

**Black hole initial data on hyperboloidal slices**Luisa T. Buchman,<sup>1,2</sup> Harald P. Pfeiffer,<sup>1,3</sup> and James M. Bardeen<sup>4</sup><sup>1</sup>*Theoretical Astrophysics, California Institute of Technology, Pasadena, California 91125, USA*<sup>2</sup>*Center for Relativity, University of Texas at Austin, Austin, Texas 78712, USA*<sup>3</sup>*Canadian Institute for Theoretical Astrophysics, University of Toronto, Toronto, Ontario M5S 3H8, Canada*<sup>4</sup>*Physics Department, University of Washington, Seattle, Washington 98195, USA*

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We generalize Bowen-York black hole initial data to hyperboloidal constant mean curvature slices which extend to future null infinity. We solve this initial value problem numerically for several cases, including unequal mass binary black holes with spins and boosts. The singularity at null infinity in the Hamiltonian constraint associated with a constant mean curvature hypersurface does not pose any particular difficulties. The inner boundaries of our slices are minimal surfaces. Trumpet configurations are explored both analytically and numerically.

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**I. INTRODUCTION**

For asymptotically flat spacetimes, gravitational radiation is well-defined only at future null infinity ( $I^+$ ) [1–3]. Consequently, numerical relativists extracting gravitational waves from binary black hole simulations ideally would like to include  $I^+$  in their computational domains, so that the Bondi news function [1,4] (which contains the gravitational wave information) can be computed. Additionally, extending the simulation to  $I^+$  would make it unnecessary to deal with gravitational wave extraction at a finite distance, or with artificial outer boundaries on a truncated domain, two very complicated aspects of black hole simulations (see, e.g. [5–14]).

Null infinity can be included in the computational domain via a compactified radial coordinate on the null hypersurfaces of the characteristic initial value problem [15], but the null hypersurfaces are subject to caustic singularities, particularly in the strong fields around black holes. The caustics can be avoided by Cauchy-characteristic matching, but an appealing alternative is solving the Cauchy problem on conformally compactified hyperboloidal spacelike slices. These behave like conventional 3 + 1 slicing in the vicinity of the sources, but smoothly become asymptotically null as they approach null infinity at a finite coordinate distance [16–18]. Friedrich [17] derived a system of symmetric hyperbolic evolution equations based on the Bianchi identities for the conformal Weyl tensor which are regular at null infinity on hyperboloidal hypersurfaces provided certain smoothness conditions are satisfied. This system has been used with some, but limited, success in numerical calculations (see, e.g. [19]). More recently, evolution schemes have been proposed which directly evolve the conformal metric and the conformal factor through the conformally compactified Einstein equations on hyperboloidal hypersurfaces [20,21]. They are not manifestly regular at  $I^+$ , but in [21], Moncrief and Rinne derive regularity conditions based on the constraint equations which deal successfully with the

singularity of the conformal factor at  $I^+$  and can be imposed in a numerical implementation.

Numerical evolution schemes on hyperboloidal hypersurfaces extending to  $I^+$  require initial data. The initial value problem on asymptotically null hyperboloidal constant mean curvature (CMC) slices turns out to be remarkably similar to the corresponding problem on asymptotically flat *zero* mean curvature slices, and can be attacked similarly to the well-known conformally flat Bowen-York [22] initial data. (The Bowen-York initial data presented in [22] also forms the basis of puncture data [23]<sup>1</sup>). Therefore, in this paper, we shall refer to the initial data constructed as *hyperboloidal Bowen-York data*.

There are four main points to this paper: (i) to lay down the formalism for constructing hyperboloidal Bowen-York black hole initial data on CMC slices containing one or more black holes with arbitrary masses, spins and boosts; (ii) to give rules for choosing the various free parameters entering the formalism; (iii) to understand the physical meaning of the free parameters; and (iv) to discuss the physical interpretation of the constructed solutions, which is different for CMC slices than for traditional maximal slices.

Part of the initial data construction is the numerical solution of an elliptic equation arising from the Hamiltonian constraint. On hyperboloidal slices, this equation is formally singular at the outer boundary  $I^+$ . Nevertheless, the employed spectral elliptic solver [24] does not exhibit any problems while computing the solution.

Our initial data is formulated with a singularity-avoiding minimal surface boundary condition, on an Einstein-Rosen

<sup>1</sup>Puncture data differs from inversion symmetric data in the handling of the black hole singularities; we do not discuss the puncture treatment of singularities on CMC slices, although all our results concerning the constraint equations will carry over to such a treatment.

bridge connecting two asymptotically flat ends. In the limit in which the conformal radius approaches zero, a “trumpet” configuration is formed in which the Einstein-Rosen bridge becomes infinitely long in proper distance. Hyperboloidal slicings that contain trumpets have been examined in Refs. [25,26]. Trumpet initial data is of interest to numerical relativists using the moving puncture approach [27,28] because puncture initial data evolve quickly toward a trumpet configuration [29–31].

The organization of this paper is as follows: Sec. II presents and analyzes the formalism for constructing hyperboloidal Bowen-York initial data. Specifically, Sec. II A presents the initial value formalism on CMC slices and Sec. II B gives the particulars in the special case of a Schwarzschild black hole. Section II C discusses the parameter space yielding a minimal surface at the inner boundary of a Schwarzschild black hole in CMC slicing. Section II D presents the construction of initial data for single black holes, with and without Bowen-York spin and boost parameters. The physical interpretation of the Bowen-York parameters is also discussed. In Sec. II E, the methods presented for single black holes are generalized to multiple black holes. Section II F details the solution to the Hamiltonian constraint equation when the inner boundary is a trumpet. Section III presents our numerical solutions for hyperboloidal Bowen-York initial data. Specifically, Secs. III A, III B, and III C give solutions for single black holes that are spherically symmetric, spinning, or boosted, respectively. Section III D presents numerical results for two unequal mass black holes with arbitrarily oriented spins and boosts. Finally, Sec. IV discusses our results. An Appendix is included, in which conditions are derived for inversion symmetry on both CMC and maximal slices.

## II. ANALYTICS

### A. Initial value formalism on CMC slices

With a standard 3 + 1 decomposition [32,33], the space-time metric is written as

$$ds^2 = -\alpha^2 dt^2 + g_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \quad (1)$$

where  $g_{ij}$  represents the induced metric on the  $t = \text{const}$  hypersurface  $\Sigma_t$ ,  $\alpha$  the lapse-function and  $\beta^i$  the shift-vector. The extrinsic curvature is defined as

$$K_{\ell m} \equiv \frac{1}{2} \mathcal{L}_{(4)_n} g_{\ell m}, \quad (2)$$

where  $\mathcal{L}_{(4)_n}$  is the Lie derivative along the hypersurface unit-normal  ${}^{(4)}n^\mu$ . In Eq. (2), as throughout this paper, we employ the sign convention of Wald [34], resulting in a *positive* mean curvature  $K \equiv g_{ij}K^{ij}$  for the cases considered. This sign convention differs from the conventions of Misner, Thorne, and Wheeler [35], which are more commonly used in numerical relativity (as, for instance, in Ref. [36]).

Einstein’s vacuum constraint equations are

$$\mathcal{R} + K^2 - K_{ij}K^{ij} = 0, \quad (3)$$

and

$$\nabla_j(K^{ij} - g^{ij}K) = 0, \quad (4)$$

where  $\nabla_i$  is the covariant derivative with respect to  $g_{ij}$ , and  $\mathcal{R}$  is the Ricci scalar associated with  $g_{ij}$ .

We now perform a conformal transformation which plays a dual role. On one hand, it allows a conformal compactification (Penrose [37]), placing  $I^+$  at a *finite* value of a compactified radial coordinate, absorbing the resulting metric singularities into the conformal factor. On the other hand, it allows the Einstein constraints to be recast as elliptic equations following the standard procedure of the conformal method [38,39].

The conformal metric  $\tilde{g}_{ij}$  and conformal factor  $\Omega$  are given by

$$g_{ij} = \Omega^{-2} \tilde{g}_{ij}, \quad g^{ij} = \Omega^2 \tilde{g}^{ij}. \quad (5)$$

The conformal metric is assumed to be regular at  $I^+$  in compactified coordinates. This implies that  $\Omega = 0$  at  $I^+$  since, in compactified coordinates, the physical metric is singular there. Comparing Eq. (5) with the more widely used definition  $g_{ij} = \psi^4 \tilde{g}_{ij}$ , we see that  $\Omega = \psi^{-2}$ . The advantage of the definition (5) is that  $\Omega$  is *finite* at  $I^+$ . The equations in the rest of this section mirror the standard conformal method [38,39], with the replacement  $\Omega \leftrightarrow \psi^{-2}$ .

The trace-free extrinsic curvature is defined as

$$A^{ij} = K^{ij} - \frac{1}{3} g^{ij} K, \quad (6)$$

where  $K^{ij} = g^{i\ell} g^{jm} K_{\ell m}$ . The physical trace-free extrinsic curvature  $A^{ij}$  and its conformal counterpart  $\tilde{A}^{ij}$  are related by

$$A_{ij} = \Omega \tilde{A}_{ij}, \quad A^{ij} = \Omega^5 \tilde{A}^{ij}. \quad (7)$$

Substituting Eq. (6) into Eq. (4) and using the CMC condition  $\nabla_j K = 0$  gives

$$\nabla_j A^{ij} = 0. \quad (8)$$

This can be rewritten as

$$\Omega^5 \tilde{\nabla}_j \tilde{A}^{ij} = 0, \quad (9)$$

where  $\tilde{\nabla}_i$  is the covariant derivative with respect to  $\tilde{g}_{ij}$ . Because of the CMC condition, the momentum constraint has decoupled from the Hamiltonian constraint, and can be solved first. Standard methods for solving the momentum constraint are given in Ref. [40]. In this paper, we assume a flat conformal metric  $\tilde{g}_{ij}$ .

Substituting the solution  $\tilde{A}^{ij}$  of the momentum constraint into the Hamiltonian constraint (3) and expressing it in terms of conformal quantities, one finds

$$\tilde{\nabla}^2 \Omega - \frac{3}{2\Omega} (\tilde{\nabla} \Omega)^2 + \frac{\Omega}{4} \tilde{\mathcal{R}} + \frac{K^2}{6\Omega} - \frac{\Omega^5}{4} \tilde{A}_{ij} \tilde{A}^{ij} = 0. \quad (10)$$

Here  $\tilde{\nabla}^2$  denotes the covariant Laplacian with respect to  $\tilde{g}_{ij}$  and  $\tilde{\mathcal{R}}$  is the scalar curvature of  $\tilde{g}_{ij}$ . Note that some terms in Eq. (10) are singular at  $I^+$  where  $\Omega = 0$ . Assuming  $\tilde{A}_{ij}$  is finite at  $I^+$ , any regular solution satisfying the boundary condition

$$\Omega|_{I^+} = 0 \quad (11)$$

must also satisfy

$$(\tilde{\nabla} \Omega)^2|_{I^+} = \left(\frac{K}{3}\right)^2. \quad (12)$$

## B. The Schwarzschild black hole in CMC slicing

Let us now discuss this initial value formalism in the particular case of a Schwarzschild black hole, recasting already established results [41–43] in our language. In spherical coordinates  $(r, \theta, \phi, t)$ , with  $t$  constant on CMC slices and  $r$  the areal radius, the metric is given by [41–43]

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{f^2} dr^2 - \frac{2a}{f} dt dr + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (13)$$

where the functions  $f = f(r)$  and  $a = a(r)$  are

$$f(r) = \left(1 - \frac{2M}{r} + a^2\right)^{1/2}, \quad a(r) = \frac{Kr}{3} - \frac{C}{r^2}. \quad (14)$$

With these definitions,  $a(r) = {}^{(4)}n^r$ , where  ${}^{(4)}n^r$  is the radial component of the hypersurface-normal  ${}^{(4)}n^\mu$ . The constant  $M$  is the mass of the black hole. The constant  $C$  represents an additional one-parameter degree of freedom in the choice of spherically symmetric CMC hypersurfaces in the Schwarzschild metric. The lapse-function  $\alpha$  of the metric (13) equals  $f$ , and the only nonvanishing component of the shift is  $\beta^r = -af$ .

The coordinate transformation between Schwarzschild coordinates  $(r, \theta, \phi, T)$  and the CMC coordinates  $(r, \theta, \phi, t)$  (again, see [41–43]) is

$$t = T - \int \frac{a(r)}{(1 - \frac{2M}{r})f(r)} dr = u + \int \frac{dr}{(1 - \frac{2M}{r})} - \int \frac{a(r)}{(1 - \frac{2M}{r})f(r)} dr, \quad (15)$$

where  $u$  is the Eddington-Finkelstein retarded null coordinate. With  $K > 0$ , constants of integration can be chosen so that  $t \rightarrow u$  as  $r \rightarrow \infty$ .

The hypersurfaces  $\Sigma_t$  of constant coordinate value  $t$  will play a central role in this paper. For  $K = 0$ , these hypersurfaces extend to spacelike infinity. For  $K \neq 0$ ,  $\Sigma_t$  becomes asymptotically null for large radius. If  $K > 0$ , this

hypersurface intersects future null infinity, because  $1/f \rightarrow 0$  and  $a/f \rightarrow +1$  in the limit  $r \rightarrow \infty$ . For  $K < 0$ , it intersects past null infinity. We are interested in hypersurfaces approaching future null infinity, and will therefore require

$$K > 0 \quad (16)$$

throughout this paper.

The constant  $C$  determines whether the  $\Sigma_t$  hypersurface intersects the black hole horizon or the white hole horizon. Considering radial null rays on the horizon,  $r = 2M$ , we find that for

$$C > \frac{8}{3}KM^3, \quad (17)$$

$\Sigma_t$  intersects the black hole horizon (i.e.  ${}^{(4)}n^r < 0$  for  $r \leq 2M$ ). If the inequality is reversed, the hypersurface enters the white hole region, where the light cone is tilted toward increasing  $r$ , and excision is not possible without allowing causal propagation from the excision boundary to the interior of the computational domain.

As with any spherically symmetric metric, a radial coordinate transformation  $r \rightarrow R = R(r)$  can be used to make the spatial sector of Eq. (13) conformally flat, i.e.  $\Omega^{-2}[dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2)]$ . Here,

$$\Omega = \frac{R}{r}, \quad (18)$$

and  $R$  is determined by the ordinary differential equation

$$\frac{dR}{dr} = \frac{R}{rf}. \quad (19)$$

Because  $f \sim r$  for large  $r$ ,  $R$  remains *finite* as  $r \rightarrow \infty$ . Denoting its limiting value by  $R_+$ , we find

$$\frac{R(r)}{R_+} = \exp\left(-\int_r^\infty \frac{dr'}{r'f(r')}\right). \quad (20)$$

As  $r \rightarrow \infty$  (or equivalently  $R \rightarrow R_+$ ),  $\Omega \rightarrow 0$ , in agreement with the boundary condition Eq. (11).

Finally, transforming  $(R, \theta, \phi)$  to Cartesian coordinates  $x^i$ , we can express the space-time metric Eq. (13) as

$$ds^2 = \Omega^{-2}[-\tilde{\alpha}^2 dt^2 + \delta_{ij}(dx^i + \tilde{\beta}^i dt)(dx^j + \tilde{\beta}^j dt)]. \quad (21)$$

Here the conformal lapse  $\tilde{\alpha}$  and conformal shift  $\tilde{\beta}^i$  are given by

$$\tilde{\alpha} = \Omega f \quad \text{and} \quad \tilde{\beta}^i = -\Omega a n^i, \quad (22)$$

where  $n^i = x^i/R$ . Because  $\Omega \sim R_+/r$  whereas  $f \sim a \sim r$  as  $r \rightarrow \infty$ ,  $\tilde{\alpha}$ , and  $\tilde{\beta}^i$  are *finite* at  $I^+$ . Therefore, the conformal space-time metric inside the square brackets in Eq. (21) is regular, in addition to its spatial part being flat.

The trace-free extrinsic curvature of the  $\Sigma_t$  hypersurface in the coordinates  $(t, x^i)$  takes the form  $A^{ij} = \Omega^5 \tilde{A}^{ij}$ , with

$$\tilde{A}^{ij} = \frac{C}{R^3}(3n^i n^j - \delta^{ij}). \quad (23)$$

The sign difference in Eq. (23) relative to earlier work [e.g. Eq. (52d) of [36]] arises because of the different sign-convention for  $A_{ij}$  [see the discussion after Eq. (2)]. The conformal trace-free extrinsic curvature satisfies the momentum constraint Eq. (9). Indices on conformal spatial tensors such as  $\tilde{A}^{ij}$  are raised or lowered with the conformal spatial metric; in our coordinates, this is simply the Kronecker delta, so the components of such tensors are identical irrespective of the index location.

We finish this section by noting that Eq. (20) determines only  $R(r)/R_+$ ; the freedom remains to rescale  $R$  by a real constant  $\eta$ ,

$$R \rightarrow \eta R. \quad (24a)$$

This rescaling induces further rescalings,

$$\Omega \rightarrow \eta \Omega, \quad (24b)$$

$$\tilde{A}^{ij} \rightarrow \eta^{-3} \tilde{A}^{ij}, \quad (24c)$$

$$\tilde{A}_{ij} \rightarrow \eta^{-3} \tilde{A}_{ij}. \quad (24d)$$

Equations (24) represent a coordinate transformation and do not affect the physical initial data. For the *zero* mean curvature case,  $K = 0$ , the hypersurface asymptotes to spatial infinity, where it is natural to impose the boundary condition  $\Omega \rightarrow 1$ , so  $R/r \rightarrow 1$  as  $R \rightarrow \infty$ . With  $K > 0$ , the coordinate scale is set by the arbitrary choice of the value of  $R$  at future null infinity,  $R_+$ . Only after the Hamiltonian constraint is solved can  $R$  and  $\Omega$  be rescaled to make the maximum value of  $\Omega$  equal one. If  $K$  is very close to zero, this rescaling makes  $\Omega$  very close to the  $K = 0$  solution almost everywhere.

### C. Minimal surfaces and trumpets

Later in this paper, we will construct black hole initial data with minimal surface boundary conditions in the interior of the black hole(s). As preparation, let us discuss the presence and location of minimal surfaces in the hypersurfaces  $\Sigma_t$  of the metric Eq. (13). Our treatment here extends earlier similar discussions [41–43].

The sphere  $r = \text{const}$  is a minimal 2-sphere within  $\Sigma_t$  if the function  $f$  defined in Eq. (14) vanishes. This can be seen most easily by noting that  $r$  is the areal radius, and that from Eq. (19),  $dr/dR = fr/R$ , which vanishes for  $f = 0$ . For any values  $M > 0$ ,  $K$  and  $C$ ,  $f(r) > 0$  for  $r > 2M$ . Furthermore, the inequalities Eqs. (16) and (17) imply that  $f$  is strictly positive,  $f > 0$ , at the black hole horizon  $r = r_H \equiv 2M$ . Therefore, a minimal surface in  $\Sigma_t$  will always lie inside the horizon, if it exists.

Vacuum general relativity possesses a rescaling freedom, and this freedom will be inherited by the function  $f$  and any equations that describe minimal surfaces. The most common way to incorporate this freedom is to rescale all dimensionful quantities by the mass  $M$ . In particular,

Eq. (14) can be written in terms of the dimensionless variables  $r/M$ ,  $KM$ , and  $C/M^2$  as

$$f^2 = 1 - \frac{2}{r/M} + \left[ \frac{(KM)(r/M)}{3} - \frac{C/M^2}{(r/M)^2} \right]^2. \quad (25)$$

Using  $M$  to make dimensionless variables is fine for Schwarzschild, but in the absence of spherical symmetry, the mass  $M$  is not known in advance. On CMC hypersurfaces,  $K$  is a free parameter and can be used to form an alternative and more widely applicable set of dimensionless variables,  $Kr$ ,  $K^2C$ , and  $KM$ , such that

$$f^2 = 1 - \frac{2KM}{Kr} + \left[ \frac{Kr}{3} - \frac{K^2C}{(Kr)^2} \right]^2. \quad (26)$$

A minimal surface at radius  $r_{\text{ms}}$  satisfies

$$1 - \frac{2KM}{Kr_{\text{ms}}} + \left[ \frac{Kr_{\text{ms}}}{3} - \frac{K^2C}{(Kr_{\text{ms}})^2} \right]^2 = 0. \quad (27)$$

This equation will play a central role in the remainder of this section.

Figure 1 plots  $f^2$  for different values of  $C/M^2$  for a fixed value of  $KM$ . For large  $C/M^2$  (or equivalently  $K^2C$ ), no root and no minimal surface exists. At some critical value  $C_T/M_T^2$ ,  $f$  touches zero, indicated by the filled circle. For  $C/M^2 < C_T/M_T^2$ , minimal surfaces exist at radii  $r_{\text{ms}}/M$ , as indicated by the open circles in Fig. 1. The critical value  $C_T/M_T^2$  delineates the region of parameters for which  $\Sigma_t$  contains a minimal surface. At this critical point,  $f$  varies *linearly* in  $r - r_T$ , passing through zero at  $r_T$ . From Eq. (13), the radial proper separation within the slice is  $ds = dr/f$ . Because  $f$  vanishes linearly at  $r_T$ , this point is an infinite proper distance away from any point  $r > r_T$ : this configuration is often called a *trumpet* [26,29–31]. In con-

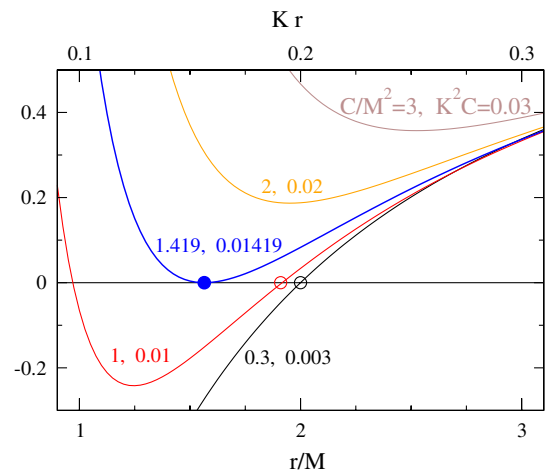


FIG. 1 (color online). Radial function  $f(r)$  for different values of  $K$  and  $C$ . Plotted is the square  $f^2$  for  $KM = 1/10$  and several different choices for  $C$ . Each curve is labeled by its value of  $C/M^2$  and its value for  $K^2C$ . The bottom axis shows radius in units of  $r/M$ , and the top axis in units of  $Kr$ .

trast, away from the critical point (i.e. at the open circles in Fig. 1),  $f$  approaches zero proportionally to  $\sqrt{r - r_{\text{ms}}}$ , and the proper distance to the minimal surface is finite.

For trumpets,  $f^2 = 0$  and  $\partial_r f^2 = 0$ ; the parameter values that lead to trumpets can be written down in terms of the areal radius of the trumpet,  $r_T$ :

$$K_T M_T = \frac{2r_T/M_T - 3}{(r_T/M_T)\sqrt{(r_T/M_T)(2 - r_T/M_T)}}, \quad (28a)$$

$$\frac{C_T}{M_T^2} = \frac{(r_T/M_T)^2(3 - r_T/M_T)}{3\sqrt{(r_T/M_T)(2 - r_T/M_T)}}, \quad (28b)$$

where  $3/2 < r_T/M_T < 2$ .

The preceding discussions determine the region of parameters  $M$ ,  $K$ ,  $C$ , for which minimal surfaces exist. Taking the scaling invariance into account, this region can be represented on a two-dimensional plot as given in Fig. 2. The blue solid and red dashed lines in the top panel, for instance, are given by a parametric plot of Eqs. (28) and (17). The unshaded wedge-shaped region between these two lines represents the allowed parameter choices which lead to a CMC hypersurface containing a minimal surface and intersecting the black hole.

Alternatively, one can compute trumpet-configurations using the dimensionless variables indicated in Eq. (26).

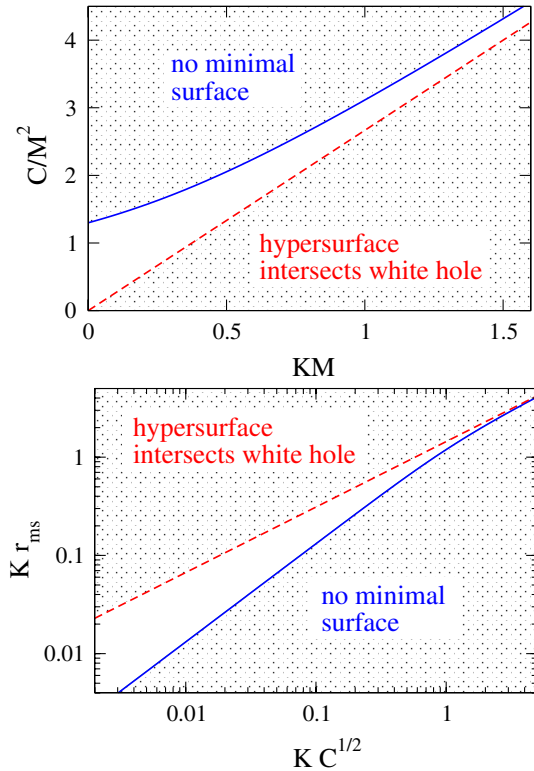


FIG. 2 (color online). Parameter choices that result in hypersurfaces  $\Sigma_t$  containing a minimal surface. The two panels correspond to two different choices of dimensionless variables. Parameter values on the blue solid lines represent trumpet configurations.

From the equations  $f^2 = 0$  and  $\partial_r f^2 = 0$ , one eliminates  $KM$  to obtain a third order polynomial that relates  $K_T r_T$  and  $K_T C_T^{1/2}$ . This polynomial has only one positive real root, which for  $K_T^2 C_T < 2/3$  can be written as

$$(K_T r_T)^2 = \frac{\frac{3}{2} K_T^2 C_T}{\cos[\frac{1}{3} \arccos(\frac{3}{2} K_T^2 C_T)]}. \quad (29)$$

For  $K_T^2 C_T > 2/3$ , the trigonometric functions in Eq. (29) should be replaced by their hyperbolic counterparts. To find the trumpet solution, substitute  $K_T r_T$  back into Eq. (26):

$$K_T M_T = \frac{K_T r_T}{2} \left( 1 + \left[ \frac{K_T r_T}{3} - \frac{K_T^2 C_T}{(K_T r_T)^2} \right]^2 \right). \quad (30)$$

Substituting  $K_T r_T$  from Eq. (29),  $K_T M_T$  is a function of  $K_T^2 C_T$  alone. Finally, dividing Eq. (29) by Eq. (30) yields the value of  $r_T/M_T$ . All these parameters for trumpet hypersurfaces are plotted in Fig. 3. To make easy contact with dimensionless quantities normalized by  $M$ , the top horizontal axis of this plot is labeled by  $KM$ . For  $K C^{1/2} \ll 1$ , the data plotted in the lower panel of Fig. 3 is proportional to  $K_T C_T^{1/2}$ :

$$K_T r_T = 3^{1/4} K_T C_T^{1/2} + \mathcal{O}((K_T^2 C_T)^{3/2}), \quad (31)$$

$$K_T M_T = \frac{2}{3^{3/4}} K_T C_T^{1/2} + \mathcal{O}(K_T^2 C_T). \quad (32)$$

For given choices  $K$  and  $K^2 C$ , minimal surfaces only exist at radii  $r_{\text{ms}}$  larger than  $r_T$  given by Eq. (29). This is indicated by the solid blue line in the lower panel of Fig. 2.

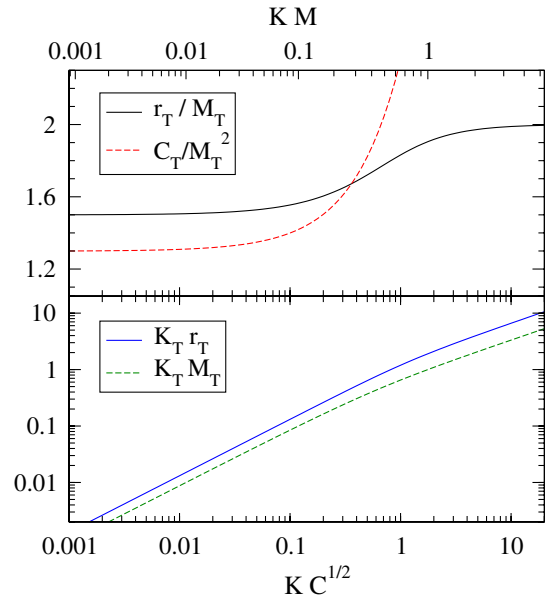


FIG. 3 (color online). Properties of the trumpet hypersurfaces, parametrized by the dimensionless parameter  $K C^{1/2}$ .

To obtain an upper limit on  $Kr_{\text{ms}}$ , we recall that all minimal surfaces must lie inside the horizon,  $r_{\text{ms}} < r_H = 2M$ . Combining this with Eq. (17) results in

$$Kr_{\text{ms}} < 2MK < (3K^2C)^{1/3}, \quad (33)$$

which is indicated by the red dashed line in the lower panel of Fig. 2.

Finally, Fig. 4 presents another view of the two-dimensional set of “good” parameter choices that we first indicated in Fig. 2. As in the lower panel of Fig. 2, we shall use  $KC^{1/2}$  to parametrize the horizontal axis. However, we use Eq. (20) in the form

$$\ln \frac{R_{\text{ms}}}{R_+} = - \int_0^{1/(Kr_{\text{ms}})} \frac{du}{\sqrt{u^2 - 2KM u^3 + (\frac{1}{3} - K^2 C u^3)^2}} \quad (34)$$

to convert the vertical axis to the ratio  $R_{\text{ms}}/R_+$  of conformal radius of the minimal surface and the conformal radius of  $I^+$ . Figure 4 shows the following data: First, the *thick black line* corresponds to parameters for which the minimal surface coincides with the black hole horizon. Setting  $Kr_{\text{ms}} = 2KM$  in Eq. (27) and solving for  $Kr_{\text{ms}}$  shows that this line is parametrized by

$$Kr_{\text{ms}} = (3K^2C)^{1/3}, \quad (35)$$

with  $Kr_{\text{ms}}$  mapped to  $R_{\text{ms}}/R_+$  by Eq. (34). The *red dashed lines* are contours of constant values  $KM$ . Each of these lines is obtained by keeping  $KM$  fixed and varying  $Kr_{\text{ms}}$  between its lower bound, the trumpet value  $K_{\text{T}}r_{\text{T}}$  [obtained from inverting Eq. (28a)] and its maximal value  $2KM$ . For each choice of  $KM$  and  $Kr$ , Eq. (27) is solved for  $K^2C$ , and the resulting data plotted as a parametric plot. Finally, the *thin blue lines* in Fig. 4 represent lines of constant ratio  $Kr_{\text{ms}}/(2KM) = r_{\text{ms}}/r_H \equiv \gamma$ , i.e. lines where the areal radius of the minimal surface is a fixed fraction of the areal radius of the horizon. Replacing  $Kr_{\text{ms}}$  by  $2\gamma KM$  in Eq. (27), we can solve for  $KC^{1/2}$  as a function of  $KM$ . The thin blue lines are then obtained as a parametric plot  $(KC^{1/2}, Kr)$  as  $KM$  is varied.

Trumpet initial conditions are obtained from Fig. 4 through the limit  $R_{\text{ms}}/R_+ \rightarrow 0$ , i.e. by going “down.” Note that the red  $KM = \text{const}$  contours become vertical in this limit. Their value as a function of  $KC^{1/2}$  is then given in the lower panel of Fig. 3.

The significance of the axes employed in Fig. 4 is that both axes represent quantities that are freely specifiable when constructing CMC hypersurfaces within the initial

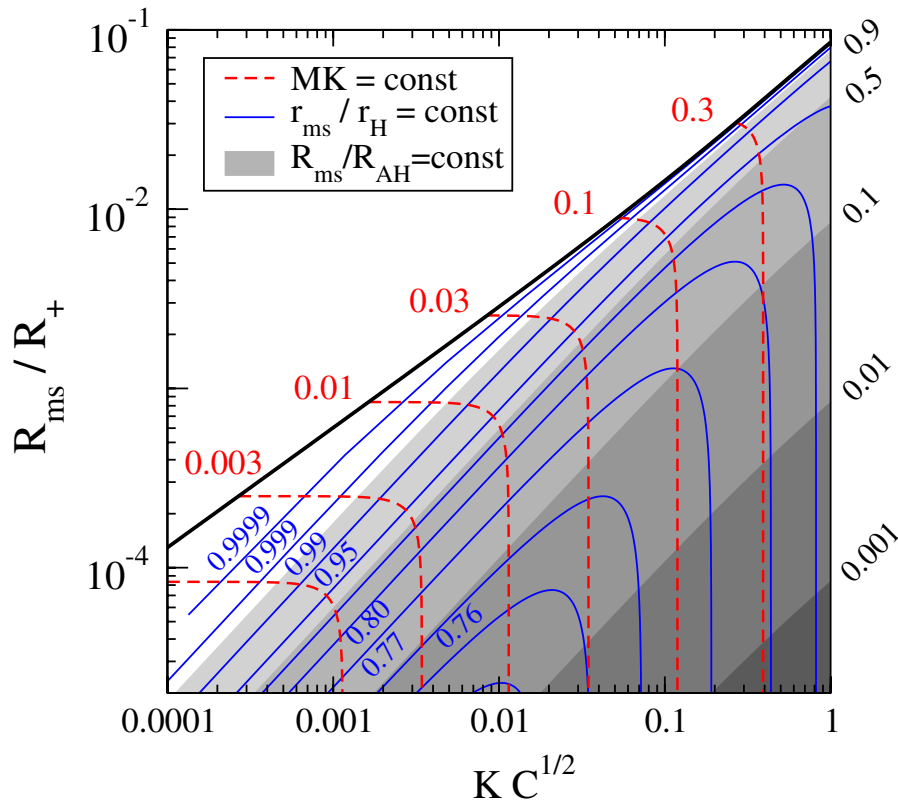


FIG. 4 (color online). Properties of CMC hypersurfaces viewed in the  $R_{\text{ms}}/R_+$  vs  $KC^{1/2}$  plane. CMC hypersurfaces with a minimal surface must lie below the thick black line, and as this line is approached, the minimal surface approaches the black hole horizon. The dashed red lines are lines of constant  $MK$ , with values given by the red numbers next to the lines. The thin blue lines represent constant values of  $\gamma = r_{\text{ms}}/r_H$ . The shaded areas are contours of constant value of  $R_{\text{ms}}/R_{\text{AH}}$ ; the shade of grey changes, from top to bottom, at values 0.9, 0.5, 0.1, 0.01, and 0.001. Trumpet hypersurfaces represent the limit  $R_{\text{ms}}/R_+ \rightarrow 0$ .

value formalism of general relativity. One use of Fig. 4 is to first pick values of  $K$  and  $M$ , fixing a particular contour of  $KM$ . One then chooses how trumpetlike the initial conditions should be; that is, how far to go down along the contour. Alternatively, one can choose a certain ratio  $\gamma = r_{\text{ms}}/r_H$ . In either case, one can then read off the corresponding values of  $R_{\text{ms}}/R_+$  and  $K\sqrt{C}$  to get the remaining initial value parameters.

#### D. CMC initial data for single black holes

The properties of CMC slices of the Schwarzschild spacetime, as described in Secs. II B and II C, mesh nicely with the conformal method of solving the Einstein constraint equations, which was outlined in Sec. II A. We shall first discuss the spherically symmetric case, for which there is a one-to-one correspondence. Subsequently, we will generalize to a single spinning or boosted black hole.

##### 1. Spherical symmetry

The CMC metric is conformally flat, so we shall use a flat conformal metric for the initial value problem,

$$\tilde{g}_{ij} = f_{ij}, \quad (36)$$

where  $f_{ij}$  is the flat space metric. The extrinsic curvature of the CMC metric has the correct scaling with conformal factor [compare Eqs. (7) and (23)], so we shall adopt Eq. (23) as the freely specifiable trace-free extrinsic curvature, with the constant  $C$  yet to be determined. The radial coordinate  $R$  ranges from a finite value  $R_+$  representing  $I^+$  to some smaller value, for instance  $R_{\text{ms}}$  at a minimal surface (assuming a minimal surface exists), so we shall adopt a computational domain with inner radius  $R_1$  and outer radius  $R_2$ . At  $I^+$ , the conformal factor vanishes, resulting in the boundary condition

$$\Omega = 0, \quad R = R_2, \quad (37)$$

which identifies  $R_2$  with  $R_+$ . At the inner boundary, we shall impose a minimal surface boundary condition,

$$\frac{d\Omega}{dR} = \frac{\Omega}{R}, \quad R = R_1, \quad (38)$$

so that  $R_1$  will coincide with  $R_{\text{ms}}$ .

With the choices Eqs. (36)–(38), we are now left with choosing the four numbers  $\{K, C, R_1, R_2\}$ . Solution of the Hamiltonian constraint Eq. (10) will then result in a complete initial data set with a certain mass  $M$ . Figure 4 is useful for informed choices for the numbers  $\{K, C, R_1, R_2\}$ . For instance, we can first decide on a mass  $M$  (say,  $M = 1$ ) and a mean curvature  $K$  (say,  $K = 0.01$ ). This selects one of the red dashed curves in Fig. 4. We can now choose a suitable value of  $r_{\text{ms}}/r_H$  by considering the intersection of the red dashed lines with the blue contours (say,  $r_{\text{ms}}/r_H = 0.9$ ), and read off the values for  $KC^{1/2}$  and  $R_{\text{ms}}/R_+$  (in our example  $KC^{1/2} = 0.0105$ ,  $R_{\text{ms}}/R_+ = 0.000554$ ), which

determine the values for  $C$  and  $R_1/R_2$ . An overall scaling of  $R_1$  and  $R_2$  remains, because the coordinate transformation  $r \rightarrow R$  for CMC slices is determined only up to an overall rescaling [see the discussion after Eq. (20)]. Thus, we are free to set, for instance,  $R_1 = 1$ .

##### 2. Single black hole with spin and boost

A relatively simple class of nonspherically symmetric initial data on maximal slices, with  $K = 0$ , was proposed by Bowen and York [22,44]. It assumes conformal flatness of the spatial metric and a solution  $\tilde{A}_{ij}$  of the conformal momentum constraint Eq. (9) characterized by a “spin” vector  $S^i$  and two “boost” vectors  $P^i, Q^i$ . On the asymptotically flat maximal slices, with the boundary condition  $\Omega \rightarrow 1$  at spatial infinity,  $S^i$  is in fact the physical angular momentum and  $P^i$  is the physical linear momentum of the system, as defined by Arnowitt-Deser-Misner (ADM) surface integrals at spatial infinity [32]. The second boost vector  $Q^i$  is introduced to allow for inversion symmetry about a minimal surface, and can be thought of as the three-momentum of the black hole as viewed from the asymptotically flat space on the other side of the Einstein-Rosen bridge associated with the minimal surface.

Since Eq. (9) is linear and identical on maximal and CMC slices, we can add the Bowen-York terms to Eq. (23) with the result<sup>2</sup>:

$$\begin{aligned} \tilde{A}_{ij} = & \frac{C}{R^3} [3n_i n_j - \delta_{ij}] - \frac{3}{R^3} [\varepsilon_{ik\ell} S^k n^\ell n_j + \varepsilon_{j\ell k} S^k n^\ell n_i] \\ & - \frac{3}{2R^2} [P_i n_j + P_j n_i + P^k n_k (n_i n_j - \delta_{ij})] \\ & + \frac{3}{2R^4} [Q_i n_j + Q_j n_i + Q^k n_k (\delta_{ij} - 5n_i n_j)]. \end{aligned} \quad (39)$$

Here  $n^i$  is a unit three-vector in the outward radial direction. The coefficient  $C$  of the spherically symmetric first term has been normally taken to be zero in papers on the initial value problem on maximal hypersurfaces. In this paper, we refer to  $\tilde{A}_{ij}$  given above in Eq. (39) as the *generalized Bowen-York solution*.

Note that  $\tilde{A}^{ij}$  is not invariant under the rescaling of  $R$  [see Eqs. (24)]. A scale-invariant effective source term in Eq. (10) is

$$R^6 \tilde{A}_{ij} \tilde{A}^{ij} \equiv W(R, \theta, \varphi). \quad (40)$$

The properties of  $W$  are important for determining questions such as the inversion symmetry of the hypersurface about the minimal surface (see the Appendix). Also, the invariance of  $W$  under the rescaling freedom  $R \rightarrow \eta R$  implies that the parameters must scale as

<sup>2</sup>The sign differences between Eq. (39) and earlier papers arise because of the different sign convention for  $A_{ij}$  [see Eq. (2)].

$$C \rightarrow C, \quad S^i \rightarrow S^i, \quad P^i \rightarrow \eta^{-1} P^i, \quad Q^i \rightarrow \eta Q^i. \quad (41)$$

The physical interpretation of the Bowen-York parameters is not necessarily the same on CMC hypersurfaces as on maximal hypersurfaces. CMC hypersurfaces are not asymptotically flat. Identification of the physical energy, linear momentum, and angular momentum of the system on asymptotically null hypersurfaces is a nontrivial matter, in general [45]. In particular, the scaling dependence of the boost parameters means that these cannot be interpreted as physical momenta.

One can argue that in the limit of small  $K$  and  $C$  ( $KM \ll 1$  and  $C \sim (8M^2/3)(KM) \ll 1$ ), the geometry in the vicinity of the black hole should be similar to that of solutions of the zero mean curvature initial value problem found, for example, by Cook [46,47]. For hyperboloidal slices, the conformal factor  $\Omega$  is generally approximately constant at intermediate distances,  $\Omega M \ll R \ll \Omega/K$ . (As can be seen in Fig. 7,  $\Omega$  is smaller near and inside the black hole, where  $\Omega \sim R/M$ , and decreases toward zero approaching  $I^+$ ,  $\Omega < KR$ .) These intermediate distances are sufficiently close to the black hole that the CMC slice still resembles a maximal slice, but far enough away to be considered in the asymptotic regime. If  $R$  is rescaled such that  $\Omega \approx 1$  in this intermediate regime, then the ADM formulas for energy, linear momentum, and angular momentum should be at least roughly valid. Therefore, one might identify the scaling invariant  $\Omega_{\max} P^i$  as a quasilocal linear momentum and  $S^i$  as a quasilocal angular momentum. However, these ‘‘quasilocal’’ values may not match the correct physical values at future null infinity if gravitational radiation is present outside the plateau region. As a surrogate for the mass of the black hole, we use the ‘‘irreducible mass’’  $M_{\text{irr}}$ , defined in the usual way from the area of the apparent horizon  $A_{\text{AH}}$ ,

$$M_{\text{irr}} \equiv \left( \frac{A_{\text{AH}}}{16\pi} \right)^{1/2}. \quad (42)$$

If the initial data is axisymmetric, then the physical angular momentum can be calculated precisely using standard techniques. The generalized Bowen-York solution for a single black hole is axisymmetric provided that the boost is zero or aligned with the spin vector. Then the coordinates can be chosen so that only the  $z$  components of the spin and boost vectors are nonzero. The solution for the conformal factor will be axisymmetric for our minimal surface inner boundary condition, since the minimal surface is assumed to be a coordinate sphere. Choose spherical polar coordinates  $(R, \theta, \varphi)$  in the conformal flat space, so that the axial Killing vector  $\rightarrow \xi = \partial/\partial\varphi$ . Then the Komar angular momentum within a coordinate sphere is

$$\begin{aligned} J &= \frac{1}{8\pi} \int_0^\pi \int_0^{2\pi} \alpha \sqrt{g} \xi^{R;t} d\theta d\varphi \\ &= -\frac{1}{8\pi} \int_0^\pi \int_0^{2\pi} \tilde{A}_\varphi^R R^2 \sin(\theta) d\theta d\varphi = S_z. \end{aligned} \quad (43)$$

In much of the earlier work on maximal hypersurfaces, considerable emphasis has been placed on obtaining inversion-symmetric solutions of the initial value problem (see [22,48,49]). Given a minimal surface at  $R = R_{\text{ms}}$ , inversion symmetry requires that  $\Omega(R, \theta, \varphi) = (R_{\text{ms}}/R)^2 \Omega(R_{\text{ms}}^2/R, \theta, \varphi)$ . Solutions of Eq. (10) for  $\Omega$  with minimal surface boundary conditions can be continued with inversion symmetry to  $R < R_{\text{ms}}$  if and only if  $W$  as defined in Eq. (40) is inversion symmetric. For  $C = 0$ , the usual story is that the boost terms are inversion symmetric if  $Q^i = \pm R_{\text{ms}}^2 P^i$ , with only the minus sign applicable if spin is also present. We show in the Appendix that with  $C \neq 0$ , inversion symmetry requires the *plus* sign when only boost is present, and that no inversion symmetry is possible with both boost and spin unless the boost and spin vectors are colinear. Inversion symmetry is desirable primarily for simplifying excision boundary conditions during evolution. On CMC hypersurfaces, the generic absence of inversion symmetry requires rethinking how excision will be handled, and it may be desirable to just set  $Q^i = 0$ , as also done in [23].

The location of the apparent horizon relative to the minimal surface is an important issue. The inner boundary of the computational domain should be inside the apparent horizon. In spherical symmetry, the apparent horizon will always be outside or on the minimal surface, provided that a solution of the Hamiltonian constraint exists. However, in the presence of a boost, the apparent horizon can straddle the minimal surface [47]. Care should be taken in the choice of  $R_{\text{ms}}$  as the boost is increased, in order to avoid this problem.

## E. Multi-black hole solutions

The goal of this section is to construct a CMC hypersurface with  $N$  black holes, with masses (approximately)  $M_\alpha$ , Bowen-York parameters  $\vec{P}_\alpha$  and  $\vec{S}_\alpha$  at coordinate locations  $\vec{c}_\alpha$ . Here  $\alpha$  labels the black holes.

The setup for single black holes in Sec. II D can be generalized by choosing one excision boundary for each black hole, with radius  $R_\alpha$  centered at  $\vec{c}_\alpha$ . Because the momentum constraint is linear, the extrinsic curvature can be taken as the superposition of  $N$  copies of Eq. (39), each one centered at the appropriate  $\vec{c}_\alpha$ ,

$$\tilde{A}_{ij} = \sum_\alpha (\tilde{A}_{ij}^{\alpha C} + \tilde{A}_{ij}^{\alpha P} + \tilde{A}_{ij}^{\alpha Q} + \tilde{A}_{ij}^{\alpha S}). \quad (44)$$

If the black holes are sufficiently widely separated, and if the outer boundary is sufficiently far away, we expect that close to each of these black holes, the solution is a perturbation of the single black hole case.



In the asymptotically flat case, the conformal factor is close to unity, except very close to each black hole. Therefore, the coordinate distance  $|\vec{c}_\alpha - \vec{c}_\beta|$  between the black holes  $\alpha$  and  $\beta$  is a convenient and reasonably accurate approximation of the proper separation between the black hole horizons. Because of the rescaling freedom discussed in Eqs. (24),  $\Omega$  may not be close to unity on the hyperboloidal slices considered here, and therefore the coordinate distance may deviate significantly from the proper separation.

There is only *one* global value  $K$  and *one* value  $R_+$  for the whole multiblack hole configuration, whereas each black hole has its “own” constants  $C_\alpha$ ,  $R_\alpha$ , as well as  $P_\alpha^i$  and  $S_\alpha^i$ . Therefore, the interesting question arises of how to use  $C_\alpha$  and  $R_\alpha$  to control properties of the individual black holes, for instance their masses, given *fixed* values for  $K$  and  $R_+$ . Assuming that the presence of  $P_\alpha^i$  and  $S_\alpha^i$  will only mildly perturb the case of the spherically symmetric black hole, we can use Fig. 4 to address this question. A given  $K$  and a (desired) value for  $M = M_\alpha$  places the solution on a particular  $KM = \text{const}$  contour. Given a desired value for  $\gamma \equiv r_{\text{ms}}/r_H$ , a unique point in this figure is determined, and one can read off  $R_{\text{ms}}/R_+$  and  $KC^{1/2}$  and then compute  $R_\alpha = R_{\text{ms}}$  and  $C_\alpha = C$ .

This procedure can be simplified in the particularly interesting limit  $KM \ll 1$ . Consider a minimal surface with  $3/4 < r_{\text{ms}}/r_H \equiv \gamma < 1$ , and with a given value of  $KM$ . Substituting  $KM$  and  $Kr_{\text{ms}} = 2\gamma KM$  into Eq. (27), one can solve for  $K^2C$  and compute  $C$ . The result is

$$C \approx g_C(\gamma)M^2, \quad KM \ll 1 \quad (45)$$

with  $g_C(\gamma) = 4\gamma\sqrt{\gamma - \gamma^2}$ . We also find numerically that

$$\frac{R_{\text{ms}}}{R_+} \approx g_R(\gamma)KM, \quad KM \ll 1. \quad (46)$$

Therefore, if one knows the coefficients  $g_C$  and  $g_R$  for the desired ratio  $\gamma = r_{\text{ms}}/r_H$ , one can immediately compute the values for  $C$  and  $R_{\text{ms}}/R_+$  from Eqs. (45) and (46). These coefficients  $g_C(\gamma)$  and  $g_R(\gamma)$  are plotted in Fig. 5.

### F. Trumpet inner boundary

In the absence of spherical symmetry, a trumpet solution of the Hamiltonian constraint equation can be understood as the limit  $R_{\text{ms}} \rightarrow 0$ . That is, the Hamiltonian constraint Eq. (10) is solved with the boundary condition  $\partial\Omega/\partial R = \Omega/R$  at  $R = 0$ . The consequence of this boundary condition is that  $\Omega = 0$  and  $\Omega/R$  is finite at  $R = 0$ , which in turn means that the proper distance from finite  $R$  to  $R = 0$ ,  $\int_0^R dR/\Omega$ , is infinite. In this section, we show how the singularities in the equation determine the nonsingular solution at the trumpet inner boundary and derive the behavior of the solution close to the trumpet boundary.

Note that a necessary condition for a nonsingular solution is that the second “boost” vector  $Q^i = 0$ . Otherwise, the  $W$  source term in the Hamiltonian constraint blows up

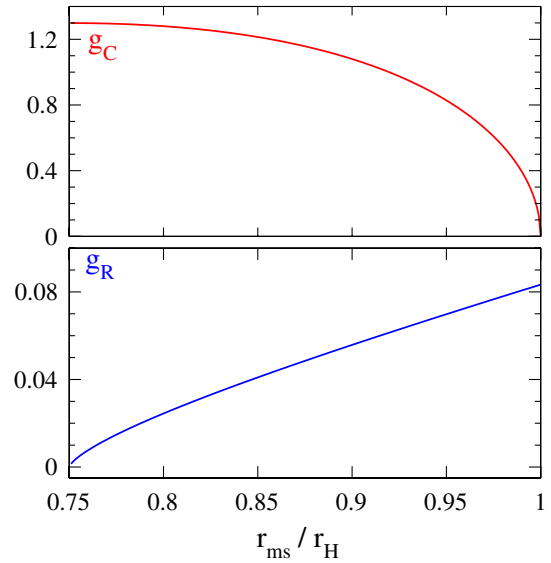


FIG. 5 (color online). The functions  $g_C$  and  $g_R$ , which are the asymptotic values of  $C/M^2$  and  $R_{\text{ms}}/(R_+KM)$  in the limit  $KM \rightarrow 0$ . These functions are useful for choosing excision radius and  $C$  when constructing CMC initial data for black holes with given  $K$ ,  $M$ , and outer boundary radius  $R_+$ .

at  $R = 0$ . This is a very reasonable condition which follows automatically from inversion symmetry in the limit  $R_{\text{ms}} \rightarrow 0$  (see the Appendix), and reflects the fact that the other side of the Einstein-Rosen bridge is infinitely far away from any point with  $R > 0$ .

We begin by rewriting Eq. (10) with  $U \equiv \Omega/R$  as the dependent variable. The new form of the equation is

$$\begin{aligned} R^2 \frac{\partial^2 U}{\partial R^2} + 4R \frac{\partial U}{\partial R} + 2U + \hat{\Delta}U \\ = \frac{3}{2U} \left[ \left( U + R \frac{\partial U}{\partial R} \right)^2 + \hat{\nabla}U \cdot \hat{\nabla}U - \left( \frac{K}{3} \right)^2 + \frac{1}{6} U^6 W \right], \end{aligned} \quad (47)$$

where  $\hat{\Delta}$  is the Laplacian operator and  $\hat{\nabla}$  the gradient operator on the unit two-sphere, and  $W$  is defined by Eq. (40). We have assumed a conformally flat spatial metric.

Now let  $U = U_0(\theta, \varphi) + R^\alpha U_1(\theta, \varphi) + \dots$  and  $W = W_0(\theta, \varphi) + RW_1(\theta, \varphi) + \dots$ . From the expression for  $W$  in Eq. (A2), we see that

$$W_0 = 6(C^2 + 3\sin^2(\psi)S^i S_i) \quad (48)$$

for any boost  $P^i$ , where  $\psi$  is the angle with the spin direction, equal to  $\theta$  if the spin is along the polar axis. Unless the boost is nonzero,  $W = W_0$  at all  $R$  and  $W_1 = 0$ .

In zeroth order, we get

$$\hat{\Delta}U_0 = -\frac{1}{2}U_0 + \frac{3}{2U_0} \left[ \hat{\nabla}U_0 \cdot \hat{\nabla}U_0 - \left( \frac{K}{3} \right)^2 + W_0 U_0^6 \right], \quad (49)$$

which has a unique solution regular everywhere on the unit sphere for any  $K > 0$ , any value of  $C > 0$ , and any spin vector  $S^i$ . Uniqueness can be demonstrated using a method of Moncrief [50] applied to the quasilinear form of the equation obtained by the change of variable  $U_0 \rightarrow 1/V^2$ .

In the absence of spin,  $U_0$  is independent of angle and  $U_0^2$  is the solution of the cubic equation

$$C^2(U_0^2)^3 - \frac{1}{3}U_0^2 - \left(\frac{K}{3}\right)^2 = 0. \quad (50)$$

The only positive real root if  $3K^2C/2 \leq 1$  is

$$U_0^2 = \frac{2}{3C} \cos\left[\frac{1}{3}\cos^{-1}\left(\frac{3}{2}K^2C\right)\right]. \quad (51)$$

The trigonometric functions are replaced by the corresponding hyperbolic functions if  $3K^2C/2 \geq 1$ .

The next-to-leading terms in Eq. (47) give an equation for  $U_1$ :

$$\begin{aligned} \hat{\Delta}U_1 - 3\frac{\hat{\nabla}U_0 \cdot \hat{\nabla}U_1}{U_0} + \left(\alpha^2 + 1 - \frac{3}{2}W_0U_0^4 + \frac{\hat{\Delta}U_0}{U_0}\right)U_1 \\ = \frac{3}{2}U_0^5W_1R^{1-\alpha}. \end{aligned} \quad (52)$$

If  $W_1 \neq 0$ , the solution of the inhomogeneous equation with  $\alpha = 1$  gives the leading contribution to  $U_1$ . There is also a unique lowest value of  $\alpha$  for which the homogeneous equation has a nontrivial solution regular everywhere on the unit sphere. This solution to the homogeneous equation, times  $R^\alpha$ , will, with a coefficient undetermined by the trumpet boundary condition, contribute to  $U - U_0$ . The coefficient is fixed by the requirement that the global solution for  $\Omega$  satisfy the  $\Omega = 0$  boundary condition at future null infinity.

If the spin is zero,  $W_0 = 6C^2$  and the homogeneous  $\alpha$  is the solution of the algebraic equation

$$\alpha^2 + 1 = 9C^2U_0^4 = 4\cos^2\left[\frac{1}{3}\cos^{-1}\left(\frac{3}{2}K^2C\right)\right]. \quad (53)$$

In the range  $0 \leq 3K^2C/2 \leq 1$  of most interest, Eq. (53) implies  $\sqrt{2} \leq \alpha \leq \sqrt{3}$ . For larger  $K^2C$ , the trigonometric functions are replaced by hyperbolic functions and  $\alpha$  continues to increase.

In practical terms, there is very little difference between a solution satisfying the exact trumpet boundary condition and a solution satisfying the minimal surface boundary condition with a very small, but nonzero,  $R_{\text{ms}}$ . Very small means that  $R_{\text{ms}}/R_{\text{AH}} \ll 1$ . For the Schwarzschild case,  $R_{\text{ms}}/R_+$  should be far below the heavy black line in Fig. 4.

Finally, all of this discussion has been in the context of single black holes. With multiple black holes, each trumpet boundary must be treated separately and matched to the global solution on a surface surrounding the black hole. The analysis right at the trumpet boundary is not affected by the presence of other black holes, since the  $R^6$  factor in  $W$  kills the finite contribution of the other black holes to the conformal traceless extrinsic curvature at  $R = 0$ .

### III. NUMERICAL RESULTS

In this section, we numerically construct a variety of hyperboloidal initial data sets using the generalized Bowen-York solution. These results are obtained with a pseudospectral elliptic solver that is part of the Spectral Einstein Code,  $\mathcal{S}_{\text{PEC}}$ . This solver is described in detail in Ref. [24]. The desired solution is expanded in terms of spherical harmonics and Chebyshev polynomials. Truncation at some finite expansion order results in an algebraic system of equations for the expansion coefficients or, equivalently, for the values of the solution at the collocation points. This system is solved with a Newton-Raphson technique, employing the preconditioned generalized minimal residual method (GMRES) [51] to solve the linearized system of equations at each iteration using the software package PETSC [52–54]. The  $\mathcal{S}_{\text{PEC}}$  elliptic solver has been used on a wide variety of formulations of the initial value problem (see, e.g. [55,56]), including puncture initial data [57,58] (which also uses the Bowen-York extrinsic curvature). Below, we present results of convergence tests of our initial data.

#### A. Spherical symmetry

As a first test, we reproduce the analytically known spherically symmetric solutions discussed in Sec. II B, using the numerical approach described in Sec. II D 1. We choose  $M = 0.85$ ,  $K = 0.1$ ,  $r_{\text{ms}}/r_H = 0.8$ , and  $R_+ = 100$ . The relations shown in Fig. 4 then imply  $C = 1.0086$  and  $R_{\text{ms}} = 0.127$ . From these numbers, only  $K$ ,  $C$ ,  $R_{\text{ms}} = R_1$ , and  $R_+ = R_2$  are used in the numerical solution; the other numbers are used only when computing the analytical solution with which to compare. Figure 6 shows convergence of the numerical solution to the analytic solution.

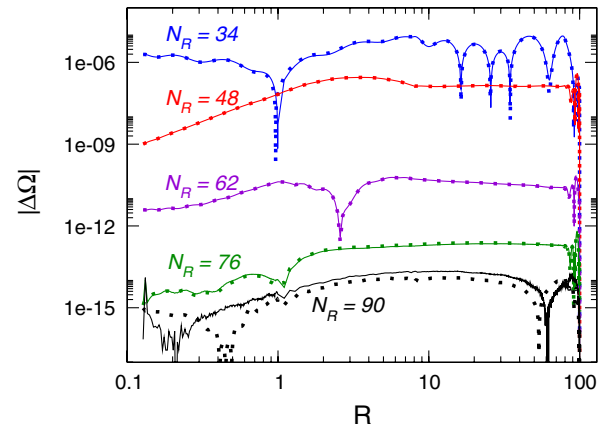


FIG. 6 (color online). Convergence of the numerical solution of the Hamiltonian constraint for a Schwarzschild black hole. Plotted are five different resolutions  $N_R$ , where  $N_R$  is the number of radial collocation points. Solid lines show the difference from the analytic solution, dotted lines the difference from the numerical solution at the next higher resolution. The highest resolution has  $N_R = 104$ .

The solid lines plot the differences between the numerically determined  $\Omega$  and the analytic solution of Eq. (34) computed with MATHEMATICA. As we increase the resolution of the elliptic solver, we find exponential convergence to the analytic solution. For the generic examples considered later in this paper (which include spin, boost, and two black holes), no analytic solutions are known. Therefore, in Fig. 6, we also present an estimate of the numerical error which does not rely on knowledge of the analytic solution. Specifically, the dotted lines show the differences between the numeric solutions at two successive resolutions. As can be seen, these track very closely the error obtained from comparing the lower resolution run to the analytic solution.

In our second example, we explore a solution which is very close to the trumpet configuration. Recall that for a trumpet, for  $R$  close to  $R = 0$ , the solution behaves as  $\Omega = U_0 R$  with  $U_0$  given in Eq. (51). We choose parameters  $C = 1$ ,  $K = 0.1$ ,  $R_{\text{ms}} = 10^{-6}$ , and  $R_+ = 100$  which, as can be seen from Fig. 4, result in an inner boundary which is very close to the trumpet limit. Because application of the minimal surface condition in this case proved numerically problematic (presumably due to dividing by the very small number  $R_{\text{ms}}$ ), we use the Dirichlet condition  $\Omega = U_0 R_{\text{ms}}$  at the inner boundary.

The numerical solution of the Hamiltonian constraint equation for this example is shown in Fig. 7. One sees that in the region  $10^{-6} \leq R \leq 1$ , the conformal factor  $\Omega$  is proportional to  $R$ , with  $U_0$  the constant of proportionality. Furthermore, within this range of conformal radius, the proper area of coordinate spheres ( $4\pi r^2$ , where  $r$  is the Schwarzschild radius) is approximately constant, which is consistent with the long cylinder of the trumpet. The

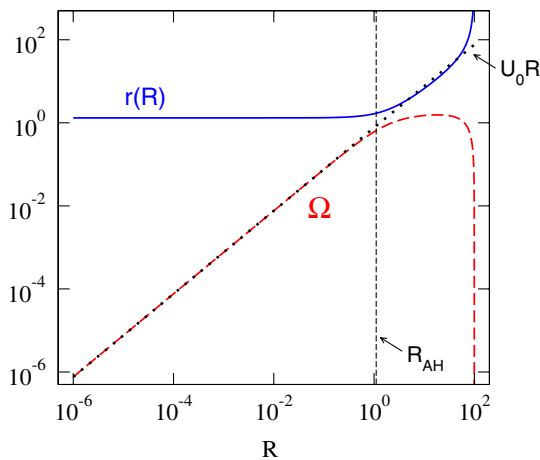


FIG. 7 (color online). Numerical solution for a Schwarzschild black hole with the inner boundary very close to the trumpet limit. The dashed red line shows the conformal factor  $\Omega$ , and the solid blue line shows the Schwarzschild radius (calculated from the proper area of coordinate spheres) as a function of the conformal radius. The dotted black line is  $U_0 R$ , which equals  $\Omega$  for a trumpet, with  $U_0$  given in Eq. (51). The vertical dashed line locates the apparent horizon.

location of the apparent horizon  $r_H = 2M = 1.69$  is shown as a vertical dashed line in this figure.

## B. Single spinning black hole

Here, we construct a single spinning black hole, with no boosts. We take the Bowen-York spin parameter  $S^i = (0, 0, S)$  and solve the Hamiltonian constraint for the conformal factor  $\Omega$ , with varying  $S$ . The solution is axisymmetric, so  $S^i$  represents the total angular momentum of the black hole (see Sec. IID 2).

In the absence of boosts,  $W$  [Eq. (40)] reduces to  $W_0$  as defined by Eq. (48), with  $\psi = \theta$ . The radial behavior of  $\Omega$  will be rather similar to the spherically symmetric solution with the same parameters as long as the spherically symmetric  $C^2$  is replaced by the solid angle average of  $W/6$ , which we denote by

$$C_{\text{eff}}^2 \equiv C^2 + 2S^2. \quad (54)$$

We use  $S/C_{\text{eff}}$  as a dimensionless measure of the importance of spin. Note that  $0 \leq S/C_{\text{eff}} < 1/\sqrt{2}$  as  $S/C$  varies from zero to infinity. We find that the irreducible mass varies less with spin keeping  $C_{\text{eff}}$  constant than when keeping  $C$  constant, particularly for large spins, as shown in Fig. 8. The constant values of  $C$  and  $C_{\text{eff}}$  are 1.0086, with  $K = 0.1$ ,  $R_{\text{ms}} = 0.127$ , and  $R_+ = 100$ .

In Fig. 9, we study how the nonzero spin distorts the intrinsic geometry of the apparent horizon and compare the results to an analogous distortion computed from the analytic Kerr solution. The solid lines of Fig. 9 show the maximum and minimum of the Ricci scalar  ${}^{(2)}\mathcal{R}M_{\text{irr}}^2$  computed from the 2-metric induced on the apparent horizon of the spinning black hole. The dashed lines show the maximum and minimum values calculated from the analytic Kerr solution, taken from Eq. (B1) of Ref. [58]. Note that

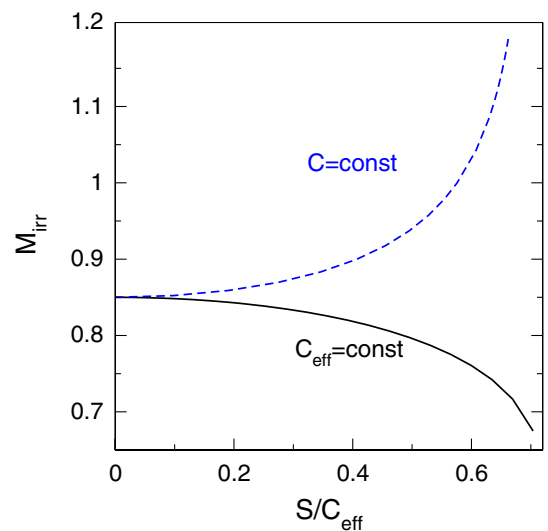


FIG. 8 (color online). Irreducible mass versus dimensionless spin for a single black hole spinning around the  $z$ -axis.

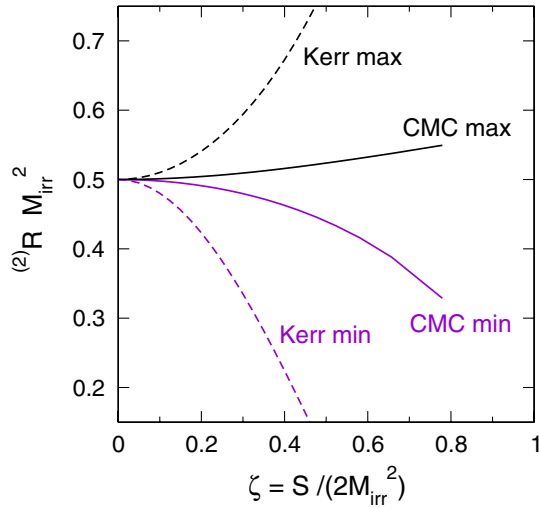


FIG. 9 (color online). Dimensionless scalar curvature of the apparent horizon versus spin for a single black hole. The solid lines are the maximum and minimum numerical values computed on CMC slices. The dashed lines are the maximum and minimum analytic values for a Kerr black hole (these lines continue to the maximal Kerr value  $\zeta = 1$ , where the maximum and minimum are 2 and  $-1/2$ , respectively).

deviations from 0.5 are deviations from a spherical geometry. We see that the apparent horizon distortion is much less for our conformally flat initial data than it is for Kerr. The CMC data plotted in Fig. 9 is the same shown in Fig. 8, with the CMC curves terminating at  $\max(S/C_{\text{eff}}) = 1/\sqrt{2}$ .

The horizontal axis in Fig. 9 is the spin-extremality parameter

$$\zeta = \frac{S}{2M_{\text{irr}}^2} \quad (55)$$

as introduced in Ref. [58]. A maximally spinning Kerr black hole has  $\zeta = 1$ , and the CMC sequence considered here allows values as large as  $\zeta \approx 0.78$ . In Sec. IV, we shall place this number into the context of results on zero mean curvature slices.

### C. Single boosted black hole

Next, we construct single, nonspinning, boosted black holes. We shall vary  $P^i$  and shall choose  $Q^i = +R_{\text{ms}}^2 P^i$ , in order to make the black hole spacetime inversion symmetric (see the Appendix for details). As we vary the boost, we keep the irreducible mass of the constructed black holes constant by a suitable choice of  $R_{\text{ms}}$ . Specifically,  $M_{\text{irr}} = 0.85$  and the remaining CMC parameters are chosen to be  $R_+ = 100$ ,  $K = 0.1$ , and  $C = 1.0086$ .

First, we compare initial data sets for an unboosted black hole and for a boosted black hole with  $P\Omega_{\text{max}}/M_{\text{irr}} = 1.77$ . Figure 10 shows the coordinate locations of both the apparent horizon and the minimal surface for these two cases. The apparent horizon remains an approximate coordinate sphere, although its coordinate radius is reduced

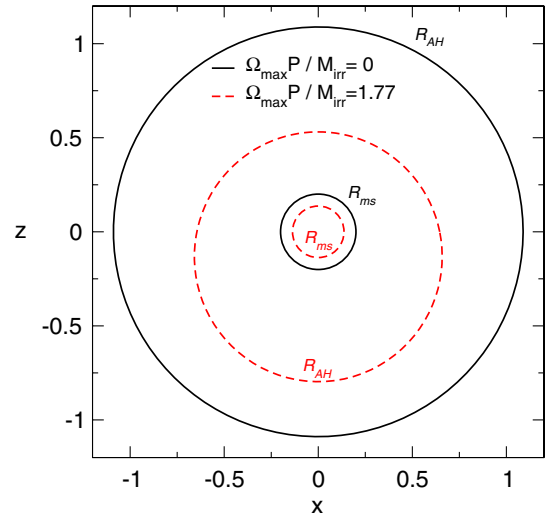


FIG. 10 (color online). Coordinate locations of the apparent horizons and minimal surfaces, cut through the  $x$ - $z$  plane, for a nonspinning, unboosted black hole (solid black circles), and for a nonspinning black hole boosted in the  $z$  direction (dashed red circles).

(recall that  $M_{\text{irr}}$  is identical for the boosted and unboosted data set, which was achieved by reducing  $R_{\text{ms}}$  for the boosted case). Furthermore, the apparent horizon is offset from the excision sphere in a direction opposite to the boost  $P^i$ , analogous to the behavior of asymptotically flat inversion symmetric Bowen-York initial data [47].

To investigate the intrinsic geometry of the apparent horizon for the boosted black hole, we compute the Ricci scalar  $(2)\mathcal{R}M_{\text{irr}}^2$  from the 2-metric induced on the apparent horizon. Figure 11 plots this quantity; it is axisymmetric (as it must be), is maximum at the poles along the  $z$ -axis

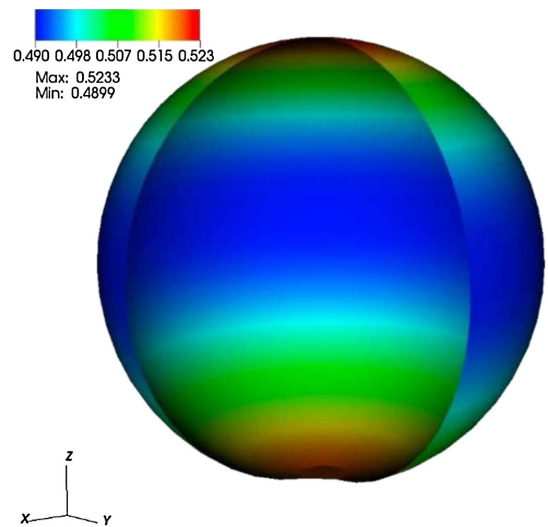


FIG. 11 (color online). 2D Ricci scalar on the apparent horizon surface of a black hole with  $\Omega_{\text{max}}P/M_{\text{irr}} = 1.77$ . In this view, a wedge-shaped region has been removed from the front.

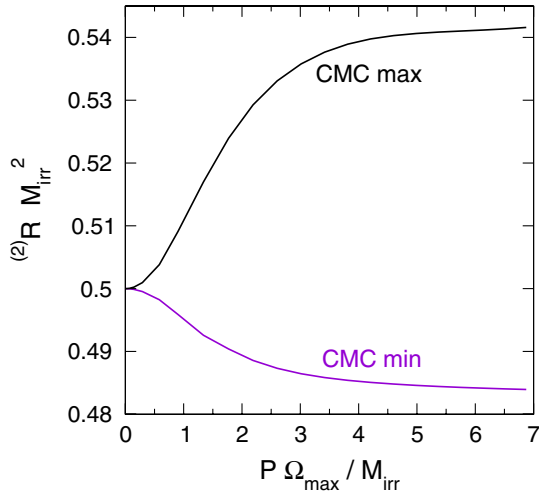


FIG. 12 (color online). Dimensionless intrinsic geometry of the apparent horizon (2D Ricci scalar, times the irreducible mass squared) versus  $P\Omega_{\max}/M_{\text{irr}}$  for a single nonspinning black hole, with the ratio  $R_{\text{AH}}/R_{\text{ms}}$  kept fairly constant. Shown are the maximum and minimum numerical values computed on CMC slices.

and minimum along the equatorial region. (Recall that  ${}^{(2)}\mathcal{R}M_{\text{irr}}^2 = 0.5$  for a spherical geometry.) The numerically computed spin of this black hole is indeed zero, to machine precision.

Figure 12 shows the minimum and maximum of  ${}^{(2)}\mathcal{R}M_{\text{irr}}^2$  as the boost parameter is varied in the range  $0 \leq \Omega_{\max}P/M_{\text{irr}} \leq 6.87$ . The minimum and maximum values of  ${}^{(2)}\mathcal{R}M_{\text{irr}}^2$  when  $\Omega_{\max}P/M_{\text{irr}} = 1.77$  agree with those shown in Fig. 11.

#### D. Binary black hole initial data

To demonstrate the generality of the approach that we have presented, we shall construct initial data for two black holes with mass-ratio approximately 2:1 and nonzero, arbitrarily oriented Bowen-York spin and boost parameters. First we describe how we obtain input parameters for the elliptic solver corresponding to our particular physical parameters. First, we choose  $\gamma \equiv r_{\text{ms}}/r_H = 0.8$ , which singles out a particular line of constant  $\gamma$  in Fig. 4, for each black hole. Next, we pick  $KM$  for each black hole so that (i) its minimal surface is at least partially down the throat of the trumpet, which is near the turnover of the  $\gamma = 0.8$  curve and (ii) Eqs. (45) and (46) hold, i.e. before the turnover. With these criteria in mind, we choose  $K$  (a global parameter) to be 0.05 and the masses of black holes A and B to be, respectively,  $M_A = 2/3$  and  $M_B = 1/3$ . Finally, from Eq. (46) and Fig. 5, we find  $R_{\text{ms}}/R_+ = 8.1 \times 10^{-4}$  for hole A and  $R_{\text{ms}}/R_+ = 4.1 \times 10^{-4}$  for hole B.

We fix the overall length scale by setting  $R_+ = 300$ . This places the excision radii at  $R_{\text{ms}} = 0.244$  and  $0.122$  for holes A and B, respectively, and the apparent horizon radii  $R_{\text{AH}} \approx 1$  (because from Fig. 4,  $R_{\text{ms}}/R_{\text{AH}} \approx 0.2$ ). The co-

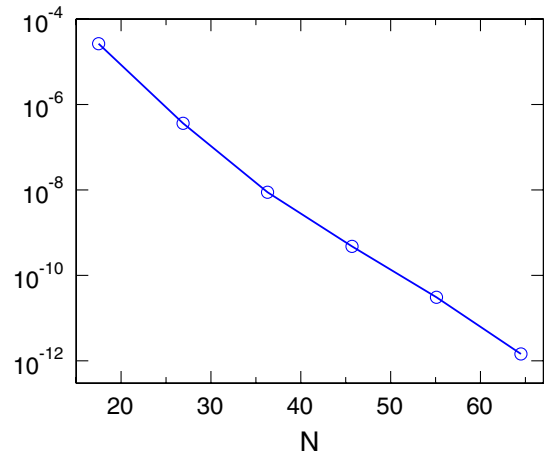


FIG. 13 (color online). Convergence of the elliptic solver for the unequal mass binary black hole example shown in Fig. 14. Shown is the volume L2-norm of the residual of  $\Omega$  as  $N$  is increased, where  $N$  is the cube root of the total number of collocation points.

ordinate locations of the two holes are then chosen to be  $(x_A, y_A, z_A) = (10, 0, 0)$  and  $(x_B, y_B, z_B) = (-20, 0, 0)$ , and the center of mass of the holes is at the origin of the coordinate system.

We take the spins to be  $S_A^i = (0, 0, S_A)$  and  $S_B^i = (S_B, 0, 0)$ . Since we are adding significant spins, it is necessary to set  $C_{\text{eff}}$  [defined in Eq. (54)] equal to  $g_C M^2$  for each hole, giving 0.569 for hole A and 0.142 for hole B. We take  $S_A = 0.4$  and  $S_B = 0.1$ , then giving  $C_A = 0.0613$  and  $C_B = 0.0128$ .

We take the boost parameters of the two black holes to be equal and opposite in the  $y$  direction, with magnitude  $P_A = P_B = 0.067$ . This gives approximate speeds of  $v_A = \Omega_{\max}P_A/M_A = 0.24$  and  $v_B = \Omega_{\max}P_B/M_B = 0.48$ , with  $M_A = 2/3$  and  $M_B = 1/3$  as given above. With  $C_A$  and  $C_B$  not equal to zero, and  $P^i$  not colinear with  $S^i$ , inversion symmetry is not possible (refer to the Appendix). Thus, we set  $Q_A^i = Q_B^i = 0$ .

Figure 13 shows exponential convergence of the volume L2-norm of the residual for the solution  $\Omega$  of the elliptic solver in this example, as the resolution of the numerical grid is increased. In addition, we have calculated the irreducible masses of the two holes and find values of 0.53 for hole A and 0.27 for hole B. This gives a mass ratio of 1.96.

Figure 14 shows the conformal factor on the full computational domain for the mass ratio 2:1 boosted, spinning binary black holes described above. The dark blue color at the outer edge shows that  $\Omega = 0$  at null infinity (to machine precision). In the middle, there is a pronglike feature, the tips of which are the two black holes. It is evident that the conformal factor becomes quite small in the vicinity of the two black holes.

Since our calculation of input parameters assumes spherical symmetry when our holes in fact have appre-

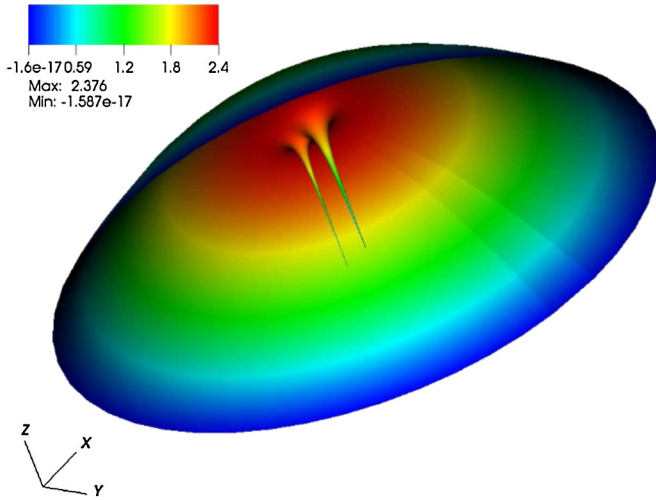


FIG. 14 (color online). Conformal factor  $\Omega$  for a spinning, boosted binary black hole, with mass ratio 2:1. The inner boundaries are the minimal surfaces of the two black holes, and the outer boundary is null infinity. The maximum of  $\Omega$  is red, and equals 2.38. The minimum is dark blue, which is zero to machine precision.

cial spins and boosts, one expects the irreducible masses to differ somewhat from the values used for calculating the input parameters. This is indeed what we find (0.53 vs 0.67 for hole A and 0.27 vs 0.33 for hole B). Finally, we find that the intrinsic geometry of each hole is distorted by the same amount. In particular, the minimum and maximum values of  ${}^{(2)}\mathcal{R}M_{\text{irr}}^2$  are, respectively, 0.37 and 0.54 for each hole.

#### IV. DISCUSSION

In this paper, we have considered the conformal method on CMC hyperboloidal slices, focusing on generalizing the traditional Bowen-York data. There are two key aspects that make Bowen-York data easy to construct. First, for *constant* mean curvature (no matter whether  $K = 0$  or  $K \neq 0$ ), the momentum constraint decouples from the Hamiltonian constraint, and the former can be solved first. Second, with *conformal flatness*, the momentum constraint simplifies to such an extent that analytical solutions are known: the symmetric, trace-free, divergence-free tensors with appropriate radial fall-off. Interestingly, the second aspect carries over from zero mean curvature to nonzero mean curvature, with the conformal factor now playing the dual role of turning the Hamiltonian constraint into an elliptic equation as well as compactifying  $I^+$ . The general Bowen-York conformal traceless extrinsic curvature still solves the momentum constraint analytically, where the only change necessary is the addition of a spherically symmetric divergence-free tensor [the first term in Eq. (39)].

Compared to the zero mean curvature case as usually formulated, the hyperboloidal initial value problem has

more free parameters, most notably the constants  $K$  (the mean curvature) and  $C$  (the coefficient of the spherically symmetric contribution to the conformal traceless extrinsic curvature), though a nonzero  $C$  is also consistent with zero mean curvature. Both these constants, as well as the minimal surface coordinate radius, have to be chosen carefully, and a significant portion of Sec. II is devoted to working out permissible choices, and their consequences on the initial data under construction.

As in the zero mean curvature case, hyperboloidal Bowen-York initial data trivially extends to multiple black holes with different spin and boost parameters for each black hole. Once again, one must be careful to choose the constants  $C$  (one for each black hole) and the radii of the excision boundaries, and Sec. II E gives simple rules how to do this. However, it is worth noting one significant difference. A single black hole *must* be centered at the origin of the conformal coordinates on a CMC hypersurface to be precisely Schwarzschild, since the outer boundary condition is imposed at a finite coordinate radius. A displaced black hole is not spherically symmetric.

For hyperboloidal slices, the elliptic equation for the Hamiltonian constraint, Eq. (10), is singular at the outer boundary  $I^+$ , where  $\Omega \rightarrow 0$ . Perhaps surprisingly, we have not encountered any difficulties when numerically solving this equation, for either single or binary black hole initial data. This is without any attempt to isolate and explicitly cancel the singular terms in the equation at future null infinity, as was advocated in [18]. We suspect that the absence of numerical difficulties is related to the simple Dirichlet boundary condition  $\Omega|_{I^+} = 0$ , and to the fact that the singular terms force the solution to also satisfy the von Neumann condition  $(\partial\Omega/\partial R)_{I^+} = -K/3$ , which follows from Eq. (12), implying spherical symmetry to at least first order in an expansion away from null infinity. The freedom in the solution at the outer boundary necessary to accommodate a global solution of the elliptic equation also satisfying an inner boundary condition at  $R_{\text{ms}}$  resides in a higher order term in the expansion of  $\Omega$  away from  $I^+$ . Our spectral code never evaluates the Hamiltonian constraint Eq. (10) right at  $I^+$ . Rinne [59] has also had no difficulty in solving the same elliptic equation with a finite difference code, as part of a constrained evolution scheme on CMC hypersurfaces [21].

The Hamiltonian constraint equation is also singular at the inner boundary in the special case of a trumpet, for which  $\Omega = 0$  at  $R = 0$ . This is a more challenging numerical problem, as discussed in Sec. II F. Singular terms include some inside the Laplacian operator. Their cancellation again uniquely determines the normal derivative of  $\Omega$  there, but now that will have angular dependence if the spin is nonzero. The solution does not have a simple expansion in integral powers of  $R$  at the boundary, which makes it more of a challenge for our spectral methods to have the accuracy required to deal with the singularities in

the equation. Still, we were able to approach very close to the trumpet limit, at least in the spherically symmetric case, by using the analytic solution for  $\Omega/R$  at the boundary to formulate the boundary condition as a Dirichlet condition on  $\Omega$  at a small, but nonzero,  $R$  and by adding extra collocation points near the boundary. If need be, reformulating the Hamiltonian constraint as an equation for  $\Omega/R$ , as in Eq. (47), in a domain near the inner boundary and explicitly canceling the singular terms at the boundary should make it possible to manage exact trumpet boundary conditions numerically.

One interesting aspect of hyperboloidal Bowen-York data lies in the physical interpretation of the spin parameter  $S^i$  and the boost parameter  $P^i$ . On asymptotically flat hypersurfaces (i.e.  $K = 0$ ), one can evaluate the ADM integrals and find that the Bowen-York parameter  $P^i$  agrees with the ADM linear momentum, and that  $S^i$  agrees with the ADM angular momentum. For hyperboloidal slices, the ADM formulas are not applicable. Nevertheless, a single unboosted spinning black hole, because it is axisymmetric, has a well-defined angular momentum which agrees with the spin parameter  $S^i$  [see Eq. (43)]. The relationship of the boost parameters to the linear momentum is less clear. The conformal compactification leaves a rescaling freedom  $R \rightarrow \eta R$  unspecified [see Eq. (24)], and as argued in Sec. IID 2, the boost parameter rescales as  $P^i \rightarrow \eta^{-1} P^i$ , so that the vector  $P^i$  by itself has no physical meaning. However, one can define a scale invariant quantity,  $\Omega_{\max} P^i$ , which may be considered a “quasilocal” linear momentum, at least when  $K$  is small. The proper interpretation of boosts on CMC hypersurfaces requires further analysis.

When angular momentum is defined, we can consider the question of how large spins can be constructed with hyperboloidal Bowen-York data. In Sec. IIIB, we have considered a sequence of black hole initial data with increasing spins, and Fig. 9 shows that black holes have been constructed with spin-extremality parameter<sup>3</sup>  $\zeta = S/(2M_{\text{irr}}^2) \approx 0.78$ . In contrast, standard Bowen-York data for a single spinning black hole allows  $\zeta \lesssim 0.83$  (Fig. 2 of Ref. [58]), whereas conformally flat conformal thin sandwich data allows  $\zeta \lesssim 0.56$  along the easily accessible lower branch of solutions, and  $\zeta \lesssim 0.87$  along the upper branch (Fig. 7 of Ref. [58]). We thus see that hyperboloidal Bowen-York initial data allows similarly large spins as the standard Bowen-York initial data (this includes the widely used puncture initial data [23] as a special case). We have not tested the sensitivity of the maximum achievable  $\zeta$  to variations of the other Bowen-York parameters  $C$ ,  $K$ ,  $R_{\text{ms}}$  of spinning black holes, but do not expect it to be large as long as  $KM_{\text{irr}}$  is reasonably small.

<sup>3</sup>We avoid the more widely used spin measure  $\chi = S/M^2$ , with  $M$  the Christoudoulou mass, because  $\chi \leq 1$  due to the definition of the Christoudoulou mass, and because the Christoudoulou mass only has physical meaning for Kerr black holes.

The simplifying assumptions of Bowen-York initial data appear to limit the ability to push towards near-extremal spins  $\zeta \approx 1$ . To construct larger spins, one would have to give up these simplifying assumptions, most notably conformal flatness. An approach based on the extended conformal thin sandwich (XCTS) equations similar to Ref. [58] seems very promising. Note that for the Schwarzschild spacetime, the *space-time* metric can be conformally rescaled, resulting in conformal lapse and shift functions which are finite at  $I^+$  [see Eq. (21)]. Thus, it seems quite likely that the XCTS equations rewritten in suitably rescaled variables can be used to construct more sophisticated hyperboloidal initial data. The XCTS approach has another interesting feature. In this approach, the spins and boosts of the black holes are implemented by *boundary conditions* at the black hole horizons [36,60,61]; the region of the initial data hypersurface close to the black holes should only be mildly affected by the “warping up” of the CMC hypersurface at large radii as it approaches  $I^+$ . Therefore, within the XCTS framework, it might be easier to interpret a boost. This will be a topic of future research.

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## APPENDIX: THE INVERSION SYMMETRY OF THE CMC INITIAL VALUE PROBLEM

In discussions of the conformally flat initial value problem, there has been considerable interest in inversion symmetric initial value data (see [62] and references therein). The issue of inversion symmetry arises when the initial hypersurface contains a minimal surface at a conformal radius  $R = R_{\text{ms}}$ . As  $R$  decreases below this value, the physical radius  $r$  increases and becomes infinite in the limit  $R \rightarrow 0$ . The hypersurface may or may not be symmetric under the inversion transformation  $R \rightarrow R_{\text{ms}}^2/R$ . Imposing inversion symmetry on the initial data, and requiring that it be preserved during subsequent evolution,

can lead to relatively simple excision boundary conditions at  $R = R_{\text{ms}}$ . Inversion symmetry was discussed in the original Bowen and York paper [22], and has been exploited in much of the numerical work based on the Bowen-York class of solutions to the initial value problem on maximal hypersurfaces. In this Appendix, we show that the conditions on the solution for the conformal traceless extrinsic curvature tensor which lead to inversion symmetry are the same on CMC hypersurfaces as they are on maximal hypersurfaces, noting, however, that the most general Bowen-York solution of the conformal momentum constraint equation does not admit inversion symmetry.

We start with the Hamiltonian constraint equation in the form given in Eq. (47) as an equation for the scale-invariant variable  $U \equiv \Omega/R$ . A rearrangement of terms gives a form in which the possibility of inversion symmetry is manifest

$$R \frac{\partial}{\partial R} \left( R \frac{\partial U}{\partial R} \right) + \hat{\Delta} U + \frac{1}{2} U = \frac{3}{2U} \left[ \left( R \frac{\partial U}{\partial R} \right)^2 + \hat{\nabla} U \cdot \hat{\nabla} U - \left( \frac{K}{3} \right)^2 + \frac{1}{6} U^6 W \right], \quad (\text{A1})$$

where  $\hat{\Delta}$  is the Laplacian operator and  $\hat{\nabla}$  is the gradient operator on the unit two-sphere. The only term not obviously symmetric under the inversion transformation is the term involving the source term  $W$ . If and only if  $W$  is inversion symmetric,  $W(R, \theta, \varphi) = W(R_{\text{ms}}^2/R, \theta, \varphi)$ , will the solution for  $U$ , subject to the minimal surface condition  $\partial U / \partial R = 0$  at  $R = R_{\text{ms}}$ , be inversion symmetric,  $U(R, \theta, \varphi) = U(R_{\text{ms}}^2/R, \theta, \varphi)$ .

The generalized Bowen-York solution for  $\tilde{A}_{ij}$  is given in Eq. (39). From this, we find

$$\begin{aligned} W &= R^6 \tilde{A}_{ij} \tilde{A}^{ij} \\ &= \frac{9}{2} R^2 [P^k P_k + 2(P^i n_i)(P^j n_j)] \\ &\quad + \frac{9}{2} R^{-2} [Q^k Q_k + 2(Q^i n_i)(Q^j n_j)] + 6C^2 \\ &\quad - 9[P^k Q_k - 4(P^i n_i)(Q^j n_j)] \\ &\quad + 18(\varepsilon_{ijk} S^j n^k)(\varepsilon^{imn} S_m n_n) \\ &\quad - 18C \left[ R(P^k n_k) + \frac{1}{R}(Q^k n_k) \right] \\ &\quad - 18 \left[ R(\varepsilon_{ijk} P^i S^j n^k) - \frac{1}{R}(\varepsilon_{ijk} Q^i S^j n^k) \right]. \quad (\text{A2}) \end{aligned}$$

Under an inversion transformation, the first two terms on the right-hand side of Eq. (A2) transform into each other provided that  $Q^i = \pm R_{\text{ms}}^2 P^i$ . The next three terms do not depend on  $R$  and are therefore trivially inversion-symmetric. Symmetry of the second to last square-bracket requires  $Q^i = +R_{\text{ms}}^2 P^i$ , while symmetry of the last square-bracket requires  $Q^i = -R_{\text{ms}}^2 P^i$  unless it vanishes because the boost and spin vectors are colinear. Without any restrictions on the Bowen-York parameters ( $C, P^i, S^i$ ), there is no choice of the  $Q^i$  which guarantees an inversion-symmetric  $W$  and therefore no guarantee of an inversion-symmetric solution of the Hamiltonian constraint equation. This result does not depend on the value of  $K$ . If we set  $C = 0$ , we recover the inversion symmetry result as usually stated for maximal hypersurfaces, that the ‘‘minus’’ form of inversion symmetry applies for general spin and boost.

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