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From twistor string theory to recursion relations

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Witten's twistor string theory gives rise to an enigmatic formula [1] known as the "connected prescription" for tree-level Yang-Mills scattering amplitudes. We derive a link representation for the connected prescription by Fourier transforming it to mixed coordinates in terms of both twistor and dual twistor variables. We show that it can be related to other representations of amplitudes by applying the global residue theorem to deform the contour of integration. For six and seven particles we demonstrate explicitly that certain contour deformations rewrite the connected prescription as the Britto-Cachazo-Feng-Witten representation, thereby establishing a concrete link between Witten's twistor string theory and the dual formulation for the S matrix of the $\mathcal{N} = 4$ SYM recently proposed by Arkani-Hamed *et al.* Other choices of integration contour also give rise to "intermediate prescriptions." We expect a similar though more intricate structure for more general amplitudes.

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I. INTRODUCTION

Witten's twistor string theory proposal [2] launched a series of developments that have greatly expanded our understanding of the mathematical structure of scattering amplitudes over the past several years, particularly in the maximally supersymmetric Yang-Mills theory (SYM). The most computationally useful technology to have emerged from subsequent developments is the Britto-Cachazo-Feng-Witten (BCFW) on-shell recursion relation [3,4], the discovery of which initiated a vast new industry for the computation of amplitudes. Building on [5], two recent papers [6,7] have paved the way for a return to twistor space by showing that the BCFW recursion has a natural formulation there. Here, we bring this set of developments full circle by demonstrating a beautiful connection between the original twistor string proposal and the dual formulation for the S matrix of SYM recently proposed by Arkani-Hamed *et al.* [8]. In particular, we show a concrete relation between the former and the BCFW representation of amplitudes.

Our specific focus is on the connected prescription [1] due to Roiban and the authors (see also [9–12]), a fascinating but mysterious formula which has been conjectured to encode the entire tree-level S matrix of SYM:

$$\mathcal{T}_{n,k}(\mathcal{Z}) = \int [d\mathcal{P}]_{k-1} d^n\sigma \prod_{i=1}^{n} \frac{\delta^{3|4}(\mathcal{Z}_i - \mathcal{P}(\sigma_i))}{\sigma_i - \sigma_{i+1}}. \quad (1)$$

Here, $\mathcal{P}(\sigma)$ denotes a $\mathbb{P}^{3|4}$-valued polynomial of degree $k-1$ in σ and $[d\mathcal{P}]_{k-1}$ is the natural measure on the space of such polynomials. We review further details shortly but pause to note that this formula simply expresses the content of Witten's twistor string theory: the N^{k-2}MHV superamplitude is computed as the integral of an open string current algebra correlator over the moduli space of degree $k-1$ curves in supertwistor space $\mathbb{P}^{3|4}$.

The formula (1) manifests several properties that scattering amplitudes must possess, including conformal invariance and cyclic symmetry of the superamplitude, both of which are hidden in other representations such as BCFW. It is also relatively easy to show that it possesses the correct soft and collinear-particle singularities, as well as (surprisingly) parity invariance [1,13]. Despite these conceptual strengths the connected prescription has received relatively little attention over the past five years because it has resisted attempts to relate it directly to the more computationally useful BCFW recursion relation.

Here, we remedy this situation by showing for the first time a direct and beautiful relation between the connected prescription (1) and the BCFW recursion. Specifically, we demonstrate explicitly for $n = 6, 7$ (and expect a similar though more intricate story for general n) that different choices of integration contour in (1) compute different, but equivalent, representations of tree-level amplitudes.[1] The privileged contour singled out by the delta functions appearing in (1) computes the connected prescription representation in which the n particle N^{k-2}MHV amplitude is expressed as a sum of residues of the integrand over the roots of a polynomial of degree $\left\langle {n-3 \atop k-2} \right\rangle$ (where $\left\langle {a \atop b} \right\rangle$ are Eulerian numbers). Different representations of tree-level amplitudes, including BCFW representations as well as intermediate prescriptions similar to those of [14,16], are all apparently encoded in various residues of the integrand $\mathcal{T}_{n,k}$ and are computed by choosing various appropriate contours. The equivalence of different representations follows from the global residue theorem, a multidimensional analogue of Cauchy's theorem.

[1] It has been argued in [14] that the connected prescription can also be related to the Cachazo-Svrček-Witten representation [15] by a contour deformation in the moduli space of curves.

The integrand $\mathcal{T}$ has many residues in common with

$$\mathcal{L}_{n,k}(\mathcal{W}) = \int [dC]_{k\times n} \prod_{i=1}^{n} \frac{\delta^{4|4}(C_{\alpha i}\mathcal{W}_i)}{(i, i+1, \ldots, i+k-1)} \quad (2)$$

recently written down by Arkani-Hamed *et al.* [8]. Here, $[dC]_{k\times n}$ is the measure on the space of $k \times n$ matrices *modulo* left-multiplication by $GL(k)$ and $(m_1, \cdots, m_k)$ denotes the minor obtained from C by keeping only columns $m_1, \ldots, m_k$. Residues of both $\mathcal{T}$ and $\mathcal{L}$ compute BCFW representations of tree amplitudes. In addition, $\mathcal{T}$ also computes various other tree-level representations while $\mathcal{L}$ evidently computes parity-conjugate P(BCFW) representations at tree level as well as leading singularities of loop amplitudes. It is natural to wonder whether there exists some richer object $\mathcal{D}$ (for "dual"), which contains information about various connected and disconnected representations of amplitudes at tree level and at all loops. This could help shed further light on twistor string theory at the loop level.

It is not yet known which contour computes which object from the integrand $\mathcal{L}$. In contrast, as mentioned above, the connected prescription $\mathcal{T}$ comes equipped with a certain privileged contour that calculates the tree amplitude. Various other contours that compute different representations of the same amplitude can be easily determined by applying the global residue theorem. We hope that a better understanding of the relation between $\mathcal{L}$ and $\mathcal{T}$ may allow us to transcribe information about the privileged contour from the latter to the former.

In Sec. II, we review the connected prescription for computing scattering amplitudes and derive its link representation by Fourier transforming it to mixed $\mathcal{Z}$, $\mathcal{W}$ variables. In Sec. III, we demonstrate the precise relation between the connected prescription, BCFW and intermediate representations of all six- and seven-particle amplitudes.

II. LINKING THE CONNECTED PRESCRIPTION

Let us begin by reviewing some details of the connected prescription formula (1) for the color-stripped n particle N^{k-2}MHV scattering amplitude. The $4|4$ component homogeneous coordinates for the ith particle in $\mathbb{P}^{3|4}$ are $\mathcal{Z}_i = (\lambda_i^\alpha, \mu_i^{\dot\alpha}, \eta_i^A)$ with $\alpha, \dot\alpha = 1, 2$ and $A = 1, 2, 3, 4$. In split signature $--++$ the spinor helicity variables $\lambda_i^\alpha, \tilde\lambda_i^{\dot\alpha}$ can be taken as independent real variables and the twistor transform realized in the naive way as a Fourier transform from $\tilde\lambda_i^{\dot\alpha}$ to $\mu_i^{\dot\alpha}$.

As emphasized in [1] (see also [12]) the integral (1) must be interpreted as a contour integral in a multidimensional complex space. The delta functions specify the contour of integration according to the usual rule

$$\int d^m z h(\vec z) \prod_{i=1}^{m} \delta(f_i(\vec z)) = \sum h(\vec z)\left[\det \frac{\partial f_i}{\partial z_j}\right]^{-1} \quad (3)$$

with the sum taken over the set of $\vec z_*$ satisfying $f_1(\vec z_*) = \cdots = f_m(\vec z_*) = 0$. In practice, the calculation of any n particle N^{k-2}MHV amplitude therefore reduces to the problem of solving certain polynomial equations that appear to have $\left\langle {n-3 \atop k-2} \right\rangle$roots in general.

To write the connected formula slightly more explicitly we first express the delta functions on $\mathbb{P}^{3|4}$ in terms of homogeneous coordinates via the contour integral

$$\delta^{3|4}(\mathcal{Z} - \mathcal{Z}') = \int \frac{d\xi}{\xi} \delta^{4|4}(\mathcal{Z} - \xi \mathcal{Z}'). \quad (4)$$

Next, we parameterize the degree $k-1$ polynomial $\mathcal{P}$ in terms of its k $\mathbb{C}^{4|4}$-valued supercoefficients $\mathcal{A}_d$ as

$$\mathcal{P}(\sigma) = \sum_{d=0}^{k-1} \mathcal{A}_d \sigma^d. \quad (5)$$

Using these ingredients (1) may be expressed as

$$\mathcal{A}(\mathcal{Z}) = \int \frac{d^{4k|4k}\mathcal{A} d^n\sigma d^n\xi}{\mathrm{vol}GL(2)} \prod_{i=1}^{n} \frac{\delta^{4|4}(\mathcal{Z}_i - \xi_i \mathcal{P}(\sigma_i))}{\xi_i(\sigma_i - \sigma_{i+1})}, \quad (6)$$

where we have indicated that the integrand and measure are invariant under a $GL(2)$ acting as Möbius transformations of the σ_i combined with a simultaneous compensating reparameterization of the curve $\mathcal{P}(\sigma)$. This symmetry must be gauge fixed in the usual way.

Motivated by [7] we now consider expressing the connected prescription (1) in a mixed representation where some of the particles are specified in terms of the $\mathcal{Z}$ variables as above, while others are specified in terms of the variables $\mathcal{W} = (\tilde\mu^{\dot a}, \tilde\lambda^{\dot a}, \tilde\eta_A)$ related by Fourier transform

$$\mathcal{F}(\mathcal{W}) = \int d^{4|4}\mathcal{Z} F(\mathcal{Z}) e^{i\mathcal{W}\cdot\mathcal{Z}}, \quad (7)$$

where $\mathcal{W}\cdot\mathcal{Z} = \tilde\mu \cdot \lambda - \mu \cdot \tilde\lambda + \eta \cdot \tilde\eta$. A particularly convenient choice for the N^{k-2}MHV amplitude is to leave precisely k particles in terms of $\mathcal{Z}$ and transform the rest to $\mathcal{W}$. This replaces the $4n|4n$ delta functions in (6) with

$$\prod_i \exp(i\xi_i \mathcal{W}_i \cdot \mathcal{P}(\sigma_i)) \prod_J \delta^{4|4}(\mathcal{Z}_J - \xi_J \mathcal{P}(\sigma_J)). \quad (8)$$

Here and in all that follows, it is implicit that sums or products over i run over the subset of the n particles expressed in the $\mathcal{W}$ variables, while sums or products over J run over the particles expressed in terms of $\mathcal{Z}$'s.

The utility of our choice is that there are now precisely as many delta functions as supermoduli $\mathcal{A}$, which moreover can be integrated out trivially since they appear linearly inside delta functions. This operation sets

$$\mathcal{P}(\sigma) = \sum_J \frac{\mathcal{Z}_J}{\xi_J} \prod_{K\neq J} \frac{\sigma_K - \sigma}{\sigma_K - \sigma_J}, \quad (9)$$

which is easily seen to satisfy $\mathcal{P}(\sigma_J) = \mathcal{Z}_J/\xi_J$. The resulting expression for the integral may be cleaned up with the help of the change of variables

$$x_i = \xi_i \prod_K (\sigma_K - \sigma_i), \qquad x_J^{-1} = \xi_J \prod_{K \neq J} (\sigma_K - \sigma_J), \quad (10)$$

which (ignoring for the moment overall signs) transforms (6) into an integral that can be put into the form of a link representation

$$\mathcal{A}(\mathcal{W}_i, \mathcal{Z}_J) = \int dc_{iJ} U(c_{iJ}) e^{ic_{iJ}\mathcal{W}_i \cdot \mathcal{Z}_J} \quad (11)$$

(as introduced in [7]) with the integrand given by

$$U(c_{iJ}) = \frac{1}{\mathrm{vol}GL(2)} \int \prod_{a=1}^{n} \frac{d\sigma_a dx_a}{x_a(\sigma_a - \sigma_{a+1})} \times \prod_{i,J} \delta\left(c_{iJ} - \frac{x_i x_J}{\sigma_J - \sigma_i}\right). \quad (12)$$

Note that this expression still requires $GL(2)$ gauge fixing. Usually this is accomplished by freezing four variables σ_1, σ_2, σ_3, x_1 to arbitrary values with the Jacobian

$$\frac{1}{\mathrm{vol}GL(2)} \int d\sigma_1 d\sigma_2 d\sigma_3 dx_1 = x_1(\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3)(\sigma_3 - \sigma_1). \quad (13)$$

Consequently, in (12) there are effectively only $2n - 4$ integration variables and $k(n - k - 4)$ delta functions, so that after integrating out the x's and σ's there remain in U a net $(k - 2)(n - k - 2)$ delta functions.

Henceforth we will work with component amplitudes rather than superamplitudes. The formula (11) still holds with the same $U(c_{iJ})$ given in (12), but with $\mathcal{W}_i$ and $\mathcal{Z}_J$ now replaced, respectively, by W_i and Z_J. Futhermore, we find it convenient to always choose all negative helicity gluons to be expressed in terms of Z variables and all positive helicity gluons to be expressed in terms of W variables.

As emphasized in [7] an important feature of the link representation is that returning to physical space is simple because the Fourier transforms $\mu_i^{\dot\alpha} \to \tilde\lambda_i^{\dot\alpha}$, $\tilde\mu_i^{\alpha} \to \lambda_i^{\alpha}$ turn the exponential factors in (11) into

$$\prod_i \delta^2(\lambda_i^\alpha - c_{iJ}\lambda_J^\alpha) \prod_J \delta^2(\tilde\lambda_J^{\dot\alpha} + c_{iJ}\tilde\lambda_i^{\dot\alpha}). \quad (14)$$

For given kinematics $(\lambda_i^\alpha, \tilde\lambda_i^{\dot\alpha})$ these equations fix the $k(n - k)$ c_{iJ} as linear functions of $(k - 2)(n - k - 2)$ remaining free parameters denoted τ_γ. Finally, we obtain the physical space amplitude in terms of U as

$$\mathcal{A}(\lambda, \tilde\lambda) = J\delta^4\left(\sum p_i\right) \int d^{(k-2)(n-k-2)}\tau U(c_{iJ}(\tau_\gamma)), \quad (15)$$

where J is the Jacobian from integrating out (14). We will always implicitly choose for simplicity a parameterization of $c_{iJ}(\tau_\gamma)$ for which $J = 1$. Before proceeding let us again emphasize that each $c_{iJ}(\tau_\gamma)$ is linear in the τ's.

III. EXAMPLES

For the trivial case of maximally helicity violating amplitudes ($k = 2$) the remaining integrations are easily carried out, leading to

$$U^{--+\cdots+} = \frac{1}{c_{31}c_{n2}} \prod_{i=3}^{n-1} \frac{1}{c_{i,i+1:1,2}} \quad (16)$$

in terms of $c_{ij;KL} = c_{iK}c_{jL} - c_{iL}c_{jK}$. The $\overline{\mathrm{MHV}}$ case $k = n - 2$ yields the same result with $c_{ab} \to c_{ba}$. When transformed to physical space using (15) these yield, respectively, the Parke-Taylor formula and its conjugate.

A. Six-point amplitudes

Next we consider the six-particle alternating helicity amplitude, for which we find from (12) the representation

$$U^{+-+-+-} = \frac{1}{c_{14}c_{36}c_{52}} \delta(S_{135:246}), \quad (17)$$

where S refers to the sextic polynomial

$$S_{ijk:rst} = c_{is}c_{kt}c_{jk:rs}c_{ij:tr} - c_{it}c_{ks}c_{jk:tr}c_{ij:rs}. \quad (18)$$

In this example the appearance of $\delta(S_{135:246})$ can be understood as follows: we are trying to express nine variables c_{iJ} in terms of eight variables (the x's and σ's) by solving the delta-function equations

$$c_{iJ} = \frac{x_i x_J}{\sigma_J - \sigma_i}. \quad (19)$$

A solution to this overconstrained set of equations for the c_{iJ} exists if and only if the sextic $S_{135:246}$ vanishes.

From (17) we arrive at the expression

$$A^{+-+-+-} = \int d\tau \frac{1}{c_{14}c_{36}c_{52}} \delta(S_{135:246}) \quad (20)$$

for the physical space amplitude. In this case $S_{135:246}$ is quartic in the single τ parameter. By choosing numerical values for the external kinematics and summing over the four roots of $S_{135:246}$ one can verify that (20) reproduces the correct amplitude.

Now consider more generally the object

$$\frac{1}{c_{14}c_{36}c_{52}} \frac{1}{S_{135:246}} \quad (21)$$

as a function of τ. The contour integral of this object around the four zeroes of $S_{135:246}$ evidently computes the alternating helicity six-particle amplitude. But (21) has three other poles located at the vanishing of c_{14}, c_{36} or c_{52}. By Cauchy's theorem we know that the sum of these three residues computes minus the amplitude,

$$A^{+-+-+-} = -\int d\tau \frac{1}{S_{135:246}} \delta(c_{14}c_{36}c_{52}). \qquad (22)$$

Since the c_{iJ} are linear in τ it is simple to calculate the corresponding residues analytically, and one obtains

$$\frac{[13]^4\langle 46\rangle^4}{[12][23]\langle 45\rangle\langle 56\rangle s_{123}\langle 6|5+4|3]\langle 4|5+6|1]} + (i \to i+2) + (i \to i+4), \qquad (23)$$

which is the BCFW representation for the amplitude.

Analysis of the other two independent six-particle helicity configurations proceeds along the same lines with link representations obtained from (12):

$$U^{+++---} = \frac{c_{25}}{c_{12:45}c_{23:56}} \delta(S_{123:456}), \qquad (24)$$

$$U^{++-+--} = \frac{c_{16}}{c_{13}c_{46}c_{12:56}} \delta(S_{124:356}). \qquad (25)$$

In each case the connected presentation expresses the amplitude as a sum over the four roots of the quartic $S_{ijk:lmn}$ in the τ plane, which a simple application of Cauchy's theorem relates to a sum over simple linear roots, which compute the BCFW representation of the amplitude.

B. Seven-point amplitudes

For the seven-particle split-helicity amplitude we find

$$U^{++++---} = \frac{c_{25}c_{26}c_{36}c_{37}}{c_{12:56}c_{34:67}} \delta(S_{123:567})\delta(S_{234:567}). \qquad (26)$$

There are now two τ variables, and the locus where both of the delta functions vanish consists of 14 isolated points in $\mathbb{C}^2$. The coordinates of these points are determined by the vanishing of a polynomial, which is a product of one of degree 11 and three of degree 1. The three linear roots do not contribute because the numerator factors in (26) vanish there. Therefore, (26) represents the amplitude as a sum over the roots of a degree 11 polynomial, as expected for the connected prescription for $n = 7$, $k = 3$.

To proceed we must use the multidimensional analogue of Cauchy's theorem known as the global residue theorem:

$$\oint_{f_1=\cdots=f_n=0} d^n z \frac{h(z)}{f_1(z)\cdots f_n(z)} = 0 \qquad (27)$$

when $h(z)$ is a polynomial of degree less than $\sum \deg f_i - (n+1)$, so that it has no poles at finite z and the integrand falls off sufficiently fast to avoid a pole at infinity.

To apply (27) to (26) we consider the integrand

$$\frac{c_{25}c_{26}c_{36}c_{37}}{c_{12:56}c_{34:67}} \frac{1}{S_{123:567}S_{234:567}}. \qquad (28)$$

There are seven independent ways of grouping the terms in the denominator into a product $f_1 f_2$. The choice

$$f_1 = c_{12:56}S_{234:567}, \qquad f_2 = c_{34:67}S_{123:567} \qquad (29)$$

is particularly nice: in this application of the global residue theorem all 11 poles at the locus $S_{123:567} = S_{234:567} = 0$ contribute as do the roots located at

$$c_{12:56} = S_{123:567} = 0, \qquad (30)$$

$$c_{34:67} = S_{234:567} = 0, \qquad (31)$$

$$c_{12:56} = c_{34:67} = 0, \qquad (32)$$

which amazingly turn out to each consist of a single linear root. The global residue theorem expresses the connected representation of the amplitude as (minus) the sum of these three linear roots, which a simple calculation reveals as precisely the three terms contributing to the BCFW representation of the amplitude.

Equally amazing is the choice

$$f_1 = S_{123:567}, \qquad f_2 = c_{12:56}c_{34:67}S_{234:567}. \qquad (33)$$

This contour computes the sum of residues at 15 poles; 11 of those are the connected prescription poles, which we know compute the correct physical amplitude, while the others consist of a single linear root together with four quartic roots. Schematically then this global residue theorem identity expresses

$$A^{++++---} = \sum 11 \text{ roots} = -\sum 4 \text{ roots} - 1 \text{ roots}. \qquad (34)$$

We interpret the right-hand side of this equation as an "intermediate" prescription [14,16], obtained by BCFW decomposing $A^{++++---}$ once into the product of a three-particle amplitude with a split-helicity six-particle amplitude, and then computing the latter via the connected prescription as a sum over four roots.

We end by tabulating link representations for the remaining independent seven-particle helicity amplitudes

$$\begin{aligned} U^{+++-+--} &= \frac{c_{26}c_{27}c_{25:46}}{c_{12:46}c_{23:67}} \delta(S_{125:467})\delta(S_{235:467}), \\ U^{++-++--} &= \frac{c_{23}c_{56}c_{57}c_{25:36}}{c_{53}c_{12:36}c_{45:67}} \delta(S_{125:367})\delta(S_{245:367}), \\ U^{++-+-+-} &= \frac{c_{17}c_{43}c_{14:57}}{c_{47}c_{63}c_{12:57}} \delta(S_{124:357})\delta(S_{146:357}). \end{aligned} \qquad (35)$$

As usual we interpret $\delta(u) = 1/u$ in the integrand with the delta functions indicating the preferred contour, which computes the connected prescription representation of the amplitude.

IV. CONCLUSION

It is obviously of great interest to extend the analysis of this paper showing the concrete relation between the connected prescription and the BCFW on-shell recursion beyond the examples considered here. Link representations for any desired amplitude may be obtained straightforwardly from the general formula (12). Shortly after our paper appeared, explicit formulas for certain classes of helicity configurations were presented in the overlapping paper [17].

The difficulty in extending the contour deformation analysis to further cases is that the integrals in the link representations no longer localize at discrete points in the τ variables. For example, for $n = 8$, $k = 3$ the link representation for the connected prescription involves three delta functions of sextics, but it is easy to check that the locus where the three sextics vanish is actually a line in $\mathbb{C}^3$ rather than isolated points, so it is difficult to interpret the integral via (3). We emphasize that no such difficulty exists when the connected prescription is formulated in terms of the original variables x and σ, where for $n = 8$, $k = 3$ one indeed finds 26 isolated roots in (x, σ) space.

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