Energy distribution of massless particles on black hole backgrounds with generalized uncertainty principle

Zhong-Heng Li*

Department of Physics, Zhejiang University of Technology, Hangzhou 310032, China

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We derive new formulas for the spectral energy density and total energy density of massless particles in a general spherically symmetric static metric from a generalized uncertainty principle. Compared with blackbody radiation, the spectral energy density is strongly damped at high frequencies. For large values of r, the spectral energy density diminishes when r grows, but at the event horizon, the spectral energy density vanishes and therefore thermodynamic quantities near a black hole, calculated via the generalized uncertainty principle, do not require any cutoff parameter. We find that the total energy density can be expressed in terms of Hurwitz zeta functions. It should be noted that at large r (low local temperature), the difference between the total energy density and the Stefan-Boltzmann law is too small to be observed. However, as r approaches an event horizon, the effect of the generalized uncertainty principle becomes more and more important, which may be observable. As examples, the spectral energy densities in the background metric of a Schwarzschild black hole and of a Schwarzschild black hole plus quintessence are discussed. It is interesting to note that the maximum of the distribution shifts to higher frequencies when the quintessence equation of state parameter w decreases.

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In quantum mechanics, the product of the uncertainty in position and the uncertainty in momentum must be equal to or exceed a fundamental limit proportional to Planck's constant. This is called the Heisenberg uncertainty principle, which shows that, in a six-dimensional phase space, no particle can be localized into a region smaller than $(2\pi\hbar)^3$, so this is the size of a fundamental element. Hence the volume in the phase space for each discrete quantum state is $(2\pi\hbar)^3$. Although the cell of the volume was first established from the study of a nonrelativistic gas [1], Bose [2] showed that, for an ultrarelativistic gas, the cell takes the same form as that in nonrelativistic cases. From which one can show that, for a perfect relativistic gas, the energy distribution has a blackbody spectrum and obeys Stefan-Boltzmann law. In fact, this follows if one does not consider gravitational interactions. But, if one does, the Heisenberg uncertainty principle is found to be modified to [3–18]

$$\Delta X \Delta P \geq \frac{\hbar}{2}[1 + \lambda(\Delta P)^2], \tag{1.1}$$

which, as is easily verified, implies a finite minimal uncertainty $(\Delta X)_{\min} = \hbar\sqrt{\lambda}$. Equation (1.1) is called the generalized uncertainty principle, which has found strong support from string theory [3,4], loop quantum gravity [5], and noncommutative geometry [6,7]. The generalized uncertainty principle has both low-energy (quantum mechanical) and high-energy (quantum gravity) limits. Based on this principle, the volume of a phase-space cell is changed from $(2\pi\hbar)^3$ into $(2\pi\hbar)^3(1 + \lambda P^2)^3$ [19], where P is the 3-momentum of a particle.

There has been much attention devoted to resolving the corrections to thermodynamical quantities in various spacetime backgrounds via the generalized uncertainty principle in recent years. Such as black hole [20–35], universe [36–42], dark energy [43], and brane world [44–46] backgrounds. However, despite extensive discussion, exact thermodynamic relations with the generalized uncertainty principle are still lacking. In this paper, we discuss a perfect relativistic gas on black hole backgrounds with the generalized uncertainty principle and show that the energy distribution is described by new formulas. This is considered for one reason. From the Heisenberg uncertainty principle, one can obtain the energy distribution formulas in curved spacetime as will be shown in the next section. Unfortunately, calculations generally lead to divergent expressions due to the infinite growth of the density of states close to an event horizon. But a generalized uncertainty principle provides a minimal length scale, which may play the role of a natural cutoff. Therefore we expect the use of a generalized uncertainty principle to be natural as it removes the divergence near event horizons.

In the next section, we discuss the thermodynamical properties of a perfect relativistic gas in a general spherically symmetric static metric, assuming the generalized uncertainty principle, and give the density of the spectral frequency distribution of the energy. In Sec. III, we further show basic properties of the spectral energy distribution in the background metric of a Schwarzschild black hole and of a Schwarzschild black hole plus quintessence. In Sec. IV, we derive a new formula for the total energy density from the spectral frequency distribution and show

*ZHLI@zjut.edu.cn

that the deviation from the Stefan-Boltzmann law is due to the presence of the generalized uncertainty principle. Finally, Sec. V contains a further discussion of the new formulas and some concluding remarks.

II. SPECTRAL ENERGY DISTRIBUTION

In this section, we discuss the density of the spectral frequency distribution of the energy from the modified number of states. Based on the generalized uncertainty principle, the number of quantum states of the system in a phase volume $\mathbf{d^3Xd^3P}$ reads [19,47–49] (we use Planck units)

$$\frac{\gamma \mathbf{d^3Xd^3P}}{(2\pi)^3(1+\lambda P^2)^3}, \tag{2.1}$$

where γ is the degeneracy due to the spin. As already mentioned in the introduction, the concept of cells in phase space can be applied not only to a nonrelativistic gas but also a relativistic gas. The physical reason can be traced back to the universality of the Heisenberg uncertainty principle. Therefore, we naturally suppose that Eq. (2.1) is also applicable to both a nonrelativistic gas and a relativistic gas, since the generalized uncertainty principle is valid for these two cases.

We now consider a spacetime with metric

$$ds^2 = B(r)dt^2 - B^{-1}(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{2.2}$$

This includes as special cases the Schwarzschild, Reissner-Nordström, Barriola-Vilenkin, and (anti-)de Sitter spacetime background or any combination of these, where the event horizons are determined by $B = 0$. At the WKB level, the 3-momentum of a free massless particle in this spacetime can be written as

$$P^2 = P_iP^i = BP_r^2 + \frac{1}{r^2}P_\theta^2 + \frac{1}{r^2\sin^2\theta}P_\varphi^2 = \frac{\omega^2}{B}, \tag{2.3}$$

where ω is an angular frequency. Equation (2.3) tells us that, near an event horizon, the momentum of a particle approaches infinity; i.e., its de Broglie wavelength is ever-increasingly blueshifted. Hence the WKB approximation or the relativistic Hamilton-Jacobi equation is justified. In fact, earlier works [50,51] have shown that the WKB approximation is valid in the neighborhood of a black hole.

Using Eqs. (2.1) and (2.3) and Planck's distribution, we obtain the energy of particles with frequencies between ω and $\omega + d\omega$:

$$dU_\omega = d\omega \int \frac{\gamma\omega^3}{2\pi^2B^2(1+\lambda\frac{\omega^2}{B})^3}\frac{4\pi r^2 dr}{e^{\beta\omega}\pm 1}. \tag{2.4}$$

In Eq. (2.4), β is the inverse temperature at large distance. The plus sign corresponds to the Fermi case, while the minus sign corresponds to the Bose case.

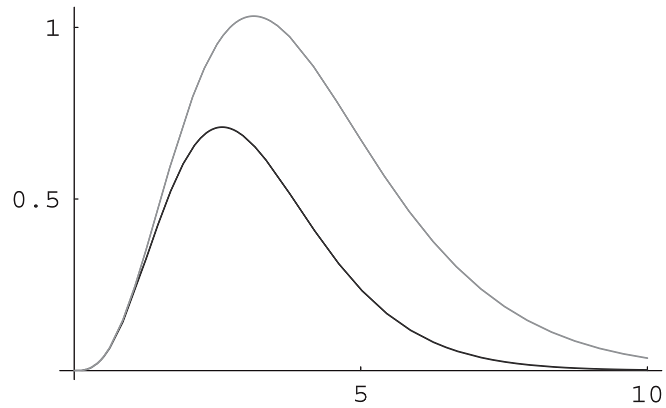

FIG. 1. The spectral energy density ρ_ω versus the frequency ω. The heavy curve and the light curve describe the behaviors based on the generalized uncertainty principle and the Heisenberg uncertainty principle, respectively.

On the other hand, the energy dU_ω can be given by the integral [52–54]

$$dU_\omega = d\omega \int \rho_\omega(r)4\pi r^2 dr. \tag{2.5}$$

Here $\rho_\omega(r)d\omega$ is the energy density in the frequency range from ω to $\omega + d\omega$. Comparing this with Eq. (2.4), we find that the density of the spectral frequency distribution of the energy of a perfect relativistic gas can be written as

$$\rho_\omega(r) = \frac{\gamma\omega^3}{2\pi^2B^2(1+\lambda\frac{\omega^2}{B})^3}\frac{1}{e^{\beta\omega}\pm 1}. \tag{2.6}$$

In this paper, $\rho_\omega(r)$ is called the spectral energy density for short.

Compared with blackbody radiation, the spectral energy density is modified at high frequencies as shown in Fig. 1. Note that when $\lambda = 0$, formula (2.6) becomes

$$\rho_\omega(r) = \frac{\gamma\omega^3}{2\pi^2B^2}\frac{1}{e^{\beta\omega}\pm 1}, \tag{2.7}$$

which is the spectral energy distribution in curved spacetime with the usual Heisenberg uncertainty principle. Obviously, in this case, the spectral energy density diverges at an event horizon ($B = 0$). To control divergences, 't Hooft [50] introduced a cutoff parameter, which is interpreted as the position of a "brick wall." However, substituting $B = 0$ into Eq. (2.6) gives $\rho_\omega = 0$, and therefore it is not necessary to introduce the brick-wall model when the generalized uncertainty principle is taken into account. In Minkowski spacetime $B = 1$, our result reduces to the case discussed by Chang *et al.* [19].

III. SOME EXAMPLES

In this section, we will further show basic properties of the spectral energy distribution via some specific examples.

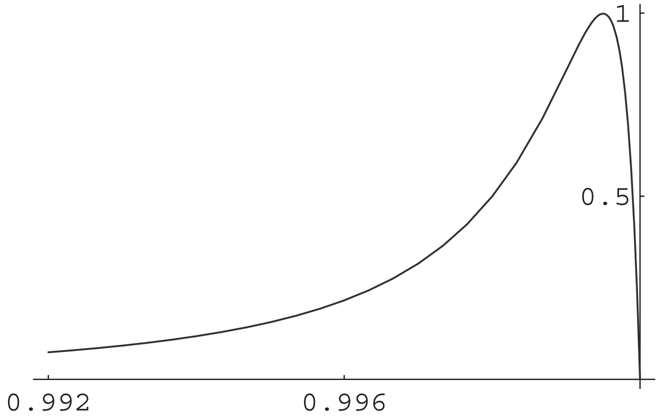

FIG. 2. The spectral energy density ρ_ω versus x for fixed frequency ω on the Schwarzschild spacetime background.

A. Schwarzschild black hole

For the Schwarzschild black hole background, $B(r)$ is given by

$$B(r) = 1 - \frac{r_H}{r}, \tag{3.1}$$

where $r_H = 2M$ is the event horizon radius. The inverse Hawking temperature β_H is expressed in terms of the event horizon radius via the relation $\beta_H = 4\pi r_H$.

By choosing the inverse temperature β to correspond to the Hawking inverse temperature β_H, the spectral energy density for the Schwarzschild spacetime background can be written as

$$\rho_\omega = \frac{\gamma\omega^3(1-x)}{2\pi^2(1-x+\lambda\omega^2)^3}\frac{1}{e^{4\pi r_H\omega} \pm 1}. \tag{3.2}$$

Here a dimensionless parameter $x = r_H/r$ is introduced which maps the region $r_H \leq r < \infty$ into the finite interval $1 \geq x > 0$.

We have drawn the behavior of the spectral energy density for fixed frequency in Fig. 2. On it, we see that, for large values of r, the spectral energy density diminishes when r grows, but at the event horizon $x = 1$, the spectral energy density vanishes. This shows that thermodynamic quantities near a black hole, calculated via the generalized uncertainty principle, do not require any cutoff parameter.

B. Schwarzschild black hole surrounded by quintessence

Nowadays, astrophysical data lead one to believe that the expansion of the Universe is accelerating [55], which implies that most of the energy of the Universe is some sort of dark energy, with a ratio of pressure to density less than $w = -1/3$. It is possible that the dark energy density has a time-dependent w. When $-1 < w < -1/3$, the Universe is in the quintessence phase.

The spacetime metric for a Schwarzschild black hole surrounded by quintessence gives [56]

$$B(r) = 1 - \frac{2M}{r} - \left(\frac{r_0}{r}\right)^{3w+1}. \tag{3.3}$$

The event horizons are determined by

$$1 - \frac{2M}{r} - \left(\frac{r_0}{r}\right)^{3w+1} = 0. \tag{3.4}$$

The mass parameter M can be written in terms of the other metric parameters when recognizing that $B(r_H) = 0$, yielding

$$M = \frac{r_H}{2}\left[1 - \left(\frac{r_0}{r_H}\right)^{3w+1}\right]. \tag{3.5}$$

The Hawking inverse temperature β_H is given by

$$\beta_H = 4\pi r_H\left[1 + 3w\left(\frac{r_0}{r_H}\right)^{3w+1}\right]^{-1}. \tag{3.6}$$

Note that, if and only if

$$\frac{r_0}{r_H} > e = 2.718\ldots, \tag{3.7}$$

the Hawking temperature is positive for any w in $(-1, -\frac{1}{3})$.

The spectral energy density at inverse temperature β_H is given by

$$\rho_\omega = \frac{\gamma\omega^3[1 - x(1-\alpha^{3w+1}) - \alpha^{3w+1}x^{3w+1}]}{2\pi^2[1 - x(1-\alpha^{3w+1}) - \alpha^{3w+1}x^{3w+1} + \lambda\omega^2]^3} \times \frac{1}{e^{4\pi r_0\omega/\alpha(1+3w\alpha^{3w+1})} \pm 1}. \tag{3.8}$$

Similarly to the previous subsection, the dimensionless parameters x and α denote $x = r_H/r$ and $\alpha = r_0/r_H$, respectively.

We see from Fig. 3 that the maximum of the distribution shifts to higher frequencies with decreasing w.

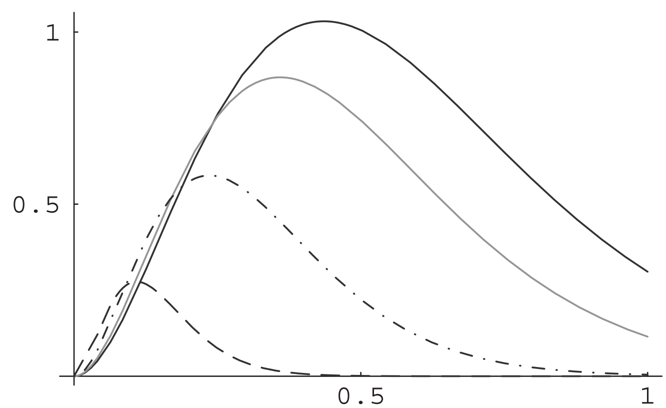

FIG. 3. The spectral energy density ρ_ω versus the frequency ω for fixed x in the spacetime metric of a Schwarzschild black hole surrounded by quintessence. The heavy solid curve is for $w = -1$, the light solid curve is for $w = -2/3$, the dotted-dashed curve is for $w = -1/2$, and the dashed curve is for $w = -2/5$.

IV. TOTAL ENERGY DENSITY

It is known that, in Minkowski spacetime and with the Heisenberg uncertainty principle, the total energy density of blackbody radiation is proportional to the fourth power of the temperature, which is the Stefan-Boltzmann law. In curved spacetime (2.2), the corresponding energy density can be derived by direct integration of the distribution (2.7). The result takes the same form as that in flat spacetime if the temperature is replaced by the local temperature [57]

$$T(r) = \frac{1}{\beta\sqrt{B}}. \tag{4.1}$$

We now turn to consider the total energy density in curved spacetime with the generalized uncertainty principle. In this case, the total energy density is obtained by integrating (2.6) over all frequencies:

$$\rho(r) = \int_0^\infty \frac{\gamma\omega^3}{2\pi^2 B^2(1+\lambda\frac{\omega^2}{B})^3}\frac{d\omega}{e^{\beta\omega}\pm 1}. \tag{4.2}$$

In the integral we put $x = \beta\omega/2\pi$ and transform it as follows:

$$\begin{aligned}\rho(r) &= 8\pi^2\gamma\int_0^\infty \frac{T^4(r)x^3}{[1+4\pi^2\lambda T^2(r)x^2]^3}\frac{dx}{e^{2\pi x}\pm 1} \\ &= -\frac{\gamma T^3(r)}{2\lambda}\frac{d}{dT(r)}\bigg[\frac{1}{[4\pi^2\lambda T^2(r)]^2} \\ &\quad\times\int_0^\infty \frac{xdx}{[(\frac{1}{2\pi\sqrt{\lambda}T(r)})^2 + x^2](e^{2\pi x}\pm 1)}\bigg].\end{aligned} \tag{4.3}$$

The integral is calculated from the formulas

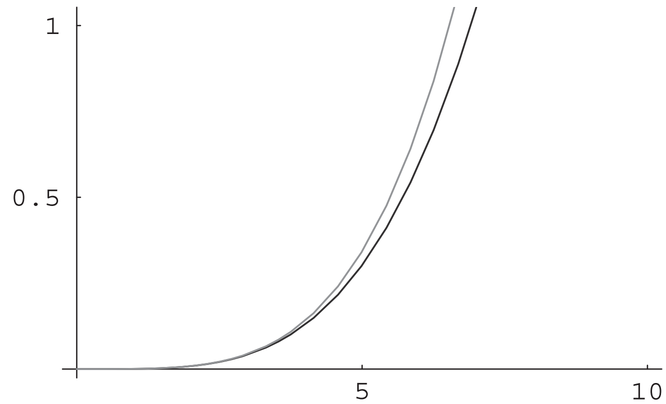

FIG. 4. The total energy density ρ versus the local temperature $T(r)$. The heavy curve and the light curve describe the behaviors of total energy density based on the generalized uncertainty principle and the Heisenberg uncertainty principle, respectively.

$$\int_0^\infty \frac{xdx}{(\alpha^2+x^2)^2(e^{2\pi x}-1)} = -\frac{1}{8\alpha^3} - \frac{1}{4\alpha^2} + \frac{1}{4\alpha}\psi'(\alpha) \tag{4.4}$$

and

$$\int_0^\infty \frac{xdx}{(\alpha^2+x^2)^2(e^{2\pi x}+1)} = \frac{1}{4\alpha^2} - \frac{1}{4\alpha}\psi'\left(\alpha+\frac{1}{2}\right), \tag{4.5}$$

where $\psi(\alpha)$ is the Euler psi function and the prime denotes the derivative with respect to its argument.

We finally find that the total energy density in curved spacetime with the generalized uncertainty principle can be written as

$$\rho(r) = \begin{cases} -\frac{\gamma}{16\pi^2\lambda^2}(1+\frac{\pi\sqrt{\lambda}T(r)}{2}) + \frac{\gamma}{64\pi^3\sqrt{\lambda^5}T(r)}[3\zeta(2,\frac{1}{2\pi\sqrt{\lambda}T(r)}) - \frac{1}{\pi\sqrt{\lambda}T(r)}\zeta(3,\frac{1}{2\pi\sqrt{\lambda}T(r)})] & \text{(bosons)},\\ \frac{\gamma}{16\pi^2\lambda^2} - \frac{\gamma}{64\pi^3\sqrt{\lambda^5}T(r)}[3\zeta(2,\frac{1}{2}+\frac{1}{2\pi\sqrt{\lambda}T(r)}) - \frac{1}{\pi\sqrt{\lambda}T(r)}\zeta(3,\frac{1}{2}+\frac{1}{2\pi\sqrt{\lambda}T(r)})] & \text{(fermions)},\end{cases} \tag{4.6}$$

where $\zeta(n+1,\alpha) = (-1)^{n+1}\psi^{(n)}(\alpha)/n!$ is Hurwitz zeta function.

From formula (4.6), it is clear that the total energy density is determined only by the local temperature $T(r)$ but is not proportional to the fourth power of the temperature. In other words, the total energy density does not take the form of the Stefan-Boltzmann law. However, the deviation of the total energy density from the Stefan-Boltzmann law appears only at a high local temperature, because the effect is too small to be observable at a low local temperature as shown in Fig. 4.

For $\sqrt{\lambda}T(r) \ll 1$, formula (4.6) gives

$$\rho(r) = \begin{cases} \frac{\pi^2}{30}\gamma T^4(r) - \frac{4\pi^4}{21}\lambda\gamma T^6(r) & \text{(bosons)},\\ \frac{7\pi^2}{240}\gamma T^4(r) - \frac{31\pi^4}{168}\lambda\gamma T^6(r) & \text{(fermions)}.\end{cases} \tag{4.7}$$

It should be noted that, when $\lambda = 0$, formula (4.6) reduces to the form proportional to the fourth power of the temperature and the theory agrees with that performed in curved spacetime with the Heisenberg uncertainty principle.

Here we give two other forms for the total energy density, which are useful in some problems.

(1) The gamma-function representation:

$$\rho(r) = \begin{cases} -\frac{\gamma}{16\pi^2\lambda^2}\left(1+\frac{\pi\sqrt{\lambda}T(r)}{2}\right) + \frac{3\gamma}{64\pi^3\sqrt{\lambda^5}T(r)}\sum_{n=1}^{\infty}\frac{1}{n}\frac{\Gamma(n)\Gamma(\frac{1}{2\pi\sqrt{\lambda}T(r)})}{\Gamma(n+\frac{1}{2\pi\sqrt{\lambda}T(r)})}\left(1-\frac{2}{3}\frac{H_{n-1}^{(1)}}{2\pi\sqrt{\lambda}T(r)}\right) & \text{(bosons)}, \\ \frac{\gamma}{16\pi^2\lambda^2} - \frac{3\gamma}{64\pi^3\sqrt{\lambda^5}T(r)}\sum_{n=1}^{\infty}\frac{1}{n}\frac{\Gamma(n)\Gamma(\frac{1}{2}+\frac{1}{2\pi\sqrt{\lambda}T(r)})}{\Gamma(n+\frac{1}{2}+\frac{1}{2\pi\sqrt{\lambda}T(r)})}\left(1-\frac{2}{3}\frac{H_{n-1}^{(1)}}{2\pi\sqrt{\lambda}T(r)}\right) & \text{(fermions)}. \end{cases} \tag{4.8}$$

(2) The beta-function representation:

$$\rho(r) = \begin{cases} -\frac{\gamma}{16\pi^2\lambda^2}\left(1+\frac{\pi\sqrt{\lambda}T(r)}{2}\right) + \frac{3\gamma}{64\pi^3\sqrt{\lambda^5}T(r)}\sum_{n=1}^{\infty}\frac{1}{n}B\left(n,\frac{1}{2\pi\sqrt{\lambda}T(r)}\right)\left(1-\frac{2}{3}\frac{H_{n-1}^{(1)}}{2\pi\sqrt{\lambda}T(r)}\right) & \text{(bosons)}, \\ \frac{\gamma}{16\pi^2\lambda^2} - \frac{3\gamma}{64\pi^3\sqrt{\lambda^5}T(r)}\sum_{n=1}^{\infty}\frac{1}{n}B\left(n,\frac{1}{2}+\frac{1}{2\pi\sqrt{\lambda}T(r)}\right)\left(1-\frac{2}{3}\frac{H_{n-1}^{(1)}}{2\pi\sqrt{\lambda}T(r)}\right) & \text{(fermions)}. \end{cases} \tag{4.9}$$

Here $\Gamma(x)$ is the gamma function and $B(x, n)$ is the beta function which are related to Hurwitz zeta function by [58]

$$\zeta(2,x) = \sum_{n=1}^{\infty}\frac{1}{n}\frac{\Gamma(n)\Gamma(x)}{\Gamma(n+x)}, \tag{4.10}$$

$$\zeta(3,x) = \sum_{n=1}^{\infty}\frac{1}{n}\frac{\Gamma(n)\Gamma(x)}{\Gamma(n+x)}H_{n-1}^{(1)}, \tag{4.11}$$

and

$$B(n,x) = \frac{\Gamma(n)\Gamma(x)}{\Gamma(n+x)}, \tag{4.12}$$

where $H_n^{(k)}$ is the generalized harmonic numbers defined by

$$H_n^{(k)} = \sum_{j=1}^{n}\frac{1}{j^k}. \tag{4.13}$$

V. DISCUSSION AND CONCLUSION

We have considered a perfect gas consisting of massless particles in static spherically symmetric metrics. The spectral energy density and the total energy density with the generalized uncertainty principle are given by formulas (2.6) and (4.6), respectively. The spectral energy density $\rho_\omega(r)$ has a maximum at a frequency ω_m, which satisfies

$$\frac{e^\chi(3-\chi)\pm 3}{\chi^2[e^\chi(3+\chi)\pm 3]} = \frac{\lambda}{\beta^2 B} \tag{5.1}$$

with $\chi = \beta\omega_m$. From Eq. (5.1) it is clear that, when $\lambda = 0$, the spectral energy density follows from the Heisenberg uncertainty principle ω_m, which does not depend on r dominating and the theory agrees with that performed in Minkowski spacetime. Hence, the dependence on r becomes more and more relevant when approaching the event horizon.

On the other hand, from Eq. (4.6) we find that at large $T(r)$, in the Bose case, the total energy density reduces to the form proportional to the local temperature, while in the Fermi case it becomes constant. In either case, the result is very different from the Stefan-Boltzmann law. This means that, as r approaches an event horizon, the effect of the generalized uncertainty principle becomes more and more important, which may be observable.

ACKNOWLEDGMENTS

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