

Solutions of coupled BPS equations for two-family Calogero and matrix models

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 (Received 19 August 2009; published 13 October 2009)

We consider a large N , two-family Calogero and matrix model in the Hamiltonian, collective-field approach. The Bogomol'nyi limit appears and the solutions to the coupled Bogomol'nyi-Prasad-Sommerfeld equations are given by the static soliton configurations. We find all solutions close to constant and construct exact one-parameter solutions in the strong-weak dual case. Full classification of these solutions is presented.

DOI: 10.1103/PhysRevD.80.085014

PACS numbers: 03.65.Sq, 05.45.Yv, 11.10.Kk, 11.15.Pg

I. INTRODUCTION

Recently, a specific duality-based generalization of the Hermitian matrix model has been analyzed [1]. The existence of two collective fields allows one to interpret the generalized matrix model as a two-family Calogero model, implying their equivalence in the collective-field approach. The soliton sector of these models was studied in Refs. [1,2]. The multivortex solutions of the coupled Bogomol'nyi-Prasad-Sommerfeld (BPS) equations were interpreted as giant gravitons [3]. Moreover, the multivortex singular solutions have motivated the authors of Refs. [1,2] to propose a realization of open-closed string duality [4]. The same BPS equations were studied in Refs. [5–7], and no multivortex solutions were found.

The purpose of this paper is to give a systematic perturbative method for solving coupled BPS equations and find all solutions close to the constant solution. In addition, we present the construction of all one-parameter exact solutions which can be continuously connected to the constant solution, in the strong-weak dual variant of the model.

The outline of the paper is as follows. In Sec. II, we briefly sketch the collective-field derivation of coupled BPS equations of the two-family Calogero model. In Sec. III, we propose a perturbative method to find the periodic solutions close to constant. In Sec. IV, we present the construction of a one-parameter solution. Section V is devoted to detailed classification of all solutions, which can be continuously connected with the constant solution, in terms of a new, more suitable parameter. Finally, in Sec. VI, we present our discussion and conclusions.

II. TWO-FAMILY MODELS IN THE COLLECTIVE-FIELD APPROACH

The Hamiltonian of the two-family Calogero system [8] reads

$$H = -\frac{1}{2m_1} \sum_{i=1}^{N_1} \frac{\partial^2}{\partial x_i^2} + \frac{\lambda_1(\lambda_1 - 1)}{2m_1} \sum_{i \neq j}^{N_1} \frac{1}{(x_i - x_j)^2} - \frac{1}{2m_2} \sum_{\alpha=1}^{N_2} \frac{\partial^2}{\partial x_\alpha^2} + \frac{\lambda_2(\lambda_2 - 1)}{2m_2} \sum_{\alpha \neq \beta}^{N_2} \frac{1}{(x_\alpha - x_\beta)^2} + \frac{1}{2} \left(\frac{1}{m_1} + \frac{1}{m_2} \right) \lambda_{12}(\lambda_{12} - 1) \sum_{i=1}^{N_1} \sum_{\alpha=1}^{N_2} \frac{1}{(x_i - x_\alpha)^2}. \quad (1)$$

Here, the first family contains N_1 particles of mass m_1 at positions x_i , $i = 1, 2, \dots, N_1$, and the second one contains N_2 particles of mass m_2 at positions x_α , $\alpha = 1, 2, \dots, N_2$. All particles interact via two-body inverse-square potentials.

The interaction strength between particles of the first and the second families is parametrized by λ_{12} . The interaction strengths within each family are parametrized by the coupling constants λ_1 and λ_2 , respectively. In Eq. (1), we imposed the restriction that there be no three-body interactions, which requires [9–11]

$$\frac{\lambda_1}{m_1^2} = \frac{\lambda_2}{m_2^2} = \frac{\lambda_{12}}{m_1 m_2}. \quad (2)$$

It follows from (2) that

$$\lambda_{12}^2 = \lambda_1 \lambda_2. \quad (3)$$

In addition, we restrict ourselves to the so-called strong-weak dual variant of the model in which the coupling parameters are related by

$$\lambda_1 \lambda_2 = \lambda_{12} = 1. \quad (4)$$

In Ref. [5], we studied the collective-field theory of the two-family Calogero model given by Hamiltonian (1). The corresponding collective Hamiltonian is

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$$\begin{aligned}
 H_{\text{coll}} = & \frac{1}{2m_1} \int dx \rho_1(x) (\partial_x \pi_1(x))^2 + \frac{1}{2m_1} \int dx \rho_1(x) \\
 & \times \left(\frac{\lambda_1 - 1}{2} \frac{\partial_x \rho_1}{\rho_1} - \lambda_1 \pi \rho_1^H(x) - \lambda_{12} \pi \rho_2^H(x) \right)^2 \\
 & + \frac{1}{2m_2} \int dx \rho_2(x) (\partial_x \pi_2(x))^2 + \frac{1}{2m_2} \int dx \rho_2(x) \\
 & \times \left(\frac{\lambda_2 - 1}{2} \frac{\partial_x \rho_2}{\rho_2} - \lambda_2 \pi \rho_2^H(x) - \lambda_{12} \pi \rho_1^H(x) \right)^2, \quad (5)
 \end{aligned}$$

which is a straightforward generalization of (1). Here ρ_1 and ρ_2 are the collective density fields of the first and the second family, respectively, and π_1 and π_2 are their conjugate momenta. The collective fields ρ_1 and ρ_2 are normalized as

$$\int dx \rho_1(x) = N_1, \quad \int dx \rho_2(x) = N_2. \quad (6)$$

Designation $\rho^H(x)$ denotes the Hilbert transform of $\rho(x)$:

$$\rho^H(x) = \frac{1}{\pi} \int \frac{dy \rho(y)}{y - x}. \quad (7)$$

The Hamiltonian (5) is essentially the sum of two positive terms. Its zero-energy classical solutions are zero-momentum, and therefore time-independent, configurations of the collective fields, which are also solutions of the coupled BPS equations

$$\frac{\lambda_1 - 1}{2} \frac{\partial_x \rho_1}{\rho_1} - \lambda_1 \pi \rho_1^H(x) - \lambda_{12} \pi \rho_2^H(x) = 0, \quad (8)$$

$$\frac{\lambda_2 - 1}{2} \frac{\partial_x \rho_2}{\rho_2} - \lambda_2 \pi \rho_2^H(x) - \lambda_{12} \pi \rho_1^H(x) = 0. \quad (9)$$

Finding the general exact solutions of these coupled equations for arbitrary couplings and masses is still an open problem, which we briefly discuss in Sec. VI. Our main interest here is simpler and concerns investigating whether the above pair of equations is exactly solvable for a special choice of parameters (4).

We observe that ρ_1 and ρ_2 are connected in a very simple way. Namely, by multiplying the second equation (9) with λ_1 and subtracting it from the first equation (8), we easily get

$$\partial_x \ln(\rho_1 \rho_2) = 0, \quad \text{or} \quad \rho_1 \rho_2 = c, \quad (10)$$

where c is some positive constant. This is a direct consequence of the special choice (4).

Now, eliminating ρ_2 in terms of ρ_1 , we can concentrate only on the first equation (8):

$$\frac{\lambda_1 - 1}{2} \frac{\partial_x \rho_1}{\rho_1} = \lambda_1 \pi \rho_1^H(x) + \pi c \left(\frac{1}{\rho_1} \right)^H, \quad (11)$$

which upon introducing abbreviations $a = \frac{2\lambda_1 \pi}{\lambda_1 - 1}$ and $b =$

$\frac{2c\pi}{\lambda_1 - 1}$ takes the form

$$\frac{d}{dx} \ln \rho(x) = a \rho^H(x) + b \left(\frac{1}{\rho(x)} \right)^H. \quad (12)$$

III. SOLUTIONS CLOSE TO CONSTANT

By having in mind the fact that the Hilbert transform of a constant is zero [12], it is evident that there always exists a uniform solution of Eq. (12):

$$\rho(x) = \rho_0. \quad (13)$$

Now we are going to construct the most general solutions of Eq. (12) which are continuously connected with constant solution (13).

They are of the form

$$\rho(x) = \rho_0 \sum_{i=0}^{\infty} \varepsilon^i \varphi_i(x), \quad (14)$$

where ε is a small parameter ($\varepsilon \ll 1$) and $\varphi_0(x) = 1$.

Expanding $\frac{d}{dx} \ln \rho$ and $(\frac{1}{\rho})^H$ in powers of small parameter ε , we find the infinite set of recursive equations for $\varphi_i(x)$ of the form:

$$\varphi_1' = k_0 \varphi_1^H, \quad (15a)$$

$$\varphi_2' = k_0 \varphi_2^H + \frac{1}{2} (\varphi_1^2)' + \frac{b}{\rho_0} (\varphi_1^2)^H, \quad (15b)$$

$$\begin{aligned}
 \varphi_3' = & k_0 \varphi_3^H + (\varphi_1 \varphi_2)' + \frac{2b}{\rho_0} (\varphi_1 \varphi_2)^H - \frac{1}{3} (\varphi_1^3)' \\
 & - \frac{b}{\rho_0} (\varphi_1^3)^H, \quad (15c)
 \end{aligned}$$

⋮

where

$$k_0 = a \rho_0 - \frac{b}{\rho_0} \quad (16)$$

and the prime denotes derivative with respect to x . The solutions up to the second order in ε expansion are given (up to a phase) by

$$\varphi_1(x) = c_1 \cos k_0 x, \quad (17a)$$

$$\varphi_2(x) = c_2 \cos 2k_0 x + \tilde{c}_1 \cos k_0 x, \quad (17b)$$

where constants c_1 and c_2 are interrelated via

$$c_2 = \frac{a \rho_0}{2k_0} c_1^2, \quad (18)$$

and c_1 , and \tilde{c}_1 are arbitrary. Note that \tilde{c}_1 can be removed by redefinition of parameter $\varepsilon \rightarrow \tilde{\varepsilon} = \varepsilon + \frac{\tilde{c}_1}{c_1} \varepsilon^2$, up to the second order in ε .

In obtaining φ_1 and φ_2 , we have used well-known Hilbert transform formulas [12]

$$(\cos kx)^H = -\text{sign}k \sin kx, \quad (19a)$$

$$(\sin kx)^H = \text{sign}k \cos kx. \quad (19b)$$

Relations (17a) and (17b) suggest that $\rho(x)$ is a periodic function. Therefore, we restrict our search for solutions $\rho(x)$ to periodic functions. In this respect, we accommodate our expansion (14):

$$\rho(x) = \rho_0 \sum_{i=0}^{\infty} \varepsilon^i C_i \cos ikx. \quad (20)$$

Inserting this expansion into the infinite set of recursive relations (15), we get

$$C_i = \sum_{m=0}^{\infty} C_{i,2m} \varepsilon^{2m}, \quad (21a)$$

$$k = \sum_{m=0}^{\infty} k_{2m} \varepsilon^{2m}, \quad (21b)$$

$$C_{i,0} = C_{1,0} \left(\frac{C_{1,0}}{2k_0} a \rho_0 \right)^{i-1}. \quad (21c)$$

All other periodic solutions can be brought to the form (20) by a suitable change of parameter ε .

Let us now define truncated function

$$\rho_n(x) = \rho_0 \sum_{i=0}^n \varepsilon^i [C_i \cos ikx]_{(n-i)}, \quad (22)$$

where an index in parentheses denotes the order of the polynomial in parameter ε . Then it is easy to see that

$$\varepsilon^n \varphi_n(x) = \rho_n(x) - \rho_{n-1}(x). \quad (23)$$

In this way we can rederive relations (17a) and (17b) and obtain, for example, $\varphi_3(x)$:

$$\begin{aligned} \varepsilon^3 \varphi_3(x) &= \varepsilon^3 C_{3,0} \cos 3k_0 x + \varepsilon C_{1,0} [\cos(k_0 + k_2 \varepsilon^2)x]_{(2)} \\ &+ \varepsilon [C_{1,2} \varepsilon^2 - C_{1,0}] \cos k_0 x, \end{aligned} \quad (24)$$

where

$$C_{3,0} = C_{1,0} \left(\frac{C_{1,0}}{2k_0} a \rho_0 \right)^2, \quad (25)$$

$$k_2 = \frac{1}{2} k_0 \left(\frac{b}{a \rho_0^2} \right)^2 \left(\frac{C_{1,0}}{2k_0} a \rho_0 \right)^2. \quad (26)$$

In the next section, we shall give the analytic form for the periodic solution $\rho(x)$, $k > 0$, and the aperiodic solutions in the limit $k \rightarrow 0_+$.

Finally, we note that in the special case $b = 0$, the expansion (20) can be easily summed. Namely, in this case

$$k = k_0 = a \rho_0, \quad (27)$$

and choosing $C_{1,0} = 1$ we find

$$C_i = 2^{1-i}, \quad i \geq 1,$$

$$\rho(x) = \rho_0 \left[1 + \sum_{i=1}^{\infty} \frac{\varepsilon^i}{2^{i-1}} c_i \cos ikx \right], \quad (28)$$

$$= \rho_0 \frac{\sqrt{1 - e^2}}{1 - e \cos k_0 x}, \quad e = \frac{\varepsilon}{1 + \frac{\varepsilon^2}{4}} \leq 1. \quad (29)$$

Note that the expansion in e is different from the expansion in ε , but they are both of the type given by Eq. (20) and describe the same solution. The one-parameter solutions given in (29) coincide with corresponding solutions in Refs. [2,7,13].

IV. CONSTRUCTION OF ONE-PARAMETER SOLUTIONS

Let us now show that there exists a particular one-parameter family of solutions to (12) which is not necessarily close to constant. To this end we define a family of functions $R_\varepsilon(x)$ depending on parameter ε and wave vector k :

$$R_\varepsilon(x) = \frac{\sqrt{1 - \varepsilon^2}}{1 - \varepsilon \cos kx}, \quad (30)$$

where $|\varepsilon| \leq 1$, $\varepsilon \in \mathbf{R}$, and $k > 0$.

These functions satisfy

$$\overline{R_\varepsilon(x)} = \lim_{L \rightarrow \infty} \frac{1}{2L} \int_{-L}^L dx R_\varepsilon(x) = 1. \quad (31)$$

We can easily verify that R_ε can be written as

$$\begin{aligned} R_\varepsilon(x) &= 1 + \frac{\gamma e^{ikx}}{1 - \gamma e^{ikx}} + \frac{\gamma e^{-ikx}}{1 - \gamma e^{-ikx}} \\ &= \frac{1 - \gamma^2}{1 + \gamma^2 - 2\gamma \cos kx}, \end{aligned} \quad (32)$$

where $|\gamma| < 1$, $\gamma \in \mathbf{R}$, and $\varepsilon = \frac{2\gamma}{1 + \gamma^2}$.

Since function $\frac{e^{ikx}}{1 - \gamma e^{ikx}}$ is analytic in the x -upper-half-plane for $k > 0$ and vanishes at infinity in that half-plane, then

$$\left(\frac{e^{ikx}}{1 - \gamma e^{ikx}} \right)^H = i \frac{e^{ikx}}{1 - \gamma e^{ikx}}. \quad (33)$$

Similarly, we have

$$\left(\frac{e^{-ikx}}{1 - \gamma e^{-ikx}} \right)^H = -i \frac{e^{-ikx}}{1 - \gamma e^{-ikx}}. \quad (34)$$

Making use of (32)–(34), we obtain

$$(R_\varepsilon)^H = -\frac{\varepsilon \sin kx}{1 - \varepsilon \cos kx} = \frac{1}{k} (\ln R_\varepsilon)'. \quad (35)$$

This is nothing but the one-family Calogero model BPS equation. Namely, for $b = 0$, Eq. (12) reduces to

$$\frac{\rho'}{\rho} = a\rho^H, \quad (36)$$

implying that corresponding one-parameter solutions are in fact given by

$$\rho(x) = \rho_0 R_\varepsilon(x), \quad (37)$$

where $k = a\rho_0 > 0$.

We are now in a position to construct a class of one-parameter solutions of coupled BPS equations (8) and (9). Let us proceed in a few steps. First note that

$$\left(\frac{R_\varepsilon}{R_\eta}\right)^H = -\frac{R_\varepsilon}{R_\eta} \left(\frac{R_\eta}{R_\varepsilon}\right)^H, \quad (38)$$

$$R_\eta(x) = \frac{\sqrt{1-\eta^2}}{1-\eta \cos kx}, \quad (39)$$

where $|\eta| \leq 1$, $\eta \in \mathbf{R}$. Using this important relation we construct a general solution of Eq. (12) given by two equivalent forms

$$\rho(x) = s \frac{R_\varepsilon(x)}{R_\eta(x)} = r_0 R_\varepsilon(x) + \alpha, \quad (40)$$

where $s > 0$, $r_0, \alpha \in \mathbf{R}$, and $\overline{\rho(x)} = \rho_0$.

Namely, after inserting solution (40) into Eq. (12), we find

$$\begin{aligned} (\ln \rho)' &= k[R_\varepsilon - R_\eta]^H \\ &= \frac{as}{\sqrt{1-\eta^2}} \left(1 - \frac{\eta}{\varepsilon}\right) R_\varepsilon^H + \frac{b}{s\sqrt{1-\varepsilon^2}} \left(1 - \frac{\varepsilon}{\eta}\right) R_\eta^H. \end{aligned} \quad (41)$$

The construction (40) satisfies Eq. (12), provided that

$$k = \frac{as(\varepsilon - \eta)}{\varepsilon\sqrt{1-\eta^2}} = \frac{b(\varepsilon - \eta)}{s\eta\sqrt{1-\varepsilon^2}} > 0. \quad (42)$$

From the $\overline{\rho(x)} = \rho_0$ condition, we find

$$\rho_0 = r_0 + \alpha = \frac{s(1-\frac{\eta}{\varepsilon})}{\sqrt{1-\eta^2}} + \frac{s\eta\sqrt{1-\varepsilon^2}}{\varepsilon\sqrt{1-\eta^2}}. \quad (43)$$

Solving Eqs. (42) and (43), we find parameters s and η as functions of ε and $B = \frac{b}{a\rho_0}$. The solutions are

$$\begin{aligned} s(B, \varepsilon) &= \sqrt{\frac{b\varepsilon}{a\eta} \frac{\sqrt{1-\eta^2}}{\sqrt{1-\varepsilon^2}}} \\ &= \rho_0 \left[1 + \frac{\varepsilon^2}{2} B(1-B) + \mathcal{O}(\varepsilon^4) \right], \end{aligned} \quad (44)$$

$$\eta(B, \varepsilon) = B\varepsilon + \frac{1}{2}B(1-B)^2\varepsilon^3 + \mathcal{O}(\varepsilon^5), \quad (45)$$

where η satisfies the equation

$$B[\varepsilon - \eta(1 - \sqrt{1-\varepsilon^2})]^2 = \varepsilon\eta\sqrt{1-\eta^2}\sqrt{1-\varepsilon^2}. \quad (46)$$

Parameter k is fixed due to (42):

$$\begin{aligned} k &= a\rho_0 - \frac{b}{s} = a\rho_0\sqrt{B} \frac{\varepsilon - \eta}{\sqrt{\varepsilon\eta\sqrt{(1-\varepsilon^2)(1-\eta^2)}}} \\ &= k_0 \left[1 + \frac{1}{2}B^2\varepsilon^2 + \mathcal{O}(\varepsilon^4) \right]. \end{aligned} \quad (47)$$

Let us now discuss some special values of parameters and their implications. For $b = 0$ and $a > 0$, we have $B = 0$, $\eta = 0$, and $k = a\rho_0 > 0$. The solution (40) reduces to

$$\rho(x) = \rho_0 \frac{\sqrt{1-\varepsilon^2}}{1-\varepsilon \cos kx}, \quad (48)$$

found in Sec. III. For $a = 0$, we have $\varepsilon = 0$, $k = -\frac{b}{s} > 0$, and $s = \rho_0\sqrt{1-\eta^2}$. Note that $k > -\frac{b}{\rho_0}$. The corresponding solution reads

$$\rho(x) = \rho_0(1 - \eta \cos kx). \quad (49)$$

Finally, for $a \neq 0$ and $b \neq 0$, we find

$$\rho(x) = \rho_0 + \frac{k}{a} \left(\frac{\sqrt{1-\varepsilon^2}}{1-\varepsilon \cos kx} - 1 \right). \quad (50)$$

In other words, the general solution (50) is proportional to a solution (39) shifted by a constant $\alpha = \rho_0 - \frac{k}{a}$.

A couple of limiting cases of solution (50) are worth mentioning. If we let ε tend to zero, we obtain the small ε expansion (14), which was introduced in Sec. III. Note that, by choosing $C_{1,0} = \frac{k_0}{a\rho_0}$, both expansions of k (21b) and (47) are equal up to the second order in ε .

If we let k tend to zero (or, equivalently, let the period $\frac{2\pi}{k} \rightarrow \infty$) with $\eta \neq \varepsilon$, and $\eta, \varepsilon \rightarrow 1$, we obtain aperiodic solutions

$$\rho(x) = \rho_0 \frac{\frac{2(1-\eta)}{k^2} + x^2}{\frac{2(1-\varepsilon)}{k^2} + x^2}. \quad (51)$$

Let us note that, if $k \leq 0$, $\rho(x) = \rho_0$.

We will clarify and analyze these solutions in more detail in the next section.

V. SOLUTIONS OF TWO COUPLED BPS EQUATIONS

Having found the solution of Eq. (11), we are now in a position to find the pair of solutions $\rho_1(x)$ and $\rho_2(x)$ of coupled BPS equations (8) and (9). It is important to take into account the normalization conditions

$$\overline{\rho_1(x)} = \rho_{10} = \lim_{L \rightarrow \infty} \frac{N_1}{2L}, \quad (52)$$

$$\overline{\rho_2(x)} = \rho_{20} = \lim_{L \rightarrow \infty} \frac{N_2}{2L}, \quad (53)$$

which follow from the constraints (6). This means that both averages ρ_{10} and ρ_{20} are fixed. Namely, it is important to note that the numbers of particles N_1 and N_2 and the length of the system L are simultaneously taken to infinity, keeping the particle densities ρ_{10} and ρ_{20} fixed. Thus from

$$\rho_1(x) = s_1 \frac{R_\varepsilon(x)}{R_\eta(x)}, \quad (54)$$

$$\rho_2(x) = s_2 \frac{R_\eta(x)}{R_\varepsilon(x)}, \quad (55)$$

we easily get

$$c = \rho_1(x)\rho_2(x) = s_1 s_2. \quad (56)$$

We use the relation (43) to write the following expressions for ρ_{10} and ρ_{20} :

$$\rho_{10} = \frac{s_1}{\sqrt{1-\eta^2}} \left[1 - \frac{\eta}{\varepsilon} (1 - \sqrt{1-\varepsilon^2}) \right], \quad (57)$$

$$\rho_{20} = \frac{s_2}{\sqrt{1-\varepsilon^2}} \left[1 - \frac{\varepsilon}{\eta} (1 - \sqrt{1-\eta^2}) \right]. \quad (58)$$

Hence, from (56)–(58), we obtain

$$c = \rho_{10}\rho_{20} \frac{\sqrt{(1-\varepsilon^2)(1-\eta^2)}}{\left[1 - \frac{\eta}{\varepsilon} (1 - \sqrt{1-\varepsilon^2}) \right] \left[1 - \frac{\varepsilon}{\eta} (1 - \sqrt{1-\eta^2}) \right]}. \quad (59)$$

Therefore, after fixing both ρ_{10} and ρ_{20} , the parameters c and $b = \frac{2c\pi}{\lambda_1 - 1}$ become dependent on ε . Hence, Eq. (46) becomes

$$\mu[\varepsilon - \eta(1 - \sqrt{1-\varepsilon^2})] = \eta - \varepsilon(1 - \sqrt{1-\eta^2}), \quad (60)$$

where

$$\mu = \frac{\rho_{20}}{\lambda_1 \rho_{10}}. \quad (61)$$

Note that

$$\begin{aligned} k &= a\rho_{10} - \frac{b}{s_1} = \frac{2\pi}{\lambda_1 - 1} (\lambda_1 \rho_{10} - s_2) \\ &= \frac{2\pi}{\lambda_2 - 1} (\lambda_2 \rho_{20} - s_1) > 0 \end{aligned} \quad (62)$$

implies

$$\frac{s_2}{\lambda_1 \rho_{10}} \leq 1, \quad \frac{s_2}{\rho_{20}} \leq \frac{1}{\mu}. \quad (63)$$

Equation (60) has two solutions for η :

$$\frac{\eta_\pm}{\varepsilon} = \frac{(\mu + 1)^2 - \mu(\mu + 1)\sqrt{1-\varepsilon^2} \pm \sqrt{2\mu(\mu + 1)[1 - \varepsilon^2 - \sqrt{1-\varepsilon^2}] + 1}}{(\mu + 1)^2 + \mu^2(1 - \varepsilon^2) + \varepsilon^2 - 2\mu(\mu + 1)\sqrt{1-\varepsilon^2}}, \quad (64)$$

with restriction $\eta_\pm < \varepsilon$.

Let us now classify solutions $\rho_1(x)$ and $\rho_2(x)$ with respect to values of parameter μ .

(i) *Case* $\mu = 0$, $0 \leq \varepsilon \leq 1$.—From (62) and (64) it follows that

$$\eta_- = 0, \quad s_1 = \rho_{10}, \quad s_2 = \rho_{20}\sqrt{1-\varepsilon^2} = 0, \quad (65)$$

$$c = \rho_{10}\rho_{20}\sqrt{1-\varepsilon^2} = 0, \quad \text{and} \quad k = \frac{2\pi\lambda_1\rho_{10}}{\lambda_1 - 1} > 0. \quad (66)$$

It is obvious that this case corresponds to $\rho_{20}/\rho_{10} \rightarrow 0$, or the $\lambda_1 \rightarrow \infty$ limit. In the latter case (the strong coupling limit), $a = 2\pi$, $b = \frac{2\pi c}{\lambda_1 - 1} \rightarrow 0$, $\varepsilon \rightarrow 1$, $k = 2\pi\rho_{10}$, and solutions $\rho_1(x)$ and $\rho_2(x)$ are given by

$$\rho_1(x) = \rho_{10} \frac{\sqrt{1-\varepsilon^2}}{1 - \varepsilon \cos(2\pi\rho_{10}x)} \rightarrow \sum_{n \in \mathbb{Z}} \delta\left(x - \frac{n}{\rho_{10}}\right). \quad (67)$$

$$\rho_2(x) = \rho_{20}[1 - \cos(2\pi\rho_{10}x)]. \quad (68)$$

(ii) *Case* $0 < \mu < 1$, $0 \leq \varepsilon \leq 1$.—Solutions $\rho_1(x)$ and $\rho_2(x)$ are close to constants ρ_{10} and ρ_{20} for $\varepsilon \ll 1$. For $\varepsilon = 1$ they are given by

$$\rho_1(x) = \frac{\lambda_1 - 1}{\lambda_1} \sum_{n \in \mathbb{Z}} \delta\left(x - \frac{2\pi}{k_0} n\right), \quad (69)$$

$$\rho_2(x) = \rho_{20} \frac{(\mu + 1)[(\mu + 1)^2 - 1](1 - \cos k_0 x)}{\mu[(\mu + 1)^2 + 1][1 - \frac{(\mu + 1)^2 - 1}{(\mu + 1)^2 + 1} \cos k_0 x]}, \quad (70)$$

where

$$k_0 = \frac{2\pi\lambda_1}{\lambda_1 - 1} \rho_{10}. \quad (71)$$

(iii) *Case $\mu = 1$.*—The allowed range for ε is $\sqrt{3}/2 \leq \varepsilon \leq 1$. The function $\frac{\eta_-}{\varepsilon}$ in that range reads

$$\frac{\eta_-}{\varepsilon} = \frac{3}{5 - 4\sqrt{1 - \varepsilon^2}}. \quad (72)$$

In the interval $0 \leq \varepsilon \leq \sqrt{3}/2$, $\eta_- = \varepsilon$, while for the interval $\sqrt{3}/2 \leq \varepsilon \leq 1$, $\eta_+ = \varepsilon$. Hence solutions $\rho_1(x)$ and $\rho_2(x)$ reduce to constants ρ_{10} and ρ_{20} , respectively.

For $\varepsilon = 1$, the solutions are given by

$$\rho_1(x) = \frac{\lambda_1 - 1}{\lambda_1} \sum_{n \in \mathbb{Z}} \delta\left(x - \frac{2\pi}{k_0} n\right), \quad (73)$$

$$\rho_2(x) = 1.2\rho_{20} \frac{(1 - \cos k_0 x)}{(1 - 0.6 \cos k_0 x)}. \quad (74)$$

(iv) *Case $\mu > 1$.*—There are two solutions for $\frac{\eta_{\pm}}{\varepsilon}$ if $\varepsilon_{\text{cr}} \leq \varepsilon \leq 1$, where

$$\varepsilon_{\text{cr}} = \frac{1}{2} \sqrt{3 + \sqrt{1 - \frac{2}{\mu(\mu + 1)}}}. \quad (75)$$

At $\varepsilon = \varepsilon_{\text{cr}}$ we find $\eta_+ = \eta_- < \varepsilon_{\text{cr}}$. At $\varepsilon = 1$ we have

$$\frac{\eta_-}{\varepsilon} = \frac{(\mu + 1)^2 - 1}{(\mu + 1)^2 + 1} \quad \text{and} \quad \frac{\eta_+}{\varepsilon} = 1. \quad (76)$$

For $\eta_- = \frac{(\mu+1)^2-1}{(\mu+1)^2+1}$, $s_1 = \rho_{10}(\mu + 1)$, $s_2 = 0$, $c = 0$, and $k_0 = \frac{2\pi\lambda_1}{\lambda_1-1} \rho_{10}$, solutions $\rho_1(x)$ and $\rho_2(x)$ are periodic of type (69) and (70).

The solutions $\rho_1(x)$ and $\rho_2(x)$ are aperiodic for $\eta_+ \rightarrow 1$, $\varepsilon \rightarrow 1$, $s_1 = \mu\rho_{10}$, $s_2 = \frac{\rho_{20}}{\mu}$, $c = \rho_{10}\rho_{20}$, $\mu = \sqrt{(1 - \eta^2)/(1 - \varepsilon^2)}$, and $k \rightarrow 0_+$ and reduce to

$$\rho_1(x) = \rho_{10} \frac{\frac{2(1-\eta)}{k^2} + x^2}{\frac{2(1-\varepsilon)}{k^2} + x^2}, \quad (77)$$

$$\rho_2(x) = \rho_{20} \frac{\frac{2(1-\varepsilon)}{k^2} + x^2}{\frac{2(1-\eta)}{k^2} + x^2}. \quad (78)$$

These solutions were already described in Refs. [5,14]. For $\lambda_1 < 1$, the first solution (77) behaves like the hole in the condensate ρ_{10} , and the second one (78) behaves like the particle above the condensate ρ_{20} . The roles are interchanged for $\lambda_1 > 1$.

We point out that there are no other aperiodic solutions which can appear from our construction $\rho_1(x) = s_1 \frac{R_\varepsilon(x)}{R_\eta(x)}$ and $\rho_2(x) = \frac{c}{\rho_1(x)}$.

VI. DISCUSSION AND CONCLUSION

The perturbative method developed in Sec. III can be applied to a wide class of coupled BPS equations. In the general case for $\lambda_1\lambda_2 = \lambda_{12}^2 \neq 1$, it can be shown that the coupled BPS equations (8) and (9) lead to the following relation between ρ_1 and ρ_2 :

$$(\lambda_1 - 1)\lambda_{12} \frac{\rho_1'}{\rho_1} = (\lambda_2 - 1)\lambda_1 \frac{\rho_2'}{\rho_2}. \quad (79)$$

This means that $\rho_2(x) = \tilde{c}\rho_1^\kappa(x)$, where

$$\kappa = \frac{(\lambda_1 - 1)\lambda_{12}}{(\lambda_2 - 1)\lambda_1}. \quad (80)$$

Hence, the most general coupled BPS equations reduce to

$$\begin{aligned} (\ln\rho_1)' &= a\rho_1^H + b(\rho_1^\kappa)^H, & \rho_1(x) &\geq 0, \\ \overline{\rho_1(x)} &= \rho_{10}, \end{aligned} \quad (81)$$

where $a = \frac{2\pi\lambda_1}{\lambda_1-1}$ and $b = \frac{2\pi\lambda_{12}}{\lambda_1-1} \tilde{c}$. The solutions of this equation possess the following general properties:

- (i) There are constant solutions $\rho_1(x) = \rho_{10} = \text{const}$.
- (ii) Solutions close to constant ρ_{10} :

$$\begin{aligned} \rho_1(x) &= \rho_{10} + \varepsilon\varphi_1(x), & |\varepsilon\varphi_1(x)| &\ll \rho_{10}, \\ \overline{\varphi_1(x)} &= 0, \end{aligned} \quad (82)$$

where $\varphi_1(x)$ satisfies

$$\varphi_1' = (a\rho_{10} + \kappa b\rho_{10}^\kappa)\varphi_1^H \quad (83)$$

and $k_0 = a\rho_{10} + \kappa b\rho_{10}^\kappa > 0$. It is easy to see that the solution φ_1 is given up to a phase by

$$\varphi_1(x) = c_1 \cos k_0 x. \quad (84)$$

- (iii) Generally, there exist one-parameter ε solutions containing constant solution ρ_{10} .
- (iv) There are periodic solutions if $k > 0$.
- (v) Aperiodic solutions appear if $k \rightarrow 0_+$.

Our perturbative method allows one to calculate higher order terms in the expansion (14). However, we have not found the solutions in closed analytic form for $\lambda_{12}^2 \neq 1$. We have found the exact solutions only in two cases: $\lambda_{12} = +1$ and $\lambda_{12} = -1$.

In the case $\lambda_1\lambda_2 = 1$, $\lambda_{12} = 1$, we have $\kappa = -1$, and in Sec. V we have found and classified all solutions of the form:

$$\rho_1(x) = (\rho_{10} - \lambda_2 s_2) R_\varepsilon + \lambda_2 s_2 \geq 0, \quad (85)$$

$$\rho_2(x) = (\rho_{20} - \lambda_1 s_1) R_\eta + \lambda_1 s_1 \geq 0, \quad (86)$$

where $\rho_1(x)\rho_2(x) = s_1s_2 = c \geq 0$ and s_1 , s_2 , and η are given by Eqs. (57), (58), and (64), respectively.

Finally, we point out that, in the special case $\lambda_1\lambda_2 = 1$, $\lambda_{12} = -1$, and $\kappa = 1$, the new exact duality appears with nice properties and a physical interpretation connecting particles and antiparticles [8,13,15].

In conclusion, we have studied the two-family Calogero model on line in the limit in which each family contains a large number of particles. We have found that, in the strong-weak dual case, there exists only one nonperiodic soliton-antisoliton, topological solution (77) and (78), and periodic, stationary waves solutions (50). Our collective-field approach can be analogously applied to the two-

family Sutherland model on a circle of perimeter length L . However, the Hilbert transform must be modified in order to take into account the compact support of the Sutherland model. Namely, the standard kernel $P\frac{1}{x-y}$ should be replaced by the $\cot\frac{\pi}{L}(x-y)$ kernel [16]. We hope to report on these issues in a separate publication.

ACKNOWLEDGMENTS

This work was supported by the Ministry of Science and Technology of the Republic of Croatia under Contract No. 098-0000000-2865.

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