

**Deriving the ADM 3 + 1 evolution equations from the second variation of arclength**

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We will show that the ADM 3 + 1 evolution equations, for a zero shift vector, arise naturally from the equations for the second variation of arclength.

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**I. INTRODUCTION**

Discussions of the dynamics of general relativity often begin with the ADM 3 + 1 evolution equations [1]. These equations describe the second time derivatives of the spatial metric in terms of other quantities such as the lapse function and the extrinsic and Riemann curvatures. If by some means we happen to have a local solution (in time) of those equations then we could imagine computing the arclength for short geodesic segments. What then would we get for the value of the second time derivative of that arclength? This question has been discussed many times elsewhere [2,3] but under a different name—the second variation of arclength. Clearly the second variation of arclength and the ADM 3 + 1 evolution equations must be related. The purpose of this paper is to establish that relationship. The result is not unexpected—the equations for the second variation of arclength can be used to recover the standard ADM 3 + 1 evolution equations with zero shift vector.

There is value in this presentation beyond the purely pedagogical—the results presented here provide strong theoretical support for an approach to numerical relativity being developed by the author [4–7]. This method is known as smooth lattice relativity and is closely related to the Regge calculus [8–10]. Both methods use a lattice to describe the metric but they differ most notably in the way they treat the curvatures. In the Regge calculus the metric is piecewise flat with the result that the curvatures are distributions on the two-dimensional subspaces known as bones (or hinges) while on a smooth lattice we allow the metric to vary smoothly in the neighborhood of any vertex. This allows all the usual tools of differential geometry to be applied to the smooth lattice. In particular, we can easily compute the Riemann and extrinsic curvatures in terms of the geodesic arclengths of the lattice and thus, using Eqs. (3.3) and (3.4), or (2.5), the second time derivatives of the leg lengths. This makes the study of dynamics on a smooth lattice quite simple in principle (though as with any numerical method in general relativity the practical aspects are far from trivial). Attempts have been made to adapt the ADM 3 + 1 equations to the Regge calculus [11,12] but progress has been slow. A much more promising scheme, for the Regge calculus, is due to Sorkin [13] with later

development by Barrett *et al.* [14] and Gentle and Miller [15].

**II. FIRST AND SECOND VARIATIONS**

Discussions on the first and second variations normally arise when asking questions about geodesics such as: Is the geodesic that joins two points unique? Is it the shortest geodesic? How far can the geodesic be extended before it fails to be the shortest geodesic? The mathematical theory that answers these questions is very elegant and has previously found its way into general relativity as a tool in studying the global properties of spacetime [16]. Hawking and Penrose [17,18] made extensive use of the first and second variations of nonspacelike geodesics in their singularity theorems. In contrast, our interest in the first and second variations is that they provide a natural setting in which to ask different questions of geodesics: How can the first and second time derivatives (of the arclength) be computed? And how are they related to the curvature tensors? As already noted in the Introduction these questions will lead to the standard ADM 3 + 1 evolution equations with a zero shift vector. But first we need to introduce some basic notation and to make clear the class of curves we will be working with.

Choose a point  $i$  and a small neighborhood of  $i$  in which the spacetime is nonsingular. All of the curves we are about to construct will have a finite length and will lie totally within this neighborhood. Through  $i$  construct a timelike curve  $C_i$  with affine parameter  $\eta$ . From  $C_i$  we can construct a nearby curve  $C_j$  by dragging  $C_i$  sideways a short distance (i.e. drag  $C_i$  along a short spacelike vector field defined on  $C_i$ ). We now have two nearby timelike curves  $C_i$  and  $C_j$  (the point  $j$  on  $C_j$  can be easily identified—it has the same  $\eta$  value as  $i$  has on  $C_i$ ). We will assume that the two curves  $C_i$  and  $C_j$  are sufficiently close that, for any given  $\eta$ , we can construct a unique geodesic that joins the two curves.

Consider now the family of geodesics generated by allowing  $\eta$  to vary. This family of geodesics (actually, segments of geodesics) will cover a two-dimensional subspace (like a taut ribbon) which we will denote by  $S$ . We will introduce coordinates on  $S$  in a rather obvious way. Consider a point  $P$  on  $S$ . There will be exactly one spacelike geodesic (of  $S$ , by assumption) that passes through  $P$ .

The point  $P$  will be located some fraction,  $\lambda$ , along the geodesic from  $C_i$  to  $C_j$ . Thus the points on  $C_i$  will have  $\lambda = 0$  while those points on  $C_j$  will have  $\lambda = 1$ . We will take the other coordinate for  $P$  to be the value of  $\eta$  that identifies this geodesic from all others (in  $S$ ). The coordinates for  $P$  are then taken to be  $(\lambda, \eta)$ . This situation is displayed in Fig. 1.

Consider now a global coordinate system,  $x^\mu$ , for the spacetime. Then  $S$  can also be described by functions of the form  $x^\mu(\lambda, \eta)$ . We now define a pair of vectors  $\eta^\mu$  and  $\lambda^\mu$  by

$$\eta^\mu = \frac{\partial x^\mu}{\partial \eta}, \quad \lambda^\mu = \frac{\partial x^\mu}{\partial \lambda}, \quad (2.1)$$

and a pair of unit vectors  $n^\mu$  and  $m^\mu$  by

$$\eta^\mu = \frac{1}{N} n^\mu, \quad m^\mu = \frac{1}{M} \lambda^\mu, \quad (2.2)$$

where  $N$  and  $M$  are scalar functions that ensure that the vectors are indeed unit vectors. Clearly the vector  $\eta^\mu$  is tangent to the  $\lambda = \text{const}$  curves while  $\lambda^\mu$  is tangent to the  $\eta = \text{const}$  curves (and both vectors will, in general, be neither unit nor orthogonal, despite appearances in Fig. 1). It is rather easy to show that  $M = ds/d\lambda = L_{ij}$ , where  $s$  is the proper distance along the geodesic and  $L_{ij}$  is the length of that geodesic. Recall that  $ds/d\lambda$  is constant along a geodesic while  $L_{ij} = \int_0^1 (ds/d\lambda) d\lambda$  and thus  $L_{ij} = ds/d\lambda$ . Next, using the requirement that  $m^\mu$  be a unit vector leads immediately to  $M = ds/d\lambda = L_{ij}$  as claimed. Later, when we specialize to the ADM 3 + 1 formulation in Sec. III we

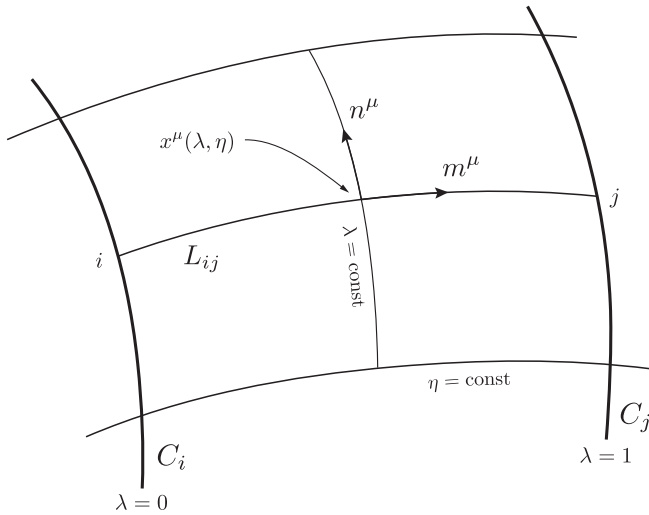


FIG. 1. This figure displays the two-dimensional surface  $S$  constructed from the pair of timelike worldlines  $C_i$  and  $C_j$ . The curve connecting  $i$  to  $j$  is a spacelike geodesic with length  $L_{ij}$ . Along these geodesics  $\eta = \text{const}$ . Note that the tangent vectors  $n^\mu$  and  $m^\mu$  are unit vectors but they need not be mutually orthogonal.

shall see that  $N$  is the usual lapse function associated with the time coordinate  $\eta$ .

We can now state clearly the equations for the arclength and their variations.

*Arclength:*

$$L_{ij} = \int_i^j \left( g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda} \frac{\partial x^\nu}{\partial \lambda} \right)^{1/2} d\lambda. \quad (2.3)$$

*First variation:*

$$\frac{dL_{ij}}{d\eta} = [m_{;\mu} \eta^\mu]_i^j = \int_i^j m_{;\mu} m^\nu \eta^\mu{}_{;\nu} ds. \quad (2.4)$$

*Second variation:*

$$\begin{aligned} \frac{d^2 L_{ij}}{d\eta^2} &= [\eta^\alpha{}_{;\mu} \eta^\mu m_\alpha]_i^j - \int_i^j R_{\mu\alpha\nu\beta} m^\mu m^\nu \eta^\alpha \eta^\beta ds \\ &+ \int_i^j (\eta_{\mu;\alpha} \eta^\mu{}_{;\nu} m^\alpha m^\nu - (m_{;\mu} m^\nu \eta^\mu{}_{;\nu})^2) ds. \end{aligned} \quad (2.5)$$

For ease of reference we have included a proof of the above equations in the Appendixes. See also [2,3] for more details.

### III. THE ADM EVOLUTION EQUATIONS: PART 1

Consider a typical Cauchy surface  $\Sigma$  and suppose that the pair of timelike curves  $C_i$  and  $C_j$  intersect  $\Sigma$  at the points  $i$  and  $j$ , respectively. At  $i$  we have two vectors  $n^\mu$ , the unit normal to  $\Sigma$  and  $m^\mu$ , the unit tangent to the geodesic that connects  $i$  to  $j$ . If we construct a third unit vector  $m'^\mu$  as a linear combination of  $n^\mu$  and  $m^\mu$ ,

$$m'^\mu \cosh\theta = m^\mu + n^\mu \sinh\theta,$$

we can, by careful choice of the boost angle  $\theta$ , ensure that  $m'^\mu$  is tangent to  $\Sigma$ . That is, we require  $\theta$  such that  $0 = n_{\mu} m'^\mu$ . This arrangement is shown in Fig. 2. In what follows we will be looking at the behavior of various expressions in the case where  $L_{ij}$  is small. So our present task is to ask how does  $\theta$  vary with  $L_{ij}$ ? The first observation is trivial:  $\theta \rightarrow 0$  as  $L_{ij} \rightarrow 0$ . Now from  $0 = n_{\mu} m'^\mu$  we

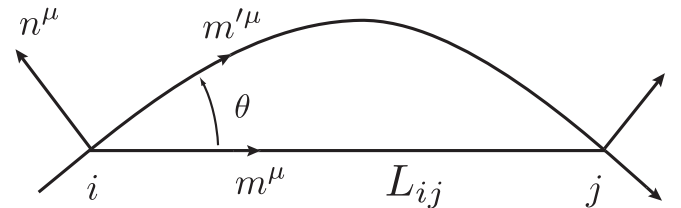


FIG. 2. In this figure the lower (straight) curve is the geodesic that joins  $i$  to  $j$ . The upper curve (which is not shown in Fig. 1) arises from the intersection of the Cauchy surface with the two-dimensional surface  $S$ . The unit vectors  $n^\mu$  and  $m'^\mu$  are orthogonal. Note that, in general,  $\eta$  is *not* constant on each Cauchy surface.

have

$$\sinh\theta = n_{\mu}m^{\mu},$$

and thus across the leg we have

$$[\sinh\theta]_i^j = [n_{\mu}m^{\mu}]_i^j = \frac{dL_{ij}}{d\eta}.$$

If we choose  $L_{ij}$  to be sufficiently small then we can be sure that the geodesic (that joins  $i$  to  $j$ ) intersects  $\Sigma$  only at  $i$  and  $j$  (see Fig. 3). From this constraint we observe that  $\theta_i$  and  $\theta_j$  must be of opposite signs and thus

$$|\sinh\theta_i| + |\sinh\theta_j| = \left| \frac{dL_{ij}}{d\eta} \right|.$$

Thus each term on the left must be of order  $\mathcal{O}(dL_{ij}/d\eta)$ , that is

$$\theta = \mathcal{O}\left(\frac{dL_{ij}}{d\eta}\right) \quad \text{as } L_{ij} \rightarrow 0.$$

### A. The first variation

Our aim in this section is to recast the expressions for the first and second variations in terms of the familiar ADM data, the lapse, shift, and extrinsic curvatures.

The extrinsic curvature,  $K_{\mu\nu}$ , can be defined in a number of ways (see [1]), such as

$$NK_{\mu\nu} = -Nn_{\mu;\nu} - \perp(N_{,\mu})n_{\nu},$$

where  $\perp$  is the projection operator ( $\perp^{\mu}_{\nu} = h^{\mu}_{\nu} = \delta^{\mu}_{\nu} + n^{\mu}n_{\nu}$ ). Then

$$\eta^{\mu}_{;\nu} = (Nn^{\mu})_{;\nu} = N_{,\nu}n^{\mu} - \perp(N^{,\mu})n_{\nu} - NK^{\mu}_{\nu},$$

and thus

$$\begin{aligned} m^{\mu}m^{\nu}(Nn_{\mu})_{;\nu} &= m^{\mu}m^{\nu}(N_{,\nu}n_{\mu} + Nn_{\mu;\nu}) \\ &= (m^{\nu}N_{,\nu})\sinh\theta - NK_{\mu\nu}m^{\mu}m^{\nu}. \end{aligned}$$

This can now be substituted into the integral for the first variation (2.4)

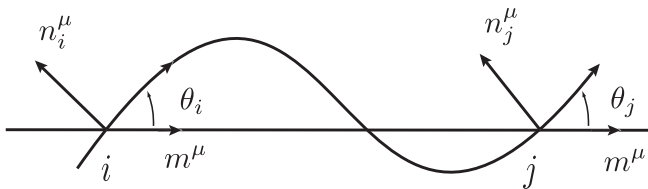


FIG. 3. This is a situation that we explicitly exclude. In this case the points  $i$  and  $j$  are so far apart that the geodesic intersects the Cauchy surface at points other than  $i$  and  $j$ . In this case  $\theta_i$  and  $\theta_j$  have the same signs, contrary to the assumptions made in the text.

$$\begin{aligned} \frac{dL_{ij}}{d\eta} &= \int_i^j m^{\mu}m^{\nu}\eta_{\mu;\nu}ds \\ &= \int_i^j (m^{\nu}N_{,\nu}\sinh\theta - NK_{\mu\nu}m^{\mu}m^{\nu})ds. \end{aligned}$$

Recall that we are dealing with short geodesic segments. Thus we can use any of a number of methods to estimate the integral. To be specific, we will chose a midpoint rule (see [19]) which leads to

$$\frac{dL_{ij}}{d\eta} = (m^{\nu}N_{,\nu}\sinh\theta)L_{ij} - (NK_{\mu\nu}m^{\mu}m^{\nu})L_{ij} + \mathcal{O}(L^2),$$

where each term is evaluated at the midpoint of the geodesic. But since  $\theta = \mathcal{O}(dL/d\eta)$ , we see that the first term is of order  $\mathcal{O}(L^2)$  and thus

$$\frac{dL_{ij}}{d\eta} = -(NK_{\mu\nu}m^{\mu}m^{\nu})L_{ij} + \mathcal{O}(L^2). \quad (3.1)$$

Notice that  $m^{\mu}$  is the unit tangent vector at the midpoint of the geodesic that joins  $i$  to  $j$  and thus we have

$$m^{\mu}L_{ij} = x_j^{\mu} - x_i^{\mu} + \mathcal{O}(L^3). \quad (3.2)$$

So, if we put  $\Delta x_{ij}^{\mu} = x_j^{\mu} - x_i^{\mu}$  we can rewrite (3.1) as

$$\frac{dL_{ij}^2}{d\eta} = -2NK_{\mu\nu}\Delta x_{ij}^{\mu}\Delta x_{ij}^{\nu} + \mathcal{O}(L^3). \quad (3.3)$$

### B. The second variation

Once again we use the basic definition of the extrinsic curvature to express the terms appearing in the second variation in an ADM form. We will do the calculations by splitting our previous expression for the second variation (2.5) into the following pieces:

$$\begin{aligned} \frac{d^2L_{ij}}{d\eta^2} &= J_1 + J_2 + J_3 + J_4, \\ J_1 &= [\eta^{\alpha}_{;\mu}\eta^{\mu}m_{\alpha}]_i^j, \\ J_2 &= -\int_i^j R_{\mu\alpha\nu\beta}m^{\mu}m^{\nu}\eta^{\alpha}\eta^{\beta}ds, \\ J_3 &= \int_i^j \eta_{\mu;\alpha}\eta^{\mu}_{;\nu}m^{\alpha}m^{\nu}ds, \\ J_4 &= -\int_i^j (m_{\mu}m^{\nu}\eta^{\mu}_{;\nu})^2ds. \end{aligned}$$

#### 1. The second term

We start with this term as it requires very little work. We simply substitute  $\eta^{\mu} = Nn^{\mu}$  and approximate the integral via a midpoint rule leading to

$$\begin{aligned} J_2 &= - \int_i^j R_{\mu\alpha\nu\beta} m^\mu m^\nu \eta^\alpha \eta^\beta ds \\ &= -N^2 R_{\mu\alpha\nu\beta} m^\mu m^\nu n^\alpha n^\beta L_{ij} + \mathcal{O}(L^2). \end{aligned}$$

### 2. The fourth term

Here we use  $m^\mu m^\nu \eta_{\mu;\nu} = -NK_{\mu\nu} m^\mu m^\nu + \mathcal{O}(L)$  [the error term arises from the  $n_\mu m^\mu = \sinh\theta = \mathcal{O}(L)$  terms]. Thus we are led to

$$\begin{aligned} J_4 &= - \int_i^j (m^\mu m^\nu \eta_{\mu;\nu})^2 ds \\ &= - \int_i^j (-NK_{\mu\nu} m^\mu m^\nu + \mathcal{O}(L))^2 ds \\ &= -(NK_{\mu\nu} m^\mu m^\nu)^2 L_{ij} + \mathcal{O}(L^2) \\ &= -\frac{1}{L_{ij}} \left( \frac{dL_{ij}}{d\eta} \right)^2 + \mathcal{O}(L^2), \end{aligned}$$

where we have used (3.1) in the second-to-last line. The remaining terms are not so easily dealt with.

### 3. The third term

For the third term the details are as follows:

$$\begin{aligned} J_3 &= \int_i^j \eta_{\mu;\alpha} \eta^\mu{}_{;\nu} m^\alpha m^\nu ds \\ &= \int_i^j (N_{,\alpha} n_\mu + N n_{\mu;\alpha}) (N_{,\beta} n^\mu + N n^\mu{}_{;\beta}) m^\alpha m^\beta ds \\ &= \int_i^j (-N_{,\alpha} m^\alpha)^2 + N^2 n_{\mu;\alpha} n^\mu{}_{;\beta} m^\alpha m^\beta ds \\ &= \int_i^j (-N_{,\alpha} m^\alpha)^2 + (\perp(N_{,\mu}) n_\alpha + K_{\mu\alpha}) (\perp(N_{,\mu}) n_\beta + K^\mu{}_\beta) m^\alpha m^\beta ds \\ &= \int_i^j (-N_{,\alpha} m^\alpha)^2 + N^2 K_{\mu\alpha} K^\mu{}_\beta m^\alpha m^\beta + \mathcal{O}(L) ds. \end{aligned}$$

The error term  $\mathcal{O}(L)$  in the last line arises from terms of the form  $n_\mu m^\mu = \sinh\theta = \mathcal{O}(L)$ . Now we use the midpoint rule, once again, to obtain

$$\begin{aligned} J_3 &= \int_i^j \eta_{\mu;\alpha} \eta^\mu{}_{;\nu} m^\alpha m^\nu ds \\ &= -(N_{,\alpha} m^\alpha)^2 L_{ij} + N^2 K_{\mu\alpha} K^\mu{}_\beta m^\alpha m^\beta L_{ij} + \mathcal{O}(L^2). \end{aligned}$$

### 4. The first term

Finally, we turn to the first term  $[\eta^\alpha{}_{;\mu} \eta^\mu m_\alpha]_i^j$ . Using the same substitutions as we have used before and also using  $N n^\mu N_{,\mu} = dN/d\eta$ , we obtain

$$J_1 = [\eta^\alpha{}_{;\mu} \eta^\mu m_\alpha]_i^j = \left[ \frac{1}{N} \frac{dN}{d\eta} \eta^\mu m_\mu \right]_i^j + [NN_{,\mu} m^\mu]_i^j.$$

We choose to write this result as a sum of two terms each of

the form  $[\cdot \cdot \cdot]_i^j$  so that we can deal with each term separately. In the first term we have  $(1/N)dN/d\eta$ , which varies slowly over the short geodesic and thus may be taken as a constant [plus an error term of order  $\mathcal{O}(L)$ ]; thus we have

$$\begin{aligned} \left[ \frac{1}{N} \frac{dN}{d\eta} \eta^\mu m_\mu \right]_i^j &= \frac{1}{N} \frac{dN}{d\eta} [\eta^\mu m_\mu]_i^j + [\eta^\mu m_\mu]_i^j \mathcal{O}(L) \\ &= \frac{1}{N} \frac{dN}{d\eta} \frac{dL_{ij}}{d\eta} + \mathcal{O}(L^2). \end{aligned}$$

For the second term we use a Taylor series expansion

$$\begin{aligned} [NN_{,\mu} m^\mu]_i^j &= \frac{d}{ds} (NN_{,\mu} m^\mu) L_{ij} + \mathcal{O}(L^2) \\ &= (NN_{,\mu} m^\mu)_{;\alpha} m^\alpha L_{ij} + \mathcal{O}(L^2) \\ &= (N_{,\mu} m^\mu)^2 L_{ij} + NN_{;\alpha\beta} m^\alpha m^\beta L_{ij} + \mathcal{O}(L^2). \end{aligned}$$

The appearance of the term  $N_{;\alpha\beta}$  is encouraging—it reminds us of the similar term in the ADM equations. We can improve on this situation. Notice that  $m'^\mu = m^\mu + \mathcal{O}(L)$  and thus

$$N_{;\alpha\beta} m^\alpha m^\beta = N_{;\alpha\beta} m'^\alpha m'^\beta + \mathcal{O}(L).$$

However,  $m'^\mu$  is tangent to  $\Sigma$ ; thus we also have

$$\begin{aligned} N_{;\alpha\beta} m^\alpha m^\beta &= N_{|\alpha\beta} m'^\alpha m'^\beta + \mathcal{O}(L) \\ &= N_{|\alpha\beta} m^\alpha m^\beta + \mathcal{O}(L), \end{aligned}$$

where the vertical stroke denotes covariant differentiation with respect to the 3-metric intrinsic to  $\Sigma$ .

Combining these two results we obtain our final estimate for the first term in the second variation

$$\begin{aligned} [\eta^\alpha{}_{;\mu} \eta^\mu m_\alpha]_i^j &= \frac{1}{N} \frac{dN}{d\eta} + (N_{,\mu} m^\mu)^2 L_{ij} \\ &\quad + NN_{|\alpha\beta} m^\alpha m^\beta L_{ij} + \mathcal{O}(L^2). \end{aligned}$$

Now we can reassemble the pieces. The result is

$$\begin{aligned} \frac{d^2 L_{ij}}{d\eta^2} &= \frac{1}{N} \frac{dN}{d\eta} \frac{dL_{ij}}{d\eta} - \frac{1}{L_{ij}} \left( \frac{dL_{ij}}{d\eta} \right)^2 \\ &\quad + N^2 K_{\mu\alpha} K^\mu{}_\beta m^\alpha m^\beta L_{ij} + NN_{|\alpha\beta} m^\alpha m^\beta L_{ij} \\ &\quad - N^2 R_{\mu\alpha\nu\beta} m^\mu m^\nu n^\alpha n^\beta L_{ij} + \mathcal{O}(L^2). \end{aligned}$$

We are almost finished; we just need to do a little bit of tidying up. We multiply both sides by  $L_{ij}/N$  and noting that

$$\begin{aligned} \frac{d^2 L_{ij}^2}{d\eta^2} &= 2 \left( \frac{dL_{ij}}{d\eta} \right)^2 + 2L_{ij} \frac{d^2 L_{ij}}{d\eta^2}, \\ \frac{d}{d\eta} \left( \frac{1}{N} \frac{dL_{ij}^2}{d\eta} \right) &= -\frac{1}{N^2} \frac{dN}{d\eta} \frac{dL_{ij}^2}{d\eta} + \frac{1}{N} \frac{d^2 L_{ij}^2}{d\eta^2}, \end{aligned}$$

we can rewrite the above equation as

$$\begin{aligned} \frac{d}{d\eta} \left( \frac{1}{N} \frac{dL_{ij}^2}{d\eta} \right) &= 2N_{|\alpha\beta} \Delta x_{ij}^\alpha \Delta x_{ij}^\beta + 2N(K_{\mu\alpha} K^\mu{}_\beta \\ &\quad - R_{\mu\alpha\nu\beta} n^\mu n^\nu) \Delta x_{ij}^\alpha \Delta x_{ij}^\beta + \mathcal{O}(L^3), \end{aligned} \quad (3.4)$$

where we have also used  $\Delta x_{ij}^\mu = m^\mu L_{ij} + \mathcal{O}(L^3)$ .

For a review, see Eq. (3.3) for the result we previously obtained for the first time derivative.

#### IV. THE ADM EVOLUTION EQUATIONS: PART 2

This completes the first stage of the construction. We have successfully expressed the first and second variations in terms of the extrinsic and Riemann curvatures. Our second and final stage will, among other things, introduce the metric tensor as a replacement for the geodesic arclengths. As we shall soon see, this is not a difficult task. The most notable change is not in the symbols, from  $L_{ij}^2$  to  $g_{\mu\nu}$ , but in the structure of the equations. We will be reworking an equation defined over a geodesic segment into a new equation defined at a point.

Consider a typical geodesic segment with end points  $i$  and  $j$ . The timelike worldlines  $C_i$  and  $C_j$  generated by  $i$  and  $j$  are, by assumption, orthogonal to the Cauchy surfaces. Thus we can use these curves to propagate the spatial coordinates of each Cauchy surface forward in time. This means that the spatial coordinates of any point on  $C_i$  are constant along  $C_i$  and thus  $0 = d\Delta x_{ij}^\mu/d\eta$ .

We now introduce the metric by estimating  $L_{ij}$  using a midpoint rule for  $\int ds$ ,

$$\begin{aligned} L_{ij} &= \int_i^j \left( g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda} \frac{\partial x^\nu}{\partial \lambda} \right)^{1/2} M d\lambda \\ &= \left( g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda} \frac{\partial x^\nu}{\partial \lambda} \right)^{1/2} M + \mathcal{O}(L^2), \end{aligned}$$

where each term on the right-hand side is evaluated at the midpoint of the geodesic. But we have previously shown (2.1), (2.2), and (3.2) that  $\partial x^\mu/\partial \lambda = m^\mu L_{ij} = \Delta x_{ij}^\mu + \mathcal{O}(L^3)$ . We can use this to estimate  $L_{ij}^2$  as

$$L_{ij}^2 = g_{\mu\nu} \Delta x_{ij}^\mu \Delta x_{ij}^\nu + \mathcal{O}(L^3).$$

We can go one step further by noting that  $g_{\mu\nu} = h_{\mu\nu} - n_\mu n_\nu$  and  $n_\mu \Delta x_{ij}^\mu = L_{ij} \sinh\theta = \mathcal{O}(L^2)$  and thus to leading order in  $L$  we have

$$g_{\mu\nu} \Delta x_{ij}^\mu \Delta x_{ij}^\nu = h_{\mu\nu} \Delta x_{ij}^\mu \Delta x_{ij}^\nu + \mathcal{O}(L^4),$$

which, when substituted into the above, leads to

$$L_{ij}^2 = h_{\mu\nu} \Delta x_{ij}^\mu \Delta x_{ij}^\nu + \mathcal{O}(L^3). \quad (4.1)$$

It is now just a short step to the finish line. First substitute (3.3) into (3.4) and then (4.1) into (3.3) and finally take the  $\Delta x_{ij}^\mu$  terms out through the time derivatives. Then notice that the  $\Delta x_{ij}^\mu$  are arbitrary and that the coefficients of

$\Delta x_{ij}^\mu \Delta x_{ij}^\nu$  are symmetric in  $\mu\nu$  and purely spatial. This allows us to cancel the  $\Delta x^\mu$  from both sides of the equations after which we can safely let  $L \rightarrow 0$  (the details of this series of substitutions and eliminations are excluded as they follow very standard lines). As expected the final result is nothing other than the familiar ADM evolution equations with a zero shift vector

$$\begin{aligned} \frac{dh_{\mu\nu}}{d\eta} &= -2NK_{\mu\nu}, \\ \frac{dK_{\mu\nu}}{d\eta} &= -N_{|\mu\nu} - N(K_{\mu\alpha} K^\alpha{}_\nu - R_{\mu\alpha\nu\beta} n^\alpha n^\beta). \end{aligned}$$

#### APPENDIX A: THE FIRST VARIATION

We know that the mixed partial derivatives of  $x^\mu(\lambda, \eta)$  must commute; thus we must have

$$\lambda^\mu{}_{,\nu} \eta^\nu = \eta^\mu{}_{,\nu} \lambda^\nu,$$

and for a symmetric connection (which we are using) we also have

$$\lambda^\mu{}_{;\nu} \eta^\nu = \eta^\mu{}_{;\nu} \lambda^\nu,$$

which can be reexpressed, in terms of the unit vectors  $n^\mu$  and  $m^\mu$ , as

$$(Nn^\mu)_{;\nu} (Mm^\nu) = (Mm^\mu)_{;\nu} (Nn^\nu). \quad (A1)$$

Finally, as the vector  $m^\mu$  is the unit tangent to an  $\eta = \text{const}$  geodesic, we have

$$0 = m^\mu{}_{;\nu} m^\nu,$$

and

$$0 = \frac{\partial^2 x^\mu}{\partial \lambda^2} + \Gamma_{\alpha\beta}^\mu \frac{\partial x^\alpha}{\partial \lambda} \frac{\partial x^\beta}{\partial \lambda}.$$

We will use the above equations frequently in the following discussions.

Here we consider the geodesic arclength and its first time derivative,

$$\begin{aligned} L_{ij} &= \int_0^1 \frac{ds}{d\lambda} d\lambda = \int_0^1 \left( g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda} \frac{\partial x^\nu}{\partial \lambda} \right)^{1/2} d\lambda, \\ \frac{dL_{ij}}{d\eta} &= \frac{d}{d\eta} \int_0^1 \frac{ds}{d\lambda} d\lambda = \int_0^1 \frac{\partial}{\partial \eta} \left( g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda} \frac{\partial x^\nu}{\partial \lambda} \right)^{1/2} d\lambda. \end{aligned}$$

Note that the path  $x^\mu(\lambda, \eta)$  in each of these integrals is a geodesic and that  $\eta$  is constant along the geodesic. The second integral in the last equation above can be readily evaluated using standard techniques (expand the  $\eta$  derivative, swap orders of mixed derivatives, integrate by parts, and impose the geodesic equation). The result is

$$\frac{dL_{ij}}{d\eta} = \frac{1}{L_{ij}} \left[ g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda} \frac{\partial x^\nu}{\partial \lambda} \right]_i^j = [m_\mu \eta^\mu]_i^j,$$

where we have taken the small liberty of replacing the



limits 0 and 1 with the more suggestive labels  $i$  and  $j$ . This is an elegant result—it shows that for a geodesic segment,  $dL_{ij}/d\eta$  can be computed from data defined only at the end points of the geodesic. This may seem simple but it hides a significant complexity—the data involved can only be found by solving a two-point boundary value problem.

Despite this compact and elegant form for the first time derivative, we will now develop an alternative integral expression that happens to be better suited to our later calculations of the second time derivative. Consider for the moment the quantity  $Q$  defined by

$$Q = \int_i^j m_\mu m^\nu (Nn^\mu)_{;\nu} ds,$$

with the integration path being, as expected, an  $\eta = \text{const}$  geodesic. We will now show that  $Q = dL_{ij}/d\eta$ . We begin by writing  $ds = (ds/d\lambda) = M d\lambda$  and using the commutation relation (A1) to obtain

$$Q = \int_i^j m_\mu \eta^\nu (Mm^\mu)_{;\nu} d\lambda.$$

Now expand the covariant derivative and use  $1 = m_\mu m^\mu$  and  $0 = m_{\mu;\nu}$  to obtain

$$\begin{aligned} Q &= \int_i^j \eta^\nu M_{;\nu} d\lambda = \int_i^j \frac{\partial}{\partial \eta} \left( \frac{ds}{d\lambda} \right) d\lambda \\ &= \frac{d}{d\eta} \int_i^j \frac{ds}{d\lambda} d\lambda = \frac{dL_{ij}}{d\eta}. \end{aligned}$$

Thus we have shown Eq. (2.4). Our challenge now is to compute the second time derivative. This proceeds in a manner similar to Eq. (2.4) though it is a tad lengthy.

## APPENDIX B: THE SECOND VARIATION

To compute the second derivative we need only apply  $d/d\eta$  to (2.4). This leads to

$$\begin{aligned} \frac{d^2 L_{ij}}{d\eta^2} &= \frac{d}{d\eta} \int_i^j m_\mu m^\nu \eta^\mu_{;\nu} ds \\ &= \int_i^j \frac{\partial}{\partial \eta} (m_\mu m^\nu \eta^\mu_{;\nu} M) d\lambda \\ &= \int_i^j (m_\mu \eta^\mu_{;\nu} \lambda^\nu)_{;\alpha} \eta^\alpha d\lambda \\ &= \int_i^j (m_{\mu;\alpha} \eta^\mu_{;\nu} \lambda^\nu + m_\mu \eta^\mu_{;\nu;\alpha} \lambda^\nu \\ &\quad + m_\mu \eta^\mu_{;\nu} \lambda^\nu_{;\alpha}) \eta^\alpha d\lambda. \end{aligned}$$

We will apply various manipulations to the three main parts of this integral and we will make extensive use of the geodesic equations,  $0 = m^\mu_{;\nu} m^\nu$ , the commutation relations,  $\lambda^\mu_{;\nu} \eta^\nu = \eta^\mu_{;\nu} \lambda^\nu$ , and the observations that  $m^\mu$  is a unit vector along the geodesic.

We start by splitting the above integral into three pieces

$$\begin{aligned} I_1 &= \int_i^j m_{\mu;\alpha} \eta^\mu_{;\nu} \lambda^\nu \eta^\alpha d\lambda, \\ I_2 &= \int_i^j m_\mu \eta^\mu_{;\nu;\alpha} \lambda^\nu \eta^\alpha d\lambda, \\ I_3 &= \int_i^j m_\mu \eta^\mu_{;\nu} \lambda^\nu_{;\alpha} \eta^\alpha d\lambda, \end{aligned}$$

which we will now attempt to simplify.

*Integral  $I_1$ :* Put  $\lambda^\nu = m^\nu M$  and  $m_{\mu;\alpha} M = (m_\mu M)_{;\alpha} - m_\mu M_{;\alpha}$  and then use the commutation rule on  $\lambda_{\mu;\alpha} \eta^\alpha$  to obtain

$$I_1 = \int_i^j \eta_{\mu;\alpha} \eta^\mu_{;\nu} m^\alpha m^\nu M d\lambda - \int_i^j m_\mu M_{;\alpha} \eta^\alpha \eta^\mu_{;\nu} m^\nu d\lambda.$$

Consider the second integral in this pair and denote it by  $I_4$ . Since  $m^\mu$  is a unit vector we can slide a factor of  $m_\theta m^\theta$  inside  $M_{;\alpha}$ , like this

$$\begin{aligned} I_4 &= \int_i^j m_\mu (m_\theta m^\theta M)_{;\alpha} \eta^\alpha \eta^\mu_{;\nu} m^\nu d\lambda \\ &= \int_i^j m_\mu (m_{\theta;\alpha} m^\theta M + m_\theta (m^\theta M)_{;\alpha}) \eta^\alpha \eta^\mu_{;\nu} m^\nu d\lambda. \end{aligned}$$

The term  $m_{\theta;\alpha} m^\theta$  is zero since  $m^\mu$  is a unit vector while the remaining term is ripe for a commutation operation. This leads to

$$\begin{aligned} I_4 &= \int_i^j m_\mu m_\theta \eta^\theta_{;\alpha} m^\alpha M \eta^\mu_{;\nu} m^\nu d\lambda \\ &= \int_i^j (m_\mu m^\nu \eta^\mu_{;\nu})^2 M d\lambda. \end{aligned}$$

So our final expression for  $I_1$  is

$$I_1 = \int_i^j (\eta_{\mu;\alpha} \eta^\mu_{;\nu} m^\alpha m^\nu - (m_\mu m^\nu \eta^\mu_{;\nu})^2) M d\lambda.$$

*Integral  $I_3$ :* We step out of sequence here because one term arises in this computation that will be useful when we tackle the second integral  $I_2$ .

This integral is slightly easier to work with than the first integral and it will give rise to the Riemann tensor. The main device used here is to swap the order of the second partial derivatives on  $\eta^\mu_{;\nu;\alpha}$  balanced by the addition of the Riemann tensor. Thus we have

$$\begin{aligned} I_3 &= \int_i^j m_\mu \eta^\mu_{;\nu;\alpha} m^\nu \eta^\alpha M d\lambda \\ &= \int_i^j m_\mu (\eta^\mu_{;\alpha;\nu} + R^\mu{}_{\rho\alpha\nu} \eta^\rho) m^\nu \eta^\alpha M d\lambda \\ &= I_5 - \int_i^j R_{\mu\alpha\nu\beta} m^\mu m^\nu \eta^\alpha \eta^\beta M d\lambda, \end{aligned}$$

where we have introduced a fifth integral,

$$I_5 = \int_i^j m_{\mu} \eta^{\mu}_{;\alpha;\nu} m^{\nu} \eta^{\alpha} M d\lambda.$$

*Integral  $I_2 + I_5$ :* As we shall soon see, the integrand for  $I_2 + I_5$  can be combined to form a total derivative and thus the integration is trivial. We start by forming the sum  $I_2$  and  $I_5$

$$I_2 + I_5 = \int_i^j (\eta_{\alpha;\mu} \eta^{\mu}_{;\beta} m^{\alpha} m^{\beta} + \eta^{\mu}_{;\alpha;\nu} m_{\mu} \eta^{\alpha} m^{\nu}) ds,$$

where  $ds = M d\lambda$ . By careful inspection of the integrand, while noting the geodesic conditions,  $0 = m^{\mu}_{;\nu} m^{\nu}$ , it is not hard to see that the integrand can also be written as  $(\eta^{\alpha}_{;\mu} \eta^{\mu} m_{\alpha})_{;\nu} m^{\nu}$ . Thus we have

$$I_2 + I_5 = \int_i^j (\eta^{\alpha}_{;\mu} \eta^{\mu} m_{\alpha})_{;\nu} m^{\nu} ds = [\eta^{\alpha}_{;\mu} \eta^{\mu} m_{\alpha}]_i^j.$$

Our job is done, and all of the integrals have been evaluated as far as possible—all that remains is to combine the above results. This leads to Eq. (2.5). This last integral can be simplified slightly by introducing

$$v^{\mu} = \eta^{\mu} - \eta_{\rho} m^{\rho} m^{\mu},$$

which leads to

$$\begin{aligned} \frac{d^2 L_{ij}}{d\eta^2} &= [\eta^{\alpha}_{;\mu} \eta^{\mu} m_{\alpha}]_i^j - \int_i^j R_{\mu\alpha\nu\beta} m^{\mu} m^{\nu} \eta^{\alpha} \eta^{\beta} ds \\ &+ \int_i^j v_{\mu;\alpha} v^{\mu}_{;\nu} m^{\alpha} m^{\nu} ds. \end{aligned}$$

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