BPS operators from the Wilson loop in the 3-dimensional supersymmetric Chern-Simons theory

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(Received 20 July 2009; published 2 October 2009)

We consider the small deformation of the pointlike Wilson loop in the 3-dimensional $\mathcal{N}=6$ superconformal Chern-Simons theory. By Taylor expansion of the pointlike Wilson loop in powers of the loop variables, we obtain the BPS operators that correspond to the excited string states of the dual IIA string theory on the pp wave background. The BPS conditions of the Wilson loop constrain both the loop variables and the forms of the operators obtained in the Taylor expansion.

DOI: 10.1103/PhysRevD.80.086001 PACS numbers: 11.25.Tq, 11.30.Pb

I. INTRODUCTION

The dual gravity interpretation of the supersymmetric Wilson loop in the $D=4$ $\mathcal{N}=4$ super Yang-Mills (SYM) theory has been important in the context of the AdS/CFT correspondence [1]. The supersymmetric Wilson loop contains the 6 scalar fields Φ^i, and the expectation value of the Wilson loop is protected from the UV divergence [2]. The expectation value of the circular Wilson loop was obtained by analyzing the string minimal surface in the anti–de Sitter space [3]. Furthermore, the supersymmetric Wilson loop in the Berenstein-Maldacena-Nastase (BMN) sector can be described by the dual IIB string theory on the pp wave background. The BMN correspondence [4] is the duality between the infinite strings of operators in $\mathcal{N}=4$ SYM and the excited string states in the dual IIB string theory on the pp wave background. In Ref. [5], it was conjectured that the 1/2-Bogomol'nyi-Prasad-Sommerfield (BPS) pointlike Wilson loop $W(C_0)$ for C_0 shrinking to a spacetime point is mapped to the vacuum state of the dual IIB string theory on the pp wave background. The functional derivatives of the Wilson loop are mapped to the excited string states. We have summarized the results obtained in Ref. [5] in Table I.

In this paper, motivated by the work in Ref. [5], we consider the dual IIA string theory description of the supersymmetric Wilson loop in the recently proposed $D=3$ $\mathcal{N}=6$ Chern-Simons-matter theory [Aharony-Bergman-Jafferis-Maldacena (ABJM) theory] [6,7]. The ABJM theory is the low-energy effective theory of the N M2-branes at the singularity of the orbifold $\mathbb{C}^4/\mathbb{Z}_k$.[1] This theory can be analyzed by using the dual IIA string theory on the $\mathrm{AdS}_4\times CP^3$ spacetime and on its Penrose limit [8–12] in the parameter regime

$$\sqrt{\lambda}\gg 1, \qquad e^{2\phi}\sim\frac{N^{1/2}}{k^{5/2}}\ll 1, \tag{1.1}$$

where k is the Chern-Simons level, N is the rank of the gauge group, $\lambda=N/k$ is 't Hooft coupling, and ϕ is the dilaton. The first one in (1.1) implies that, in the dual type IIA string theory, the radius of curvature is much larger than 1 in the string unit, and the second one implies that we take the small string coupling limit, to suppress the quantum corrections.

The supersymmetric Wilson loop in the ABJM theory was proposed in the literature [13–15]. The Wilson loop contains a product of the bifundamental scalars on the exponent. It was shown that the straight line and circular Wilson loops preserve 1/6 of the ABJM supersymmetry.

The main purpose of our paper is to study the dual IIA string theory description of the pointlike Wilson loop that has enhanced 1/3 supersymmetry. We show that by deforming the pointlike Wilson loop, we can obtain the BPS operators that correspond to the excited string states of the dual IIA string theory on the pp wave background.[2] The BPS conditions of the Wilson loop, (3.10) and (3.11), give the constraint on both the loop variables and the forms of the BPS operators. By following the conjecture in Ref. [5], we give maps from the functional derivatives of the Wilson loop to the dual IIA string excited states.

The Penrose limit of the dual gravity theory is given by the following limit:

$$N, J\to\infty \quad \text{with} \quad \lambda'=\frac{\lambda}{J^2} \text{ fixed}, \tag{1.2}$$

where J is the charge of the infinite strings of operators under the $U(1)$ subgroup of $SU(4)$ R symmetry. By determining the function $h(\lambda)$ that appears in the dispersion relation, the gauge/gravity correspondence has been proved up to the curvature corrections to the pp wave background [17].

The content of this paper is as follows: In Sec. II, we consider the pointlike Wilson loop in the ABJM theory and compare it with the vacuum state of the dual IIA superstring theory. In Sec. III, we obtain the BPS conditions for the Wilson loop. We show that the pointlike Wilson loop is 1/3 BPS and the supersymmetry generator preserved by the pointlike Wilson loop is the same as that preserved by the infinite chain dual to the IIA string vacuum state. In

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[1] In the special case of the gauge group $SU(2)\times SU(2)$, the ABJM theory has $SO(8)$ enhanced global R symmetry.

[2] See [16] on the deformation of the Wilson loop operator in the $\mathcal{N}=4$ SYM as well as in the YM theory.

TABLE I. The relation between the functional derivatives and the IIB string oscillation modes $\alpha^{\mu}_{(n)}$ and $\alpha^{4+a}_{(n)}$. In the table, $\mu = 0, 1, 2, 3$, $a = 1, 2, 3, 4$, and $Z = \Phi^5 + i\Phi^6$.

SYM side Functional derivatives	Corresponding operators	Dual IIB string side
$\cdots$	$W(C_0)$	$\|0; p^+\rangle$
$\int ds e^{2\pi i n s/t} \frac{\delta}{\delta x^{\mu}(s)}$	$D_{\mu} Z$	$\alpha^{\mu}_{(n)}$
$\int ds e^{2\pi i n s/t} \frac{\delta}{\delta y^{a}(s)}$	Φ^a	$\alpha^{4+a}_{(n)}$

Sec. IV, we solve the BPS equations satisfied by the loop variables and expand the Wilson loop in powers of the independent loop variables. Thus, we obtain the maps from the functional derivatives of the Wilson loop to the dual IIA string excited states.

II. THE SUPERSYMMETRIC WILSON LOOP IN THE ABJM THEORY

The ABJM theory is the 3-dimensional $\mathcal{N} = 6$ supersymmetric Chern-Simons theory with the gauge group $U(N) \times U(N)$. The fields in the ABJM theory are the $U(N) \times U(N)$ gauge fields A_m and $\hat{A}_m$, the bifundamental bosonic fields Y^I [$Y^I = (A_1, A_2, \bar{B}_1, \bar{B}_2)$] and the bifundamental spinors $\psi_{I\alpha}$, where I ($I = 1, \ldots, 4$) is the index of $SU_R(4)$ R symmetry and α ($\alpha = 1, 2$) is the $(2 + 1)$-dimensional spinor index.

According to Ref. [13], the supersymmetric Wilson loop in the ABJM theory becomes[3]

$$W[C] = \mathrm{Tr}\left[P \exp i \oint_C ds (\dot{x}^m(s) A_m + M_I^J(s) Y^I Y_J^{\dagger}) \right], \quad (2.1)$$

where $x^m(s)$ describes the path C on $R^{1,2}$ and the function $M_I{}^J(s)$, determined by the supersymmetry (SUSY), will be the coordinate of the transverse space $\mathbb{C}^4/\mathbb{Z}_k$. We assume that $M_I{}^J(s)$ is a 4×4 real matrix.[4]

[3]In Ref. [14], they also obtain the Wilson loop with the gauge field $\hat{A}_m$. We do not consider the Wilson loop with $\hat{A}_m$ here; our Wilson loop breaks the parity symmetry of the ABJM theory (see [11,18]).

[4]We can give $M_I{}^J(s)$ a $U_R(1)$ charge. $U_R(1)$ is a subgroup of $SU_R(4)$ R symmetry. In the Wilson loop (2.1), the $U_R(1)$ symmetry that rotates A_1 and B_1 by $\alpha = \exp(i\varphi/2)$ also operates on $M_I{}^J(s)$ as follows:

$$V^{-1} M V = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \bar{\alpha} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \begin{pmatrix} \bar{\alpha} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} m_{11} & \alpha m_{12} & \alpha^2 m_{13} & \alpha m_{14} \\ \bar{\alpha} m_{21} & m_{22} & \alpha m_{23} & m_{24} \\ \bar{\alpha}^2 m_{31} & \bar{\alpha} m_{32} & m_{33} & \bar{\alpha} m_{34} \\ \bar{\alpha} m_{41} & m_{42} & \alpha m_{43} & m_{44} \end{pmatrix}.$$

In Sec. IV, we use this $U_R(1)$ symmetry to classify the loop variables.

We consider the pointlike Wilson loop whose path C_0 shrinks to the point $x^m = x_0^m$ ($\dot{x}^m = 0$). We set $M^I{}_J$ in a nilpotent matrix,

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (2.2)$$

By expanding the exponential part of the Wilson loop, we obtain the infinite sum of the local operator as follows:

$$W[C_0] = \mathrm{Tr}[\exp(itA_1 B_1(x_0))]$$
$$= \sum_{J=0}^{\infty} \frac{(it)^J}{J!} \mathrm{Tr}[(A_1 B_1)^J](x_0), \quad (2.3)$$

where t describes the periodicity of the loop; we identify $s = 0$ with $s = t$. In higher order of J, Eq. (2.2) includes the infinite strings of the operator $A_1 B_1$ that correspond to the vacuum state of the dual IIA string theory. In Sec. IV, by deforming (2.3), we also obtain the BPS operators that correspond to the dual IIA string excited states.

In next section, we analyze the SUSY preserved by the Wilson loop (2.1) and (2.3).

III. SUPERSYMMETRY

In Sec. III A, we shortly review the SUSY transformation of the supersymmetric Wilson loop in the ABJM theory. In Sec. III B, we derive the BPS conditions satisfied by the supersymmetric Wilson loop. In Sec. III C, we show that the pointlike Wilson loop (2.3) is 1/3 BPS.

A. Supersymmetry transformation of the Wilson loop

The $\mathcal{N} = 6$ SUSY generator described by the superspace coordinate is ω_{IJ},[5] which transforms as the antisymmetric representation of $SU_R(4)$ and satisfies the following relations:

$$(\omega_{IJ,\alpha})^* = \omega_{\alpha}^{IJ}, \qquad \omega_{\alpha}^{IJ} = \tfrac{1}{2} \epsilon^{IJKL} \omega_{KL,\alpha}, \quad (3.1)$$

$$\omega^{IK} \omega_{KJ} = \delta^I{}_J \epsilon^i \epsilon_i, \quad (3.2)$$

where ϵ^i ($i = 1, \ldots, 6$) are Majorana spinors, which are also the $\mathcal{N} = 6$ SUSY generator. The $\mathcal{N} = 6$ SUSY transformations are given by

$$\delta Y^I = i \omega^{IJ} \psi_J, \quad (3.3)$$

$$\delta Y_I^{\dagger} = i \psi^{\dagger J} \omega_{IJ}, \quad (3.4)$$

$$\delta A_m = -(Y^I \psi^{J\dagger} \gamma_m \omega_{IJ} + \omega^{IJ} \gamma_m \psi_I Y_J^{\dagger}), \quad (3.5)$$

$$\delta \hat{A}_m = \psi^{I\dagger} Y^J \gamma_m \omega_{IJ} + \omega^{IJ} \gamma_m Y_I^{\dagger} \psi_J, \quad (3.6)$$

[5]ω_{IJ} is obtained by using the Clebsch-Gordan decomposition of the 6 Majorana spinors ϵ_i ($i = 1, \ldots, 6$), which are also the $\mathcal{N} = 6$ SUSY generators (see Appendix B).

where the convention of the spinors is the same as given in Refs. [7,19].

We consider the SUSY transformation of the Wilson loop as follows:

$$e^{i(\omega_{IJ}Q^{IJ}+\omega^{IJ}\bar{Q}_{IJ})}W[C]e^{-i(\omega_{IJ}Q^{IJ}+\omega^{IJ}\bar{Q}_{IJ})} = W[C] + \delta W[C], \tag{3.7}$$

where Q^{IJ} is the SUSY generator. The Wilson loop preserves a part of the SUSY in the ABJM theory when $\delta W[C] = 0$ for an arbitrary s.

By using the condition $\delta W[C] = 0$, we can show that the SUSY generator ω_{AB} preserved by the Wilson loop satisfies the following equations as given in Ref. [13]:

$$\omega^{\alpha}_{AB}\gamma_{m\alpha\beta}\dot{x}^m - iM_B{}^K\omega_{KA,\beta} = 0, \tag{3.8}$$

$$\omega^{AB,\alpha}\gamma_{m\alpha\beta}\dot{x}^m - iM_I{}^A\omega^{IB}_{\beta} = 0. \tag{3.9}$$

Note that the complex conjugate of (3.8) gives (3.9) when $M_I{}^J$ is a Hermitian matrix.

B. BPS conditions

When we contract (3.8) and (3.9) by using $\epsilon_{\alpha\beta}$ and the charge conjugation matrix $\hat{C}$ (see Appendix A) or when we multiply (3.8) by $\gamma_n\dot{x}^n - iM^T$ from the right,[6] we obtain the following BPS conditions:

$$4\dot{x}^2(s) + M_I{}^K(s)M_K{}^I(s) = 0, \tag{3.10}$$

$$\det(\dot{x}^2(s) + M^T(s)M^T(s)) = \det(\dot{x}^2(s) + M(s)M(s)) = 0. \tag{3.11}$$

(3.10) and (3.11) are the necessary condition to preserve a part of SUSY: The straight line and (2.3) satisfy (3.10). Equation (3.10) is similar to the BPS conditions $\dot{x}^2 + \dot{y}^2 = 0$ satisfied by the Wilson loop in the $d = 4$, $\mathcal{N} = 4$ SYM.

C. Supersymmetry preserved by the pointlike Wilson loop

We can show easily that the pointlike Wilson loop (2.3) preserves 1/3 of the SUSY in the ABJM theory. For $\dot{x}_m = 0$, Eqs. (3.8) and (3.9) are given by

$$M_B{}^K\omega_{KA,\beta} = 0, \tag{3.12}$$

$$M_I{}^B\omega^{IA}_{\beta} = 0. \tag{3.13}$$

By solving these equations, we obtain $\omega_{13} = \omega_{23} = \omega_{24} = \omega_{34} = 0$.[7] The pointlike Wilson loop preserves the SUSY ω_{12} and ω_{14} which are not constrained by (3.12) and (3.13). Note that the Wilson loop over the straight path is 1/6 BPS. We guess that there is an enhancement of the SUSY when we shrink the loop to a point.

We want to explain why the Wilson loop in the ABJM theory is 1/3 BPS or 1/6 BPS instead of 1/2 BPS. In the dual IIA string theory side, the Wilson loop is described by the fundamental string on $AdS_4 \times CP^3$ spacetimes. From the supersymmetry analysis [14] of the Killing spinors, it has been known that the fundamental string dual to the straight Wilson loop is not localized at CP^3 but smeared along CP^3: The smeared string preserves less SUSY. So, we guess that a similar phenomenon happens for our pointlike Wilson loop.

IV. BPS OPERATORS FROM THE WILSON LOOP

In this section, we show that the BPS operators arise in the double series expansion of the pointlike Wilson loop operator (2.3) in powers of the loop variables $\delta x^m(s)$ and $\delta M_I{}^J(s) = m_{IJ}(s)$ (see also [5]).

First, we consider the Wilson loop fluctuated near the point $x^m(s) = x^m_0$. We parametrize $M_I{}^J(s)$ by

$$M_I{}^J(s) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} m_{11} & m_{12} & 0 & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix}, \tag{4.1}$$

where we fix the gauge freedom of the parameter s by imposing $M_1{}^3 = 1$, and for convenience, omit the label s in $m_{IJ}(s)$. Since the loop coordinates $\delta x^m(s)$ and $m_{IJ}(s)$ should be periodic about s, they can be rewritten as follows:

$$\begin{aligned} \delta x^m(s) &= \sum_{n=-\infty}^{\infty} \delta x^m_{(n)} e^{2\pi i n s/t}, \\ m_{IJ}(s) &= \sum_{n=-\infty}^{\infty} m_{(n)IJ} e^{2\pi i n s/t} \quad \text{for } (I,J) \neq (3,1). \end{aligned} \tag{4.2}$$

In the Fourier space, the loop variables have zero modes.

The loop variables $\delta x^m(s)$ and $\delta M_I{}^J(s)$ are not independent but are related by the BPS condition (3.10) and (3.11). By substituting (4.1) into the matrix $X_{ab} = (\delta\dot{x}^2 + M^2)_{ab}$ and the BPS condition (3.10) and (3.11), we obtain the following matrix elements and the following equation:

[6] When we multiply (3.9) by $\gamma_n\dot{x}^n - iM$ from the right, we obtain the same equation.

[7] The superconformal group satisfied by (2.3) is $SU(2|2)$ generated by ω_{12} and ω_{14} as given in Ref. [9].

$$
\begin{aligned}
&X_{11} = \delta\dot{x}^2 + {m_{11}}^2 + m_{12}m_{21} + m_{31} + m_{14}m_{41}, \qquad X_{12} = m_{11}m_{12} + m_{12}m_{22} + m_{32} + m_{14}m_{42},\\
&X_{13} = m_{11} + m_{12}m_{23} + m_{33} + m_{14}m_{43}, \qquad X_{14} = m_{11}m_{14} + m_{12}m_{24} + m_{34} + m_{14}m_{44},\\
&X_{21} = m_{21}m_{11} + m_{22}m_{21} + m_{23}m_{31} + m_{24}m_{41}, \qquad X_{22} = \delta\dot{x}^2 + m_{12}m_{21} + {m_{22}}^2 + m_{23}m_{32} + m_{24}m_{42},\\
&X_{23} = m_{21} + m_{22}m_{23} + m_{23}m_{33} + m_{24}m_{43}, \qquad X_{24} = m_{21}m_{14} + m_{22}m_{24} + m_{23}m_{34} + m_{24}m_{44},\\
&X_{31} = m_{31}m_{11} + m_{32}m_{21} + m_{33}m_{31} + m_{34}m_{41}, \qquad X_{32} = m_{31}m_{12} + m_{32}m_{22} + m_{33}m_{32} + m_{34}m_{42},\\
&X_{33} = \delta\dot{x}^2 + m_{31} + m_{23}m_{32} + {m_{33}}^2 + m_{34}m_{43}, \qquad X_{34} = m_{31}m_{14} + m_{32}m_{24} + m_{33}m_{34} + m_{34}m_{44},\\
&X_{41} = m_{41}m_{11} + m_{42}m_{21} + m_{43}m_{31} + m_{44}m_{41}, \qquad X_{42} = m_{41}m_{12} + m_{42}m_{22} + m_{43}m_{32} + m_{44}m_{42},\\
&X_{43} = m_{41} + m_{42}m_{23} + m_{43}m_{33} + m_{44}m_{43}, \qquad X_{44} = \delta\dot{x}^2 + m_{14}m_{41} + m_{24}m_{42} + m_{34}m_{43} + {m_{44}}^2,
\end{aligned}
\tag{4.3}
$$

$$
\begin{aligned}
\det X = {} & X_{11}X_{22}X_{33}X_{44} - X_{11}X_{22}X_{34}X_{43} + X_{11}X_{32}X_{43}X_{24} - X_{11}X_{32}X_{23}X_{44} + X_{11}X_{42}X_{23}X_{34} - X_{11}X_{42}X_{33}X_{24}\\
& - X_{21}X_{12}X_{33}X_{44} + X_{21}X_{12}X_{34}X_{43} - X_{21}X_{32}X_{43}X_{14} + X_{21}X_{32}X_{13}X_{44} - X_{21}X_{42}X_{13}X_{34} + X_{21}X_{42}X_{33}X_{14}\\
& + X_{31}X_{12}X_{23}X_{44} - X_{31}X_{12}X_{43}X_{24} + X_{31}X_{22}X_{43}X_{14} - X_{31}X_{22}X_{13}X_{44} + X_{31}X_{42}X_{13}X_{24} - X_{31}X_{42}X_{23}X_{14}\\
& - X_{41}X_{12}X_{23}X_{34} + X_{41}X_{12}X_{33}X_{24} - X_{41}X_{22}X_{33}X_{14} + X_{41}X_{22}X_{13}X_{34} - X_{41}X_{32}X_{13}X_{24} + X_{41}X_{32}X_{23}X_{14} = 0,
\end{aligned}
\tag{4.4}
$$

$$
m_{31} = \frac{4(\delta\dot{x})^2 + \sum_{(i,j)\neq(3,1)} m_{ij}m_{ji}}{2}. \tag{4.5}
$$

A special solution of (4.4) is given by[8]

$$
m_{22} = m_{44} = m_{24} = m_{42} = 0, \tag{4.6}
$$

$$
m_{32} = -m_{11}m_{12}, \quad m_{34} = -m_{11}m_{14}, \quad m_{41} = -m_{43}m_{33}, \tag{4.7}
$$

$$
m_{21} = -m_{23}m_{33}, \qquad m_{11} = -m_{33}, \tag{4.8}
$$

$$
\delta\dot{x}^2 = 0, \tag{4.9}
$$

$$
\Rightarrow X_{12} = X_{14} = X_{43} = X_{23} = X_{42} = X_{24} = X_{22} = X_{44} = 0. \tag{4.10}
$$

In the Fourier space, the relations (4.5), (4.6), (4.7), (4.8), (4.9), and (4.10) are rewritten as follows:

$$
m_{32(n)} = -m_{11(n)}m_{12(-n)}, \qquad m_{34(n)} = -m_{11(n)}m_{14(-n)},
$$
$$
m_{41(n)} = -m_{43(n)}m_{33(-n)}, \tag{4.11}
$$

$$
m_{21(n)} = -m_{23(n)}m_{33(-n)}, \qquad m_{11(n)} = -m_{33(n)}, \tag{4.12}
$$

$$
\delta x^0_n \delta x^0_{-n} = \delta x^1_n \delta x^1_{-n} + \delta x^2_n \delta x^2_{-n} \quad (\text{for } n \neq 0), \tag{4.13}
$$

$$
m_{31(n)} = m_{11(n)}m_{11(-n)}. \tag{4.14}
$$

Note that, while the zero modes $\delta x^m_{(0)}$ are the independent parameters, $\delta x^m_{(n)}$ is not independent.

Recall that the loop variable has the $U_R(1)$ charges as given in Table II (see footnote 4). The loop variable also has the dimension Δ since the dimension of the Wilson loop becomes zero. Since the charge $Q(=\Delta - J)$ of the loop variables is well-defined for (4.6), (4.7), (4.8), (4.9), and (4.10), we expand the Wilson loop fluctuated at x^m_0 up to one in powers of the charge Q of the loop variable and up to one impurity as follows:

$$
\begin{aligned}
W[C] = {} & W[C_0] + \int_0^t ds \sum_{Q(m_{ij})=1/2} m_{ij}(s) \frac{\delta W[C]}{\delta m_{ij}(s)}\bigg|_{C=C_0}\\
& + \int_0^t ds \delta x^m(s) \frac{\delta W[C]}{\delta x^m(s)}\bigg|_{C=C_0}\\
& + \int_0^t ds m_{11}(s)\left(\frac{\delta W[C]}{\delta m_{11}(s)} - \frac{\delta W[C]}{\delta m_{33}(s)}\right)\bigg|_{C=C_0} + \cdots,
\end{aligned}
\tag{4.15}
$$

where we used (4.6), (4.7), and (4.8). Here, $\sum_{Q(m_{kl})=n}$ means that we sum the terms in which the charge of m_{ij} is n; for $n = 1/2$, we sum the terms containing m_{12}, m_{14}, m_{23}, and m_{43}.

We introduce a new functional derivative as follows:

$$
\frac{\delta}{\delta m'} = \frac{\delta}{\delta m_{11}} - \frac{\delta}{\delta m_{33}}. \tag{4.16}
$$

By substituting the Fourier transformation of the loop variables (4.2) into the resulting formula, we obtain

$$
\begin{aligned}
W[C] = {} & W[C_0] + \sum_{Q(m_{ij})=1/2} \sum_n m_{ij(n)} \int_0^t ds \frac{\delta W[C]}{\delta m_{ij}(s)}\bigg|_{C=C_0}\\
& \times e^{2\pi i n s/t} + \sum_n \delta x^m_{(n)} \int_0^t ds \frac{\delta W[C]}{\delta x^m(s)}\bigg|_{C=C_0} e^{2\pi i n s/t}\\
& + \sum_n m_{11(n)} \int_0^t ds \frac{\delta W[C]}{\delta m'(s)}\bigg|_{C=C_0} e^{2\pi i n s/t} + \cdots.
\end{aligned}
\tag{4.17}
$$

The functional derivative of (2.3) contained in (4.17) generates the impurity operator (in the sense of the spin chain)

[8] We can solve (3.11) about m_{31} since (3.11) is the equation of second degree about m_{31}. However, we cannot expand the denominator of the solution in powers of the loop variables.

TABLE II. $U_R(1)$ charge of the loop variables ($i = 1, 2, 3, 4$).

	m_{13}	m_{12}, m_{14}, m_{23}, m_{43}	δx^m, m_{ii}, m_{24}, m_{42}	m_{21}, m_{41}, m_{32}, m_{34}	m_{31}
J	1	1/2	0	$-1/2$	-1
$Q(=\Delta - J)$	0	1/2	1	3/2	2

as follows [20,21]:

$$\frac{\delta W(C)}{\delta x^m(s)}\bigg|_{C=C_0} = i\,\mathrm{Tr}[(F_{mn}(x(s))\dot{x}^n(s) + (D_m Y^I Y_J^\dagger) \times (x(s))M_I{}^J(s))w_s^{s+t}(C)]|_{C=C_0} = i\sum_{J=0}^{\infty}\frac{(it)^J}{J!}\,\mathrm{Tr}[(D_m A_1 B_1)(x_0)(A_1 B_1(x_0))^J], \tag{4.18}$$

$$\frac{\delta W(C)}{\delta M_K{}^L(s)}\bigg|_{C=C_0} = i\sum_{J=0}^{\infty}\frac{(it)^J}{J!}\,\mathrm{Tr}[Y^K Y_L^\dagger(x_0)(A_1 B_1(x_0))^J], \tag{4.19}$$

where $w_{s'}^{s''}(C)$ describes the Wilson line along the path C from $s = s'$ to $s = s''$ and the covariant derivative D_m including A_m operates on $Y^I Y_J^\dagger$ so that $Y^I Y_J^\dagger$ is an adjoint field. Note that (4.18) and (4.19) do not depend on the loop parameter s.

By using (4.18) and (4.19), we find that the right side of (4.17) becomes the zero mode contribution as follows:

$$\begin{aligned}&\sum_{J=0}^{\infty}\frac{(it)^J}{J!}\,[\mathrm{Tr}[(A_1B_1)^J](x_0)\\ &+ it\sum_{Q(m_{ij})=1/2} m_{ij(0)}\,\mathrm{Tr}[Y^i Y_j^\dagger(A_1B_1)^J](x_0)\\ &+ itm_{11(0)}\,\mathrm{Tr}[(Y^1Y_1^\dagger - Y^3Y_3^\dagger)(A_1B_1)^J](x_0)\\ &+ it\delta x^m_{(0)}\,\mathrm{Tr}[(D_mA_1B_1)(A_1B_1)^J](x_0)].\end{aligned} \tag{4.20}$$

In higher order of J, Eq. (4.20) contains the following infinite strings of the operator A_1B_1, the bifundamental scalars $Y^iY_j^\dagger$, and the covariant derivative D_m:

$$\begin{aligned}&\mathrm{Tr}\,[(A_1B_1)^J](x_0), \qquad \mathrm{Tr}[Y^iY_j^\dagger(A_1B_1)^J](x_0),\\ &\mathrm{Tr}[(D_mA_1B_1)(A_1B_1)^J](x_0),\end{aligned} \tag{4.21}$$

$$\mathrm{Tr}\,[(Y^1Y_1^\dagger - Y^3Y_3^\dagger)(A_1B_1)^J](x_0), \tag{4.22}$$

where $(i, j) = (1, 2), (1, 4), (2, 3), (4, 3)$ and we separated the operators coupled to the independent loop variables. These BPS operators protected by the supersymmetry correspond to the vacuum state and the excited states of the dual IIA string theory on the pp wave background (see [8,9]).

By following the correspondence between (4.21) and the dual IIA string excited states, and by following the conjecture in Ref. [5], we map the functional derivatives of the Wilson loop to the dual IIA string excited states in Table III.

Before ending this section, we want to comment on the relation between our Taylor expansion of the pointlike Wilson loop and strong coupling expansion: The Wilson loop should be expanded in terms of $1/\sqrt{\lambda}$ [22] and have a convergent region of the expansion. The operators obtained in the Taylor expansion should be normalized by using their 2-point function. However, the appearance of strong coupling expansion is not clear in both our pointlike Wilson loop and that of $\mathcal{N} = 4$ SYM [5]. On the other hand, our Taylor expansion of the Wilson loop (4.20) is similar to the mode expansion of the wave function in the string field theory (SFT) as was pointed out in the work of Ref. [5]. Since the SFT is independent of the string coupling, (4.20) may be independent of the 't Hooft coupling.

V. DISCUSSIONS

In this paper, motivated by the paper [5], we discussed the dual IIA string description of a slightly deformed pointlike Wilson loop. By expanding the pointlike Wilson loop in powers of the loop variables, we obtained the BPS operators[9] that correspond to the excited string states of the IIA string theory on the pp wave background. Our new result was the impurity operator $Y^1Y_1^\dagger - Y^3Y_3^\dagger$ that was not determined from the analysis in the gravity side [8]. By following the conjecture in Ref. [5], we gave the maps from the functional derivatives of the Wilson loop to the dual IIA string excited states in Table III.

The BPS conditions of the Wilson loop (3.10) and (3.11) constrained both the loop variables and the forms of the BPS operators, though we did not prove the uniqueness of solution of the BPS conditions. In Appendix C, we also obtained the BMN operators [23] in higher order terms of the Taylor expansion (C1). It will also be important to compute the anomalous dimension [24] of the BMN operators (C15) and (C16).

Since our Wilson loop is pointlike, it is important to construct the straight line BPS Wilson loop that connects the pointlike 1/3-BPS Wilson loop to the 1/6-BPS Wilson loop in the ABJM theory. We leave it to future work.

We also want to comment on the Wilson loop in the ABJM theory from the viewpoint of 11-dimensional theory

[9]As was pointed out in Ref. [9], we need to include the fermions to complete the BPS multiplet that the impurity operators belong to, though the gauge/gravity duality could be proven by using only the bosonic string action.

TABLE III. The relation between the functional derivatives and the IIA supergravity modes (the zero modes of the string oscillation operator). In the table, $m = 0, 1, 2$, $(i, j) = (1, 2), (1, 4), (2, 3), (4, 3)$, and $k = 5, 6, 7, 8$.

ABJM side		Dual IIA superstring side
Dual superstring side	IIACorresponding operators	
...	$W(C_0)$	$\|0; p^+\rangle$
$\int dse^{2\pi ins/t}\frac{\delta}{\delta x^m(s)}$	$D_m A_1 B_1$	$\alpha^{m+1}_{(0)}$
$\int dse^{2\pi ins/t}\frac{\delta}{\delta m'(s)}$	$Y^1 Y_1^\dagger - Y^3 Y_3^\dagger$	$\alpha^4_{(0)}$
$\int dse^{2\pi ins/t}\frac{\delta}{\delta m_{ij}(s)}\|_{Q=1/2}$	$Y^i Y_j^\dagger$	$\alpha^k_{(0)}$

(M theory). It has been known that in the dual IIA supergravity side, the Wilson loop is described by the fundamental string on the $AdS_4 \times CP^3$ spacetime. Then, the dual fundamental string should be described by the M2-brane wrapped on the M-circle: The 3-form $C_{(3)}$ in the 11-dimensional supergravity reduces to the Neveu-Schwarz–Neveu-Schwarz field $B_{\mu\nu}$ coupling to the IIA string. Thus, it is also interesting to analyze the dynamics of the Wilson loop by using the loop equation [25].[10] As the dynamics of the Wilson loop describe the dynamics of the dual string, the loop equation will also describe the dynamics of the M2-brane wrapped on the M-circle. When we analyze the loop equation, however, we need to extend the analysis to include supersymmetry (see also [21]).

ACKNOWLEDGMENTS

Especially, we thank M. Fukuma, H. Hata, H. Kawai, and K. Yoshida for the discussions and helpful comments. We thank K. Katayama for the discussions. We thank T. Nishioka and T. Takayanagi for the helpful comments on this manuscript.

APPENDIX A: CHARGE CONJUGATION MATRIX

In Appendix A, we introduce the charge conjugation matrix of both $SO(6)$ and $SO(4)$ and construct the reducible representation of $SO(6)$ gamma matrices by using the reducible representation of $SO(4)$ gamma matrices.

The charge conjugation matrix constructed by $SO(6)$ gamma matrices satisfies the following relations:

$$C\gamma^i C^{-1} = -\gamma^{iT}, \tag{A1}$$

$$\begin{aligned} C^T &= \eta C, \\ (C\gamma^{i_1 i_2 \ldots i_n})^T &= \gamma^{i_n T} \ldots \gamma^{i_1 T} \eta C \\ &= \eta(-1)^{(1/2)n(n+1)} C\gamma^{i_1 i_2 \ldots i_n}, \end{aligned} \tag{A2}$$

where (A1) is the definition of the charge conjugation matrix and η is a constant determined later.[11] (A2) shows that the matrix defined on the right-hand side of (A2) transforms as the symmetric tensor or the antisymmetric tensor in terms of the $SO(6)$ spinor index. Since the product representation in Table IV must be consistent with the decomposition $8_s \otimes 8_s \to 36_S \oplus 28_A$, η must be $+1$.

C defined in (A1) is the metric of the spinor index as follows:

$$\begin{gathered} C = C_{\alpha\beta} = C^T, \qquad C^{-1} = (C^{-1})^{\alpha\beta}, \\ C_{\alpha\beta}(\gamma^i)^\beta{}_\gamma = (C\gamma^i)_{\alpha\gamma}, \end{gathered} \tag{A3}$$

$$C\gamma^i = (C\gamma^i)^T, \qquad C_{\alpha\beta} g^\alpha{}_\gamma g^\beta{}_\delta = C_{\gamma\delta}, \tag{A4}$$

where g is the spinor representation of $SO(6)$.[12]

We introduce the $SO(4)$ charge conjugation matrix $\hat{C}$ that satisfies the following relations:

$$\hat{C}\rho^k\hat{C}^{-1} = -(\rho^k)^T, \tag{A5}$$

$$\hat{C}^T = -\hat{C}. \tag{A6}$$

We decompose the $SO(6)$ Dirac gamma matrices and the charge conjugation matrix as follows:

$$\gamma^k = \begin{pmatrix} 0 & \rho^k \\ \rho^k & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} 0 & \rho^5 \\ \rho^5 & 0 \end{pmatrix}, \qquad \gamma^6 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{A7}$$

$$C = \begin{pmatrix} 0 & \hat{C} \\ -\hat{C} & 0 \end{pmatrix}, \tag{A8}$$

where ρ^k is the $SO(4)$ Dirac gamma matrix. Here the matrices defined in (A7) and (A8) satisfy the Clifford algebra and the definition of the charge conjugation matrix.

APPENDIX B: CLEBSCH-GORDAN DECOMPOSITION OF THE 6 MAJORANA SPINORS

We show the Clebsch-Gordan decomposition of the 6 $SO(1, 2)$ Majorana spinors ϵ_i, which transform as the vector representation of $\text{spin}(6) \sim SO(6)$, and show the equality (3.2). We know the following equivalence:

$$\text{spin}\,(6) \sim SU(4), \tag{B1}$$

$$6_v \sim 6_A, \tag{B2}$$

$$4_s \sim 4_v. \tag{B3}$$

We introduce the new matrix C_2 and 4_s spinor index I and J as follows:

[10] The loop equation for the pure Chern-Simons theory is slightly different from that of the YM theory.

[11] These equations are satisfied for any $SO(N)$.

[12] We can show the second equation in (A4) by expanding the exponential of g in powers of the generator of the Lorentz algebra.

TABLE IV. The product representation of the 8_S spinor representation.

n	0	1	2	3	4	5	6
${}_6C_n$	1	6	15	20	15	6	1
$(-1)^{(1/2)n(n+1)}$	+	−	−	+	+	−	−

$$\gamma^i = \begin{pmatrix} 0 & \hat{\gamma}^i_{I\dot{J}} \\ \hat{\gamma}^i_{\dot{I}J} & 0 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 0 & i\hat{C}\rho_5 \\ -i\hat{C}\rho_5 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \hat{C}_{2,I\dot{J}} \\ \hat{C}_{2,\dot{I}J} & 0 \end{pmatrix}$$

$$(I, J = 1 \sim 4), \tag{B4}$$

$$C_2\gamma^i = \begin{pmatrix} (\hat{C}_2\hat{\gamma}^i)_{IJ} & 0 \\ 0 & (\hat{C}_2\hat{\gamma}^i)_{\dot{I}\dot{J}} \end{pmatrix}. \tag{B5}$$

By using (A4), we can show that $\hat{C}_2\hat{\gamma}^i$ transforms as an antisymmetric tensor in terms of the index I, J.

We describe the Clebsch-Gordan decomposition of $\hat{C}_2\hat{\gamma}^i$ as follows:

$$v_{IJ} \to v^i \equiv \sum_{I<J} (\hat{C}_2\hat{\gamma}^i)_{IJ} v^{IJ}, \tag{B6}$$

$$\omega_{IJ} = \epsilon_i (\hat{C}_2\hat{\gamma}^i)_{IJ}. \tag{B7}$$

We choose the basis of the $SO(4)$ gamma matrices as follows:

$$\rho^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \rho^2 = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix},$$
$$\rho^3 = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}, \qquad \rho^4 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \tag{B8}$$

$$\hat{C} = \rho^2\rho^3, \qquad \hat{C}_2 = \rho^4\rho^1. \tag{B9}$$

After a $SO(6)$ permutation of the $SO(6)$ gamma matrices γ^i including a change of the signs, we obtain the supersymmetry generator ω_{IJ}, which satisfies (3.1) and (3.2) (see also [19]).

It is convenient to introduce the following equations:

$$(C\gamma^i)^* = -\gamma^i C^*, \tag{B10}$$

$$(C\gamma^7)^* = \gamma^7 C^*. \tag{B11}$$

By using the $SO(6)$ gamma matrices C and C_2, we can show (3.2) as follows:

$$\epsilon_i (C_2\gamma^i)\epsilon^i (C_2^*\gamma^{*i}) \tag{B12}$$

$$= \epsilon^i\epsilon_j (C\gamma^7\gamma^i\gamma^7 C^*\gamma^{*j}) \tag{B13}$$

$$= \epsilon^i\epsilon_j C\gamma^i\gamma^j C^* = \epsilon^i\epsilon_i CC^* = \epsilon^i\epsilon_i 1, \tag{B14}$$

where $\gamma^7 = \gamma^1\gamma^2\gamma^3\gamma^4$.

APPENDIX C: HIGHER ORDER TERMS IN THE EXPANSION OF THE POINTLIKE WILSON LOOP

In this section, we compute higher order terms in the expansion (4.15). We expand the pointlike Wilson loop up to 2 in powers of the charge Q as follows:

$$\begin{aligned}
W[C] = W[C_0] &+ \int_0^t ds \sum_{Q(m_{IJ})=1/2} m_{ij}(s) \frac{\delta W[C]}{\delta m_{ij}(s)} \bigg|_{C=C_0} + \int_0^t ds\, \delta x^m(s) \frac{\delta W[C]}{\delta x^m(s)} \bigg|_{C=C_0} + \int_0^t ds \Big(m_{11}(s) \frac{\delta W[C]}{\delta m_{11}(s)} \\
&+ m_{33}(s) \frac{\delta W[C]}{\delta m_{33}(s)} \Big) \bigg|_{C=C_0} + \frac{1}{2} \int_0^t ds_1 \int_0^t ds_2 \sum_{Q(m_{ij}+m_{kl})=1} m_{ij}(s_1) m_{kl}(s_2) \frac{\delta^2 W[C]}{\delta m_{ij}(s_1)\delta m_{kl}(s_2)} \bigg|_{C=C_0} \\
&+ \int_0^t ds \sum_{Q(m_{ij})=3/2} m_{ij}(s) \frac{\delta W[C]}{\delta m_{ij}(s)} \bigg|_{C=C_0} + \int_0^t ds_1 \int_0^t ds_2 \sum_{Q(m_{ij})=1/2} m_{ij}(s_1)\delta x^n(s_2) \frac{\delta^2 W[C]}{\delta x^n(s_2)\delta m_{ij}(s_1)} \bigg|_{C=C_0} \\
&+ \frac{1}{2} \int_0^t ds_1 \int_0^t ds_2 \sum_{Q(m_{ij}+m_{kl})=3/2} m_{ij}(s_1) m_{kl}(s_2) \frac{\delta^2 W[C]}{\delta m_{ij}(s_1)\delta m_{kl}(s_2)} \bigg|_{C=C_0} + \int_0^t ds m_{31}(s) \frac{\delta W[C]}{\delta m_{31}(s)} \bigg|_{C=C_0} \\
&+ \frac{1}{2} \int_0^t ds_1 \int_0^t ds_2\, \delta x^m(s_1)\delta x^n(s_2) \frac{\delta^2 W[C]}{\delta x^m(s_1)\delta x^n(s_2)} \bigg|_{C=C_0} \\
&+ \int_0^t ds_1 \int_0^t ds_2 \sum_{Q(m_{IJ})=1} m_{IJ}(s_1)\delta x^n(s_2) \frac{\delta^2 W[C]}{\delta x^n(s_2)\delta m_{IJ}(s_1)} \bigg|_{C=C_0} \\
&+ \frac{1}{2} \int_0^t ds_1 \int_0^t ds_2 \sum_{Q(m_{ij}+m_{kl})=2} m_{ij}(s_1) m_{kl}(s_2) \frac{\delta^2 W[C]}{\delta m_{ij}(s_1)\delta m_{kl}(s_2)} \bigg|_{C=C_0} + \cdots,
\end{aligned} \tag{C1}$$

where $\sum_{Q(m_{ij}+m_{kl})=n}$ means that we sum the terms in which the charge of $m_{ij}m_{kl}$ is n. Recall that the loop variables satisfy the relation (4.11), (4.12), (4.13), and (4.14) as follows:

$$m_{32(n)} = -m_{11(n)}m_{12(-n)}, \qquad m_{34(n)} = -m_{11(n)}m_{14(-n)},$$
$$m_{41(n)} = -m_{43(n)}m_{33(-n)}, \tag{C2}$$

$$m_{21(n)} = -m_{23(n)}m_{33(-n)}, \qquad m_{11(n)} = -m_{33(n)}, \tag{C3}$$

$$\delta x^0_n \delta x^0_{-n} = \delta x^1_n \delta x^1_{-n} + \delta x^2_n \delta x^2_{-n} \qquad (\text{for } n \neq 0), \tag{C4}$$

$$m_{31(n)} = m_{11(n)}m_{11(-n)}. \tag{C5}$$

Because of (C4), the spacetime coordinates are twisting for $n \neq 0$. By substituting the Fourier transformation of the loop variables (4.2) into (C1) and by using (C2)–(C5), we obtain

$$\begin{aligned}
W[C] = W[C_0] &+ \sum_{Q(m_{ij})=1/2} \sum_n m_{ij(n)} \int_0^t ds \frac{\delta W[C]}{\delta m_{ij}(s)} \bigg|_{C=C_0} e^{2\pi i n s/t} + \sum_n \delta x^m_{(n)} \int_0^t ds \frac{\delta W[C]}{\delta x^m(s)} \bigg|_{C=C_0} e^{2\pi i n s/t} \\
&+ \sum_n m_{11(n)} \int_0^t ds \frac{\delta W[C]}{\delta m'(s)} \bigg|_{C=C_0} e^{2\pi i n s/t} + \sum_{Q(m_{ij}+m_{kl})=1} \sum_{n_1,n_2} m_{ij(n_1)}m_{kl(n_2)} \int_0^t ds_1 \int_0^t ds_2 \frac{\delta^2 W[C]}{\delta m_{ij}(s_1)\delta m_{kl}(s_2)} \bigg|_{C=C_0} \\
&\times e^{2\pi i(n_1 s_1 + n_2 s_2)/t} + \sum_{Q(m_{ij})=1/2} \sum_{n_1,n_2} \delta x^n_{(n_1)} m_{ij(n_2)} \int_0^t ds_1 \int_0^t ds_2 \frac{\delta^2 W[C]}{\delta x^n(s_2)\delta m_{ij}(s_1)} \bigg|_{C=C_0} e^{2\pi i(n_1 s_1 + n_2 s_2)/t} \\
&+ \sum_{n_1,n_2} m_{11(n_1)}m_{12(n_2)} \bigg(\int_0^t ds_1 \int_0^t ds_2 \frac{\delta^2 W[C]}{\delta m'(s_1)\delta m_{12}(s_2)} \bigg|_{C=C_0} e^{2\pi i(n_1 s_1 + n_2 s_2)/t} - \int_0^t ds \frac{\delta W[C]}{\delta m_{32}(s)} \bigg|_{C=C_0} e^{2\pi i(n_1+n_2)s/t} \bigg) \\
&+ \sum_{n_1,n_2} m_{11(n_1)}m_{14(n_2)} \bigg(\int_0^t ds_1 \int_0^t ds_2 \frac{\delta^2 W[C]}{\delta m'(s_1)\delta m_{14}(s_2)} \bigg|_{C=C_0} e^{2\pi i(n_1 s_1 + n_2 s_2)/t} - \int_0^t ds \frac{\delta W[C]}{\delta m_{34}(s)} \bigg|_{C=C_0} \bigg) e^{2\pi i(n_1+n_2)s/t} \\
&+ \sum_{n_1,n_2} m_{11(n_1)}m_{43(n_2)} \bigg(\int_0^t ds_1 \int_0^t ds_2 \frac{\delta^2 W[C]}{\delta m'(s_1)\delta m_{43}(s_2)} \bigg|_{C=C_0} e^{2\pi i(n_1 s_1 + n_2 s_2)/t} + \int_0^t ds \frac{\delta W[C]}{\delta m_{41}(s)} \bigg|_{C=C_0} e^{2\pi i(n_1+n_2)s/t} \bigg) \\
&+ \sum_{n_1,n_2} m_{11(n_1)}m_{23(n_2)} \bigg(\int_0^t ds_1 \int_0^t ds_2 \frac{\delta^2 W[C]}{\delta m'(s_1)\delta m_{23}(s_2)} \bigg|_{C=C_0} e^{2\pi i(n_1 s_1 + n_2 s_2)/t} + \int_0^t ds \frac{\delta W[C]}{\delta m_{21}(s)} |_{C=C_0} e^{2\pi i(n_1+n_2)s/t} \bigg) \\
&+ \frac{1}{2} \sum_{m,n} \sum_{n_1,n_2} \delta x^m_{(n_1)} \delta x^n_{(n_2)} \int_0^t ds_1 \int_0^t ds_2 \frac{\delta^2 W[C]}{\delta x^m(s_1)\delta x^n(s_2)} \bigg|_{C=C_0} e^{2\pi i(n_1 s_1 + n_2 s_2)/t} \\
&+ \frac{1}{2} \sum_{n_1,n_2} m_{11(n_1)}m_{11(n_2)} \bigg(\int_0^t ds_1 \int_0^t ds_2 \frac{\delta^2 W[C]}{\delta m'(s_2)\delta m'(s_1)} \bigg|_{C=C_0} e^{2\pi i(n_1 s_1 + n_2 s_2)/t} + \int_0^t ds \frac{\delta W[C]}{\delta m_{31}(s)} \bigg|_{C=C_0} e^{2\pi i(n_1+n_2)s/t} \bigg) \\
&+ \cdots,
\end{aligned} \tag{C6}$$

where the loop variables $\delta x^m_{(n)}$ (for $n \neq 0$) are not independent. Note that, in the context of the spin chain, the impurity D_m coupling with $\delta x^m_{(n)}$ mixes with the fermions: We guess that the Wilson loop containing the fermions will be needed to explain the full excited string spectrum in the IIA string theory. The two functional derivatives of the pointlike Wilson loop are given by

$$\begin{aligned}
\frac{\delta^2 W(C)}{\delta x^n(s_2)\delta x^m(s_1)} \bigg|_{C=C_0} &= \frac{\delta}{\delta x^n(s_2)} i\,\mathrm{Tr}[(F_{ml}(x(s_1))\dot{x}^l(s_1) + D_m Y^I Y^\dagger_J(x(s_1))M^J_I(s_1))w^{s_1^+ + t}_{s_1^+}(C)]|_{C=C_0} \\
&= i\,\mathrm{Tr}[\delta(s_1 - s_2)D_{(n}D_{m)}A_1B_1(x_0)w^{s_1^+ + t}_{s_1^+}(C_0)] + i\,\mathrm{Tr}[F_{mn}(x_0)\delta'(s_1 - s_2)w^{s_1^+ + t}_{s_1^+}(C_0)] \\
&\quad - \mathrm{Tr}[D_m A_1B_1(x_0)w^{s_2}_{s_1^+}(C_0)D_n A_1B_1(x_0)w^{s_1^+ + t}_{s_2^+}(C_0)],
\end{aligned} \tag{C7}$$

$$\begin{aligned}
\frac{\delta^2 W(C)}{\delta M_K{}^L(s_2)\delta x^m(s_1)} \bigg|_{C=C_0} &= \frac{\delta}{\delta M_K{}^L(s_2)} i\,\mathrm{Tr}[(F_{ml}(x(s_1))\dot{x}^l(s_1) + D_m Y^I Y^\dagger_J(x(s_1))M_I{}^J(s_1))w^{s_1^+ + t}_{s_1^+}(C)]|_{C=C_0} \\
&= i\,\mathrm{Tr}[\delta(s_1 - s_2)D_m Y^K Y^\dagger_L(x_0)w^{s_1^+ + t}_{s_1^+}(C_0)] - \mathrm{Tr}[D_m A_1B_1(x_0)w^{s_2}_{s_1^+}(C_0)Y^K Y^\dagger_L(x_0)\cdot w^{s_1^+ + l}_{s_2^+}(C_0)],
\end{aligned} \tag{C8}$$

$$\frac{\delta^2 W(C)}{\delta M_K{}^L(s_2)\delta M_I{}^J(s_1)}\bigg|_{C=C_0} = \frac{\delta}{\delta M_K{}^L(s_2)} i\,\mathrm{Tr}[Y^I Y_J^\dagger(x(s_1)) w_{s_1}^{s_1+t}(C)]|_{C=C_0} = -\mathrm{Tr}[Y^I Y_J^\dagger(x_0) w_{s_1}^{s_2}(C_0) Y^K Y_L^\dagger(x_0) w_{s_2}^{s_1+l}(C_0)]. \tag{C9}$$

Next, we introduce the following integral:

$$F_2(n,k,J) = \frac{1}{(J-k)!k!}\int_0^1 d\tilde{s}\tilde{s}^k(1-\tilde{s})^{J-k}e^{2\pi i n\tilde{s}} \sim \frac{1}{JJ!}\exp\left(\frac{2\pi i n k}{J}\right) \qquad \text{(in the large } J \text{ limit).} \tag{C10}$$

The derivation of the second line in (C10) is given in the appendix of [5]. By substituting (C7)–(C9) into (C6) and by transforming the parameter (s_1, s_2) into $(s_1/t, (s_2 - s_1)/t)$ for the double integral about s_1 and s_2, we obtain the local operator expression of the remaining terms as follows: The $Q = 1$ terms in which a product of the loop variables has the charge $Q = 1$ become

$$\sum_{Q(m_{ij}+m_{lm})=1}\sum_J\sum_{n_1}(it)^{J+2}m_{ij(n_1)}m_{lm(-n_1)}\left[\sum_{k=0}^J \mathrm{Tr}[Y^i Y_j^\dagger (A_1B_1)^k Y^l Y_m^\dagger \cdot (A_1B_1)^{J-k}\right](x_0)F_2(n_1,k,J). \tag{C11}$$

The $Q = 3/2$ terms become

$$\begin{aligned}
&+\sum_J\sum_{n_1}(it)^{J+2}m_{11(n_1)}m_{12(-n_1)}\left[\sum_{k=0}^J \mathrm{Tr}[(Y^1Y_1^\dagger - Y^3Y_3^\dagger)(A_1B_1)^k Y^1 Y_2^\dagger (A_1B_1)^{J-k}](x_0)\cdot F_2(n_1,k,J)\right.\\
&\left.-\frac{1}{(J+1)!}\mathrm{Tr}[Y^3Y_2^\dagger(A_1B_1)^{J+1}](x_0)\right] + \sum_J\sum_{n_1}(it)^{J+2}m_{11(n_1)}m_{14(-n_1)}\left[\sum_{k=0}^J \mathrm{Tr}[(Y^1Y_1^\dagger - Y^3Y_3^\dagger)(A_1B_1)^k Y^1 Y_4^\dagger (A_1B_1)^{J-k}]\right.\\
&\left.\times(x_0)\cdot F_2(n_1,k,J) - \frac{1}{(J+1)!}\mathrm{Tr}[Y^3Y_4^\dagger(A_1B_1)^{J+1}](x_0)\right] + \sum_J\sum_{n_1}(it)^{J+2}m_{11(n_1)}m_{43(-n_1)}\left[\sum_{k=0}^J \mathrm{Tr}[(Y^1Y_1^\dagger - Y^3Y_3^\dagger)\right.\\
&\left.\times(A_1B_1)^k Y^4 Y_3^\dagger (A_1B_1)^{J-k}](x_0)\cdot F_2(n_1,k,J) + \frac{1}{(J+1)!}\mathrm{Tr}[Y^4Y_1^\dagger(A_1B_1)^{J+1}](x_0)\right]\\
&+\sum_J\sum_{n_1}(it)^{J+2}m_{11(n_1)}m_{23(-n_1)}\left[\sum_{k=0}^J \mathrm{Tr}[(Y^1Y_1^\dagger - Y^3Y_3^\dagger)(A_1B_1)^k Y^2 Y_3^\dagger (A_1B_1)^{J-k}](x_0)\cdot F_2(n_1,k,J)\right.\\
&\left.+\frac{1}{(J+1)!}\mathrm{Tr}[Y^2Y_1^\dagger(A_1B_1)^{J+1}](x_0)\right],
\end{aligned} \tag{C12}$$

where we have not written the terms dependent on δx^m since they are not independent. We obtain the $Q = 2$ terms up to two impurity as follows:

$$\begin{aligned}
&+\frac{1}{2}\sum_J\sum_{n_1}(it)^{J+2}m_{11(n_1)}m_{11(-n_1)}\left[\sum_{k=0}^J \mathrm{Tr}[(Y^1Y_1^\dagger - Y^3Y_3^\dagger)(A_1B_1)^k (Y^1Y_1^\dagger - Y^3Y_3^\dagger)\cdot(A_1B_1)^{J-k}](x_0)F_2(n_1,k,J)\right.\\
&\left.+\frac{1}{(J+1)!}\mathrm{Tr}[Y^3Y_1^\dagger(A_1B_1)^{J+1}](x_0)\right] + \cdots,
\end{aligned} \tag{C13}$$

where we have not written the terms dependent on δx^m for the same reason. Note that $Q = 2$ terms also contain the 3 impurity terms.

In the large J limit, (C11)–(C13) contain the following BMN operators:

$$\sum_{k=0}^{J} \mathrm{Tr}[Y^i Y_j^\dagger (A_1 B_1)^k Y^l Y_m^\dagger (A_1 B_1)^{J-k}] e^{2\pi i n k/J}, \tag{C14}$$

$$\begin{aligned}
&\sum_{k=0}^{J} \mathrm{Tr}[(Y^1 Y_1^\dagger - Y^3 Y_3^\dagger)(A_1 B_1)^k Y^1 Y_2^\dagger (A_1 B_1)^{J-k}] e^{2\pi i n k/J} - \mathrm{Tr}[Y^3 Y_2^\dagger (A_1 B_1)^{J+1}], \\
&\sum_{k=0}^{J} \mathrm{Tr}[(Y^1 Y_1^\dagger - Y^3 Y_3^\dagger)(A_1 B_1)^k Y^1 Y_4^\dagger (A_1 B_1)^{J-k}] e^{2\pi i n k/J} - \mathrm{Tr}[Y^3 Y_4^\dagger (A_1 B_1)^{J+1}], \\
&\sum_{k=0}^{J} \mathrm{Tr}[(Y^1 Y_1^\dagger - Y^3 Y_3^\dagger)(A_1 B_1)^k Y^4 Y_3^\dagger (A_1 B_1)^{J-k}] e^{2\pi i n k/J} + \mathrm{Tr}[Y^4 Y_1^\dagger (A_1 B_1)^{J+1}], \\
&\sum_{k=0}^{J} \mathrm{Tr}[(Y^1 Y_1^\dagger - Y^3 Y_3^\dagger)(A_1 B_1)^k Y^2 Y_3^\dagger (A_1 B_1)^{J-k}] e^{2\pi i n k/J} + \mathrm{Tr}[Y^2 Y_1^\dagger (A_1 B_1)^{J+1}],
\end{aligned} \tag{C15}$$

$$\sum_{k=0}^{J} \mathrm{Tr}[(Y^1 Y_1^\dagger - Y^3 Y_3^\dagger)(A_1 B_1)^k (Y^1 Y_1^\dagger - Y^3 Y_3^\dagger)(A_1 B_1)^{J-k}] e^{2\pi i n k/J} + \mathrm{Tr}[Y^3 Y_1^\dagger (A_1 B_1)^{J+1}], \tag{C16}$$

where (i, j) and (l, m) are equal to (1, 2), (1, 4), (2, 3), and (4, 3).

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