiral field appears on the mass shell upon integrating out one of the derivatives of the second order field equations which follow from the Lagrangian (3.2) and have (upon the use of the Bianchi identities) the form [28]

$$\frac{\delta S}{\delta A^{a\dot{b}}} = 0 \Rightarrow \partial_{\dot{c}}(F - \tilde{f})^{a\dot{b}\dot{c}} = 0 \Rightarrow (F - \tilde{f})_{a\dot{b}\dot{c}} = \frac{1}{2}\varepsilon_{\dot{b}\dot{c}\dot{a}}\varepsilon_{abc}\partial^{\dot{a}}A^{bc}, \tag{3.8}$$

$$\frac{\delta S}{\delta A^{\dot{a}\dot{b}}} = 0 \Rightarrow \partial_a F^{a\dot{b}\dot{c}} + \partial_{\dot{a}}F^{\dot{a}\dot{b}\dot{c}} = 0, \tag{3.9}$$

where $A_{ab}(x^\mu)$ is an $SO(1,2)$ antisymmetric tensor field. Then Eq. (3.8) takes the form of the duality relation

$$(F - \tilde{F})^{a\dot{b}\dot{c}} = 0 \Rightarrow F_{a\dot{b}\dot{c}} = \tilde{F}_{a\dot{b}\dot{c}}, \tag{3.10}$$

where

$$\tilde{F}_{a\dot{b}\dot{c}} \equiv \frac{1}{2}\varepsilon_{abc}\varepsilon_{\dot{b}\dot{c}\dot{a}}F^{bc\dot{a}} \tag{3.11}$$

and

$$F_{ab\dot{c}} = f_{ab\dot{c}} + \partial_{\dot{c}}A_{ab} = \partial_a A_{b\dot{c}} - \partial_b A_{a\dot{c}} + \partial_{\dot{c}}A_{ab} \tag{3.12}$$

is a complete gauge invariant $F_{ab\dot{c}}$ component of the field strength $F_{\mu\nu\lambda}$.

Substituting $F^{a\dot{b}\dot{c}}$ with its dual (3.10) and (3.12) into the Eq. (3.9) we get

$$\begin{aligned} &\partial^{\dot{a}}F_{\dot{a}\dot{b}\dot{c}} + \frac{1}{2}\varepsilon_{\dot{a}\dot{b}\dot{c}}\varepsilon_{abc}\partial^{\dot{a}}\partial^a A^{bc} = 0, \\ &\Rightarrow F_{\dot{a}\dot{b}\dot{c}} + \frac{1}{2}\varepsilon_{\dot{a}\dot{b}\dot{c}}\varepsilon_{abc}\partial^a A^{bc} = \varepsilon_{\dot{a}\dot{b}\dot{c}}f(x), \end{aligned} \tag{3.13}$$

where $f(x^a)$ is a function of only three coordinates $x^a = (x^0, x^1, x^2)$, that can always be written as the divergence of a vector $f(x) = \partial_a f^a(x)$. It can thus be absorbed by a redefinition $A_{ab} \rightarrow A_{ab} + \frac{1}{3}\varepsilon_{abc}f^c(x)$ without any effect on (3.10). As a result, Eq. (3.13) takes the form of the duality relation

$$F_{abc} = \frac{1}{6}\varepsilon_{abc}\varepsilon_{\dot{a}\dot{b}\dot{c}}F^{\dot{a}\dot{b}\dot{c}}, \tag{3.14}$$

where

$$F_{abc} = \partial_a A_{bc} + \partial_b A_{ca} + \partial_c A_{ab} \tag{3.15}$$

are components of the field strength of the $D = 6$ chiral field which do not enter the Lagrangian (3.2).

Equations (3.8) and (3.14) combine into the $SO(1,5)$ covariant self-duality condition (2.7) in which the components of the $D = 6$ antisymmetric tensor $\varepsilon_{\mu\nu\lambda\rho\sigma\delta}$ are defined as follows

$$\varepsilon_{abc\dot{a}\dot{b}\dot{c}} = -\varepsilon_{\dot{a}\dot{b}\dot{c}abc} = \varepsilon_{a\dot{b}\dot{c}bc\dot{a}} = \varepsilon_{abc}\varepsilon_{\dot{a}\dot{b}\dot{c}}. \tag{3.16}$$

### B. Symmetries of the noncovariant formulation

We have already mentioned that the Lagrangian (3.2) is invariant under the gauge transformations (3.7) only up to a total derivative, because the $A_{ab}$ component of the gauge field does not enter the Lagrangian. We can restore the complete gauge invariance of the Lagrangian by adding to it certain terms depending on $A_{ab}$ in such a way that they enter the Lagrangian as total derivatives and hence do not modify corresponding equations of motion. With these terms the action takes the form

$$\begin{aligned} S &= -\frac{1}{4}\int d^6x\Big[F_{a\dot{b}\dot{c}}(F^{a\dot{b}\dot{c}} - \tilde{F}^{a\dot{b}\dot{c}}) + \frac{1}{3}F_{\dot{a}\dot{b}\dot{c}}(F^{\dot{a}\dot{b}\dot{c}} - \tilde{F}^{\dot{a}\dot{b}\dot{c}})\Big] \\ &= \frac{1}{4}\int d^6x\Big[\tilde{F}_{ab\dot{c}}(\tilde{F}^{ab\dot{c}} - F^{ab\dot{c}}) + \frac{1}{3}\tilde{F}_{abc}(\tilde{F}^{abc} - F^{abc})\Big]. \end{aligned} \tag{3.17}$$

Since the component $A_{ab}$ enters this action under a total derivative, in addition to the conventional gauge symmetry (2.2), the action (3.17) is also invariant under the following local transformations

$$\delta A_{ab} = \Phi_{ab}(x^\mu), \tag{3.18}$$

which are analogous to the transformations (2.14) in Sec. II B.

We shall now show that, similar to the formulation of Sec. II B, the action (3.17) has a nonmanifest $D = 6$ space-time symmetry.

By construction, Eq. (3.17) is manifestly invariant under the $SO(1,2) \times SO(3)$ subgroup of the full Lorentz group $SO(1,5)$. So we should check its invariance under the transformations of the components of the gauge field $A_{\mu\nu}$ corresponding to the coset $SO(1,5)/[SO(1,2) \times SO(3)]$ which are parametrized by the $3 \times 3$ constant matrix $\lambda^a_{\dot{b}}$,

$$\begin{aligned} \delta_1 A^{a\dot{a}} &= \lambda^a_{\dot{b}}A^{\dot{b}\dot{a}} + \lambda^b_{\dot{c}}(x_b\partial^{\dot{c}} - x^{\dot{c}}\partial_b)A^{a\dot{a}}, \\ \delta_1 A^{\dot{a}\dot{b}} &= -\lambda^{\dot{a}}_a A^{a\dot{b}} + \lambda^{\dot{b}}_b A^{b\dot{a}} + \lambda^b_{\dot{c}}(x_b\partial^{\dot{c}} - x^{\dot{c}}\partial_b)A^{\dot{a}\dot{b}}. \end{aligned} \tag{3.19}$$

[For simplicity, we work in the gauge $A^{ab} = 0$, which can always be imposed by fixing one of the local symmetries (2.2) and (3.18)]. The action is not invariant under the transformations (3.19), but changes as follows

$$\delta_1 S = -\frac{1}{2}\int d^6x\lambda^{\dot{c}}_b(F_{a\dot{b}\dot{c}} - \tilde{F}_{a\dot{b}\dot{c}})(F^{ab\dot{b}} - \tilde{F}^{ab\dot{b}}). \tag{3.20}$$

This variation of the action can be compensated if the Lorentz transformations of the gauge field are accompanied by the following transformation

$$\begin{aligned} &\delta_2 A_{a\dot{b}} = \lambda^c_{\dot{d}}x^{\dot{d}}(F_{ca\dot{b}} - \tilde{F}_{ca\dot{b}}), \\ &\delta_2 A_{\dot{a}\dot{b}} = 0, \qquad (A_{ab} = 0). \end{aligned} \tag{3.21}$$

Indeed,

$$\delta_2 S = \frac{1}{2}\int d^6x\lambda^{\dot c}_b(F_{a\dot b\dot c} - \tilde F_{a\dot b\dot c})(F^{ab\dot b} - \tilde F^{ab\dot b}). \quad (3.22)$$

As a result, we conclude that the action (3.17) is invariant under the following modified $SO(1,5)/[SO(1,2)\times SO(3)]$ transformations

$$\delta A^{a\dot a} = \lambda^a_{\dot b}A^{\dot b\dot a} + \lambda^b_{\dot c}(x_b\partial^{\dot c} - x^{\dot c}\partial_b)A^{a\dot a} + \lambda^{\dot d}_c x_{\dot d}(F^{ca\dot a} - \tilde F^{ca\dot a}), \quad (3.23)$$
$$\delta A^{\dot a\dot b} = -\lambda^{\dot a}_a A^{a\dot b} + \lambda^{\dot b}_b A^{b\dot a} + \lambda^b_{\dot c}(x_b\partial^{\dot c} - x^{\dot c}\partial_b)A^{\dot a\dot b},$$

which together with the $SO(1,2)\times SO(3)$ transformations form a modified nonmanifest $D=6$ Lorentz symmetry of the action (3.17). The space-time transformations become the conventional $SO(1,6)$ Lorentz transformations on the mass shell, when the gauge field strength satisfies the self-duality condition.

### C. Alternative covariant formulation

Let us now generalize the action (3.17) in such a way that it becomes Lorentz-covariant. To this end, by analogy with the covariant formulation of Sec. II C, we introduce auxiliary fields which appear in the action in the form of projector matrices $P_\nu{}^\mu(x)$ and $\Pi_\nu{}^\mu(x)$

$$P_\nu{}^\rho P_\rho{}^\mu = P_\nu{}^\mu(x), \qquad \Pi_\nu{}^\rho\Pi_\rho{}^\mu = \Pi_\nu{}^\mu(x), \qquad \Pi_\nu{}^\mu = \delta_\nu{}^\mu - P_\nu{}^\mu. \quad (3.24)$$

In contrast to the projector (2.20), we now require that $P_\nu{}^\mu(x)$ and $\Pi_\nu{}^\mu(x)$ have the rank three and look for an action that has a local symmetry, analogous to (2.18), which allows one to gauge fix the projectors to become the constant matrices

$$P_\nu{}^\mu = \begin{pmatrix} \delta_b{}^a & 0 \\ 0 & 0 \end{pmatrix}, \qquad \Pi_\nu{}^\mu = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{\dot b}{}^{\dot a} \end{pmatrix}. \quad (3.25)$$

To construct the $SO(1,5)$ covariant generalization of the action (3.17) we first rewrite it in the form

$$S = \frac{1}{4!}\int d^6x[-F_{\mu\nu\rho}F^{\mu\nu\rho} + \mathcal{F}_{abc}\mathcal{F}^{abc} + 3\mathcal{F}_{ab\dot c}\mathcal{F}^{ab\dot c}], \quad (3.26)$$

where

$$\mathcal{F}_{\mu\nu\rho} = F_{\mu\nu\rho} - \tilde F_{\mu\nu\rho}, \qquad \mathcal{F}_{abc} = F_{abc} - \tilde F_{abc}, \qquad \mathcal{F}_{ab\dot c} = F_{ab\dot c} - \tilde F_{ab\dot c}, \qquad \text{etc.} \quad (3.27)$$

Note that the field $\mathcal{F}_{\mu\nu\rho}$ is anti-self-dual,

$$\tilde{\mathcal{F}}_{\mu\nu\rho} = \frac{1}{6}\varepsilon_{\mu\nu\rho\alpha\beta\gamma}\mathcal{F}^{\alpha\beta\gamma} = -\mathcal{F}_{\mu\nu\rho}. \quad (3.28)$$

Now, using the projectors (3.24), we construct the Lorentz-covariant generalization of (3.26)

$$S = \frac{1}{4!}\int d^6x[-F_{\mu\nu\rho}F^{\mu\nu\rho} + \mathcal{F}_{\mu\nu\rho}\mathcal{F}^{\alpha\beta\gamma}(P^\mu_\alpha P^\nu_\beta P^\rho_\gamma + 3P^\mu_\alpha P^\nu_\beta\Pi^\rho_\gamma)], \quad (3.29)$$

or, equivalently,

$$S = -\frac{1}{12}\int d^6x F_{\mu\nu\rho}\mathcal{F}^{\alpha\beta\gamma}(\Pi^\mu_\alpha\Pi^\nu_\beta\Pi^\rho_\gamma + 3\Pi^\mu_\alpha\Pi^\nu_\beta P^\rho_\gamma) = -\frac{1}{12}\int d^6x\tilde F_{\mu\nu\rho}\mathcal{F}^{\alpha\beta\gamma}(P^\mu_\alpha P^\nu_\beta P^\rho_\gamma + 3P^\mu_\alpha P^\nu_\beta\Pi^\rho_\gamma). \quad (3.30)$$

We shall now show that the action (3.29) or (3.30) has indeed the required local symmetry, provided the projectors are constructed in an appropriate way from a triplet of scalar fields $a^r(x)$ ($r=1,2,3$) being a vector with respect to the $GL(3)$ group. These scalar fields play the same role as the auxiliary field $a(x)$ of Sec. II C.

### D. Symmetries of the covariant action

Recall that the action (3.17) is invariant under the local transformations (3.18). The generalization of this symmetry to the case of the Lorentz-covariant action (3.29) is

$$\delta A_{\mu\nu} = P^\alpha_\mu P^\beta_\nu\Phi_{\alpha\beta}(x), \qquad \delta P_\mu{}^\nu = \delta\Pi_\mu{}^\nu = 0. \quad (3.31)$$

To check this and other symmetries let us perform a general variation of the action (3.29) with respect to $A_{\mu\nu}$. Using the identities

$$\varepsilon_{\mu\nu\rho\alpha\beta\gamma}P^\mu_{\mu'}P^\nu_{\nu'}P^\rho_{\rho'} = -\varepsilon_{\mu\nu\rho\mu'\nu'\rho'}\Pi^\mu_\alpha\Pi^\nu_\beta\Pi^\rho_\gamma,$$
$$\varepsilon_{\mu\nu\rho\alpha\beta\gamma}P^\mu_{[\mu'}P^\nu_{\nu'}\Pi^\rho_{\rho']} = -\varepsilon_{\mu\nu\rho\mu'\nu'\rho'}\Pi^\mu_{[\alpha}\Pi^\nu_\beta P^\rho_{\gamma]}, \quad (3.32)$$
$$P_{\mu'}{}^\mu P_{\nu'}{}^\nu\partial_\lambda P_{\mu\nu} = 0,$$

we find that

$$\delta S_{\delta A} = \frac{1}{12}\int d^6x\delta F_{\mu\nu\rho}[-F^{\mu\nu\rho} + (P^\mu_\alpha P^\nu_\beta P^\rho_\gamma + 3P^\mu_\alpha P^\nu_\beta\Pi^\rho_\gamma - \Pi^\mu_\alpha\Pi^\nu_\beta\Pi^\rho_\gamma - 3\Pi^\mu_\alpha\Pi^\nu_\beta P^\rho_\gamma)\mathcal{F}^{\alpha\beta\gamma}]$$
$$= \frac{1}{12}\int d^6x\delta F_{\mu\nu\rho}[-F^{\mu\nu\rho} + (-4P^\mu_\alpha P^\nu_\beta P^\rho_\gamma + 6P^\mu_\alpha P^\nu_\beta\delta^\rho_\gamma - \delta^\mu_\alpha\delta^\nu_\beta\delta^\rho_\gamma)\mathcal{F}^{\alpha\beta\gamma}]$$
$$= \int d^6x\delta A_{\mu\nu}\left[\frac{1}{2}\partial_\rho\mathcal{F}^{\mu\nu\rho} - \frac{1}{2}P^\mu_\alpha P^\nu_\beta\partial_\rho\mathcal{F}^{\alpha\beta\rho} - (\partial_\rho P^\mu_\alpha)P^\nu_\beta\mathcal{F}^{\alpha\beta\rho} - \partial_\rho(P^\rho_\alpha P^\mu_\beta\Pi^\nu_\gamma\mathcal{F}^{\alpha\beta\gamma})\right]. \quad (3.33)$$

For the variation of $A_{\mu\nu}$ in the form (3.31) we get

$$\delta_\Phi S = -\int d^6x \Phi_{\alpha\beta}\mathcal{F}^{\nu\sigma\lambda}P^\alpha_\mu P^\beta_\nu \Pi^\rho_\sigma \Pi^\gamma_\lambda(\partial_{[\rho}P^\mu_{\gamma]}). \quad (3.34)$$

We see that $\delta_\Phi S = 0$ if

$$\Pi_\sigma{}^\rho \Pi_\lambda{}^\gamma \partial_{[\rho}P_{\gamma]}{}^\mu P_\mu{}^\nu = 0. \quad (3.35)$$

Equation (3.35) is the main differential constraint which must be satisfied by the projector. It is solved by expressing the projector in terms of derivatives of a triplet of auxiliary scalar fields $a^r(x)$ with the index $r = 1, 2, 3$ corresponding to a three-dimensional representation of $GL(3)$. Namely,

$$P_\mu{}^\nu = \partial_\mu a^r Y^{-1}_{rs}\partial^\nu a^s, \qquad \Pi_\mu{}^\nu = \delta_\mu{}^\nu - P_\mu{}^\nu, \quad (3.36)$$

where $Y^{-1}_{rs}$ is the inverse matrix for[3]

$$Y^{rs} \equiv \partial_\rho a^r \partial^\rho a^s.$$

Thus, to satisfy the requirement of the local symmetry (3.31), the projector in the action (3.29) is taken to be in the form (3.36).

In view of the similarity of the structure of the projectors (2.20) and (3.36), one may expect that there is a local symmetry acting on $a^r(x)$ and $A_{\mu\nu}$, analogous to (2.18), which allows one to get the gauge condition (3.25) by putting

$$a^r = \delta_a{}^r x^a \quad (3.37)$$

and to recover the modified Lorentz transformation (3.21) and (3.23) of the noncovariant formulation as a compensating transformation of the local symmetry, preserving the gauge (3.25) and (3.37).

There is indeed such a local symmetry, *i.e.*

$$\delta_\varphi a^r = \varphi^r(x), \qquad \delta_\varphi A_{\mu\nu} = 2\varphi^r Y^{-1}_{rs}\partial^\gamma a^s \mathcal{F}_{\alpha\beta\gamma}P^\alpha_{[\mu}\Pi^\beta_{\nu]}, \quad (3.38)$$

where $\varphi^r(x)$ are local parameters. To check the invariance of the action under (3.38) it is also instructive to present the variation of the projector

$$\delta_\varphi P_{\mu\nu} = 2\Pi_{\rho(\mu}\partial^\rho\varphi^q Y^{-1}_{qr}\partial_{\nu)}a^r. \quad (3.39)$$

Note that the variation (3.39) preserves the constraint (3.35), which reflects the fact that the latter is solved by the projector $P_\mu{}^\nu$ having the form (3.36). A direct computation shows that the action is invariant under the variations (3.38) and (3.39). Indeed,

$$\delta_\varphi S = \int d^6x T^{\mu\nu}_r \partial_\mu \partial_\nu a^r = 0, \quad (3.40)$$

where $T^{\mu\nu}_r$ is the antisymmetric tensor of the form

$$T^{\mu\nu}_r = -T^{\nu\mu}_r = Y^{-1}_{rs}Y^{-1}_{kl}\varphi^k\partial^\sigma a^s \partial^\delta a^l(\Pi^\mu_\alpha \Pi^\nu_\beta \Pi^\rho_\gamma \mathcal{F}^{\alpha\beta\gamma}\mathcal{F}_{\rho\sigma\delta} + 2\delta^{[\mu}_{[\sigma}\Pi^{\nu]\rho}\mathcal{F}^{\lambda\tau}_{\delta]}\mathcal{F}_{\rho\alpha\beta}\Pi^\alpha_\lambda P^\beta_\tau). \quad (3.41)$$

The gauge condition (3.37) is preserved under the combined Lorentz transformations and the $\varphi$-transformation (3.38) with parameters $\lambda^a_{\dot b}$ and $\varphi = -\lambda^a_{\dot b}x^{\dot b}$, respectively,

$$\delta a^r = \delta_L a^r + \delta_\varphi a^r = 0. \quad (3.42)$$

When acting on the components of the gauge field $A_{\mu\nu}$, such a combined transformation generates the modified Lorentz transformations (3.23) of the noncovariant formulation.

### E. Coupling to gravity and supersymmetric generalization

Because of the manifest Lorentz covariance of the formulation under consideration, like in the case of the formulation of [15,16], the coupling of the chiral gauge field to gravity is straightforward. One should only replace in the action (3.29) and in all the symmetry transformations the Minkowski metric $\eta_{\mu\nu}$ with a curved $D = 6$ metric $g_{\mu\nu}(x)$. As a result the $D = 6$ chiral-field action coupled to gravity has the following form

$$S = \frac{1}{24}\int d^6x\sqrt{-g}[-F_{\mu\nu\rho}F^{\mu\nu\rho} + \mathcal{F}_{\mu\nu\rho}\mathcal{F}^{\alpha\beta\gamma}(P^\mu_\alpha P^\nu_\beta P^\rho_\gamma + 3P^\mu_\alpha P^\nu_\beta \Pi^\rho_\gamma)] + \int d^6x\sqrt{-g}R, \quad (3.43)$$

where now the projectors include the $D = 6$ metric

$$P_\mu{}^\nu = \partial_\mu a^r(\partial_\rho a^r g^{\rho\sigma}\partial_\sigma a^s)^{-1}g^{\nu\lambda}\partial_\lambda a^s, \quad \Pi_\mu{}^\nu = \delta_\mu{}^\nu - P_\mu{}^\nu. \quad (3.44)$$

#### *1.* $\mathcal{N} = (1, 0)$, $D = 6$ *tensor supermultiplet*

The simplest $\mathcal{N} = (1, 0)$ supersymmetric generalization of the chiral-field action is also straightforward. It involves the $\mathcal{N} = (1, 0)$ superpartners of $A_{\mu\nu}$ which are a scalar field $\phi(x)$ and an $SU(2)$ symplectic Majorana-Weyl fermion $\psi^I_A(x)$ ($A = 1, 2, 3, 4$; $I = 1, 2$) [44,45]

$$(\psi^I_A)^* = \bar\psi_{I\dot A} = \varepsilon_{IJ}B^B_{\dot A}\psi^J_B, \quad (3.45)$$

where the matrix $B$ is unitary and satisfies $B^*B = -1$. The $SU(2)$ indices are raised and lowered according to the following rule

$$\psi^I = \varepsilon^{IJ}\psi_J, \qquad \psi_I = \varepsilon_{IJ}\psi^J \qquad \varepsilon_{12} = -\varepsilon^{12} = 1.$$

The existence of the matrix $B$ implies that we do not need spinors with dotted indices for the fermionic action to be real. To construct the $\mathcal{N} = (1, 0)$ supersymmetric action

[3]Compare with Eq. (2.20) of Sec. II C.

one should just add to the action (3.29) or (3.30) the kinetic terms for $\psi_{AI}(x)$ and $\phi(x)$. The resulting free action is

$$S = \frac{1}{4!}\int d^6x[-F_{\mu\nu\rho}F^{\mu\nu\rho} + \mathcal{F}_{\mu\nu\rho}\mathcal{F}^{\alpha\beta\gamma}(P^\mu_\alpha P^\nu_\beta P^\rho_\gamma + 3P^\mu_\alpha P^\nu_\beta \Pi^\rho_\gamma)] - \frac{1}{2}\int d^6x(\psi_I\Gamma^\mu\partial_\mu\psi^I + \partial_\mu\phi\partial^\mu\phi). \tag{3.46}$$

It is invariant under the following supersymmetry transformations with a constant fermionic parameter $\epsilon^{AI}$

$$\delta_\epsilon\phi = \epsilon^I\psi_I, \qquad \delta_\epsilon A_{\mu\nu} = \epsilon^I\Gamma_{\mu\nu}\psi_I,$$
$$\delta_\epsilon\psi_I = \left(\Gamma^\mu\partial_\mu\phi + \frac{1}{12}\Gamma_{\mu\nu\rho}K^{\mu\nu\rho}\right)\epsilon_I, \qquad \delta_\epsilon a^r(x) = 0, \tag{3.47}$$

where

$$K^{\mu\nu\rho} = \frac{1}{2}[F^{\mu\nu\rho} + \tilde{F}^{\mu\nu\rho} + (\Pi^\mu_\alpha\Pi^\nu_\beta\Pi^\rho_\gamma + 6\Pi^{[\mu}_\alpha\Pi^\nu_\beta P^{\rho]}_\gamma - P^\mu_\alpha P^\nu_\beta P^\rho_\gamma - 6P^{[\mu}_\alpha P^\nu_\beta\Pi^{\rho]}_\gamma)\mathcal{F}^{\alpha\beta\gamma}] \equiv F^{\mu\nu\rho} + (2P^\mu_\alpha P^\nu_\beta P^\rho_\gamma - 6P^{[\mu}_\alpha P^\nu_\beta\delta^{\rho]}_\gamma)\mathcal{F}^{\alpha\beta\gamma} \tag{3.48}$$

is the self-dual tensor $K_{\mu\nu\rho} = \frac{1}{6}\varepsilon_{\mu\nu\rho\alpha\beta\gamma}K^{\alpha\beta\gamma}$. The conventions for the $D = 6$ gamma-matrices are given in the Appendix.

Note that the supersymmetry transformation (3.47) of the fermionic field is unusual. In addition to the field strength $F^{\mu\nu\rho}$ it contains terms with the anti-self-dual tensor $\mathcal{F}^{\alpha\beta\gamma}$. On the mass shell, due to the self-duality condition $\mathcal{F}^{\alpha\beta\gamma} = 0$, the supersymmetry variation of the fermions take the conventional form. Our supersymmetry transformations differ from those given in [29] (in the linear approximation of their model) by this additional contribution to the variation of the fermions, which is required for the supersymmetry of the action.

#### 2. $\mathcal{N} = (2, 0)$, $D = 6$ *tensor supermultiplet*

One can combine the supersymmetric action (3.46) with actions for other matter supermultiplets, e.g. by including into the model four more scalars and one more Majorana-Weyl spinor and thus getting the action for an $\mathcal{N} = (2, 0)$, $D = 6$ chiral tensor supermultiplet (associated with the physical fields on the M5-brane worldvolume).

The fields of the $\mathcal{N} = (2, 0)$ tensor supermultiplet transform under the $SO(5)$ R-symmetry of the $\mathcal{N} = (2, 0)$ superalgebra as follows. The tensor field is a singlet of $SO(5)$, the set of the five scalars $\phi^{\underline{m}}$, $\underline{m} = 1, \ldots, 5$ form an $SO(5)$ vector while the fermions $\psi_{IA}$ carry the index $I = 1, 2, 3, 4$ of a spinor representation of $SO(5) \sim USp(4)$ and the index $A = 1, 2, 3, 4$ of a spinor representation of $SO(1, 5) \sim Sp(4)$. The fermions satisfy the $USp(4)$-symplectic Majorana-Weyl condition analogous to (3.45)

$$(\psi^I_A)^* = \bar{\psi}_{I\dot{A}} = C_{IJ}B^B_{\dot{A}}\psi^J_B, \tag{3.49}$$

where $C_{IJ}$ is a skew-symmetric $USp(4)$-invariant tensor

$$C^{IJ}C_{JK} = \delta^I_K, \qquad C^{IJ}C_{IJ} = -4, \tag{3.50}$$

which is used to rise and lower the $USp(4)$ indices

$$\psi^I_A = C^{IJ}\psi_{IA}, \qquad \psi_{IA} = C_{IJ}\psi^J_A. \tag{3.51}$$

The antisymmetric matrices $\gamma^{\underline{m}}_{IJ} = -\gamma^{\underline{m}}_{JI}$ associated with the spinor representation of $SO(5) \sim USp(4)$ satisfy the conventional anticommutation relations

$$\gamma^{\underline{m}}_{IJ}\gamma^{\underline{n}JK} + \gamma^{\underline{n}}_{IJ}\gamma^{\underline{m}JK} = 2\delta^{\underline{mn}}\delta^K_I, \tag{3.52}$$

and the orthogonality and completeness relations

$$\gamma^{\underline{m}}_{IJ}\gamma^{\underline{n}IJ} = -4\delta^{\underline{mn}},\quad \gamma^{\underline{m}}_{IJ}\gamma^{KL}_{\underline{m}} = -2(\delta^K_I\delta^L_J - \delta^L_I\delta^K_J) - C_{IJ}C^{KL},\quad C^{IJ}\gamma^{\underline{m}}_{IJ} = 0. \tag{3.53}$$

The action

$$S = \frac{1}{24}\int d^6x[-F_{\mu\nu\rho}F^{\mu\nu\rho} + \mathcal{F}_{\mu\nu\rho}\mathcal{F}^{\alpha\beta\gamma}(P^\mu_\alpha P^\nu_\beta P^\rho_\gamma + 3P^\mu_\alpha P^\nu_\beta\Pi^\rho_\gamma)] - \frac{1}{2}\int d^6x(\psi_{IA}\Gamma^{\mu AB}\partial_\mu\psi^I_B + \partial_\mu\phi^{\underline{m}}\partial^\mu\phi_{\underline{m}}) \tag{3.54}$$

is invariant under the following $\mathcal{N} = (2, 0)$ supersymmetry variations of the fields

$$\delta_\epsilon\phi^{\underline{m}} = \epsilon^{IA}\gamma^{\underline{m}}_{IJ}\psi^J_A, \tag{3.55}$$

$$\delta_\epsilon A_{\mu\nu} = \epsilon^I\Gamma_{\mu\nu}\psi_{IB}, \tag{3.56}$$

$$\delta_\epsilon\psi_{IA} = \left(\Gamma^\mu_{AB}\gamma^{\underline{m}}_{IJ}\partial_\mu\phi_{\underline{m}}\epsilon^{JB} + \frac{1}{12}(\Gamma_{\mu\nu\rho})_{AB}K^{\mu\nu\rho}\epsilon^B_I\right), \tag{3.57}$$

$$\delta_\epsilon a^r(x) = 0. \tag{3.58}$$

As a further generalization, one can straightforwardly couple the matter supermultiplets discussed above to supergravity and construct $D = 6$ chiral supergravity actions in a form alternative to that considered in [46–49].

### F. Comparison of the two actions for the chiral field

Let us now compare the chiral field actions of Secs. II and III. For simplicity, let us consider their noncovariant versions (2.9) and (3.26). We split the $SO(5)$ indices $i, j, \ldots$ of the second term of (2.9) into the $SO(3)$ indices $\dot{a}, \dot{b}, \ldots$ and $SO(2)$ indices $I, J = 1, 2$ and try to rewrite the terms of the action (2.9) in a form in which the indices $I$, $J$ combine with the timelike index 0 into the $SO(1, 2)$ indices $a$, $b$, $c$. As a result, upon the use of the anti-self-duality of $\mathcal{F}_{\mu\nu\rho} = (F - \tilde{F})_{\mu\nu\rho}$, the action (2.9) can be rewritten in the form

$$S = -\frac{1}{4!}\int d^6x[F_{\mu\nu\rho}F^{\mu\nu\rho} - \mathcal{F}_{abc}\mathcal{F}^{abc} - 3\mathcal{F}_{ab\dot{c}}\mathcal{F}^{ab\dot{c}} + 6\mathcal{F}_{0\dot{a}\dot{b}}\mathcal{F}_0^{\dot{a}\dot{b}}]. \tag{3.59}$$

We see that (3.59) differs from the action (3.26) in the last term which is quadratic in the components $\mathcal{F}_{0\dot{a}\dot{b}}$ of the anti-self-dual part of the field strength. Since on the mass shell $\mathcal{F}_{\mu\nu\rho}$ vanishes, the two formulations are classically equivalent, as we have seen in the previous sections. It would be of interest to understand whether the difference of the two chiral-field actions off the mass shell may lead to different results upon quantization. For instance, the two formulations may complement each other when the chiral field is considered in topologically nontrivial backgrounds.

## IV. NONLINEAR MODEL FOR THE $D = 6$ CHIRAL-GAUGE FIELD FROM THE BLG ACTION REVISITED

Let us now consider the nonlinear chiral-field model of [28,29]. We shall study this model in a simplified case, in which all the scalar and spinor matter fields are put to zero, and will show that the general solution of the field equations of this model results in the $D = 6$ Hodge self-duality of a nonlinear field strength of the chiral field. We shall thus prove the assumption of the authors of [29] that the field strength is self-dual. The solution of the equations of motion will allow us to get the dual field strength components which were missing in [29], to show that they transform as scalar fields under the volume preserving diffeomorphisms and to find a form of the nonlinear action of [29] which only involves components of the field strength and, hence, is gauge-covariant.

Let us begin with a short overview of the model. It was obtained from the Bagger-Lambert-Gustavsson model by promoting its non-Abelian gauge symmetry based on a 3-algebra to an infinite-dimensional local symmetry of volume preserving diffeomorphisms in an internal three-dimensional space $\mathcal{N}_3$ whose algebra is defined by the Nambu bracket

$$\{f, g, h\} \equiv \varepsilon^{\dot{a}\dot{b}\dot{c}}\partial_{\dot{a}}f\partial_{\dot{b}}g\partial_{\dot{c}}h,$$

where $f(x^{\dot{a}})$, $g(x^{\dot{a}})$ and $h(x^{\dot{a}})$ are functions on $\mathcal{N}_3$, $x^{\dot{a}}$ are local coordinates of $\mathcal{N}_3$ and $\varepsilon^{\dot{a}\dot{b}\dot{c}}$ is the $SO(3)$-invariant antisymmetric unit tensor. The six-dimensional space-time, which is assumed to be associated with the worldvolume of an M5-brane, is a fiber bundle with the fiber $\mathcal{N}_3$ over the three-dimensional space-time of the BLG model. The six-dimensional coordinates are $x^\mu = (x^a, x^{\dot{a}})$ as defined in the previous sections.

According to [28,29], the field content of the six-dimensional model with the local symmetry of the $\mathcal{N}_3$-volume preserving diffeomorphisms comprises gauge fields $A_{a\dot{b}}(x^\mu)$ and $A^{\dot{a}} = \frac{1}{2}\varepsilon^{\dot{a}\dot{b}\dot{c}}A_{\dot{b}\dot{c}}(x^\mu)$, the five scalar fields $X^{\underline{m}}(x^\mu)$, $\underline{m} = 1, \ldots, 5$, interpreted as five bulk directions transversal to the 5-brane worldvolume, and 16 fermionic superpartners $\Psi(x^\mu)$ thereof. In what follows we shall neglect the matter fields $X^{\underline{m}}$ and $\Psi$. The fields $A_{a\dot{b}}$ and $A_{\dot{a}\dot{b}}$ are assumed to be part of the components of the $D = 6$ chiral gauge field $A_{\mu\nu}$ whose components $A_{ab}$ do not appear in the nonlinear model of [28,29].

The field $A^{\dot{a}}$ can be combined with the coordinates $x^{\dot{a}}$ to form the quantities

$$X^{\dot{a}} \equiv \frac{1}{g}x^{\dot{a}} + A^{\dot{a}}(x^\mu), \tag{4.1}$$

where $g$ is a coupling constant. $X^{\dot{a}}$ are interpreted in [28,29] as coordinates parametrizing three bulk directions orthogonal to the M2-branes and parallel to the 5-brane.

A scalar field $\Phi$ and the gauge fields $A_{a\dot{b}}$ and $A_{\dot{a}\dot{b}}$ transform under local gauge transformations with parameters $\Lambda_{\dot{a}}(x^\mu)$ and $\Lambda_a(x^\mu)$ as follows

$$\delta_\Lambda\Phi = g\xi^{\dot{c}}\partial_{\dot{c}}\Phi, \qquad \delta_\Lambda A_{\dot{a}\dot{b}} = \partial_{\dot{a}}\Lambda_{\dot{b}} - \partial_{\dot{b}}\Lambda_{\dot{a}} + g\xi^{\dot{c}}\partial_{\dot{c}}A_{\dot{a}\dot{b}}, \tag{4.2}$$

$$\delta_\Lambda A_{a\dot{b}} = \partial_a\Lambda_{\dot{b}} - \partial_{\dot{b}}\Lambda_a + g\xi^{\dot{c}}\partial_{\dot{c}}A_{a\dot{b}} + g(\partial_{\dot{b}}\xi^{\dot{c}})A_{a\dot{c}}, \tag{4.3}$$

where

$$\xi^{\dot{a}} = -\frac{1}{g}\delta_\Lambda x^{\dot{a}} = \epsilon^{\dot{a}\dot{b}\dot{c}}\partial_{\dot{b}}\Lambda_{\dot{c}} \tag{4.4}$$

so that $\partial_{\dot{a}}\xi^{\dot{a}} = \partial_{\dot{a}}\epsilon^{\dot{a}\dot{b}\dot{c}}\partial_{\dot{b}}\Lambda_{\dot{c}} \equiv 0$, which is the volume preserving condition.

Here it is worth to mention a subtle point of the construction of [29]. Namely, the quantities $X^{\dot{a}}$ defined in (4.1) transform as scalars under the volume preserving diffeomorphisms (4.2) and (4.4), though they carry the vector index $\dot{a}$. As we shall see below, this property allows one to construct gauge field strengths which transform as scalars under (4.2) and, hence, can be used to construct a gauge invariant action of the model within the line of [29].

If $\Phi_i$ ($i = 1, 2, 3$) are scalar fields with respect to the volume preserving diffeomorphisms, their Nambu bracket $\{\Phi_1, \Phi_2, \Phi_3\}$ is also a scalar field. This allows one to define a covariant derivative along the fiber $\mathcal{N}_3$ [29]

$$\mathcal{D}_{\dot{a}}\Phi = \frac{g^2}{2}\varepsilon_{\dot{a}\dot{b}\dot{c}}\{\Phi, X^{\dot{b}}, X^{\dot{c}}\} = \Big[\partial_{\dot{a}} + g(\partial_{\dot{b}}A^{\dot{b}}\partial_{\dot{a}} - \partial_{\dot{a}}A^{\dot{b}}\partial_{\dot{b}}) + \frac{g^2}{2}\varepsilon_{\dot{a}\dot{b}\dot{c}}\varepsilon^{\dot{d}\dot{e}\dot{f}}\partial_{\dot{d}}A^{\dot{b}}\partial_{\dot{e}}A^{\dot{c}}\partial_{\dot{f}}\Big]\Phi. \tag{4.5}$$

Note that

$$\mathcal{D}_{\dot a} X^{\dot b} = \frac{1}{g}\delta_{\dot a}{}^{\dot b}\,\det M = g^2\{X^{\dot 1}, X^{\dot 2}, X^{\dot 3}\}\delta_{\dot a}{}^{\dot b},$$

where

$$M_{\dot a}{}^{\dot b} = g\partial_{\dot a} X^{\dot b} = \delta_{\dot a}^{\dot b} + g\partial_{\dot a} A^{\dot b}. \tag{4.6}$$

One also defines a covariant derivative along the $x^a$ directions of the $D = 6$ space-time which acts on a scalar field $\Phi$ as follows

$$\mathcal{D}_a \Phi = \partial_a \Phi - g\{A_{a\dot b}, x^{\dot b}, \Phi\} = (\partial_a - gB_a{}^{\dot a}\partial_{\dot a})\Phi. \tag{4.7}$$

where

$$B_a{}^{\dot a} = \varepsilon^{\dot b\dot c\dot a}\partial_{\dot b} A_{a\dot c}. \tag{4.8}$$

The definition of the covariant derivative $\mathcal{D}_a$ can be extended to any tensor field $T$ on $\mathcal{N}_3$ [32]

$$\mathcal{D}_a T = (\partial_a - g\mathcal{L}_{B_a})T, \tag{4.9}$$

where $\mathcal{L}_{B_a}$ is the Lie derivative along the $\mathcal{N}_3$ vector field $(B_a)^{\dot a}$.

It follows from (4.8) that $B_a{}^{\dot a}$ is a divergenceless field

$$\partial_{\dot a} B_a{}^{\dot a} = 0, \tag{4.10}$$

which plays the role of the deformation of the Nambu-Poisson structure when the parameters of the volume preserving diffeomorphisms depend on $x^a$. Under the gauge transformations (4.2), (4.3), and (4.4), $B_a{}^{\dot a}$ transforms as follows

$$\delta B_a{}^{\dot a} = \partial_a \xi^{\dot a} + g\xi^{\dot b}\partial_{\dot b} B_a{}^{\dot a} - gB_a{}^{\dot b}\partial_{\dot b}\xi^{\dot a}. \tag{4.11}$$

Therefore, the covariant derivative $\mathcal{D}_a$, Eq. (4.9), transforms as a scalar.

Note that, since $X^{\dot a}$ is a scalar under the gauge transformations (4.2), (4.3), and (4.4), and $\varepsilon^{\dot a\dot b\dot c}$ is the invariant tensor, also the covariant derivative $\mathcal{D}_{\dot a}$ is scalar under the gauge transformations and the matrix $M_{\dot a}{}^{\dot b}$ transforms as a covariant vector (with respect to the lower index $\dot a$), i.e.

$$\delta M_{\dot a}{}^{\dot b} = \xi^{\dot c}\partial_{\dot c} M_{\dot a}{}^{\dot b} + g(\partial_{\dot a}\xi^{\dot c})M_{\dot c}{}^{\dot b}.$$

Because of this property the matrix $M_{\dot a}{}^{\dot b}$, as well as its inverse $M^{-1b}_{\dot a}$, acts as a "bridge" which converts scalar quantities, like $\mathcal{D}_{\dot a}$, into vector ones, like $\partial_{\dot a}$, and vice versa. They can also be regarded as dreibeins which relate global $SO(3)$ vector indices with $\mathcal{N}_3$ worldvolume indices. For example, the following useful identity holds for the covariant derivative (4.5) acting on a field $\Phi$

$$\mathcal{D}_{\dot a}\Phi = \det M M^{-1\dot b}_{\dot a}\partial_{\dot b}\Phi. \tag{4.12}$$

Thus, when $\Phi$ is a scalar field, the above formula demonstrates how the matrix $M^{-1\dot b}_{\dot a}$ transforms the vector $\partial_{\dot b}\Phi$ into the scalar $\mathcal{D}_{\dot a}\Phi$ (with respect to the volume preserving diffeomorphisms).

Note that, as defined in Eq. (4.5), the derivative $\mathcal{D}_{\dot a}$ acts covariantly only on the $\mathcal{N}_3$-scalar fields, but using the matrix $M_{\dot a}{}^{\dot b}$ one can generalize it to act covariantly also on the $\mathcal{N}_3$-tensor fields. For instance, the covariant derivative of a vector field $V_{\dot b}$ is

$$\hat{\mathcal{D}}_{\dot a} V_{\dot b} = \mathcal{D}_{\dot a} V_{\dot b} - (\mathcal{D}_{\dot a} M^{\dot c}_{\dot b})M^{-1\dot d}_{\dot c} V_{\dot d}. \tag{4.13}$$

One can use the covariant derivatives (4.5) and (4.7) to construct covariant field strengths of the gauge fields $A^{\dot a}$ and $A_{a\dot b}$ as follows

$$\mathcal{H}_{\dot a\dot b\dot c} + \frac{1}{g}\varepsilon_{\dot a\dot b\dot c} = \frac{1}{6}\varepsilon_{\dot f[\dot a\dot b}\mathcal{D}_{\dot c]} X^{\dot f} \tag{4.14}$$

and

$$\mathcal{H}_{a\dot a\dot b} = \varepsilon_{\dot a\dot b\dot f}\mathcal{D}_a X^{\dot f}. \tag{4.15}$$

Explicitly, the field strengths (4.14) and (4.15) have the following form

$$\begin{aligned}\mathcal{H}_{\dot 1\dot 2\dot 3} &= \partial_{\dot a} A^{\dot a} + \frac{g}{2}(\partial_{\dot a} A^{\dot a}\partial_{\dot b} A^{\dot b} - \partial_{\dot b} A^{\dot a}\partial_{\dot a} A^{\dot b}) \\ &\quad + \frac{g^2}{6}\varepsilon_{\dot a\dot b\dot c}\varepsilon^{\dot d\dot f\dot e}\partial_{\dot d} A^{\dot a}\partial_{\dot f} A^{\dot b}\partial_{\dot e} A^{\dot c}, \\ &\equiv \frac{1}{g}(\det M - 1),\end{aligned} \tag{4.16}$$

$$\begin{aligned}\mathcal{H}_{a\dot b\dot c} &= \partial_a A_{\dot b\dot c} - \partial_{\dot b} A_{a\dot c} + \partial_{\dot c} A_{a\dot b} - g\varepsilon^{\dot d\dot e\dot f}\partial_{\dot d} A_{a\dot e}\partial_{\dot f} A_{\dot b\dot c} \\ &\equiv \varepsilon_{\dot a\dot b\dot c}\mathcal{D}_a X^{\dot a}.\end{aligned} \tag{4.17}$$

The field strengths $\mathcal{H}_{\dot a\dot b\dot c}$ and $\mathcal{H}_{\dot a\dot b c}$, which by construction transform as scalars under the gauge transformations (4.2), can also be derived from the commutator of the covariant derivatives, since as was shown in [29]

$$[\mathcal{D}_{\dot a}, \mathcal{D}_{\dot b}]\Phi = -g^2\{\mathcal{H}_{\dot a\dot b\dot f}, X^{\dot f}, \Phi\}, \tag{4.18}$$

$$[\mathcal{D}_a, \mathcal{D}_{\dot b}]\Phi = -g^2\{\mathcal{H}_{a\dot b\dot f}, X^{\dot f}, \Phi\} \tag{4.19}$$

and

$$[\mathcal{D}_a, \mathcal{D}_b]\Phi = -\frac{g}{\det M}\varepsilon_{abc}\mathcal{D}_d\tilde{\mathcal{H}}^{dc\dot a}\mathcal{D}_{\dot a}\Phi. \tag{4.20}$$

Equation (4.19), in which $\Phi$ is taken to be $X^{\dot b}$ is nothing but the Bianchi identity

$$\mathcal{D}_a\tilde{\mathcal{H}}^{abc} + \mathcal{D}_{\dot a}\tilde{\mathcal{H}}^{\dot a bc} \equiv 0, \tag{4.21}$$

where $\tilde{\mathcal{H}}^{abc}$ and $\tilde{\mathcal{H}}^{ab\dot c}$ are Hodge dual of (4.16), similar to Eqs. (3.11) and (3.14) of the linear case.

In the absence of the scalar and fermion matter fields, the nonlinear chiral-field action of [29] has the following form

$$S = -\int d^6x\Big(\frac{1}{4}\mathcal{H}_{a\dot{b}\dot{c}}\mathcal{H}^{a\dot{b}\dot{c}} + \frac{1}{12}\mathcal{H}_{\dot{a}\dot{b}\dot{c}}\mathcal{H}^{\dot{a}\dot{b}\dot{c}} + \frac{1}{2}\varepsilon^{abc}B_a{}^{\dot{a}}\partial_b A_{c\dot{a}} + g\,\det B_a{}^{\dot{a}}\Big) \tag{4.22}$$

or equivalently (up to a total derivative)

$$S = -\int d^6x\Big(\frac{1}{2}(\mathcal{D}_a X^{\dot{b}})^2 + \frac{g^4}{2}\{X^{\dot{1}}, X^{\dot{2}}, X^{\dot{3}}\}^2 + \frac{1}{2g^2} + \frac{1}{2}\varepsilon^{abc}B_a{}^{\dot{a}}\partial_b A_{c\dot{a}} + g\,\det B_a{}^{\dot{a}}\Big). \tag{4.23}$$

One can compare the form (4.23) of the action (and also the complete action of [29] including the scalar and the spinor fields) with the action of the BLG model based on the volume preserving diffeomorphisms constructed in [32]. One can see that the two actions differ only by the fact that in the model of [29] the eight BLG scalars transforming as vectors of an $SO(8)$ R-symmetry are split into 3 + 5 scalars $X^{\dot{a}}$ and $X^{\underline{m}}$ ($\underline{m} = 1, \cdots, 5$), so that $SO(8)$ is broken to $SO(3) \times SO(5)$. The scalar fields $X^{\dot{a}}$ are identified, via Eq. (4.1), with three directions along $\mathcal{N}_3$ and with components $A_{\dot{a}\dot{b}}$ of the chiral gauge field. Note that both of the models are invariant under the volume preserving diffeomorphisms, because the above identification does not change the variation properties of $X^{\dot{a}}$, which remain the scalar fields, as discussed above.

The action (4.22) is invariant under the volume preserving diffeomorphisms but does not have a covariant form due to the fact that its last two (Chern-Simons) terms are not expressed in terms of the field strengths. We shall present a gauge-covariant form of the action of this model in Sec. IV B.

Varying the action (4.22) with respect to the gauge potentials $A_{a\dot{b}}$ and $A_{\dot{a}\dot{b}}$ one gets the covariant equations of motion [29]

$$\mathcal{D}_a\tilde{\mathcal{H}}^{ab\dot{c}} + \mathcal{D}_{\dot{a}}\mathcal{H}^{\dot{a}b\dot{c}} = 0, \tag{4.24}$$

$$\mathcal{D}_a\mathcal{H}^{a\dot{b}\dot{c}} + \mathcal{D}_{\dot{a}}\mathcal{H}^{\dot{a}\dot{b}\dot{c}} = 0, \tag{4.25}$$

In [29] the field strength components $\mathcal{H}_{abc}$ and $\mathcal{H}_{ab\dot{c}}$, which do not show up in the action (4.22) and equations of motion (4.24) and (4.25), were not defined, but it was assumed that they are dual, respectively, to (4.16) and (4.17), so that the whole nonlinear field strength $\mathcal{H}_{\mu\nu\rho}$ is Hodge self-dual

$$\mathcal{H}_{\mu\nu\rho} = \tilde{\mathcal{H}}_{\mu\nu\rho} \Rightarrow \mathcal{H}_{a\dot{b}\dot{c}} = \frac{1}{2}\varepsilon_{abc}\varepsilon_{\dot{b}\dot{c}\dot{a}}\mathcal{H}^{bc\dot{a}},$$
$$\mathcal{H}_{\dot{a}\dot{b}\dot{c}} = -\frac{1}{6}\varepsilon_{\dot{a}\dot{b}\dot{c}}\varepsilon^{abc}\mathcal{H}_{abc}. \tag{4.26}$$

In the next subsection we shall prove this assumption and find the explicit expressions for $\mathcal{H}_{ab\dot{c}}$ and $\mathcal{H}_{abc}$ in the following form

$$\mathcal{H}_{ab\dot{c}} = M_{\dot{c}}^{-1\dot{b}}(F_{ab\dot{b}} + g\varepsilon^{\dot{a}\dot{e}\dot{k}}\varepsilon^{\dot{d}\dot{f}\dot{g}}\varepsilon_{\dot{k}\dot{g}\dot{b}}\partial_{\dot{a}}A_{a\dot{e}}\partial_{\dot{d}}A_{b\dot{f}})$$
$$= M_{\dot{c}}^{-1\dot{d}}(F_{ab\dot{d}} + g\varepsilon_{\dot{d}\dot{a}\dot{b}}B_a{}^{\dot{a}}B_b{}^{\dot{b}}), \tag{4.27}$$

$$\frac{1}{6}\varepsilon^{abc}\mathcal{H}_{abc} = \frac{1}{1+\det M}\Big(\frac{1}{3}\varepsilon^{abc}F_{abc} - \frac{g}{2}\mathcal{H}_{a\dot{b}\dot{c}}\mathcal{H}^{a\dot{b}\dot{c}} - g\varepsilon^{abc}B_a{}^{\dot{b}}F_{bc\dot{b}} - 4g^2\det B_a{}^{\dot{b}}\Big)$$
$$= \frac{1}{2 + \frac{g}{6}\varepsilon^{\dot{a}\dot{b}\dot{c}}\mathcal{H}_{\dot{a}\dot{b}\dot{c}}} \times\Big(\frac{1}{3}\varepsilon^{abc}F_{abc} - \frac{g}{2}\mathcal{H}_{a\dot{b}\dot{c}}\mathcal{H}^{a\dot{b}\dot{c}} - g\varepsilon^{abc}B_a{}^{\dot{b}}F_{bc\dot{b}} - 4g^2\det B_a{}^{\dot{b}}\Big), \tag{4.28}$$

where $F_{ab\dot{b}}$ and $F_{abc}$ are the linear field strengths (3.12) and (3.15), respectively.

By a straightforward calculation one can show that the field strengths (4.27) and (4.28) are covariant and transform as scalars under the local gauge transformations (4.2), (4.3), and (4.4), as their dual counterparts (4.16) and (4.17) do. This is achieved by requiring the following gauge transformations of the potential $A_{ab}$

$$\delta_\Lambda A_{ab} = \partial_a\Lambda_b - \partial_b\Lambda_a + g\xi^{\dot{b}}\partial_{\dot{b}}A_{ab} + g(A_{a\dot{c}}\partial_b\xi^{\dot{c}} - A_{b\dot{c}}\partial_a\xi^{\dot{c}}). \tag{4.29}$$

Note that Eq. (4.27) is the covariant generalization of a "pre-field-strength"

$$G_{ab\dot{c}} = \partial_a A_{b\dot{c}} - \partial_b A_{a\dot{c}} + g\varepsilon_{\dot{c}\dot{a}\dot{b}}B_a{}^{\dot{a}}B_b{}^{\dot{b}} \tag{4.30}$$

introduced in [32]. The addition to (4.30) of the term $\partial_{\dot{c}}A_{ab}$ makes it to transform as a covariant vector under the gauge transformations (4.2), (4.3), (4.4), and (4.29), while multiplication by $M^{-1}$ converts this vector into the gauge scalar $\mathcal{H}_{ab\dot{c}}$.

### A. Solution of the equations of motion and the Bianchi identities

Let us now explain how one gets the field strengths (4.27) and (4.28) and the duality relations (4.26) by solving the field Eqs. (4.24) and (4.25). The derivation is similar to that in the linear case of Sec. IV, but requires more intermediate steps.

We start with Eq. (4.24) and multiply it by $M_{\dot{c}}^{-1\dot{d}}$ to get

$$M_{\dot{c}}^{-1\dot{d}}\mathcal{D}_a\tilde{\mathcal{H}}^{ab\dot{c}} + M_{\dot{c}}^{-1\dot{d}}\mathcal{D}_{\dot{a}}\mathcal{H}^{\dot{a}b\dot{c}} = 0. \tag{4.31}$$

In view of the definition (4.17) of the field strength $\mathcal{H}^{\dot{a}b\dot{c}} = -\mathcal{H}^{b\dot{a}\dot{c}}$ and the identity (4.12), the second term of this equation can be written as a total partial derivative

$$M_{\dot c}^{-1\dot d}\mathcal{D}_{\dot a}\mathcal{H}^{\dot a b\dot c} = \det M \varepsilon^{\dot c\dot a\dot f} M_{\dot c}^{-1\dot d} M_{\dot a}^{-1\dot b}\partial_{\dot b}\mathcal{D}^b X_{\dot f} = \varepsilon^{\dot d\dot b\dot c}\partial_{\dot b}(M_{\dot c}{}^{\dot f}\mathcal{D}^b X_{\dot f}) = \frac{1}{2}\varepsilon^{\dot d\dot b\dot c}\partial_{\dot b}(M_{\dot c}{}^{\dot f}\varepsilon_{\dot f\dot a\dot k}\mathcal{H}^{b\dot a\dot k}). \tag{4.32}$$

The first term of (4.31) can also be presented as a total partial derivative

$$M_{\dot c}^{-1\dot d}\mathcal{D}_a\tilde{\mathcal{H}}^{ab\dot c} = \varepsilon^{bac}M_{\dot c}^{-1\dot d}\mathcal{D}_a\mathcal{D}_c X^{\dot c} = -\varepsilon^{bac}\varepsilon^{\dot a\dot b\dot d}\partial_{\dot a}\Big(\partial_a A_{c\dot b} + \frac{g}{2}\varepsilon_{\dot b\dot c\dot f}B_a^{\dot c}B_c{}^{\dot f}\Big), \tag{4.33}$$

where $B_a^{\dot c}$ is defined in (4.8).

Substituting (4.32) and (4.33) into Eq. (4.31) we get the Bianchi-like equation which, upon taking off the total derivative (in topologically trivial spaces), produces the duality relation

$$\mathcal{H}^{\,b\dot a\dot c} = \frac{1}{2}\varepsilon^{bcd}\varepsilon^{\dot a\dot c\dot b}\mathcal{H}_{cd\dot b} \equiv \tilde{\mathcal{H}}^{b\dot a\dot c}, \tag{4.34}$$

where $\mathcal{H}_{cd\dot b}$ are, by definition, the '$cd\dot b$'-components of the nonlinear gauge field strength given in Eq. (4.27). The components $A_{ab}$ of the gauge potential have appeared in $F_{ab\dot b}$ as a result of the integration of Eq. (4.31). Substituting the above duality relation back into Eq. (4.24) we get the Bianchi identity

$$\mathcal{D}_a\tilde{\mathcal{H}}^{ab\dot c} + \mathcal{D}_{\dot a}\tilde{\mathcal{H}}^{\dot a b\dot c} = 0. \tag{4.35}$$

It is important to observe that the expression (4.27) for $\mathcal{H}_{ab\dot c}$ follows directly from the Bianchi identity (4.35), without any need of the equation of motion (4.24). Indeed, using the identity (4.12) and the explicit form (4.17) of $\mathcal{H}_{a\dot b\dot c}$, the Bianchi identity (4.35) can be rewritten as

$$[\mathcal{D}_a, \mathcal{D}_b]X^{\dot c} = -g^2\{\mathcal{H}_{ab\dot d}, X^{\dot d}, X^{\dot c}\}. \tag{4.36}$$

This expression brings the commutation relation (4.20) to the form similar to that of (4.18) and (4.19). The explicit form of Eq. (4.36) is

$$\varepsilon^{\dot a\dot b\dot c}\partial_{\dot a}(\partial_{[a}A_{b]\dot b} + \varepsilon_{\dot b\dot d\dot f}(B_a{}^{\dot d}B_b{}^{\dot b}))\partial_{\dot c}X^{\dot g} = \varepsilon^{\dot a\dot b\dot c}\partial_{\dot a}(\mathcal{H}_{ab\dot d}\partial_{\dot b}X^{\dot d})\partial_{\dot c}X^{\dot g}, \tag{4.37}$$

which yields (4.27) after integration. Therefore, Eq. (4.27) holds off the mass shell. The equation of motion (4.24) together with the Bianchi (4.35) yields

$$\mathcal{D}_{\dot a}(\tilde{\mathcal{H}}^{\dot a\dot b c} - \mathcal{H}^{\dot a\dot b c}) = 0 \tag{4.38}$$

that implies the self-duality condition (4.34), which was explicitly shown above.

We can now proceed and solve the second field Eq. (4.25). Multiplying it by $M_{\dot a}{}^{\dot d}\varepsilon_{\dot d\dot b\dot c}$ we get

$$M_{\dot a}{}^{\dot d}\varepsilon_{\dot d\dot b\dot c}\mathcal{D}_a\mathcal{H}^{a\dot b\dot c} + 2M_{\dot a}{}^{\dot d}\mathcal{D}_{\dot d}\mathcal{H}_{\dot 1\dot 2\dot 3} = 0. \tag{4.39}$$

Using the definition (4.16) of $\mathcal{H}_{\dot a\dot b\dot c}$ and the identity (4.12), one finds that the second term of this equation is a total derivative

$$2M_{\dot a}{}^{\dot d}\mathcal{D}_{\dot d}\mathcal{H}_{\dot 1\dot 2\dot 3} = \frac{1}{g}\partial_{\dot a}((\det M)^2 - 1), \tag{4.40}$$

where in the rhs we have introduced the unit constant to ensure that the integral of (4.40) does not diverge when $g \to 0$ and $\det M \to 1$.

It now remains to show that also the first term in (4.39) is a total derivative modulo the duality relation (4.34). To this end using Eqs. (4.15) and (4.27) of $\mathcal{H}_{bc\dot d}$ we rewrite this term in the following form

$$M_{\dot a}{}^{\dot d}\varepsilon_{\dot d\dot b\dot c}\mathcal{D}_a\mathcal{H}^{a\dot b\dot c} = \varepsilon^{abc}M_{\dot a}{}^{\dot d}\mathcal{D}_a\mathcal{H}_{bc\dot d} + 2M_{\dot a}{}^{\dot d}\mathcal{D}_a\Big(\mathcal{D}^a X_{\dot d} - \frac{1}{2}\varepsilon^{abc}\mathcal{H}_{bc\dot d}\Big)$$
$$= \varepsilon^{abc}\mathcal{D}_a(F_{bc\dot a} + g\varepsilon_{\dot k\dot g\dot a}B_b{}^{\dot k}B_c{}^{\dot g}) - 2g(\mathcal{D}_a\partial_{\dot a}X^{\dot d})\mathcal{D}^a X_{\dot d} + 2\mathcal{D}_a\Big(M_{\dot a}{}^{\dot d}\Big(\mathcal{D}^a X_{\dot d} - \frac{1}{2}\varepsilon^{abc}\mathcal{H}_{bc\dot d}\Big)\Big). \tag{4.41}$$

Upon some algebra we finally get

$$M_{\dot a}{}^{\dot d}\varepsilon_{\dot d\dot b\dot c}\mathcal{D}_a\mathcal{H}^{a\dot b\dot c} = \partial_{\dot a}\Big(\varepsilon^{abc}\partial_a A_{bc} - \frac{g}{2}\mathcal{H}_{a\dot b\dot c}\mathcal{H}^{a\dot b\dot c} - g\varepsilon^{abc}B_a{}^{\dot b}F_{bc\dot b} - 4g^2\det B_a{}^{\dot b}\Big) + 2\mathcal{D}_a\Big(M_{\dot a}{}^{\dot d}\Big(\mathcal{D}^a X_{\dot d} - \frac{1}{2}\varepsilon^{abc}\mathcal{H}_{bc\dot d}\Big)\Big). \tag{4.42}$$

Notice that the first term is a total derivative and the last term is proportional to the duality relation (4.34). Therefore, when the duality relation (4.34) is satisfied, Eq. (4.39) can be integrated to produce, as in the linear case of Sec. III A, the field strength $\mathcal{H}_{abc}$ given in (4.28), the duality relation

$$\mathcal{H}_{\dot a\dot b\dot c} = -\frac{1}{6}\varepsilon_{\dot a\dot b\dot c}\varepsilon^{abc}\mathcal{H}_{abc} \tag{4.43}$$

and the Bianchi identity

$$\mathcal{D}_a \tilde{\mathcal{H}}^{a\dot{b}\dot{c}} + \mathcal{D}_{\dot{a}} \tilde{\mathcal{H}}^{\dot{a}\dot{b}\dot{c}} = 0. \qquad (4.44)$$

One may ask if it is possible to get the expression (4.28) for $\mathcal{H}_{abc}$ starting from the Bianchi identity (4.44) without the use of equations of motion and, in particular, the duality relation (4.34). Unfortunately, for $\mathcal{H}_{abc}$ defined in (4.28) this seems not to be possible. Indeed, if one starts from the Bianchi relation (4.44), adds to it the null term

$$2\det M \partial_{\dot{a}} \mathcal{H}_{\dot{1}\dot{2}\dot{3}} - \frac{1}{g} \partial_{\dot{a}}((\det M)^2 - 1) = 0,$$

and repeats the previous calculation without taking into account the duality condition (4.34) one gets

$$\partial_{\dot{a}}\Big(\frac{1}{3}\varepsilon^{abc}F_{abc} - \frac{g}{2}\mathcal{H}_{a\dot{b}\dot{c}}\mathcal{H}^{a\dot{b}\dot{c}} - g\varepsilon^{abc}B_a{}^{\dot{b}}F_{bc\dot{b}} - 4g^2\det B_a{}^{\dot{b}} + \frac{1}{g}(\overset{2}{\det} M - 1)\Big) + 2\mathcal{D}_a\Big(M_{\dot{a}}{}^{\dot{d}}\Big(\mathcal{D}^a X_{\dot{d}} - \frac{1}{2}\varepsilon^{abc}\mathcal{H}_{bc\dot{d}}\Big)\Big) - \frac{1}{3}\det M \partial_{\dot{a}}(\varepsilon^{abc}\mathcal{H}_{abc} + \varepsilon^{\dot{a}\dot{b}\dot{c}}\mathcal{H}_{\dot{a}\dot{b}\dot{c}}) = 0, \qquad (4.45)$$

which is satisfied only if one uses the duality relations (4.34) and (4.43). Thus we have encountered a peculiar feature of the model under consideration that if the nonlinear $\mathcal{H}_{abc}$ has the form (4.28), the Bianchi relation (4.44) is only satisfied on the mass shell.

### B. Gauge-covariant action

The knowledge of the explicit form (4.28) of $\mathcal{H}_{abc}$ allows us to rewrite the action (4.22) in the equivalent (modulo total derivatives) but gauge-covariant form

$$S = -\int d^6x\Big(\frac{1}{8}\mathcal{H}_{a\dot{b}\dot{c}}\mathcal{H}^{a\dot{b}\dot{c}} + \frac{1}{12}\mathcal{H}_{\dot{a}\dot{b}\dot{c}}\mathcal{H}^{\dot{a}\dot{b}\dot{c}} - \frac{1}{144}\varepsilon^{abc}\mathcal{H}_{abc}\mathcal{H}_{\dot{a}\dot{b}\dot{c}}\varepsilon^{\dot{a}\dot{b}\dot{c}} - \frac{1}{12g}\varepsilon^{abc}\mathcal{H}_{abc}\Big). \qquad (4.46)$$

Note that, as one can check directly, the potential $A_{ab}$ enters the action (4.46) only under a total derivative and hence can be dropped out modulo boundary terms. This means that, as in the case of its free-field limit considered in Sec. III, the action (4.46) is invariant under the local symmetry (3.18).

Note also that the last term in (4.46) is of a Chern-Simons type and can be interpreted as a coupling of the 5-brane to the constant background field $C_3$ which has the nonzero components $C_{\dot{a}\dot{b}\dot{c}} = \frac{1}{g}\varepsilon_{\dot{a}\dot{b}\dot{c}}$ along the $x^{\dot{a}}$-directions of the 5-brane. It can thus be rewritten in the Chern-Simons form similar to that of the M5-brane action (see Eq. (5.3) below)

$$\int d^6x \frac{1}{12g}\varepsilon^{abc}\mathcal{H}_{abc} = \frac{1}{2}\int \mathcal{H}_3 \wedge C_3,$$

where the field strength $\mathcal{H}_3$ and $C_3$ are regarded as $D = 6$ three-forms. The presence of the constant background field $C_3$ explicitly breaks the $D = 6$ Lorentz invariance. It is not obvious that the action (4.46) can be invariant under a modified Lorentz symmetry similar to (3.23) of the free-field case. This issue requires additional study.

In the next section we shall briefly discuss a possibility of the construction of an alternative nonlinear generalization of the chiral-field action (3.29) which may possess (nonmanifest) Lorentz invariance and describe an M5-brane in a generic $D = 11$ background.

This completes our consideration of the nonlinear chiral-field model. We have proved that the general solution of the nonlinear Eqs. (4.24) and (4.25) is amount to the Hodge self-duality of the nonlinear field strength $\mathcal{H}_{\mu\nu\lambda}$. Thus the number of independent gauge field degrees of freedom of the nonlinear model is the same as in the linear case, *i.e.* equals to three, as was assumed in [29]. The knowledge of the explicit form of the field strengths $\mathcal{H}_{abc}$ and $\mathcal{H}_{ab\dot{c}}$ has also allowed us to fined the form (4.46) of the nonlinear action (4.22) whose Lagrangian is covariant under the volume preserving diffeomorphisms. We leave for a future the analysis of the nonlinear model in the presence of the scalar and spinor matter fields.

## V. ON THE RELATION TO THE M5-BRANE

Let us now briefly discuss the relation of the model of [28,29] to the known formulations of the M5-brane. In [29] it was shown that by performing a double dimensional reduction, the BLG model with the gauge group of 3$d$ volume preserving diffeomorphisms reduces to a five-dimensional noncommutative $U(1)$ gauge theory with a small noncommutativity parameter which can be interpreted as an effective worldvolume theory of a D4-brane in a background with a strong NS-NS gauge field $B_2$. The symmetries and the fields of these two theories are known to be related to each other by the Seiberg-Witten map [50]. Thus, the authors of [29] assumed that the BLG model with the Nambu-Poisson algebra structure and a week coupling constant can be related to an M5-brane theory in a $D = 11$ background with a constant gauge field $C_3$ (in a strong $C_3$ value limit) and proposed a Seiberg-Witten map relating the two theories.

M5-branes in a constant $C_3$ field with M2-branes ending on M5 and corresponding noncommutative (quantum) structures have been considered, e.g. in [51–53] using the formulation of [39,41] and extending the results of [54] on a self-dual string soliton on M5. From the perspective of multiple M2-branes the M5-brane in a constant $C_3$ field was studied in [35] making use of a $C$-field modified Basu-Harvey equation [55]. Recently, in [56] these M2-M5 brane systems and corresponding BPS string solutions on the M5-brane worldvolume have been studied in the framework of the model of [28,29] in the linear order of the coupling constant $g$ and an agreement with previous results have been found via the Sieberg-Witten map.

As we have seen in Sec. III F, at the quadratic order the alternative actions for the chiral field differ in a term quadratic in anti-self-dual components of the gauge field strength, so to study the relation between [28,29] and the conventional formulations of the M5-branes in more detail it should be useful to have a Lagrangian formulation of the M5-brane dynamics in which the components of the field strength of the chiral gauge field are naturally split into the $SO(1,2)\times SO(3)$ way, as has been considered in the previous sections. Let us briefly discuss how one might construct such a formulation.

In the known Lagrangian formulation of the M5-brane, the six-dimensional indices of the chiral-field strength are subject to the $1+5$ splitting (as has been explained in Sec. III). Then the self-duality condition (2.13) or its Lorentz-covariant counterpart (2.22) gets generalized to a nonlinear relation between the components of the chiral-field strength $F_{\mu\nu\rho}$ and its dual $\tilde{F}_{\mu\nu\rho}$ [36,37]. In the covariant formalism [36] the nonlinear self-duality condition has the following Born-Infeld-like form

$$\frac{1}{\sqrt{-(\partial a)^2}}H_{\mu\nu\rho}\partial^\rho a=\frac{1}{\sqrt{-\det g}}\frac{\delta\sqrt{\det(g_{\rho\sigma}+i\tilde{H}_{\rho\sigma})}}{\delta\tilde{H}_{\mu\nu}} \tag{5.1}$$

where $g_{\mu\nu}$ is an induced metric on the worldvolume of the M5-brane, $H_{\mu\nu\rho}\equiv(F+C)_{\mu\nu\rho}$ is the field strength of the M5-brane worldvolume chiral gauge field $A_{\mu\nu}$ extended with the worldvolume pullback of the antisymmetric gauge field $C_3$ of $D=11$ supergravity and

$$\tilde{H}_{\mu\nu}=\frac{1}{6\sqrt{-(\partial a)^2}}\partial^\rho a\varepsilon_{\rho\mu\nu\lambda\sigma\tau}H^{\lambda\sigma\tau}. \tag{5.2}$$

Equation (5.1) follows from the Dirac-Born-Infeld-like M5-brane action

$$\begin{aligned}S_{\mathcal{M}_6}=&\int_{\mathcal{M}_6}d^6x\Big[-\sqrt{-\det(g_{\mu\nu}+i\tilde{H}_{\mu\nu})}\\&-\frac{\sqrt{-g}}{4(\partial a)^2}\partial_\lambda a\tilde{H}^{\lambda\mu\nu}H_{\mu\nu\rho}\partial^\rho a\Big]\\&-\frac{1}{2}\int_{\mathcal{M}_6}[C_6+H_3\wedge C_3],\end{aligned} \tag{5.3}$$

where $C_6$ is the dual of the gauge potential $C_3$. It is important to notice that the dual field strength $\tilde{H}_{\mu\nu}$ (5.2) which enters the Born-Infeld part of the action (5.3) and the rhs of the self-duality condition (5.1) is invariant under the gauge transformations (2.17).

An alternative Lorentz-covariant nonlinear self-duality condition [which does not involve the auxiliary scalar field $a(x)$] was obtained from the superembedding description of the M5-brane [39,40] which was the first to produce the complete set of the M5-brane equations of motion [39].[4] The superembedding self-duality condition is formulated in terms of a conventional Hodge self-dual rank-3 field $h_{\mu\nu\rho}=\tilde{h}_{\mu\nu\rho}$ which is related to the field strength $H_{\mu\nu\rho}=(F+C)_{\mu\nu\rho}$ by the following nonlinear algebraic equation

$$\begin{gathered}(F+C)_{\mu\nu\rho}=(m^{-1})_\mu{}^\lambda h_{\lambda\nu\rho},\qquad h_{\mu\nu\rho}=\tilde{h}_{\mu\nu\rho},\\F_{\mu\nu\rho}=\partial_\mu A_{\nu\rho}+\partial_\nu A_{\rho\mu}+\partial_\rho A_{\mu\nu},\end{gathered} \tag{5.4}$$

where

$$m_\mu{}^\lambda=\delta_\mu{}^\lambda-2h_{\mu\sigma\nu}h^{\sigma\nu\lambda}. \tag{5.5}$$

In [41] it was shown that the nonlinear self-duality condition that follows from the superembedding is equivalent to the self-duality condition (5.1) resulting from the M5-brane action [more precisely, to its noncovariant counterpart when the field $a(x)$ is gauge fixed as in (2.19)]. The relation and the equivalence of the whole systems of the M5-brane equations of motion which follow from the two alternative formulations was established in [42].

Yet another derivation of the nonlinear self-duality condition based on its consistency with the M5-brane kappa-symmetry was given in [43]. This derivation is in a sense close to the one which follows from the superembedding formulation since from the point of view of the superembedding the kappa-symmetry is just a peculiar realization of a conventional local supersymmetry on the worldvolume of the branes (see [60] for a review).

The evidence that the two *a priori* different approaches, the on-shell superembedding formulation [39] (or its kappa-symmetric counterpart [43]) and the action principle of [36,37], give the equivalent interrelated descriptions of the classical dynamics of the M5-brane, points to its uniqueness and, hence, allows one to assume that any alternative formulation of the M5-brane dynamics should be related to those described above.

In particular, an appropriate nonlinear generalization of the self-duality conditions (3.10) and (3.14) [which would be alternative to (5.1)] should be related to the Lorentz-covariant superembedding self-duality condition (5.4). One can try to derive a nonlinear self-duality relation generalizing Eqs. (3.10) and (3.14) from Eq. (5.4) by performing the $(3+3)$ splitting of the six-dimensional indices of $H_{\mu\nu\rho}$ and $h_{\mu\nu\rho}$ and following the lines of

[4] Other cases in which the superembedding condition results in the dynamical equations of motion include the Type II $D=10$ superstrings and the $D=11$ M2-brane [57], and $D$-branes [58,59].

Ref. [41]. The goal is to get these conditions in the following generic form [whose rhs is invariant under the gauge transformations (3.18) or (3.31)]

$$H_{abc} = f_{abc}(\tilde{H}, \tilde{H}_{d\dot{b}}), \qquad H_{ab\dot{c}} = g_{ab\dot{c}}(\tilde{H}, \tilde{H}_{d\dot{b}}), \quad (5.6)$$

where $f_{abc}(\tilde{H}, \tilde{H}_{d\dot{b}})$ and $g_{ab\dot{c}}(\tilde{H}, \tilde{H}_{d\dot{b}})$ are tensorial functions of

$$\tilde{H} \equiv \frac{1}{6}\varepsilon^{\dot{a}\dot{b}\dot{c}}(F+C)_{\dot{a}\dot{b}\dot{c}}, \qquad \tilde{H}^{d}_{\dot{b}} \equiv \frac{1}{2}\varepsilon_{\dot{b}\dot{c}\dot{d}}(F+C)^{d\dot{c}\dot{d}}. \quad (5.7)$$

Once the explicit form of (5.6) is known, one can use it to construct an M5-brane action in a form alternative to (5.3). Such an action should be invariant under local symmetries generalizing (3.18) and (3.23) [or (3.31) and (3.38)] and should produce the nonlinear self-duality conditions (5.6). Having at hand this alternative formulation of the M5-brane dynamics one can analyze its relation to the model of [29] in a limit of a strong constant $C_3$ field. Note that one cannot directly relate the nonlinear self-dual field strength $h_{\mu\nu\lambda}$ to the self-dual field strength $\mathcal{H}_{\mu\nu\lambda}$ of the previous section, since the former is invariant under the conventional gauge transformations (2.2), while the latter is invariant under the gauge transformations (4.2), (4.3), (4.4), and (4.29) which include the volume preserving diffeomorphisms. Therefore, the gauge field potentials and the field strengths in the two formulations may only coincide at the free-field level when the coupling constant $g$ is set to zero. In the generic case the relation is not straightforward and can probably be established via a kind of the Seiberg-Witten map proposed in [29] or by generalizing results of [33]. We leave the study of these problems for a future research.

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## APPENDIX

Let $\mu, \nu = 0, \ldots, 5$ be $SO(1,5)$ Lorentz indices while $A, B = 1, 2, 3, 4$ be the corresponding spinor indices. The matrices $(\Gamma_\mu)_{AB}$ and $(\Gamma_\mu)^{AB}$ satisfy the Weyl algebra

$$(\Gamma_\mu)_{AB}(\Gamma_\nu)^{BC} + (\Gamma_\nu)_{AB}(\Gamma_\mu)^{BC} = -2\eta_{\mu\nu}\delta_A^C \quad (A1)$$

and are related to each other as follows

$$(\Gamma_\mu)_{AB} = \frac{1}{2}\varepsilon_{ABCD}(\Gamma_\mu)^{CD}, \qquad (\Gamma_\mu)^{AB} = \frac{1}{2}\varepsilon^{ABCD}(\Gamma_\mu)_{CD}, \quad (A2)$$

where $\varepsilon_{1234} = \varepsilon^{1234} = 1$.

The $\Gamma$-matrices satisfy the following identities

$$(\Gamma^\mu)_{AB}(\Gamma_\mu)^{CD} = 2(\delta_A^C\delta_B^D - \delta_A^D\delta_B^C),$$
$$(\Gamma_\mu)_{AB}(\Gamma_\nu)^{AB} = 4\eta_{\mu\nu}, \qquad (\Gamma_\mu)_{AB}(\Gamma^\mu)_{CD} = 2\varepsilon_{ABCD}. \quad (A3)$$

We define the antisymmetrized products of gamma-matrices as

$$(\Gamma_{\mu\nu})_A{}^B = \frac{1}{2}[(\Gamma_\mu)_{AC}(\Gamma_\nu)^{CB} - (\Gamma_\nu)_{AC}(\Gamma_\mu)^{CB}] = (\Gamma_{[\mu})_{AC}(\Gamma_{\nu]})^{CB},$$
$$(\Gamma_{\mu\nu\rho})_{AB} = (\Gamma_{[\mu})_{AC}(\Gamma_\nu)^{CD}(\Gamma_{\rho]})_{DB},$$
$$(\Gamma_{\mu\nu\rho\sigma})_A{}^B = (\Gamma_{[\mu})_{AC}(\Gamma_\nu)^{CD}(\Gamma_\rho)_{DE}(\Gamma_{\sigma]})^{EB}, \qquad \text{etc.} \quad (A4)$$

There is the following duality relation for these matrices,

$$\Gamma_{\mu_1\ldots\mu_k} = -(-1)^{k(k-1)/2}\frac{1}{(6-k)!}\varepsilon_{\mu_1\ldots\mu_6}\Gamma^{\mu_{k+1}\ldots\mu_6}. \quad (A5)$$

In particular,

$$\Gamma_{\alpha\beta} = \frac{1}{4!}\varepsilon_{\alpha\beta\mu\nu\rho\sigma}\Gamma^{\mu\nu\rho\sigma},$$
$$\Gamma_{\mu\nu\rho\sigma} = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma\alpha\beta}\Gamma^{\alpha\beta}, \quad (A6)$$
$$\Gamma_{\mu\nu\rho} = \frac{1}{6}\varepsilon_{\mu\nu\rho\alpha\beta\gamma}\Gamma^{\alpha\beta\gamma}.$$

One can prove the following identity

$$\Gamma_{\mu\nu\rho}\Gamma_\sigma = -(\eta_{\sigma\mu}\Gamma_{\nu\rho} + \eta_{\sigma\nu}\Gamma_{\rho\mu} + \eta_{\sigma\rho}\Gamma_{\mu\nu}) - \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma\alpha\beta}\Gamma^{\alpha\beta}. \quad (A7)$$

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