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Kaluza-Klein theory in the limit of large number of extra dimensions

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The Kaluza-Klein compactification in the limit of a large number of extra dimensions is studied. The starting point is the Einstein-Hilbert action plus cosmological constant in $4 + \mathbf{D}$ dimensions. It is shown that in the large $\mathbf{D}$ limit the effective four-dimensional cosmological constant is of order $1/\mathbf{D}$, whereas the size of the extra dimensions remains finite. A 't Hooft-like large $\mathbf{D}$ expansion of the effective Lagrangian for the Kaluza-Klein scalar and gauge fields arising from the dimensional reduction is considered. It is shown that the propagator of the scalar field associated to the determinant of the metric of the extra dimensions is strongly suppressed. This is an interesting result as in standard Kaluza-Klein theory this scalar degree of freedom is responsible for the constraint on the gauge fields which makes it impossible to recover the usual Yang-Mills equations. Moreover in the large $\mathbf{D}$ limit it turns out that the ultraviolet divergences due to the interactions between gauge and scalar fields are softened.

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I. INTRODUCTION

The Kaluza-Klein scenario aiming to recover gauge fields from pure space-time geometry is one of the most fascinating ideas of theoretical physics. In the original proposal, the attempt was to unify four-dimensional general relativity (GR) with Maxwell electrodynamics as GR in five dimensions compactified on a circle. In this way one gets Einstein's equations in four dimensions, a gauge field and also a scalar field. However, a quite serious problem in trying to make contact with gauge theory arises. The dynamics of the Maxwell field is not exactly what one would like because the extra scalar degree of freedom (which corresponds to the determinant of the metric along the extra dimension) gives rise to an extra constraint which prevents one from having both a constant scalar field and the usual Maxwell equations for the gauge field.

In order to include non-Abelian gauge fields the curvature of the extra dimensions cannot vanish. This is problematic since the product of four-dimensional Minkowski space-time with a compact manifold with non-Abelian isometry group G (the natural ground state of Kaluza-Klein compactification) is not a solution of higher dimensional GR [1]. A nonvanishing positive cosmological constant may help since solutions which are the product of a four-dimensional Lorentzian manifold of constant positive curvature with a compact manifold with non-Abelian isometry group G exist. However in this case the effective four-dimensional cosmological constant turns out to be of the same order of magnitude as the curvature of the compact space. It is therefore difficult to make contact with phenomenology if one assumes that the compact extra dimensions are characterized by a scale much smaller than the macroscopic four dimensions (a bright analysis of the problem of Kaluza-Klein compactification is in [2]; for updated reviews, see, e.g., [3–5]). Also the problem already mentioned above remains: namely, the dynamics of the Yang-Mills field is not exactly what one would like because the extra scalar degrees of freedom (and, in particular, the degree of freedom corresponding to the determinant of the metric along the extra dimension) give rise to an extra constraint absent in Yang-Mills theory. It has been recently shown [6] that in the context of Lovelock gravities many of the problems of usual Kaluza-Klein compactifications can be addressed.

One may also be interested in analyzing the quantum features of the effective Kaluza-Klein Lagrangian for scalar and gauge fields (thinking of the four-dimensional metric as a classical background on which the Kaluza-Klein scalars and the gauge fields propagate).[1] Indeed, such Lagrangian contains nonrenormalizable interactions between the scalars and the gauge fields which generate many problems in the ultraviolet (UV) limit.

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[1]This makes sense if the typical length scale of the extra dimensions is much smaller than the typical length scale of the macroscopic four-dimensional metric.

Here, we propose a framework in which the problems described above can be treated in a natural way: we will apply the well-known 't Hooft large **N** expansions [7,8][2] to the effective Kaluza-Klein Lagrangian for scalars and gauge fields (thus, in the present case, **N** will be related to the number **D** of extra dimensions).

The first attempts to obtain an expansion similar to the 't Hooft one in gravity have been performed in [11–13] and further refined in [14]. In [12] and in [14] the "small parameter" is $1/\mathbf{d}$ (**d** being the total number of space-time dimensions). While in [15,16], it has been proposed to think of the (Euclidean) four-dimensional GR as a constrained gauge theory for the $\mathbf{SO}(\mathbf{4})$ group and performs an expansion in which $\mathbf{4} \rightarrow \mathbf{N}$ is large (keeping fixed the number of space-time dimensions, that is $\mathbf{d} = 4$).

For the Kaluza-Klein compactification of GR plus cosmological constant in $4 + \mathbf{D}$ dimensions, the subject of the present paper, it is found that already at a classical level there is a nontrivial large **D** expansion whose most remarkable feature is that the effective four-dimensional cosmological constant is of the order of $1/\mathbf{D}$. At a quantum level, performing the 't Hooft-like large **D** expansion, it is found that the propagator of the scalar field corresponding to the determinant of the metric of the extra dimensions is strongly suppressed. This is a nontrivial feature as in standard Kaluza-Klein theory such scalar degree of freedom is responsible for the extra constraint on the gauge fields mentioned above which makes it impossible to recover the Yang-Mills equations when this field is constant.

Moreover, from the "large **N**" perspective, the Kaluza-Klein effective Lagrangian presents new features which are absent in the large **N** expansion of QCD or in the large **N** expansions of a model with global symmetries (such as the Gross-Neveu model; for two reviews see [10]). These novel features allow one to soften the UV problems already mentioned.[3]

Indeed, already the classical theory manifests a nontrivial large **D** scaling, so one could wonder about the justification of treating $1/\mathbf{D}$ as a coupling constant. On the other hand, as it is well known, in quantum mechanics one reaches the semiclassical regime in the limit of very high quantum numbers. In quantum field theory, the semiclassical regime is valid when the vacuum expectation value(s) of the number operator(s) of the field(s) is (are) very large (as it happens, for instance, when condensates appear). Therefore, since in the large **D** expansion the number of degrees of freedom grows polynomially with **D**, one can treat $1/\mathbf{D}$ as a small parameter for the semiclassical expansion around the Kaluza-Klein vacuum (in analogy with what happens in the large **N** expansion of the 3D Gross-Neveu model). In fact, as in quantum field theory condensates break some symmetry of the theory, the Kaluza-Klein vacuum breaks part of the symmetry of the trivial maximally symmetric vacuum. Furthermore, as will be shown in the next sections, the large **D** expansion (at least partially) solves some consistency problems of the classical Kaluza-Klein theory.

The structure of the paper is as follows: First the classical nontrivial features of the large **D** limit of Kaluza-Klein compactification which arise at a classical level are discussed. Then the basic features of the 't Hooft expansion and of some nontrivial large **D** resummations are discussed. It is found that the two most remarkable features of this expansion are the suppression of the scalar degree of freedom corresponding to the determinant of the metric and the softening of the ultraviolet divergences. In the last section the conclusions are presented.

II. THE KALUZA-KLEIN SCENARIO: A SHORT INTRODUCTION

Let us consider the Kaluza-Klein scenario in $(4 + \mathbf{D})$ dimensions whose ground state is a product manifold $M_4 \times K_{\mathbf{D}}$ ($K_{\mathbf{D}}$ being a Euclidean manifold of constant positive curvature; nice reviews on this subject are [3,4,17]). Here, we will only consider the Einstein-Hilbert action with a positive cosmological constant Λ,

$$\Lambda_{4+\mathbf{D}} = \Lambda$$

(in order to have non-Abelian gauge fields) in $(4 + \mathbf{D})$ dimensions. The ground state metric is the following direct product in which the extra-dimensional manifold is a constant curvature manifold

$$g_{(4+\mathbf{D})} = g_{\mu\nu}(x^{\mu})dx^{\mu}dx^{\nu} + \hat{g}_{ab}(y^{a})dy^{a}dy^{b},$$

where the coordinates x^{μ} are intrinsic to M_4 and y^a are **D**-dimensional coordinates intrinsic to $K_{\mathbf{D}}$. With this ansatz, the mixed components of the Einstein equation (written in the usual second order formalism)[4] involving the mixed components $G_{a\mu}$ of the $(4 + \mathbf{D})$-dimensional Einstein tensor G_{AB} are trivially satisfied. The $(4 + \mathbf{D})$-dimensional Einstein equations

$$G_{AB} = \Lambda g_{AB}$$

reduce to a four-dimensional Einstein equations for $g_{\mu\nu}$:

$$G^{(4)}_{\mu\nu} + \Lambda_4 g_{\mu\nu} = 0,$$

with an effective cosmological constant Λ_4 and to a

[2]The Veneziano limit [9], in which the ratio $\mathbf{N}/\mathbf{N}_f$ is kept fixed ($\mathbf{N}_f$ being the number of quarks flavors), was also important to further clarify several features of quark and mesons dynamics; two pedagogical reviews are [10].

[3]This framework is somehow in between the points of view of Refs. [12,14] (in which a large d expansion was considered) and the point of view proposed in [15,16] (where d is kept fixed and "$\mathbf{SO}(\mathbf{4})$ is enlarged" in such a way as to separate the "internal indices" from the "space-time indices").

[4]In the next section, the Palatini first order formalism will be considered.

D-dimensional Euclidean Einstein equations for $\hat{g}_{ab}$:

$$G^{(\mathbf{D})}_{ab} + \Lambda_{\mathbf{D}} g_{ab} = 0.$$

As is well known, a non-Abelian algebra of Killing fields is only compatible with a **D**-dimensional symmetric space of positive effective cosmological constant $\Lambda_{\mathbf{D}}$: the effective four-dimensional and **D**-dimensional cosmological constants are, respectively,

$$\Lambda_4 = \frac{2}{\mathbf{D}+2}\Lambda,$$

and

$$\Lambda_{\mathbf{D}} = \frac{\mathbf{D}-2}{\mathbf{D}+2}\Lambda,$$

so that

$$\frac{\Lambda_4}{\Lambda_{\mathbf{D}}} = \frac{2}{\mathbf{D}-2}.$$

Therefore, when **D** is large, Λ_4 is much smaller than $\Lambda_{\mathbf{D}}$: in particular, at leading order in the large **D** expansion, Λ_4 vanishes. One then sees that the large **D** expansion itself is able to keep separated the macroscopic four-dimensional scale from the typical extra-dimensional scale without any extra ingredient. This is indeed a very attractive feature of the present framework. In the following section, the nontrivial large **D** scaling of the classical theory will be deduced in the first order Palatini formalism which, for this goal, is more convenient than the second order formalism.

A. Kaluza-Klein scenarios in the first order formalism

To fully display the **D** dependence in the large **D** expansion it is convenient to introduce the following notations (we will follow [17]): let t_a be the Lie algebra generators corresponding to the Lie group G (which will play the role of the gauge group of the Kaluza-Klein gauge fields):

$$[t_a, t_b] = C^c_{ab} t_c, \qquad s^{-1}ds = e^a t_a, \qquad dss^{-1} = -\hat{e}^a t_a,$$
$$s \in G,\ e^a(Y_b) = \delta^a_b, \qquad \hat{e}^a(\hat{Y}_b) = \delta^a_b.$$

Here $\hat{Y}_a$ and Y_b represent the right and left invariant vector fields while $\hat{e}^a$ and e^a are the corresponding dual one-forms. To have a consistent Kaluza-Klein scenario one may consider the case in which the Y_a are the Killing vectors of the full $(4+\mathbf{D})$-dimensional metric.[5] The natural Kaluza-Klein ground state is a product of a four-dimensional manifold fulfilling the four-dimensional Einstein equations (with a small cosmological constant if **D** is large) times a coset manifold G/H invariant under the corresponding non-Abelian algebra of the Killing fields. The ground state metric (whose Killing vectors are the Y_a) on G/H will be written as

$$g_{G/H} = \hat{g}_{ab}\hat{e}^a\hat{e}^b. \tag{1}$$

In the ground state, $\hat{g}_{ab}$ does not depend on x: to fix the idea, one can think at the extra-dimensional manifold corresponding to the ground state of the Kaluza-Klein scalars as the **D** sphere

$$\frac{G}{H} = S^{\mathbf{D}} = \frac{SO(\mathbf{D}+1)}{SO(\mathbf{D})}. \tag{2}$$

At the semiclassical level, the large **D** expansion corresponds, from the point of view of gauge fields, to a (bit unusual as we shall see in the next section) 't Hooft expansion of the effective Kaluza-Klein Lagrangian for scalars and gauge fields with $SO(\mathbf{D})$ as the gauge group.

Eventually, the usual Kaluza-Klein ansatz for the $(4+\mathbf{D})$-dimensional metric $g_{(4+\mathbf{D})}$ reads

$$g_{(4+\mathbf{D})} = g_{\mu\nu}dx^\mu dx^\nu + \hat{g}_{ab}(x^\mu)(\hat{e}^a + A^a)(\hat{e}^b + A^b). \tag{3}$$

The above metric (3) is left unchanged by the following gauge transformations:

$$A' = u^{-1}Au + du^{-1}, \qquad (\hat{g}_{ab})' = (\hat{g}_{cd})R(u)^c_a R(u)^d_b,$$

where $u(x^\mu) \in SO(\mathbf{D})$, while the matrix $R(u)^c_a$ is in the adjoint representation in the sense that the element $u \in SO(\mathbf{D})$ induces the following transformation on generators t_a:

$$(t_a)' = t_b R(u)^b_a.$$

As a consequence, since in the 't Hooft notation the propagators of the "$SO(\mathbf{D})$ gluons" A^b are represented by a double line (as usual, the A^a fields transform in the adjoint), the scalar degrees of freedom corresponding to $\hat{g}_{ab}(x^\mu)$ [see Eq. (13)] will be represented by four lines as will be explained in more detail in the next section (a similar phenomenon also occurs in [15,16]): this is the origin of the unusual features of the 't Hooft expansion of the Kaluza-Klein effective Lagrangian.

In many field theoretical models in which the large **N** expansion is available (such as the Gross-Neveu, Yang-Mills theory, and so on) the nontrivial scaling with **N** only appears at a quantum level (see, for instance, [10]). Namely, only after computing Feynman diagrams with loops, one can recognize the possibility to perform a large **N** expansion which corresponds to a semiclassical expansion. However, in gravity (because of the fact that the extra dimensions describe in a sense the local gauge symmetry of Yang-Mills theory), already the classical equations of motion manifest a nontrivial scaling with **D** (related to the dimension of the Kaluza-Klein gauge fields).

Let ω^{AB} and e^A be the (torsion free) spin connection and the "$(4+\mathbf{D})$-bein," respectively, and the Riemann curvature two form R^{AB} is defined as

[5] The case can also be analyzed in which the Y_a are the Killing vectors of the metric only when restricted to the extra dimensions [17], but we will restrict the present analysis only to the case in which the Y_a are the Killing vectors of the total metric.

$$R^{AB} = d\omega^{AB} + \omega^{A}_{C}\omega^{CB},$$
$$A, B, C, \ldots = 1, \ldots, \mathbf{D} + 4,$$
$$\mu, \nu, \rho, \sigma = 1, \ldots, 4,$$
$$De^{A} = T^{A} = de^{A} + \omega^{A}_{C}e^{C} = 0,$$
$$a, b, c, \ldots, a_1, a_2, \ldots, i, j, k, \ldots = 1, \ldots, \mathbf{D}.$$

The Einstein-Hilbert action plus the cosmological term in $4 + \mathbf{D}$ dimensions in the first order formalism reads

$$I_{\mathbf{D}+4} = \int \bigg(\frac{c_0}{4+\mathbf{D}} e^{A_1} \cdots e^{A_{4+D}} + \frac{c_1}{\mathbf{D}+2} R^{A_1A_2} e^{A_3} \cdots e^{A_{4+D}} \bigg), \qquad (4)$$

where c_0 is proportional to the cosmological constant and c_1 to the $4 + \mathbf{D}$ Newton constant. At a classical level, no argument can be invoked which suggests that c_0 is negligible with respect to c_1 so that the "bare" classical coupling constants c_0 and c_1 scale with $\mathbf{D}$ in the same way: therefore, in the large $\mathbf{D}$ limit, c_0/c_1 is a nonvanishing finite constant (let us call such a constant $\frac{6\hat{\beta}}{\hat{\alpha}}$ for future convenience)

$$\frac{c_0}{c_1} \underset{\mathbf{D}\gg 1}{\approx} \frac{6\hat{\beta}}{\hat{\alpha}} + o(1/\mathbf{D}).$$

Let us divide the indices into two groups: $\mu, \nu, \rho, \sigma, \ldots$ represent the macroscopic Lorentzian four dimensions ($\mu, \nu, \rho, \sigma = 1, \ldots, 4$), while small Latin indices $a, b, c, \ldots, i, j, k, \ldots, a_1, a_2, \ldots$ (which will play the role of the internal indices of the Yang-Mills fields) represent the $\mathbf{D}$ compact extra dimensions ($a, b, c, \ldots, i, j, k, \ldots, a_1, a_2, \ldots = 1, \ldots, \mathbf{D}$). Thus, there are three different kinds of components of the Riemann curvature two form R^{AB}:

$$R^{\mu\nu}, \qquad R^{\mu a}, \qquad R^{ab}.$$

Roughly speaking, the components $R^{\mu\nu}$ give rise to the usual four-dimensional gravitational interaction of GR with a suitable energy-momentum tensor for the gauge and scalar fields as source, the components $R^{\mu a}$ are related to the field strength of the gauge fields (generating the corresponding equations of motion) while the components R^{ab} are related to the scalar fields and to the well-known scalar constraint on the gauge fields (the explicit decomposition of the Riemann tensor can be found, for instance, in [3,4,17]). The equations of motion corresponding to the action in Eq. (4) split as follows:

$$\begin{aligned} 0 &= E_{\mu} \\ &= \varepsilon_{\mu\nu\rho\sigma a_1\cdots a_{\mathbf{D}}} \bigg\{ c_0 \frac{(\mathbf{D}+3)(\mathbf{D}+2)}{6} (e^{\nu} e^{\rho} e^{\sigma} e^{a_1} \cdots e^{a_{\mathbf{D}}}) \\ &\quad + c_1 \mathbf{D}(\mathbf{D}-1) \bigg[\frac{R^{a_1a_2}}{6} (e^{\nu} e^{\rho} e^{\sigma} e^{a_3} \cdots e^{a_{\mathbf{D}}}) \\ &\quad + \frac{R^{\nu a_1}}{\mathbf{D}-1} (e^{\rho} e^{\sigma} e^{a_2} \cdots e^{a_{\mathbf{D}}}) + \frac{(e^{a_1} \cdots e^{a_{\mathbf{D}}} e^{\sigma})}{(\mathbf{D}-1)\mathbf{D}} R^{\nu\rho} \bigg] \bigg\}, \end{aligned}$$

$$\begin{aligned} 0 &= E_{a_1} \\ &= \varepsilon_{\mu\nu\rho\sigma a_1\cdots a_{\mathbf{D}}} \bigg\{ c_0 \frac{(\mathbf{D}+3)(\mathbf{D}+2)}{24} (e^{\mu} e^{\nu} e^{\rho} e^{\sigma} e^{a_2} \cdots e^{a_{\mathbf{D}}}) \\ &\quad + c_1 (\mathbf{D}-2)(\mathbf{D}-1) \bigg[\frac{R^{a_2a_3}}{24} (e^{\mu} e^{\nu} e^{\rho} e^{\sigma} e^{a_4} \cdots e^{a_{\mathbf{D}}}) \\ &\quad + \frac{R^{\mu a_2}}{3(\mathbf{D}-2)} (e^{\nu} e^{\rho} e^{\sigma} e^{a_3} \cdots e^{a_{\mathbf{D}}}) \\ &\quad + \frac{(e^{\rho} e^{\sigma} e^{a_2} \cdots e^{a_{\mathbf{D}}})}{2(\mathbf{D}-1)(\mathbf{D}-2)} R^{\mu\nu} \bigg] \bigg\}. \end{aligned}$$

It is then clear that when $\mathbf{D}$ is very large the above equations separate into decoupled equations for the different components $R^{\mu\nu}$, $R^{\mu a}$, and R^{ab}: the reason is that for large $\mathbf{D}$ the number of scalar field components grows faster than the number of Kaluza-Klein gauge fields while the number of four-dimensional gravitational degrees of freedom does not change.

We will assume, as is usually done in various types of large $\mathbf{N}$ expansions, that for very large $\mathbf{D}$ any field Φ can be expanded as follows:

$$\Phi = \Phi_{(0)} + \frac{1}{\mathbf{D}} \Phi_{(1)} + \frac{1}{\mathbf{D}^2} \Phi_{(2)} + \cdots,$$

such that $\forall\ k$, $\Phi_{(k)}$ does not depend on $\mathbf{D}$, where $\Phi_{(0)}$ is the leading order and the terms $\Phi_{(i)}$ for $i > 0$ can be considered as subleading corrections so that no component of R^{AB} is divergent at large $\mathbf{D}$. Indeed, such an hypothesis is the most natural one since the large $\mathbf{D}$ expansion itself provides one with a suitable tool to keep well separated the macroscopic scale of the four-dimensional directions ($\mu, \nu, \ldots$) from the compactified directions ($a_1, a_2, \ldots$).

To simplify the notation, it is convenient to define two rescaled coupling constants $\hat{\beta}$ and $\hat{\alpha}$ in terms of c_0 and c_1 as follows:

$$c_0 = \frac{6\hat{\beta}}{(\mathbf{D}+3)(\mathbf{D}+2)}, \qquad c_1 = \frac{\hat{\alpha}}{\mathbf{D}(\mathbf{D}-1)}.$$

The field equations $E_\mu = 0$ and $E_{a_1} = 0$ now read

$$0 = E_\mu = \varepsilon_{\mu\nu\rho\sigma a_1\cdots a_{\mathbf{D}}}\Big\{\hat\beta(e^\nu e^\rho e^\sigma e^{a_1}\cdots e^{a_{\mathbf{D}}}) + \hat\alpha\Big[\frac{R^{a_1a_2}}{6}(e^\nu e^\rho e^\sigma e^{a_3}\cdots e^{a_{\mathbf{D}}}) + \frac{R^{\nu a_1}}{\mathbf{D}-1}(e^\rho e^\sigma e^{a_2}\cdots e^{a_{\mathbf{D}}}) + \frac{(e^{a_1}\cdots e^{a_{\mathbf{D}}}e^\sigma)}{(\mathbf{D}-1)\mathbf{D}}R^{\nu\rho}\Big]\Big\}, \tag{5}$$

$$0 = E_{a_1} = \varepsilon_{\mu\nu\rho\sigma a_1\cdots a_{\mathbf{D}}}\Big\{\frac{\hat\beta}{4}(e^\mu e^\nu e^\rho e^\sigma e^{a_2}\cdots e^{a_{\mathbf{D}}}) + \hat\alpha\Big(1-\frac{2}{\mathbf{D}}\Big)\cdot\Big[\frac{R^{a_2a_3}}{24}(e^\mu e^\nu e^\rho e^\sigma e^{a_4}\cdots e^{a_{\mathbf{D}}}) + \frac{R^{\mu a_2}}{3(\mathbf{D}-2)}\times(e^\nu e^\rho e^\sigma e^{a_3}\cdots e^{a_{\mathbf{D}}}) + \frac{(e^\rho e^\sigma e^{a_2}\cdots e^{a_{\mathbf{D}}})}{2(\mathbf{D}-1)(\mathbf{D}-2)}R^{\mu\nu}\Big]\Big\}. \tag{6}$$

It is easy to see a very nice feature of the present large **D** framework: *a priori*, one should assume that all the components of the full Riemann tensor $R^{\mu\nu}$, $R^{\mu a}$, R^{ab} have already at a classical level a nontrivial $1/\mathbf{D}$ expansion:

$$R^{\mu\nu} = R^{\mu\nu}_{(0)} + \frac{1}{\mathbf{D}}R^{\mu\nu}_{(1)} + \cdots,$$
$$R^{\mu a} = R^{\mu a}_{(0)} + \frac{1}{\mathbf{D}}R^{\mu a}_{(1)} + \cdots,$$
$$R^{ab} = R^{ab}_{(0)} + \frac{1}{\mathbf{D}}R^{ab}_{(1)} + \cdots.$$

However, as far as $R^{\mu\nu}$ and $R^{\mu a}$ are concerned, it is consistent with the field equations to simply consider the leading terms:

$$R^{\mu\nu} = R^{\mu\nu}_{(0)}, \qquad R^{\mu a} = R^{\mu a}_{(0)},$$

while as far as R^{ab} is concerned it is enough to consider the first two terms of the expansion:

$$R^{ab} = R^{ab}_{(0)} + \frac{1}{\mathbf{D}}R^{ab}_{(1)}. \tag{7}$$

Thus, at large **D**, one gets the following decoupled equations for $R^{a_1a_2}_{(0)}$, $R^{a_1a_2}_{(1)}$, $R^{\nu a_1}$, and $R^{\nu\rho}$:

$$\varepsilon_{a_1\cdots a_{\mathbf{D}}}\Big(\frac{\hat\beta}{4}e^{a_2}\cdots e^{a_{\mathbf{D}}} + \hat\alpha\frac{R^{a_2a_3}_{(0)}}{24}(e^{a_4}\cdots e^{a_{\mathbf{D}}})\Big) = 0, \tag{8}$$

$$\varepsilon_{a_1\cdots a_{\mathbf{D}}}(e^{a_4}\cdots e^{a_{\mathbf{D}}})R^{a_2a_3}_{(1)} = 0, \tag{9}$$

$$\varepsilon_{\mu\nu\rho\sigma a_1\cdots a_{\mathbf{D}}}(e^\rho e^\sigma e^{a_2}\cdots e^{a_{\mathbf{D}}})R^{\nu a_1} = 0, \tag{10}$$

$$\varepsilon_{\mu\nu\rho\sigma}R^{\nu\rho}e^\sigma = 0. \tag{11}$$

The $R^{a_2a_3}_{(0)}$ components satisfy Euclidean Einstein equations with an effective **D**-dimensional cosmological constant given by $\frac{6\hat\beta}{\hat\alpha}$. Thus, no matter how large the actual $(4+\mathbf{D})$-dimensional cosmological constant is, the consistency of the large **D** expansion demands that the effective four-dimensional cosmological constant is of order $1/\mathbf{D}$ (indeed, the effective four-dimensional cosmological constant vanishes at leading order in the large **D** expansion). The leading correction to R^{ab} in the $1/\mathbf{D}$ expansion (namely, $R^{ab}_{(1)}$) satisfies Euclidean **D**-dimensional Einstein equations with a vanishing cosmological constant. The nontrivial large **D** scaling already present in the classical equations of motion is an interesting feature of the present framework.

III. PROPAGATORS AND 'T HOOFT EXPANSION

In the next section, some large **D** correction to the propagators of the scalar fields ρ and $\vec\pi$ will be analyzed: in order to achieve this goal, it is convenient to use the second order formalism. The $4+\mathbf{D}$-dimensional gravitational action reads

$$S_{4+\mathbf{D}} = \int\sqrt{g_{4+\mathbf{D}}}(R_{4+\mathbf{D}} + 2\Lambda_{4+\mathbf{D}}).$$

The $4+\mathbf{D}$-dimensional Ricci scalar can be expressed in terms of the four-dimensional Ricci scalar R_4, the Kaluza-Klein scalars and gauge fields as follows

$$R_{4+\mathbf{D}} = R_4 + R_{\mathbf{D}} - \frac{g^{\mu\nu}g^{\alpha\beta}}{4}\hat g_{ab}F^a_{\mu\alpha}F^b_{\nu\beta} - \nabla_\mu(\mathrm{tr}(\hat g^{-1}\nabla^\mu\hat g)) - \frac{1}{4}\mathrm{tr}((\hat g^{-1}\nabla^\mu\hat g)(\hat g^{-1}\nabla_\mu\hat g)) - \frac{1}{4}(\mathrm{tr}(\hat g^{-1}\nabla^\mu\hat g))(\mathrm{tr}(\hat g^{-1}\nabla_\mu\hat g)), \tag{12}$$

where it has been introduced as the shorthand notation $\hat g$ for the scalar Kaluza-Klein fields $\hat g_{ab}(x^\mu)$ and $R_{\mathbf{D}}$ is the Ricci scalar of the extra-dimensional manifold

$$R_{\mathbf{D}} = -\hat g^{ij}\Big(C^k_{ai}C^a_{kj} + \frac{1}{2}C^k_{li}C^l_{kj}\Big) - \hat g^{mn}C^i_{im}C^j_{jn} - \frac{\hat g_{ij}\hat g^{kp}\hat g^{mn}}{4}C^i_{km}C^j_{pn}.$$

It is apparent the origin of the UV divergences (mentioned in the Introduction) of the Kaluza-Klein Lagrangian for gauge and scalar fields (in which the four-dimensional part of the gravitational field is considered as a classical background). The two most dangerous sources of nonrenormalizable interactions are the determinant $\sqrt{g_{4+\mathbf{D}}}$ of the metric in the gravitational action[6] and the term

[6]The presence of such term ($\sqrt{g_{4+\mathbf{D}}}$, when ρ is small, is proportional to a constant plus ρ) generates nonrenormalizable interactions in which ρ multiplies the kinetic terms of the $\vec\pi$ scalar fields [defined in Eq. (13)] and of the gauge fields.

$$\frac{g^{\mu\nu}g^{\alpha\beta}}{4}\hat{g}_{ab}F^{a}_{\mu\alpha}F^{b}_{\nu\beta}$$

in the Ricci scalar. Such a term also generates nonrenormalizable interactions between the ρ field and the gauge fields as well as nonrenormalizable interactions between the $\vec{\pi}$ fields [defined in Eq. (13)] and the gauge fields.[7] For the reasons mentioned at the beginning of the next section, it is not possible to give a complete treatment of the renormalization of the above Kaluza-Klein action. However, it is interesting to stress that one of the main causes of the UV problems is the scalar degrees of freedom ρ. If one would find a sound mechanism to suppress the propagator of ρ one would also soften many of the UV divergences of the theory. We will come back to this important point in the following.

It is convenient the following decomposition of the scalar fields:

$$\hat{g} = \exp(2\hat{\rho}\mathbf{1})\exp(\vec{\pi}\cdot\vec{t}), \tag{13}$$

where $\mathbf{1}$ is the identity and $\vec{t}$ are the generators of the algebra $SO(\mathbf{D})$ in the tensor product of the adjoint representation with itself, and the matrix $\hat{g}$ has been decomposed into a factor belonging to the group $SO(\mathbf{D})$ and its determinant $\exp(2\hat{\rho}\mathbf{1})$. Thus, in the ground state both $\hat{\rho}$ and the $\vec{\pi}$ vanish so that $\hat{g}_{ab} = \delta_{ab}$. The fields $\vec{\pi}$ correspond to fluctuations which leave the determinant of $\hat{g}$ unchanged while the field $\hat{\rho}$ corresponds to fluctuations of the determinant of $\hat{g}$. The fields $\vec{\pi}$ belong to the algebra of $SO(\mathbf{D})$ and have two indices in the adjoint representation so that in the 't Hooft notation, they will be represented by four lines while the field $\hat{\rho}$ is a singlet under $SO(\mathbf{D})$.

It is worth noting here that $\hat{\rho}$ is precisely the analog of the scalar degree of freedom of the Abelian Kaluza-Klein framework (in which the reduction from five to four dimensions is considered). In particular, this implies that the extra scalar constraint which prevents one from having both a constant scalar field and the usual Yang-Mills equations for the gauge fields in the non-Abelian Kaluza-Klein framework is related to $\hat{\rho}$.

In order to assure a proper behavior of determinant of $\hat{g}$ in the large $\mathbf{D}$ limit, the $\hat{\rho}$ field will be normalized as follows:

$$\rho = \frac{2}{\mathbf{D}(\mathbf{D}-1)}\hat{\rho} \tag{14}$$

(where ρ is a field which is finite in the large $\mathbf{D}$ limit) since, in this way, when $\mathbf{D}\to\infty$ the determinant of $\hat{g}$ stays finite.

In terms of these fields, the kinetic terms of the scalars read

$$-\operatorname{tr}((\nabla^{\mu}\vec{\pi}\cdot\vec{t})(\nabla_{\mu}\vec{\pi}\cdot\vec{t})) + \left(1 - \frac{4}{(\mathbf{D}-1)^2\mathbf{D}^2}\right)\nabla_{\mu}\rho\nabla^{\mu}\rho,$$

where one recognizes, except by a constant factor, the usual kinetic terms for scalar fields.

The large $\mathbf{D}$ scaling suggests that the scalar mode ρ is subdominant with respect to the Kaluza-Klein gauge fields and $\vec{\pi}$ scalars whose numbers grow with $\mathbf{D}$. This suggests that the well-known problem which arises in Kaluza-Klein scenarios when one tries to deal with nontrivial Kaluza-Klein gauge fields but with constant scalars arises at order $1/\mathbf{D}$. As we shall explain in the next section, large $\mathbf{D}$ effects strongly suppress the propagator of the ρ field.

IV. SOME EXAMPLES OF NONTRIVIAL LARGE D RESUMMATIONS

We will now describe some nontrivial features of the large $\mathbf{D}$ expansion of the Kaluza-Klein gauge and scalar fields. The 't Hooft expansion in the Kaluza-Klein Lagrangian presents novel features due to the appearance of scalar fields represented by four internal lines in the usual large $\mathbf{N}$ notation. This leads to a resummation which softens the UV problem of the theory in a quite systematic way.

We will not try in the present paper to prove the full renormalizability of the Kaluza-Klein Lagrangian for scalars and gauge fields. As it is well known, for theory with gauge symmetry, the powerful methods of algebraic renormalization based on the Becchi-Rouet-Stora-Tyutin (BRST) symmetry have been developed (for a detailed book on these methods, see [18]). These tools allow one to prove the quantum consistency of the BRST symmetry to all order in the gauge coupling constant: the proof is recursive in the coupling constant itself. However, these techniques cannot be applied in the large $\mathbf{D}$ expansion of the Kaluza-Klein Lagrangian: the reason is that in the usual case of Yang-Mills theory the dependence of the classical action plus the gauge fixing term on the coupling constant is very simple (a polynomial). While, in the present case, the dependence of the Kaluza-Klein action on the coupling constant $1/\mathbf{D}$ is quite complicated and very far from being a simple polynomial: this prevents one from using the techniques of [18] in the present case.[8]

[7] The reason is that when one expands $\hat{g}_{ab}$ around the chosen ground state the expansion contains a term proportional to $\rho\delta_{ab}$ as well as a term proportional to $\vec{\pi}\cdot\vec{t}$ [where $\vec{\pi}$ are defined in Eq. (13) and the $\vec{t}$ are the generators of the algebra of $SO(\mathbf{D})$ in the tensor product of the adjoint representation with itself].

[8] To the best of the authors' knowledge, the renormalization procedure in the large $\mathbf{N}$ expansion has been developed only for theories with global symmetry (such as the Gross-Neveu model in three dimensions which is renormalizable at large $\mathbf{N}$ despite being nonrenormalizable in the usual perturbative expansion). When dealing with the large $\mathbf{N}$ expansion of QCD one does not worry about the renormalizability of the theory at large $\mathbf{N}$ since the theory is already known to be renormalizable by other means. Indeed (unlike the cases with global symmetries like the Gross-Neveu model), cases of gauge theories which are not renormalizable in the usual perturbative expansion but can be renormalized in the large $\mathbf{N}$ expansion are not known.

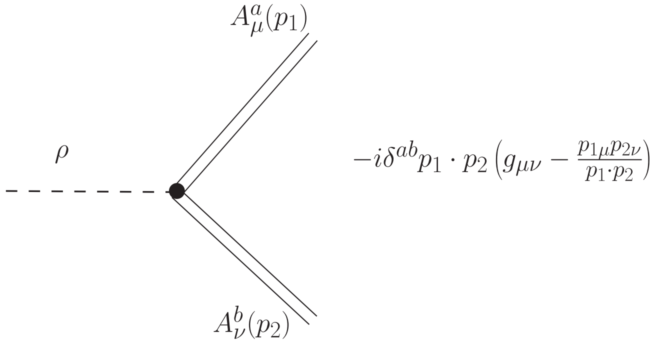

FIG. 1. Feynman rule for the $\rho A^a_\mu A^b_\nu$ interaction.

For these reasons, we will satisfy ourselves by showing that the large **D** expansion leads to a surprising softening of the UV divergences by considering two simple examples: since we are considering the UV limit, when computing the propagators and vertices the background geometry will be assumed to be flat.

A. Examples of large D corrections to the scalar propagators

Now we will discuss the simplest correction to the ρ propagator due to Kaluza-Klein gauge field loops. We will consider the expansion of $\hat{g}_{ab}$ [defined in terms of the fundamental fields ρ and $\vec{\pi}$ in Eq. (13)] around the natural ground state $\hat{g}_{ab}|_{\mathrm{GS}}$:

$$\hat{g}_{ab}|_{\mathrm{GS}} = \delta_{ab}$$

in such a way that the Feynman rules for the fundamental fields ρ, $\vec{\pi}$ and the gauge fields can be read directly from the Lagrangian equation (12).

In what follows, we will only need the bare propagators of ρ, A, and $\vec{\pi}$ which read

$$\Pi_\rho(k) = \left(1 - \frac{4}{(\mathbf{D}-1)^2\mathbf{D}^2}\right)^{-1} \frac{i}{k^2},$$

$$\Pi^{abcd}_\pi(k) = \frac{i\delta^{ab}\delta^{cd}}{k^2}, \qquad \Pi^{ab}_{A\mu\nu}(k) = \frac{-i\delta^{ab} g_{\mu\nu}}{k^2},$$

where the A propagator is in the Feynman gauge and k is the 4-momentum of the particle.

Here, we will only focus on the analysis of the vertices which do not appear in the usual Yang-Mills theory: the ones coming from the $\hat{g}_{ab} F^a_{\mu\alpha} F^b_{\nu\beta}$ term when expanding $\hat{g}_{ab}$ around the ground state. Such vertices describe nonrenormalizable interaction in normal perturbation theory and its presence could be viewed as problematic in the usual scheme. Nevertheless, the large **D** expansion leads to a surprising improvement, as we will see in the case of the "ρAA" vertex.

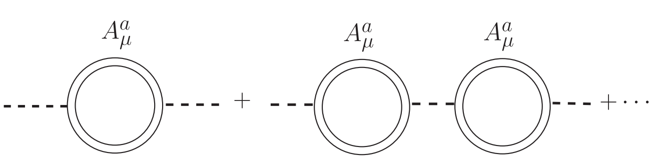

FIG. 2. Corrections to the ρ propagator due to gluon loops.

We are going to consider the contribution of such a vertex to the ρ propagator. As we show in Fig. 1, the Feynman rule for the nonrenormalizable vertex ρAA (which originates from the term $\hat{g}_{ab} F^a_{\mu\alpha} F^b_{\nu\beta}$ of the Lagrangian) is

$$-i\delta^{ab}\kappa p_1 \cdot p_2\left(g_{\mu\nu} - \frac{p_{1\mu}p_{2\nu}}{p_1 \cdot p_2}\right), \tag{15}$$

where κ is Newton's constant,[9] p_1 and p_2 are the 4-momenta of the gauge fields.

With this propagator, we can construct loop corrections to the ρ propagator as shown in Fig. 2.

Each loop contributes with a term given by

$$\Delta_\rho(p) = \mathbf{D}(\mathbf{D}-1) \times \int d^4k \left\{\frac{2[k\cdot(p-k)]^2 + k^2\cdot(p-k)^2}{k^2\cdot(p-k)^2}\right\},$$

where p is the 4-momentum of the ρ, k is the internal 4-momentum running in the loop, and the dot represents the usual Lorentz product. Notice that the integral is highly divergent but it can be regularized by the usual methods. The presence of $\mathbf{D}(\mathbf{D}-1)$ in the expression for $\Delta_\rho(p)$ can be easily understood by using the 't Hooft notation.

When we sum up all the terms we obtain the geometric series and the result reads

$$\left[\left(1 - \frac{4}{(\mathbf{D}-1)^2\mathbf{D}^2}\right)p^2 - \Delta_\rho(p)\right]^{-1}.$$

Because Δ_ρ is proportional to $\mathbf{D}(\mathbf{D}-1)$, we find that the propagator of ρ is suppressed in the large **D** limit strongly suggesting the decoupling of this degree of freedom in the UV. This is an interesting effect since, in this way, all the "nonrenormalizable" loops in which the ρ field appears are suppressed as well by such large **D** resummation. Therefore, this "large **D** resummation" leads to a clear improvement of the UV behavior of the theory.

In a similar way, one can compute the correction to the propagator of the π^{ab} fields due to gauge field loops. The Feynman rule for the nonrenormalizable vertex "πAA" (see Fig. 3), which also originates from the term $\hat{g}_{ab} F^a_{\mu\alpha} F^b_{\nu\beta}$ of the Lagrangian as shown in Fig. 4, is

$$-i\delta^{ac}\delta^{bd}\kappa p \cdot p_2\left(g_{\mu\nu} - \frac{p_{1\mu}p_{2\nu}}{p_1 \cdot p_2}\right), \tag{16}$$

where, again, p_1 and p_2 are the 4-momenta of the gauge fields.

[9] As it has been already stressed, we are interested in the UV limit of the theory. Therefore, when writing the Feynman rules, we will consider a flat background geometry.

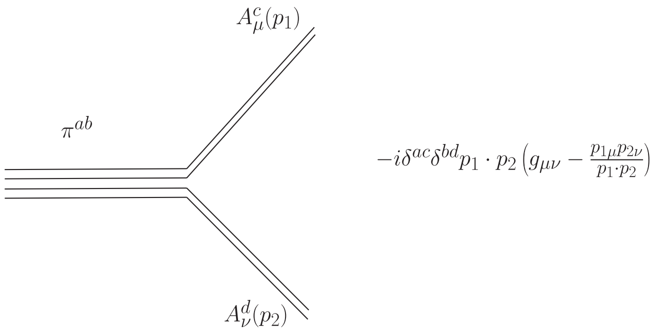

FIG. 3. Feynman rule for the $\pi^{ab}A^c_\mu A^d_\nu$ interaction.

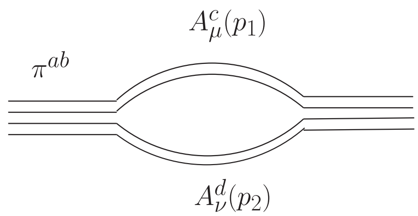

FIG. 4. First correction to the π^{ab} propagator.

The main difference with the previous case is the ''color index factor.'' Being π^{ab} is a ''four line'' field in t' Hooft diagrammatic notation (while the gauge field is a usual ''two line'' field), at the leading order there are not closed color lines in the loops as we have in Fig. 4.

Therefore, at the leading order each loop contribution is independent of **D**:

$$\Delta_{\vec{\pi}}(p) = 2\int d^4k \left\{\frac{2[k\cdot(p-k)]^2 + k^2\cdot(p-k)^2}{k^2\cdot(p-k)^2}\right\}.$$

Consequently, the $\vec{\pi}$ propagator is not directly suppressed in the large **D** limit as in the case of ρ.

Eventually, since the ghost fields must be in the same representation of the gauge group as the gauge fields, they will be represented in the 't Hooft notation by two internal lines. Consequently, it can be easily seen that they will ''suffer'' the same color corrections as the corresponding gauge fields: therefore the large **D** corrections to the ghost propagators will be the same as the large **D** correction to the gauge fields. This should be enough to guarantee a consistent semiclassical expansion.

Indeed, in the present paper only two examples of ''large **D**'' corrections have been considered. However, the qualitative effects that one should expect by including other possible vertices are consistent with the effects discussed here: the reason is that in the large **D** limit one can understand which are the dominant contributions by looking at the 't Hooft double line notation.[10]

V. CONCLUSIONS

In the present paper some very interesting features of the large **D** expansion of a Kaluza-Klein compactification in $4 + \mathbf{D}$ dimensions have been analyzed. First, it has been found that already at a classical level this model exhibits a nontrivial large **D** scaling: in particular, it has been shown that the four-dimensional effective cosmological Λ_4 constant is of order $1/\mathbf{D}$ (so that, at leading order in the large **D** expansion, Λ_4 vanishes), whereas the size of the extra dimensions remains finite. At the quantum level some features of the 't Hooft large **D** expansion of the effective Lagrangian for the scalar and gauge fields have been studied. It has been shown that the scalar degree of freedom associated with the determinant of the extra-dimensional metric (responsible for many UV divergences) is suppressed in the large **D** limit: this effect strongly indicates that the UV Kaluza-Klein divergences are softened in the large **D** expansion.

As a final remark it is worth pointing out that one could expect that this mechanism can work better than the usual perturbative expansion even for a not extremely large value of **D**. For instance, in QCD already $\mathbf{N} = 3$ is enough to trust large **N** expansion. On the other hand, it is well known that in order to encompass the standard model within the Kaluza-Klein framework we need at least seven extra dimensions.

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[10] As a matter of fact, if one looks at the double line structure of the other vertices, one can easily see, for instance, that generically large **D** effects tend to suppress the ρ propagator because of the double line structure shown in Figs. 1 and 2.

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