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Nonmarginal Lemaitre-Tolman-Bondi-like models with inverse triad corrections from loop quantum gravity

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Marginal Lemaitre-Tolman-Bondi (LTB) models with corrections from loop quantum gravity have recently been studied with an emphasis on potential singularity resolution. This paper corroborates and extends the analysis in two regards: (i) the whole class of LTB models, including nonmarginal ones, is considered, and (ii) an alternative procedure to derive anomaly-free models is presented which first implements anomaly freedom in spherical symmetry and then the LTB conditions rather than the other way around. While the two methods give slightly different equations of motion, not altogether surprisingly given the ubiquitous sprawl of quantization ambiguities, final conclusions remain unchanged: compared to quantizations of homogeneous models, bounces seem to appear less easily in inhomogeneous situations, and even the existence of homogeneous solutions as special cases in inhomogeneous models may be precluded by quantum effects. However, compared to marginal models, bouncing solutions seem more likely with nonmarginal models.

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I. INTRODUCTION

Quantum gravity changes the structure and dynamics of space-time on small distance scales, which should have implications for the final stages of matter collapse. An interesting class of models to shed light on this issue is given by Lemaître-Tolman-Bondi (LTB) space-times, which are inhomogeneous but do not show too much complexity. Classically, these models describe collapsing dust balls, containing Friedmann-Robertson-Walker solutions as special cases. They thus provide an interesting extension of models beyond homogeneity, an extension which is particularly important to understand in the case of quantum gravity.

Loop quantum gravity implies characteristic correction terms in the Hamiltonian constraint of gravity and matter. Once a combination of corrections keeping the algebra of constraints anomaly-free has been found, a canonical analysis of quantum gravitational collapse becomes possible. Finding such equations is not easy, making the investigation of implications from quantum gravity corrections highly restricted. Quantum geometry corrections result from an underlying spatial discreteness, which may cast doubt on whether they can leave the theory covariant. There are arguments at the level of the full theory [1] stating that quantum operators may be anomaly-free, but it remains unknown how to descend from this statement to anomaly-free effective space-time geometries. A phenomenological approach has thus been followed to investigate possible geometrical and physical effects of diverse corrections. Here, one inserts expected corrections in the classical constraints, suitably parametrized to reflect quantization ambiguities, and evaluates conditions under which the corrected constraints remain first class. As several articles have by now shown, it is indeed possible to have anomaly freedom even in the presence of quantum corrections resulting from spatial discreteness [2–4].

Marginal models, which are a subclass of general LTB models, have been analyzed in this spirit in Ref. [3]. This has resulted in consistent deformations which implement some types of quantum corrections without spoiling general covariance, and made possible an initial analysis of implications regarding effective pictures of collapse singularities. (At the fundamental level of dynamical difference equations in a loop quantization, spherically symmetric models are singularity-free [5] as are homogeneous models [6–8].) It turned out that there is no clear generic avoidance of either spacelike or null singularities by an obvious mechanism, in contrast to several homogeneous models of loop quantum cosmology [9] where phenomenological mechanisms such as bounces could be found easily. While this outcome is not entirely unexpected given the types of corrections analyzed in the marginal case, it does show that further analysis is required. Marginal models, after all, provide spatially flat Friedmann-Robertson-Walker models in the homogeneous limiting case which give rise to phe-

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nomenological singularity avoidance in their loop quantization (including a positive matter potential) only with holonomy corrections [10,11], which were not fully included in [3] due to technical complications. It is thus natural to extend the constructions to nonmarginal models which would provide a homogeneous model of positive (as well as negative) spatial curvature as limit. In that case, loop quantum cosmology can give rise to phenomenological singularity resolution even in the presence of inverse triad corrections alone [12], which in inhomogeneous situations are easier to control than holonomy corrections. If the behavior seen in homogeneous models should be generic and apply also to inhomogeneous situations, loop quantized nonmarginal LTB models must give rise to singularity resolution more easily than marginal ones.

To find anomaly-free versions of nonmarginal LTB models including inverse triad corrections from loop quantum gravity, we will follow two derivations. First, we will extend the methods of [3] where constraints already incorporating the LTB reduction of metric components are made anomaly-free by consistency conditions between correction functions. Secondly, we will derive a general anomaly-free system of spherically symmetric constraints, on which we then apply the LTB reduction in a second step. As we will show, the two steps of LTB reduction and deriving consistency conditions almost commute: in the end, we obtain consistent equations of motion of similar structure, although they do differ by some terms. This outcome considerably supports the constructions of [3].

Using these consistent equations, gravitational collapse can be analyzed. We are specifically interested here in the possibility of a turnaround of the collapse, or a bounce, in the corrected equations, which are suggested to exist by models where homogeneous interiors have been matched, Oppenheimer–Snyder-style, to spherically symmetric exteriors [13]. Also here, as in the marginal case but in contrast to homogeneous models, we do not find a clear indication for singularity resolution, although several extra terms do seem to make a bounce more likely. As in the marginal case, this part of the result is not conclusive since not all corrections have been included and no complete analysis has been performed. Our results thus do not mean that there is no bounce in these inhomogeneous models. But they do show that an outright treatment of inhomogeneous models is different from matching homogeneous results. In fact, we also confirm the observation of [3] that quantum corrections of the type studied here prevent the existence of an exact homogeneous limit. "Effective" homogeneous geometries thus have to be taken with care, but consistent relationships with inhomogeneous ones do provide insights in their structure [14].

II. CLASSICAL EQUATIONS

Nonmarginal LTB models [15–17] have a space-time metric given by

$$ds^2 = -dt^2 + \frac{R'^2}{1+\kappa(x)}dx^2 + R^2 d\Omega^2 \tag{1}$$

with $d\Omega^2 = d\vartheta^2 + \sin^2\vartheta d\varphi^2$ and where $\kappa \neq 0$ is a function of the radial coordinate x. (The limiting case $\kappa = 0$ is that of marginal models.) The function $R(t, x)$ can depend on both time and the radial coordinate, but not on the angular coordinates to leave the metric spherically symmetric. It is easy to see that positively curved Friedmann-Robertson-Walker models with scale factor $a(t)$ are obtained for $\kappa(x) = -x^2$ and $R(t, x) = a(t)x$.

For an application of loop quantization we use densitized triads instead of the spatial metric components, whose conjugate momenta are given in terms of the Ashtekar-Barbero connection and extrinsic curvature components. Written as a densitized vector field taking values in the Lie algebra of SU(2) with basis τ_i, the spherically symmetric densitized triad is

$$E = E^x(x)\tau_3 \sin\vartheta \frac{\partial}{\partial x} + (E^1(x)\tau_1 + E^2(x)\tau_2)\sin\vartheta \frac{\partial}{\partial\vartheta} + (E^1(x)\tau_2 - E^2(x)\tau_1)\frac{\partial}{\partial\varphi}.$$

Similarly the Ashtekar connection $A_a^i = \Gamma_a^i + \gamma K_a^i$, where Γ_a^i and K_a^i are the components of spin connection and extrinsic curvature, respectively, and γ is the Barbero-Immirzi parameter [18,19], reads

$$A = A_x(x)\tau_3 dx + (A_1(x)\tau_1 + A_2(x)\tau_2)d\vartheta + (A_1(x)\tau_2 - A_2(x)\tau_1)\sin\vartheta d\varphi + \tau_3\cos\vartheta d\varphi.$$

Introducing the U(1)-gauge invariant quantities $(E^\varphi)^2 = (E^1)^2 + (E^2)^2$ and $A_\varphi^2 = A_1^2 + A_2^2$ (see [20–22] for details), we have the symplectic structure

$$\{A_x(x), E^x(y)\} = \{\gamma K_\varphi(x), 2E^\varphi(y)\} = \{\eta(x), P^\eta(y)\} = 2G\gamma\delta(x,y)$$

or more explicitly the Poisson bracket of functions f and g is

$$\{f,g\} = 2G\int dx\left(\gamma\frac{\delta f}{\delta A_x}\frac{\delta g}{\delta E^x} + \frac{1}{2}\frac{\delta f}{\delta K_\varphi}\frac{\delta g}{\delta E^\varphi} + \gamma\frac{\delta f}{\delta\eta}\frac{\delta g}{\delta P^\eta} - \gamma\frac{\delta f}{\delta E^x}\frac{\delta g}{\delta A_x} - \frac{1}{2}\frac{\delta f}{\delta E^\varphi}\frac{\delta g}{\delta K_\varphi} - \gamma\frac{\delta f}{\delta P^\eta}\frac{\delta g}{\delta\eta}\right).$$

Compared to metric variables, we have an extra field $\eta(x)$ with momentum

$$P^\eta(x) = 2A_\varphi E^\varphi \sin\alpha = 4\,\mathrm{tr}((E^1\tau_1 + E^2\tau_2)(A_2\tau_1 - A_1\tau_2))$$

(with α defined as the angle between the internal directions of A and E components), which plays the role of a U(1)-gauge angle in the spherically symmetric theory. (This gauge angle also determines the x component of the spin connection $\Gamma_x = -\eta'$, and thus enters the Ashtekar connection by $A_x = -\eta' + \gamma K_x$ with an extrinsic curvature component K_x.)

In this situation, we have three constraints: the Gauss constraint

$$G_{\rm grav}[\lambda] = \frac{1}{2G\gamma}\int dx \lambda((E^x)' + P^\eta), \tag{2}$$

the vector constraint

$$\begin{aligned} D_{\rm grav}[N^x] &= \frac{1}{2G}\int dx N^x\Big(2E^\varphi K_\varphi' - \frac{1}{\gamma}A_x(E^x)' + \frac{1}{\gamma}\eta' P^\eta\Big) \\ &= \frac{1}{2G}\int dx N^x\Big(2E^\varphi K_\varphi' - K_x(E^x)' \\ &\quad + \frac{1}{\gamma}\eta'((E^x)' + P^\eta)\Big), \end{aligned} \tag{3}$$

and the Hamiltonian constraint

$$\begin{aligned} H_{\rm grav}[N] = -\frac{1}{2G}\int dx N|E^x|^{-1/2}(K_\varphi^2 E^\varphi + 2K_\varphi K_x E^x \\ + (1-\Gamma_\varphi^2)E^\varphi + 2\Gamma_\varphi' E^x) \end{aligned} \tag{4}$$

with $\Gamma_\varphi = -(E^x)'/2E^\varphi$ the gauge invariant angular component of the spin connection. Solving the Gauss constraint removes the pair (η, P^η) and reduces the vector constraint to the diffeomorphism constraint. After this step we can work with the canonical pairs

$$\{K_x(x), E^x(y)\} = \{K_\varphi(x), 2E^\varphi(y)\} = 2G\delta(x,y).$$

The relation to the usual spherically symmetric geometrodynamical variables

$$\{R(x), P_R(y)\} = \{L(x), P_L(y)\} = G\delta(x,y) \tag{5}$$

as used, for example, in [23,24] can be obtained directly by comparing the spatial metric

$$dq^2 = L^2 dx^2 + R^2 d\Omega^2 = \frac{(E^\varphi)^2}{|E^x|}dx^2 + |E^x| d\Omega^2 \tag{6}$$

in each set of variables and making use of the equations of motion:

$$\begin{aligned} L &= E^\varphi |E^x|^{-1/2}, \\ R &= |E^x|^{1/2}, \\ P_L &= -K_\varphi |E^x|^{1/2}, \\ P_R &= -sK_x|E^x|^{1/2} - K_\varphi E^\varphi |E^x|^{-1/2} \end{aligned} \tag{7}$$

where $s = \mathrm{sgn}(E^x)$. (The sign factor corresponds to the two possible orientations of a triad; we will mostly use $s = +1$ below.)

Specializing the general spherically symmetric metric

$$\begin{aligned} ds^2 = -N(t,x)^2 dt^2 + L^2(t,x)(dx + N^x(t,x)dt)^2 \\ + R^2(t,x) d\Omega^2 \end{aligned}$$

to the LTB form (1) requires a vanishing shift function $N^x = 0$ and lapse $N = 1$ for comoving coordinates of the dust, and on using the first equation in (7) gives the nonmarginal LTB condition

$$2\sqrt{1+\kappa(x)}E^\varphi = (E^x)' \tag{8}$$

in terms of triads. From this we can derive the spin connection component

$$\Gamma_\varphi = -\frac{(E^x)'}{2E^\varphi} = -\sqrt{1+\kappa(x)} \tag{9}$$

and its derivative $\Gamma_\varphi' = -\kappa'(x)/2\sqrt{1+\kappa(x)}$, which appear in the Hamiltonian constraint. Since the spin connection, unlike in the marginal case, is not a constant -1, the Hamiltonian constraint is different from the marginal case

$$\begin{aligned} H^{\rm class}_{\rm grav}[N] = -\frac{1}{2G}\int dx N(x)|E^x|^{-1/2} \\ \times\Big(K_\varphi^2 E^\varphi + 2K_\varphi K_x E^x - \kappa(x)E^\varphi \\ - \frac{\kappa'(x)E^x}{\sqrt{1+\kappa(x)}}\Big). \end{aligned} \tag{10}$$

If we solve the diffeomorphism constraint identically, which requires $2E^\varphi K_\varphi' - K_x(E^x)' = 0$, the LTB condition for triad variables gives rise to a condition

$$K_\varphi' = \sqrt{1+\kappa(x)}K_x \tag{11}$$

for the extrinsic curvature components. For a consistent LTB formulation, the two LTB conditions must be preserved by evolution generated by the constraint. This is indeed the case as can be seen from deriving Poisson brackets between the Hamiltonian constraint and each of the LTB conditions. For the Poisson bracket of the two LTB conditions, after smearing them with fields $\mu(x)$ and $\nu(x)$, we get

$$\begin{aligned} \Big\{\int dx \nu(x)(\sqrt{1+\kappa(x)}K_x - K_\varphi'), \int dy \mu(y)(2\sqrt{1+\kappa(x)}E^\varphi \\ - (E^x)')\Big\} = 2G\int dz \sqrt{1+\kappa(z)}(\mu\nu)'. \end{aligned} \tag{12}$$

This in general is nonvanishing, unlike in the marginal case where κ is zero. (Although we will not follow this route here, we note that this will have an impact on implementing the LTB conditions at the state level, as done in [3] for the marginal case. Another complication for such a construction is the explicit κ-dependence of the LTB conditions, which makes their integrated version used as conditions on holonomies more complicated.)

Equations of motion in this canonical formulation are derived using $\dot{E}^x = \{E^x, H^{\rm class}_{\rm grav}\}$, with a similar equation for E^φ. With these we can first eliminate K_x and K_φ, and finally E^φ using the nonmarginal LTB condition to obtain an equation entirely in terms of E^x. After replacing E^x by R^2 we obtain

$$H^{\rm class}_{\rm grav} = \frac{-2R\dot{R}\dot{R}' - \dot{R}^2R' + \kappa' R + \kappa R'}{2G\sqrt{1+\kappa(x)}}, \qquad (13)$$

which has to be equated to the matter part of the Hamiltonian for dust given by $H_{\rm dust} = -\frac{1}{2G}F'/\sqrt{1+\kappa(x)}$. (A more general canonical derivation of the gravity-dust system will be given in Sec. IV B.) Thus

$$2R\dot{R}\dot{R}' + \dot{R}^2R' - \kappa' R - \kappa R' = F' \qquad (14)$$

is the equation of motion, in agreement with the spatial derivative of $R\dot{R}^2 = \kappa(x)R + F(x)$, which is the equation obtained by solving Einstein's equation for the nonmarginal case.

III. INVERSE TRIAD CORRECTIONS FROM LOOP QUANTUM GRAVITY

We will now repeat the canonical analysis using a Hamiltonian constraint containing correction functions as they are suggested by constraint operators in loop quantum gravity. Consistency will then require conditions for the possible terms, which show how quantum corrections can be realized in an anomaly-free way. We discuss here only inverse triad corrections which are easier to implement, and which already provide insights into one of the main classes of quantum geometry corrections.

Inverse triad corrections arise from every Hamiltonian operator quantized by loop techniques, where inverse components of the densitized triad appear. Such corrections are directly related to spatial discreteness of quantum geometry since densitized triads as basic variables are quantized to flux operators with discrete spectra containing zero [25]. Since such operators do not have densely defined inverses, no direct inverse operator is available. Instead, well-defined quantizations exist based on techniques introduced in [1,26], implying corrections to the classical inverse.

In several symmetric models, inverse triad operators and the corrections they imply can be computed explicitly [27]. As an example, spherically symmetric models used in [3] give rise to a correction function of the form

$$\alpha(\Delta) = 2\frac{\sqrt{|\Delta + \gamma\ell_{\rm P}^2/2|} - \sqrt{|\Delta - \gamma\ell_{\rm P}^2/2|}}{\gamma\ell_{\rm P}^2}\sqrt{|\Delta|} \qquad (15)$$

where Δ is the size of an elementary plaquette in a discrete state underlying an LTB geometry. For corrections in inverse powers of E^x, which is proportional to the area of a spherical orbit, the relevant operators give rise to a dependence on plaquette sizes on orbits. For a nearly spherical distribution of $\mathcal{N}(E^x)$ such plaquettes making up the whole orbit, we thus have $\Delta = E^x/\mathcal{N}$. Since we refer only to the orbit size, corrections thus naturally depend on E^x only but not on E^φ. (That this is required will later be shown independently when we use anomaly freedom to rule out that α could depend on E^φ.) Such a correction function then multiplies any classical appearance of $(E^x)^{-1}$ in a Hamiltonian operator. In particular, classical divergences of inverse factors of E^x are cut off as one can see from the plot in Fig. 1. Correspondingly, the dynamics given by such a Hamiltonian will change from quantum corrections. Classically, i.e. for $\ell_{\rm P} \to 0$, we have $\alpha(E^x) = 1$, and this limit is approached for large E^x. Also this behavior of the correction function is illustrated in Fig. 1.

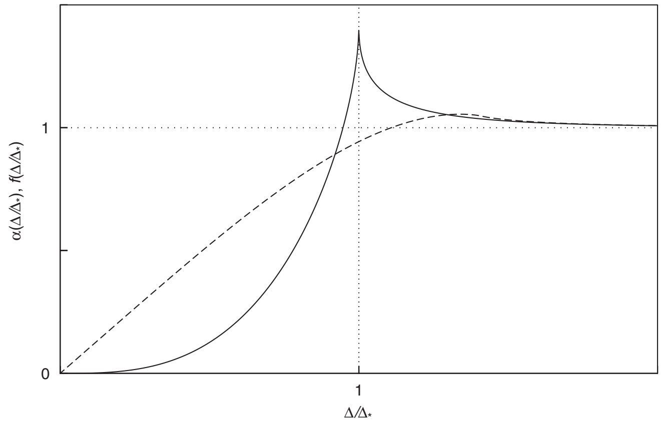

FIG. 1. The correction functions $\alpha(\Delta)$ (solid line) and $f(\Delta)$ (dashed line) where Δ is taken relative to $\Delta_* := \sqrt{\gamma/2}\ell_{\rm P}$.

For a large number $\mathcal{N}$ of discrete blocks, the scale Δ is reduced compared to E^x and corrections from α can be significant even for large E^x. Since typically $\mathcal{N}$, in relation to the underlying state, is not a constant but would depend on the size E^x, different kinds of behaviors can arise. This phenomenon of lattice refinement [28,29] is important to capture the full dynamics of quantum gravity and its elementary degrees of freedom. It is also crucial for realizing the correct scaling behavior of correction terms under changes of coordinates [30]. In this paper, we will mainly be looking at general implications in local equations of motion, where the value or behavior of $\mathcal{N}$ is not important. More detailed investigations could at some point provide restrictions on the possible form of $\mathcal{N}(E^x)$, and thus give insights in the required behavior of discrete quantum gravity states.

A. First version

We turn to the case of inverse triad corrections called "first version" in [3], where only those terms in the Hamiltonian with explicit $1/\sqrt{|E^x|}$ dependence are corrected by a factor $\alpha(E^x)$. Starting as in the marginal case, we first assume that the classical expression for the spin connection can be used, but show that this is inconsistent. The Hamiltonian, now assuming $E^x > 0$, is

$$H^I_{\rm grav}[N] = -\frac{1}{2G}\int dx N\bigg(\alpha(E^x)\frac{K_\varphi^2 E^\varphi}{\sqrt{E^x}} + 2K_\varphi K_x\sqrt{E^x} - \alpha(E^x)\frac{\kappa(x)E^\varphi}{\sqrt{E^x}} - \frac{\kappa'(x)\sqrt{E^x}}{\sqrt{1+\kappa(x)}}\bigg). \tag{16}$$

As in the marginal case, it turns out that the correction in the Hamiltonian can lead to consistent LTB-type solutions only if we also change the LTB conditions by a correction function $f(E^x)$

$$(E^x)' = 2\sqrt{1+\kappa(x)}f(E^x)E^\varphi \tag{17a}$$

and

$$K_\varphi' = \sqrt{1+\kappa(x)}f(E^x)K_x. \tag{17b}$$

These relations still solve the classical diffeomorphism constraint identically, which does not receive corrections in loop quantum gravity. The main consistency condition then is that Poisson brackets of the LTB conditions with the Hamiltonian constraint vanish.

Each of the Poisson brackets gives a differential equation for $f(E^x)$. For the LTB condition corresponding to the triad variables we obtain

$$2E^x\frac{df}{dE^x} = f(1-\alpha). \tag{18}$$

This is the same equation as found for the marginal case in [3]. For $\mathcal{N} = 1$, for instance, the solution is given by

$$f(E^x) = \frac{c_1\sqrt{E^x}e^{-\alpha/2}}{(\sqrt{E^x}+\sqrt{E^x-\gamma\ell_{\rm P}^2/2})^{1/2}(\sqrt{E^x}+\sqrt{E^x+\gamma\ell_{\rm P}^2/2})^{1/2}} \tag{19}$$

for $E^x > \gamma\ell_{\rm P}^2/2$ and

$$f(E^x) = \frac{c_2\sqrt{E^x}\exp(-\frac{1}{2}\alpha + \frac{1}{2}\arctan(\sqrt{E^x/(\gamma\ell_{\rm P}^2/2 - E^x)}))}{\sqrt{\sqrt{E^x}+\sqrt{E^x+\gamma\ell_{\rm P}^2/2}}} \tag{20}$$

for $E^x < \gamma\ell_{\rm P}^2/2$ (dashed curve in Fig. 1). Here $c_1 = 2\sqrt{e}$ and $c_2 = 2^{5/4}e^{1/2-\pi/4}\gamma^{-1/4}\ell_{\rm P}^{-1/2}$ are constants of integration fixed, respectively, by the condition $\lim_{E^x\to\infty}f(E^x)\to 1$ and by requiring that $f(E^x)$ be continuous at $E^x = \gamma\ell_{\rm P}^2/2$.

It is clear that we obtain the same equation because there is no influence of κ in the evaluation of the Poisson bracket, κ not affecting the terms containing K_x and K_φ in the Hamiltonian. However, the differential equation for $f(E^x)$ obtained by demanding that the corrected LTB condition for extrinsic curvature components is also preserved in time gives the equation

$$-2\sqrt{1+\kappa}K_\varphi K_x\sqrt{E^x}\frac{df}{dE^x} + \frac{\sqrt{1+\kappa}K_\varphi K_x f}{\sqrt{E^x}} - \frac{\alpha\sqrt{1+\kappa}K_\varphi K_x f}{\sqrt{E^x}} - \frac{\kappa' f}{2\sqrt{E^x}} + \frac{\kappa'\alpha}{2\sqrt{E^x}} = 0 \tag{21}$$

which, due to the κ' terms, is different from and in fact inconsistent with Eq. (18) obtained from the LTB condition for triads.

B. Second version

We now repeat the above procedure for the second version of the inverse triad corrections for which the Hamiltonian is corrected by $\alpha(E^x)$ in all terms and reads

$$H^{II}_{\rm grav}[N] = -\frac{1}{2G}\int dx N\frac{\alpha(E^x)}{\sqrt{E^x}}\bigg(K_\varphi^2 E^\varphi + 2K_\varphi K_x E^x - \kappa(x)E^\varphi - \frac{\kappa'(x)E^x}{\sqrt{1+\kappa(x)}}\bigg). \tag{22}$$

Again, the LTB conditions in the form

$$(E^x)' = 2\sqrt{1+\kappa(x)}g(E^x)E^\varphi \tag{23a}$$

and

$$K_\varphi' = \sqrt{1+\kappa(x)}g(E^x)K_x \tag{23b}$$

solve the diffeomorphism constraint identically. However, for the same reasons as noted above the conditions cannot be consistent: since the terms containing K_x and K_φ in the Hamiltonian do not involve κ, the Poisson bracket for the triads gives the same result as in the marginal case where the differential equation was

$$\alpha\frac{dg}{dE^x} = g\frac{d\alpha}{dE^x}, \tag{24}$$

with the solution $g(E^x) = \alpha(E^x)$. (This solution is unique with the boundary condition imposing that $g = 1$ for large arguments.)

For the Poisson bracket involving the condition on extrinsic curvature, on the other hand, the terms in the Hamiltonian involving κ are important and we get a different result:

$$- 2\sqrt{1+\kappa(x)}\alpha K_\varphi K_x\sqrt{E^x}\frac{dg}{dE^x} + 2\sqrt{1+\kappa(x)}K_\varphi K_x\sqrt{E^x} \times \frac{d\alpha}{dE^x}g - \frac{\kappa'\alpha g}{2\sqrt{E^x}} - \kappa'\sqrt{E^x}\frac{d\alpha}{dE^x}g + \frac{\kappa'\alpha}{2\sqrt{E^x}} = 0. \quad (25)$$

As in the previous case the presence of κ' terms spoils the consistency. Using $\alpha = g$ the first two terms cancel while the rest would require $d\alpha/dE^x = (\alpha - 1)/2E^x$ with a solution $\alpha = 1 + c\sqrt{E^x}$ violating the classical limit at large arguments.

C. Inclusion of corrections in the spin connection

A direct extension of the results from marginal to nonmarginal models is thus impossible. Here we have an example for the information gained by a phenomenological treatment: LTB-type solutions require additional corrections to compensate inconsistencies seen so far. Such corrections may be more difficult to derive from a full Hamiltonian, but they follow directly from a phenomenological treatment. A successful consistent implementation thus provides feedback on the full theory: additional corrections required for consistency must eventually follow from the full theory just like the primary correction α followed from inverse triad operators.

In particular, to resolve inconsistencies, we thus have to include further corrections in terms not affected yet, the chief candidate being the spin connection terms in the Hamiltonian constraint. They vanish in the marginal case, such that results from there do not provide much directions for more general models. Moreover, such terms are in fact more difficult to derive from a full Hamiltonian so that not much is known about their form. We will now look for corrections in the spin connection terms which are such that they combine with those already used to provide a consistent formulation.

1. Implementation

Classically we started with the expression $\Gamma_\varphi = -(E^x)'/2E^\varphi = -\sqrt{1+\kappa(x)}$. However, with quantum corrections to the Hamiltonian the LTB conditions are also corrected. For example, for the second version, the modified LTB condition $(E^x)' = 2\sqrt{1+\kappa}g(E^x)E^\varphi$ implies that now the spin connection is

$$\Gamma_\varphi = -\frac{(E^x)'}{2E^\varphi} = -\sqrt{1+\kappa(x)}g(E^x). \quad (26)$$

We include the additional factor of $g(E^x)$ in the Hamiltonian by replacing any occurrence of the classical spin connection $\Gamma_\varphi^{\rm class}$ from (9) with $g(E^x)\Gamma_\varphi^{\rm class}$.

Furthermore, the derivative of the spin connection then is

$$\Gamma'_\varphi = -\frac{\kappa' g(E^x)}{2\sqrt{1+\kappa}} - \sqrt{1+\kappa}(E^x)'\frac{dg}{dE^x}. \quad (27)$$

This introduces an explicit $(E^x)'$ in the LTB-reduced Hamiltonian, which would imply $\{H[N], H[M]\} \neq 0$ even though the diffeomorphism constraint has been solved identically. The system would thus be anomalous. We are finally led to incorporate another correction function multiplying Γ'_φ in a function $h(E^x)$ so that in the Hamiltonian Γ'_φ is to be replaced with $h(E^x)(\Gamma_\varphi^{\rm class})'$. The form of $h(E^x)$ will be determined by the requirement of consistency. With all the possible corrections, the new Hamiltonian in the second version is

$$H^{II}_{\rm grav}[N] = -\frac{1}{2G}\int dx N\left(\frac{\alpha K_\varphi^2 E^\varphi}{\sqrt{E^x}} + 2\alpha K_\varphi K_x\sqrt{E^x} + \frac{\alpha E^\varphi}{\sqrt{E^x}} - \frac{\alpha g^2 E^\varphi}{\sqrt{E^x}} - \frac{\kappa\alpha g^2 E^\varphi}{\sqrt{E^x}} - \frac{\kappa'\alpha h\sqrt{E^x}}{\sqrt{1+\kappa}}\right). \quad (28)$$

We now demand that this Hamiltonian Poisson commutes with the LTB conditions in (23). Here we note that the terms containing the spin connection (and its derivative) in the Hamiltonian do not contain K_x or K_φ and therefore in evaluations of the Poisson bracket with the first LTB condition there will be no changes. This leads to the same differential Eq. (24) for $g(E^x)$ as obtained earlier, implying $g(E^x) = \alpha(E^x)$. Evaluating the Poisson bracket of the Hamiltonian with the second condition and using the solution for $g(E^x)$ gives a differential equation for $h(E^x)$

$$\frac{d(\alpha h\sqrt{E^x})}{dE^x} = \frac{\alpha^2}{2\sqrt{E^x}}. \quad (29)$$

Here, we can thus have a consistent formulation for the nonmarginal case for a suitable h by correcting the spin connection terms.

We proceed in a similar manner for the first version of the inverse triad corrections. From the LTB condition for triads in (17a) we find that the spin connection would be replaced with $f(E^x)\Gamma_\varphi^{\rm class}$, and the derivative of the spin connection receives a correction function $l(E^x)$ in the form $l(E^x)(\Gamma_\varphi^{\rm class})'$. With these changes the Hamiltonian is

$$H^{I}_{\rm grav}[N] = -\frac{1}{2G}\int dx N\left(\frac{\alpha K_\varphi^2 E^\varphi}{\sqrt{E^x}} + 2K_\varphi K_x\sqrt{E^x} + \frac{\alpha E^\varphi}{\sqrt{E^x}} - \frac{\alpha f^2 E^\varphi}{\sqrt{E^x}} - \frac{\kappa\alpha f^2 E^\varphi}{\sqrt{E^x}} - \frac{\kappa' l\sqrt{E^x}}{\sqrt{1+\kappa}}\right). \quad (30)$$

Evaluating the Poisson bracket of this with the LTB conditions and equating them to zero implies that the equation for $f(E^x)$ is unchanged compared to (18). The other Poisson bracket then gives a differential equation for $l(E^x)$:

$$\frac{d(l\sqrt{E^x})}{dE^x} = \frac{\alpha f}{2\sqrt{E^x}}. \quad (31)$$

Differential equations for the correction functions h and l are difficult to solve in general for given α and f. For a near center analysis done later we will need the lowest term in h and l in a power series expansion in x. Integrating (29) and (31) keeping only the lowest order term in α and f we find the solution

$$h(x\approx 0)=\left(\frac{2}{\gamma\ell_{\rm P}^2}\right)^{3/2}\frac{R^3}{7},\qquad l(x\approx 0)=\frac{8e^{(1/2)-(\pi/4)}}{5(\gamma\ell_{\rm P}^2)^2}R^4 \tag{32}$$

valid near the center.

2. Ambiguities

The E^x-dependence of α follows from the consideration of inverse triad operators in the full quantum theory. Although it is not determined uniquely in this way (see [31,32] for a discussion), the general shape of this correction function is known well. No such arguments exist for some other correction functions such as h, whose form is thus less clear. More ambiguities are thus expected to arise for it.

A basic condition on multiplicative corrections is that they be scalar to preserve the transformation properties of corrected expressions under changing coordinates. Among the basic triad variables, E^x is the only one free of a density weight and thus can appear in correction functions in an unrestricted way. The other component E^φ, on the other hand, is a density of weight one and would have to appear in combination with other densities to result in a scalar. If only triad components are considered for the dependence, the only other density would be $(E^x)'$. Scalars made from these densities, such as $E^\varphi/(E^x)'$ are however unsuitable for corrections since they are not always finite.

In the present situation, we use the function κ for a nonmarginal LTB model, which means that we have another density, κ', at our disposal. Scalars of the form $(E^x)'/\kappa'$ or E^φ/κ' are well-defined for most functions κ of interest, and can thus arise in corrections. This enlargement of the space of acceptable variables means that additional ambiguities can arise. In the next section we will see how several of these ambiguities can be fixed by an analysis of the constraint algebra. The equations of motion will remain structurally similar, so that we proceed for now with an analysis of the equations resulting from the treatment done so far.

D. Equations of motion

Given the consistency conditions between correction functions we can derive consistent equations of motion even without having explicit solutions for the differential Eqs. (29) and (31). Once consistent constraints are available, the derivation follows the classical lines which we briefly illustrate first. The first order equation (in time) has already been worked out in (14), so that we can go on to the evolution equation. From $\dot{E}^x=\{E^x,H_{\rm grav}^{\rm class}\}$ we have

$$K_\varphi=\frac{\dot{E}^x}{2\sqrt{E^x}}. \tag{33}$$

Similarly, using $\dot{K}_\varphi=\{K_\varphi,H_{\rm grav}^{\rm class}\}$ we obtain

$$\dot{K}_\varphi=\frac{1}{2}\left(\frac{\kappa}{\sqrt{E^x}}-\frac{K_\varphi^2}{\sqrt{E^x}}\right). \tag{34}$$

Eliminating K_φ from the above two equations we obtain

$$\ddot{E}^x=\kappa+\frac{(\dot{E}^x)^2}{4E^x} \tag{35}$$

which, using $E^x=R^2$, can be written as

$$2R\ddot{R}+\dot{R}^2=\kappa. \tag{36}$$

Equations (14) and (36) are automatically consistent, which can be seen explicitly by subtracting a time derivative of (14) from a space derivative of (36).

The same procedure is then applied to constrained systems including consistent correction terms. To get the first order equation in version two we use (28) in the equations of motion $\dot{E}^a=\{E^a,H_{\rm grav}^{II}\}$ to solve for the extrinsic curvature components

$$K_\varphi=\frac{\dot{E}^x}{2\alpha\sqrt{E^x}}\quad\text{and}\quad K_x=\frac{\dot{E}^\varphi}{\alpha\sqrt{E^x}}-\frac{K_\varphi E^\varphi}{E^x}. \tag{37}$$

Using these along with the LTB condition $E^\varphi=(E^x)'/2\sqrt{1+\kappa g}$ in (28) we rewrite the Hamiltonian in terms of E^x only

$$\begin{aligned}H_{\rm grav}^{II}=-\frac{1}{2G}\Bigg(&-\frac{(\dot{E}^x)^2(E^x)'}{8\sqrt{1+\kappa}\alpha^2(E^x)^{3/2}}+\frac{\dot{E}^x(\dot{E}^x)'}{2\sqrt{1+\kappa}\alpha^2\sqrt{E^x}}\\&-\frac{(\dot{E}^x)^2(E^x)'}{2\sqrt{1+\kappa}\alpha^3\sqrt{E^x}}\frac{d\alpha}{dE^x}+(1-\alpha^2-\kappa\alpha^2)\\&\times\frac{(E^x)'}{2\sqrt{1+\kappa}\sqrt{E^x}}-\frac{\kappa'\alpha h\sqrt{E^x}}{\sqrt{1+\kappa}}\Bigg),\end{aligned} \tag{38}$$

where we have already used the condition $g=\alpha$. When equated to the dust Hamiltonian after using $E^x=R^2$ a first order equation in time ensues

$$\begin{aligned}&\frac{\dot{R}^2R'}{\alpha^2}+\frac{2R\dot{R}\dot{R}'}{\alpha^2}-\frac{2R\dot{R}^2R'}{\alpha^3}\frac{d\alpha}{dR}+(1-\alpha^2-\kappa\alpha^2)R'\\&\quad-\kappa'\alpha hR=F'.\end{aligned} \tag{39}$$

To obtain the evolution equation we use

$$\dot{K}_\varphi=\{K_\varphi,H_{\rm grav}^{II}\}=-\frac{1}{2}\frac{\alpha-\alpha g^2-\kappa\alpha g^2+\alpha K_\varphi^2}{\sqrt{E^x}} \tag{40}$$

together with $K_\varphi=\dot{E}^x/2\alpha\sqrt{E^x}$ from (37), such that

$$\ddot{E}^x-\frac{(\dot{E}^x)^2}{4E^x}-\frac{(\dot{E}^x)^2}{\alpha}\frac{d\alpha}{dE^x}=-\alpha^2(1-g^2-\kappa g^2). \tag{41}$$

With $g=\alpha$ and $E^x=R^2$ this becomes

$$2R\ddot{R} + \left(1 - 2\frac{d\log\alpha}{d\log R}\right)\dot{R}^2 = -\alpha^2(1 - \alpha^2 - \kappa\alpha^2). \quad (42)$$

It is easy to see that this equation has the correct classical limit and [using (29)] is consistent with the first order equation.

Proceeding in a similar manner for the first version of the inverse triad correction we find that the first order equation is

$$\alpha\dot{R}^2R' + 2\dot{R}\dot{R}'R + \alpha(1 - f^2 - \kappa f^2)R' - \kappa' flR = fF' \quad (43)$$

and the evolution equation

$$2R\ddot{R} + \alpha\dot{R}^2 = -\alpha(1 - f^2 - \kappa f^2). \quad (44)$$

Using (31) along with (24) one can verify explicitly that the first order and the second order equations are consistent with each other.

E. Effective density

To interpret effects from correction terms it is often useful to formulate them in terms of effective densities rather than new terms in equations of motion. As in the marginal case, we use the Misner-Sharp mass defined by

$$m = \tfrac{R}{2}(1 - \nabla_A R\nabla^A R) \quad (45)$$

where $A = (1, 2)$ corresponds to the $t - r$ manifold. Writing the metric for spherical dust collapse as

$$ds^2 = -dt^2 + L^2(t, x)dx^2 + R^2(t, x)d\Omega^2 \quad (46)$$

the equation for the Misner-Sharp mass becomes

$$m = \frac{R}{2}\left(1 + \dot{R}^2 - \frac{R'^2}{L^2}\right). \quad (47)$$

Classically $R'^2/L^2 = 1 + \kappa$ and therefore the Misner-Sharp mass is $m = (R\dot{R}^2 - \kappa R)/2$. With an effective density defined in terms of the Misner-Sharp mass by

$$\epsilon_{\rm eff} = \frac{m'}{4\pi GR^2R'} \quad (48)$$

we find that for the classical collapse it is

$$\epsilon_{\rm eff}^{\rm class} = \frac{F'}{8\pi GR^2R'}. \quad (49)$$

Here we have made use of the equation of motion $\dot{R}^2R = \kappa R + F$ which is obtained from the Hamiltonian constraint. This is in agreement with the 00 component of Einstein's equation, $G_{00} = 8\pi G\epsilon(t, x)$ where $\epsilon(t, x)$ is the dust density, implying $\epsilon = F'/8\pi GR^2R'$. Thus, classically the effective density, defined in terms of the Misner-Sharp mass, is the same as the dust density. Moreover, as expected, the expressions for the two are unchanged compared to those for the marginal case.

We now proceed in the same way to find the effective density for the first version of the inverse triad correction. With the new LTB condition $E^\varphi = (E^x)'/2\sqrt{1 + \kappa}f(E^x)$, the metric coefficient $L \equiv E^\varphi/\sqrt{E^x}$ implies $L = R'/\sqrt{1 + \kappa}f(R)$ using $E^x = R^2$. Therefore the Misner-Sharp mass as defined in (47) is now

$$m^I = \tfrac{R}{2}(1 + \dot{R}^2 - (1 + \kappa)f^2). \quad (50)$$

The corresponding effective density as implied by (48) is

$$\epsilon_{\rm eff}^I = \frac{1}{8\pi GR^2}\left(\frac{fF'}{R'} + (\alpha - 1)(3f^2 + 3\kappa f^2 - \dot{R}^2 - 1) + \frac{\kappa' flR}{R'} - \frac{\kappa' f^2R}{R'}\right) \quad (51)$$

where we have made use of (18) after substituting for E^x in terms of R, and of (43). We note that this equation has the correct classical limit.

For the second version of the inverse triad correction we have $E^\varphi = (E^x)'/2\sqrt{1 + \kappa}g(E^x)$, implying $L = R'/\sqrt{1 + \kappa}g(R)$ where as seen earlier $g(R) = \alpha(R)$. With this the Misner-Sharp mass is

$$m^{II} = \tfrac{R}{2}(1 + \dot{R}^2 - (1 + \kappa)g^2). \quad (52)$$

Using the relation $\alpha' = R'd\alpha/dR$ (where the prime denotes derivative with respect to x) along with (39) we find that the effective density is

$$\epsilon_{\rm eff}^{II} = \frac{1}{8\pi G}\left(\frac{\alpha^2F'}{R^2R'} + \frac{1 - \alpha^2}{R^2}(1 - \alpha^2 - \kappa\alpha^2) + \frac{2}{\alpha R}(\dot{R}^2 - \alpha^2 - \kappa\alpha^2)\frac{d\alpha}{dR} + \frac{\kappa'\alpha^3h}{RR'} - \frac{\kappa'\alpha^2}{RR'}\right). \quad (53)$$

As in the marginal case, these effective densities imply that the near center expansion for the mass function F can have different behavior compared to the classical case, as discussed in Sec. VI. The matter contribution to the effective density, as given by the first terms of (51) and (53), is the same as in the marginal case.

F. Quantum correction to the energy function κ?

Physically one would expect that the energy function κ, which is related to the velocity of the dust cloud, should also receive corrections after including quantum effects. To derive those, we have to find an independent definition of κ referring only to the constraints or evolution equations derived from them. One possibility, in the classical case, is to use (36) whose right-hand side only contains the energy function. Once brought into an analogous form, a corrected evolution equation can directly be used to read off a corrected energy function. Specifically for version one, where the evolution equation is given by (44), the effective energy function is

$$\kappa_{\rm eff}^I = \alpha f^2\kappa - \alpha(1 - f^2) \quad (54)$$

while for version two, where the evolution equation is given by (42), the effective energy function becomes

$$\kappa_{\rm eff}^{II} = \alpha^4\kappa - \alpha^2(1-\alpha^2). \tag{55}$$

This correction in effect would imply that the near center expansion for κ can be different for the quantum corrected equations as we will see when we come to the near center analysis below.

IV. SPHERICALLY SYMMETRIC CONSTRAINTS

We have now several versions of consistent sets of equations of motion for nonmarginal LTB models including inverse triad corrections as expected from loop quantum gravity. To make these equations consistent, we had to introduce several correction functions in different terms of the Hamiltonian constraint, which were then related to each other by consistency conditions following from the requirement that the LTB conditions be preserved. Since there is some freedom in choosing the places and forms of corrections in the constraint as well as the LTB conditions, one may question how reliable such an analysis is regarding the structure of resulting equations of motion or implications for gravitational collapse.

Before analyzing corrected equations of motion further, we now present an independent derivation which starts with a consistent set of corrected spherically symmetric constraints, and then implements the LTB reduction. As we will see, the structure of the resulting equations is nearly unchanged, while much less assumptions about different corrections are required. With these two procedures we thus demonstrate the robustness of consistently including corrections at a phenomenological level. Note that this would not have been possible had we chosen to fix the gauge generated by the Hamiltonian constraint in any way instead of dealing with the anomaly issue head on.

A. Gravitational variables and constraints

Quantum corrections due to inverse powers of the densitized triad are introduced in the Hamiltonian constraint (4) by functions which we initially assume to be of the general form $\alpha(E^x, E^\varphi)$ and $\bar\alpha(E^x, E^\varphi)$ entering the Hamiltonian constraint as

$$H_{\rm grav}^Q[N] = -\frac{1}{2G}\int dx N(\alpha|E^x|^{-1/2}K_\varphi^2E^\varphi + 2s\bar\alpha K_\varphi K_x|E^x|^{1/2} + \alpha|E^x|^{-1/2}E^\varphi - \alpha_\Gamma|E^x|^{-1/2}\Gamma_\varphi^2E^\varphi + 2s\bar\alpha_\Gamma\Gamma_\varphi'|E^x|^{1/2}). \tag{56}$$

To account for possible corrections from the quantization of the spin connection, as suggested by the previous analysis, we have also introduced functions $\alpha_\Gamma(E^x, E^\varphi)$ and $\bar\alpha_\Gamma(E^x, E^\varphi)$ in those terms. The only restriction so far is that we have the same α in the first and third term of the Hamiltonian constraint due to their common origin from the inverse $|E^x|^{-1/2}$. The two main cases of interest here are $\bar\alpha = 1$ or $\bar\alpha = \alpha$, corresponding to two versions of inverse triad corrections.

We now proceed to make the corrected constraints anomaly-free before implementing LTB conditions. (For a similar analysis for dilaton gravity, see [4].) To ensure anomaly-freedom, we must determine conditions under which the system of constraints, including its corrections in the Hamiltonian constraint, remains first class. Computing the Poisson bracket $\{H_{\rm grav}^Q[M], H_{\rm grav}^Q[N]\}$ gives

$$\{H_{\rm grav}^Q[M], H_{\rm grav}^Q[N]\} = D_{\rm grav}[\bar\alpha\bar\alpha_\Gamma|E^x|(E^\varphi)^{-2}(MN'-NM')] - G_{\rm grav}[\bar\alpha\bar\alpha_\Gamma|E^x|(E^\varphi)^{-2}(NM'-MN')\eta'] + \frac{1}{2G}\int dx(MN'-NM')(\bar\alpha\alpha_\Gamma - \alpha\bar\alpha_\Gamma)\frac{sK_\varphi(E^x)'}{E^\varphi} + \frac{1}{2G}\int dx(MN'-NM')(\bar\alpha'\bar\alpha_\Gamma - \bar\alpha\bar\alpha_\Gamma')\times\frac{2K_\varphi|E^x|}{E^\varphi}. \tag{57}$$

For a first class algebra the last two terms, which are not related to constraints, must vanish, providing conditions on the correction functions. The last term vanishes if, among other possibilities,$\bar\alpha_\Gamma \propto \bar\alpha$, upon which the third term gives $\alpha_\Gamma \propto \alpha$. To recover the classical limit, we then have

$$\alpha_\Gamma = \alpha, \qquad \bar\alpha_\Gamma = \bar\alpha. \tag{58}$$

In this case, anomaly freedom is realized with corrections to the spin connection terms to be only due to the inverse power of the densitized triad factors they contain. This may look contradictory to what we derived earlier, where additional correction functions such as h were needed. However, the previous case (where LTB conditions were used instead of the diffeomorphism constraint) implicitly makes h dependent on $(E^x)'$ as well. Comparing the correction terms we have

$$-\frac{\kappa'}{\sqrt{1+\kappa}}\alpha h = 2\bar\alpha_\Gamma\Gamma_\varphi' = -2\bar\alpha_\Gamma\left(\frac{1}{2}\frac{\kappa'}{\sqrt{1+\kappa}}g[E^x] + \sqrt{1+\kappa}\frac{dg[E^x]}{dE^x}(E^x)'\right)$$

and we can write, using $g = \alpha$

$$h = \bar{\alpha}_\Gamma + 2\frac{1+\kappa}{\kappa'}\frac{d\log\alpha}{dE^x}(E^x)'.$$

Thus, to match the current equations the correction function h used earlier must depend on $(E^x)'$, which has a density weight. (Similar considerations apply to the correction function l.) As the expression demonstrates, this is made possible since in our earlier procedure we had the function κ at our disposal in addition to the triad components. Its derivative κ' provides an extra density, which can be combined with $(E^x)'$ to provide a scalar correction function. In the current setting, by contrast, we have not yet introduced any such function by LTB conditions, and so a possible dependence on $(E^x)'$ is more restricted. The new procedure of this section is clearly less ambiguous, while the final results will be very close. This again demonstrates the robustness.

To continue with the analysis of anomaly-freedom, we compute Poisson brackets

$$\{H^Q_{\rm grav}[N], D_{\rm grav}[N^x]\} = -H^Q_{\rm grav}[N^xN'] - \frac{1}{2G}\int dx N(N^x)'E^\varphi\Big(\frac{\partial\alpha}{\partial E^\varphi}|E^x|^{-1/2}K_\varphi^2E^\varphi + 2s\frac{\partial\bar{\alpha}}{\partial E^\varphi}K_\varphi K_x|E^x|^{1/2} + \frac{\partial\alpha}{\partial E^\varphi}|E^x|^{-1/2}E^\varphi - \frac{\partial\alpha}{\partial E^\varphi}|E^x|^{-1/2}\Gamma_\varphi^2E^\varphi + 2s\frac{\partial\bar{\alpha}}{\partial E^\varphi}\Gamma'_\varphi|E^x|^{1/2}\Big). \tag{59}$$

In the case $\bar{\alpha} = \alpha$,

$$\{H^Q_{\rm grav}[N], D_{\rm grav}[N^x]\} = -H^Q_{\rm grav}[N^xN' - (\partial\log\alpha/\partial E^\varphi)E^\varphi N(N^x)'].$$

The corrected Hamiltonian $H^Q_{\rm grav}$ transforms as a scalar only if α is independent of E^φ since E^φ is the only basic quantity of density weight one. However, the vacuum algebra is first class even if α depends on E^φ. In contrast, when $\bar{\alpha} = 1$ (or more generally $\bar{\alpha} \neq \alpha$), α must be independent of E^φ. (The case $\alpha = \bar{\alpha}$ in vacuum is special because any such correction could be absorbed in the lapse function, making the algebra formally first class.)

In summary, for corrections α (and $\bar{\alpha}$) independent of E^φ we have

$$\{H^Q_{\rm grav}[N], D_{\rm grav}[N^x]\} = -H^Q_{\rm grav}[N^xN']$$

and

$$\{H^Q_{\rm grav}[M], H^Q_{\rm grav}[N]\} = D_{\rm grav}[\bar{\alpha}^2|E^x|(E^\varphi)^{-2}(MN' - NM')] - G_{\rm grav}[\bar{\alpha}^2|E^x|(E^\varphi)^{-2}\times(MN' - NM')\eta']. \tag{60}$$

To proceed, we will include matter in the form of dust as it is assumed in LTB models.

B. Dust

For a full consistency analysis based on the constraint algebra we have to use a dynamical formulation of the dust matter source, rather than a phenomenological implementation via the dust profile $F(x)$. It is convenient to use a canonical formulation for dust with stress-energy tensor $T_{\alpha\beta} = \epsilon U_\alpha U_\beta$ as developed in [33]. The dust four-velocity is given by the Pfaff form $U_\alpha = -\tau_{,\alpha} + W_kZ^k{}_{,\alpha}$, where as canonical coordinates the dust proper time τ and comoving dust coordinates Z^k with $k = 1, 2, 3$ appear. Their respective conjugate momenta will be called P and P_k. Matter contributions to the diffeomorphism and Hamiltonian constraint read

$$D_{\rm dust}[N^a] = \int d^3xN^a\tilde{D}_a = \int d^3xN^a(P\tau_{,a} + P_kZ^k{}_{,a})$$
$$H_{\rm dust}[N] = \int d^3xN\sqrt{P^2 + q^{ab}\tilde{D}_a\tilde{D}_b}. \tag{61}$$

Imposing spherical symmetry and using adapted coordinates $\Phi := Z^1$, $Z^2 = \vartheta$, $Z^3 = \varphi$ the constraints become

$$D_{\rm dust}[N^x] = 4\pi\int dxN^x(P_\tau\tau' + P_\Phi\Phi')$$
$$H_{\rm dust}[N] = 4\pi\int dxN\sqrt{P_\tau^2 + \frac{|E^x|}{(E^\varphi)^2}(P_\tau\tau' + P_\Phi\Phi')^2} \tag{62}$$

with the remaining canonical pairs

$$\{\tau, P_\tau\} = \{\Phi, P_\Phi\} = \frac{1}{4\pi}$$

whose momenta P_τ and P_Φ are defined by the relations $P = P_\tau\sin\vartheta$ (in terms of the P of the full three-dimensional theory) and $P_\Phi = -P_\tau W_1$.

For nonrotating dust, as must be the case with spherical symmetry, the constraints $P_k = 0$ can be imposed by requiring that the dust motion be described with respect to the frame orthogonal foliation, so that the state does not depend on the frame variables Z^k. As a result P_Φ is usually taken to be zero. However, we will not choose to do so until we try to solve the equations of motion.

From the form of the Hamiltonian (61) in the full theory and $q^{ab} = (\det E^c_k)^{-1}E^a_iE^b_i$, we can expect quantum corrections $\beta[E^x, E^\varphi]$ from a quantization of inverse triads inside the square root

$$H^Q_{\rm dust}[N] = 4\pi\int dxN\sqrt{P_\tau^2 + \beta\frac{|E^x|}{(E^\varphi)^2}(P_\tau\tau' + P_\Phi\Phi')^2}.$$

Also here, the form of β will be restricted by the requirement of anomaly freedom.

Adding the individual contributions, the diffeomorphism and corrected Hamiltonian constraint for the gravity-dust system are $D[N^x] = D_{\rm grav}[N^x] + D_{\rm dust}[N^x]$ and $H^Q[N] = H^Q_{\rm grav}[N] + H^Q_{\rm dust}[N]$. Now, the Poisson bracket for the matter part of the Hamiltonian with the diffeomorphism constraint is

$$\{H^Q_{\rm dust}[N], D[N^x]\} = -H^Q_{\rm dust}[N^x N'] + \int dx N (N^x)' \frac{\partial \beta}{\partial E^\varphi} \times \frac{|E^x|}{2E^\varphi} \frac{\tilde{D}_x^2}{\sqrt{P_\tau^2 + \beta |E^x| (E^\varphi)^{-2} \tilde{D}_x^2}}$$

with $\tilde{D}_x := P_\tau \tau' + P_\Phi \Phi'$. The closure of $\{H^Q[N], D[N^x]\}$ consistently imposes the condition that α and β be independent of E^φ, upon which

$$\{H^Q[N], D[N^x]\} = -H^Q[N^x N'].$$

Finally,

$$\begin{aligned}\{H^Q[N], H^Q[M]\} &= \{H^Q_{\rm grav}[M], H^Q_{\rm grav}[N]\} + \{H^Q_{\rm dust}[M], H^Q_{\rm dust}[N]\} \\ &= D_{\rm grav}[\bar{\alpha}^2 |E^x| (E^\varphi)^{-2} (MN' - NM')] \\ &\quad - G_{\rm grav}[\bar{\alpha}^2 |E^x| (E^\varphi)^{-2} (MN' - NM')\eta'] + D_{\rm dust}[\beta |E^x| (E^\varphi)^{-2} \times (MN' - NM')]\end{aligned}$$

gives the relation

$$\beta = \bar{\alpha}^2. \tag{63}$$

Note that the presence of matter makes this consistent deformation of the classical constraint algebra nontrivial: corrections can no longer be absorbed in the lapse function. We also point out that a deformation of the constraint algebra is required to implement the corrections consistently. This seems to be an interesting difference to a reduced phase space quantization which is possible in this class of models based on a deparametrization [34].

C. LTB-like solutions

Using the transformation Eqs. (7), the quantum corrected Hamiltonian in ADM variables (5) reads

$$H^Q[N] = \frac{1}{G} \int dx N \left[-\bar{\alpha} \frac{P_L P_R}{R} + (2\bar{\alpha} - \alpha) \frac{L P_L^2}{2R^2} - \alpha \frac{L}{2} - (2\bar{\alpha} - \alpha) \frac{R'^2}{2L} + \bar{\alpha} \left(\frac{R R'}{L} \right)' + 4\pi G P_\tau \left(1 + \beta \frac{\tau'^2}{L^2} \right)^{1/2} \right]$$

and the diffeomorphism constraint is

$$D[N^x] = \frac{1}{G} \int dx N^x (R' P_R - L P_L' + 4\pi G P_\tau \tau')$$

where we have already used $P_\Phi = 0$. Paralleling our treatment of LTB-reduced constraints our further analysis will be split into two different cases of correction functions. We choose to work here in ADM variables instead of triad variables, but of course identical results follow using the latter.

1. Case $\bar{\alpha} = \alpha$

The equations of motion $\dot{R} = \{R, H^Q[N] + D[N^x]\}$, $\dot{L} = \{L, H^Q[N] + D[N^x]\}$, $\dot{P}_R = \{P_R, H^Q[N] + D[N^x]\}$, $\dot{P}_L = \{P_L, H^Q[N] + D[N^x]\}$, $\dot{\tau} = \{\tau, H^Q[N] + D[N^x]\}$, and $\dot{P}_\tau = \{P_\tau, H^Q[N] + D[N^x]\}$ are, respectively,

$$P_L = \frac{R}{\alpha N} (-\dot{R} + N^x R') \tag{64}$$

$$P_R = \frac{1}{\alpha N} (-L\dot{R} - \dot{L} R + (N^x R L)') \tag{65}$$

$$\begin{aligned}\dot{P}_R = &-N\alpha \left(\frac{P_L P_R}{R^2} - \frac{L P_L^2}{R^3} \right) - N \frac{d\alpha}{dR} \left(-\frac{P_L P_R}{R} + \frac{L P_L^2}{2R^2} - \frac{L}{2} \right) - \left(N\alpha \frac{R'}{L} \right)' + N'\alpha \frac{R'}{L} - \left(N'\alpha \frac{R}{L} \right)' + N' \frac{d\alpha}{dR} \\ &\times \frac{R R'}{L} + \left(\frac{d^2\alpha}{dR^2} R + \frac{d\alpha}{dR} \right) \frac{N R'^2}{L} - \left(2N \frac{d\alpha}{dR} \frac{R R'}{L} \right)' \\ &+ (N^x P_R)' - 2\pi G N P_\tau \left(1 + \beta \frac{\tau'^2}{L^2} \right)^{-1/2} \frac{\tau'^2}{L^2} \frac{d\beta}{dR}\end{aligned} \tag{66}$$

$$\dot{P}_L = N\alpha \left(\frac{1}{2} - \frac{P_L^2}{2R^2} - \frac{R'^2}{2L^2} \right) - (N\alpha)' \frac{R R'}{L^2} + N^x P_L' + 4\pi G N \beta P_\tau \frac{\tau'^2}{L^3} \left(1 + \beta \frac{\tau'^2}{L^2} \right)^{-1/2} \tag{67}$$

$$\dot{\tau} = N \left(1 + \beta \frac{\tau'^2}{L^2} \right)^{1/2} + N^x \tau' \tag{68}$$

$$\dot{P}_\tau = \left[N\beta \frac{P_\tau \tau'}{L^2} \left(1 + \beta \frac{\tau'^2}{L^2} \right)^{-1/2} + N^x P_\tau \right]'. \tag{69}$$

To try to find an LTB-like solution, following [17] and the recent [34], we choose an embedding by coordinates such that $\tau = t$, or equivalently, from (68), $N = 1$. Setting $N^x = 0$, Eq. (69) becomes $\dot{P}_\tau = 0$ so that $P_\tau(x)$ is a function of the spatial coordinate only.

Substituting this and (64) with (65) in the diffeomorphism constraint

$$\frac{\delta D}{\delta N^x} = R' P_R - L P_L' + 4\pi G P_\tau \tau' = 0$$

gives the equation

$$\left(\frac{R'}{\alpha L}\right)^{\cdot} = 0$$

or $R'/\alpha L = \mathcal{E}$, with $\mathcal{E}(x)$ an arbitrary function of the spatial coordinate only. To make contact with the classical LTB solution we may define κ by

$$\mathcal{E} = \sqrt{1+\kappa} \tag{70}$$

and obtain the corrected LTB condition

$$L = \frac{R'}{\alpha\mathcal{E}}.$$

Note that this is exactly the form of corrections used earlier in (23), which we now have derived from the corrected constrained system without extra assumptions about the possible form of LTB solutions.

With all this, the Hamiltonian constraint

$$\alpha\left[-\frac{P_L P_R}{R} + \frac{LP_L^2}{2R^2} - \frac{L}{2} - \frac{R'^2}{2L} + \left(\frac{RR'}{L}\right)'\right] + 4\pi G P_\tau = 0$$

becomes

$$\left[\frac{R\dot{R}^2}{\alpha^2} + R(1-\alpha^2\mathcal{E}^2)\right]' = 8\pi G\mathcal{E}P_\tau. \tag{71}$$

Again, we may define F by the equation $8\pi G\mathcal{E}P_\tau = F'$, so finally the integrated Hamiltonian constraint reads

$$R\dot{R}^2 = \alpha^2 F + R\alpha^2(\alpha^2\mathcal{E}^2 - 1) + c(t) \tag{72}$$

and the second order, evolution equation from (67) is

$$2R\ddot{R} + \dot{R}^2 = 2(\dot{R}^2 + \alpha^4\mathcal{E}^2)\frac{d\log\alpha}{d\log R} - \alpha^2(1-\alpha^2\mathcal{E}^2). \tag{73}$$

This is precisely the time derivative of (72), provided $c(t) = c = \text{const}$, and shows consistency. In the limit $\alpha = 1$ we recover the classical LTB condition and evolution equation. In this limit the integration constant can be absorbed in F so we may set $c = 0$ here.

Even though the corrected LTB condition coincides with the one derived earlier, the first order and evolution equations are slightly different from the corresponding ones (39) and (42) derived before:

$$\left(\frac{R\dot{R}^2}{\alpha^2}\right)' + R'(1-\alpha^2\mathcal{E}^2) - 2\mathcal{E}\mathcal{E}'\alpha hR = F' \tag{74}$$

which unlike (71) is not a spatial derivative, and the second order equation

$$2R\ddot{R} + \dot{R}^2 = 2\dot{R}^2\frac{d\log\alpha}{d\log R} - \alpha^2(1-\alpha^2\mathcal{E}^2). \tag{75}$$

While several terms in the two sets of the equations agree, the spatial derivative of (72) differs from (74) by a term $2\mathcal{E}\mathcal{E}'\alpha R(h-\alpha) + 2\mathcal{E}^2R\alpha\alpha'$, and (73) from (75) by $2\alpha^4\mathcal{E}^2 d\log\alpha/d\log R$. It is interesting to note that the more restrictive new method of this section provides a Hamiltonian constraint which can be spatially integrated.

2. Case $\bar{\alpha} = 1$

Similarly to the previous case, choosing $N = 1$ and $N^x = 0$ gives the equations of motion

$$P_L = -R\dot{R} \tag{76}$$

$$P_R = -(RL)^{\cdot} - (1-\alpha)L\dot{R} \tag{77}$$

$$\dot{P}_L = \frac{\alpha}{2} + (2-\alpha)\left(-\frac{P_L^2}{2R^2} - \frac{R'2}{2L^2}\right) \tag{78}$$

$$\dot{P}_R = \left(\frac{LP_L^2}{R^3} - \frac{P_LP_R}{R^2}\right) - \left(\frac{R'}{L}\right)' + \frac{d\alpha}{dR}\left(\frac{LP_L^2}{2R^2} + \frac{L}{2} - \frac{R'^2}{2L}\right) + (1-\alpha)\frac{LP_L^2}{R^3} - \left((1-\alpha)\frac{R'}{L}\right)'. \tag{79}$$

Substituting the first two equations in the diffeomorphism constraint gives

$$\left(\frac{R'}{L}\right)^{\cdot} = (1-\alpha)\frac{R'\dot{R}}{RL}.$$

Inserting the ansatz (17a)

$$L = \frac{R'}{\mathcal{E}f}$$

gives the same Eq. (18) for the function $f(E^x)$ as in the previous treatment

$$R\frac{df}{dR} = f(1-\alpha). \tag{80}$$

Substituting Eqs. (76) and (77) into the Hamiltonian constraint gives

$$(2-\alpha)R'\dot{R}^2 + 2R\dot{R}\dot{R}' - 2RR'\dot{R}\frac{\dot{f}}{f} + \alpha R' - \alpha R'\mathcal{E}^2f^2 - R(\mathcal{E}^2f^2)' = 8\pi G\mathcal{E}fP_\tau$$

or using (80) and defining $F' = 8\pi G\mathcal{E}P_\tau$

$$\alpha R'\dot{R}^2 + 2R\dot{R}\dot{R}' + \alpha R'(1-\mathcal{E}^2f^2) - R(\mathcal{E}^2f^2)' = fF' \tag{81}$$

which reproduces the classical equation in the limit $\alpha = f = 1$.

The second order equation from (78) is

$$2R\ddot{R} + \alpha\dot{R}^2 = -\alpha(1+\mathcal{E}^2f^2) + 2\mathcal{E}^2f^2. \tag{82}$$

Also here, compared to (43) and (44) as obtained earlier:

$$\alpha R'\dot{R}^2 + 2R\dot{R}\dot{R}' + \alpha R'(1-\mathcal{E}^2f^2) - R(\mathcal{E}^2)'fl = fF'$$

$$2R\ddot{R} + \alpha\dot{R}^2 = -\alpha(1-\mathcal{E}^2f^2),$$

the structure of the resulting equations remains similar up to a few extra terms.

As in Secs. III E and III Fwe may interpret the effects of correction terms using effective densities and energy functions.

V. POSSIBILITY OF SINGULARITY RESOLUTION THROUGH BOUNCES

We can now use the different sets of consistent equations to analyze properties of gravitational collapse, such as the formation of singularities. Classically we have the first order equation

$$\dot{R}^2 = \kappa + \frac{F}{R}. \tag{83}$$

Compared with the marginal case where $\kappa = 0$, there exists the possibility that $\dot{R} = 0$ even for positive mass functions F. However, to conclude whether there is a bounce or not we also need to look at the evolution equation

$$2R\ddot{R} + \dot{R}^2 = \kappa \tag{84}$$

and see whether we can have $\ddot{R} > 0$ in addition to $\dot{R} = 0$. From the first equation we see that $\dot{R} = 0$ implies $\kappa + F/R = 0$ and with both F and R positive we get $\kappa < 0$; bounces would be possible only for negative κ. On the other hand, for $\dot{R} = 0$ the second order equation implies $2R\ddot{R} = \kappa$. Since $R > 0$ and $\kappa < 0$ we conclude that $\ddot{R} < 0$ and thus there is no bounce classically.

We would like to proceed in a similar manner for the quantum corrected case also. However, there the first order equation in time contains terms with spatial derivative as well (which cannot be integrated out in all cases). Therefore the analysis for the possibility of a bounce cannot necessarily be done as easily as for the classical case, a feature clearly related to the fact that we are dealing with inhomogeneous models. Furthermore, because of the inhomogeneous nature of the problem a bounce also makes the analysis difficult by the fact that after the bounce there will be the possibility of shell crossing (unless shells with larger values of x bounce at larger values of R).

We note that (39) and (43) imply, as for the classical case, that $\dot{R} = 0$ is possible in both versions of inverse triad corrections. Whether this corresponds to a bounce is what we want to analyze. We start by putting $\dot{R} = 0$ in (39) which gives

$$(1 - \alpha^2 - \kappa\alpha^2)R' - \kappa'\alpha h R = F'. \tag{85}$$

Similarly, the evolution Eq. (42) becomes

$$2R\ddot{R} = -\alpha^2(1 - \alpha^2 - \kappa\alpha^2). \tag{86}$$

Using (85) on the right of the above equation we have

$$\ddot{R} = -\alpha^2 \frac{F' + \kappa'\alpha h R}{2RR'}. \tag{87}$$

For a bounce, $\ddot{R} > 0$ which implies that $-\alpha^2(F' + \kappa'\alpha h R)/2RR'$ should be greater than zero. We need to check whether this condition can be satisfied in the nonmarginal case where, as mentioned before, there are two possibilities $\kappa > 0$ and $-1 < \kappa < 0$. In what follows we will assume that $R' > 0$, which locally around a potential bounce is a valid assumption even though a collapsing shell with $x = x_1$, say, would start expanding after it experiences a bounce: when $\dot{R}(t, x_1) = 0$ the radius of this particular shell is not changing with time. The assumption then essentially implies that a shell with $x = x_2 > x_1$, which may still be contracting, does not immediately catch up with the x_1 shell. With this assumption and because R and α are positive, the condition that (87) be positive becomes

$$\kappa' < -F'/\alpha h R \tag{88}$$

as a condition on κ' which needs to be satisfied for a bounce, implying, in particular, that for a bounce κ' has to be negative. Whether (88) can be satisfied generically is not clear and must be determined from a numerical analysis of the equations.

For the first version of the inverse triad correction, $\dot{R} = 0$ in (43) gives

$$\alpha(1 - f^2 - \kappa f^2)R' - \kappa' f l R = fF'. \tag{89}$$

The evolution Eq. (44) becomes

$$2R\ddot{R} = -\alpha(1 - f^2 - \kappa f^2) \tag{90}$$

and using (89) on the right we get

$$\ddot{R} = -f \frac{F' + \kappa' l R}{2RR'}. \tag{91}$$

This should be greater than zero for a bounce. Again because of the presence of spatial derivatives in the above expressions it is difficult to say whether there is a bounce in general and whether or not the singularity can be avoided.

With (72) we have a corrected equation which can be spatially integrated, allowing an analysis similar to the classical one. The condition $\dot{R} = 0$ at a bounce implies

$$R = \frac{F}{1 - \alpha^2\mathcal{E}^2} = \frac{F}{1 - \alpha^2 - \kappa\alpha^2}$$

which is positive provided $\mathcal{E} < 1/\alpha$. The second derivative

$$2R\ddot{R}|_{\dot{R}=0} = 2\alpha^4 \frac{d\log\alpha}{d\log R}\mathcal{E}^2 - \alpha^2(1 - \alpha^2\mathcal{E}^2)$$

can be positive under this condition only if the derivative $d\log\alpha/d\log R$ is sufficiently positive. This is not the case in semiclassical regimes (for geometries to the right of the peak in Fig. 1), where bounces are thus prohibited. The correction function is increasing to the left of the peak, which is a regime with strong quantum geometry corrections. Since we have not included all quantum corrections, a conclusion of a bounce in this regime would be unreliable. The only semiclassical option for a bounce is to have a geometry above the peak of inverse triad corrections, but have decreasing patch sizes Δ which appear as the argument of α. Thus, the number $\mathcal{N}$ of patches would have to increase sufficiently rapidly. In this regime, we have $\alpha > 1$ and thus $\mathcal{E} < 1$ by our condition for $R > 0$. Such a bounce would thus be possible only for $\kappa < 0$. (As the argument shows, without lattice refinement a bounce from inverse triad corrections would at best be possible only in the strong quantum regime.)

While bounces seem possible in the present situation, they cannot be considered generic. They require a regime where the patch number is increasing sufficiently rapidly in such a way that the patch size decreases. Since in the discrete quantum geometry of loop quantum gravity the patch size has a positive lower bound, the patch number of an orbit of fixed size cannot increase arbitrarily. Tuning then seems required to have the right behavior just when a shell is about to bounce.

A. Near center analysis regarding bounces

On the basis of a general analysis it seems difficult to conclude whether quantum corrections generically resolve the singularity in nonmarginal LTB models through bounces. The simplest possibility seems to be one where the central shell is prevented from becoming singular because of a bounce. For almost complete collapse, we should expect the relevant regime to be one of small R. In this case the subsequent study of the outer shells will become difficult due to possible shell crossings, but presumably these outer shells will not become singular either. We therefore now proceed to a near center analysis.

As in the marginal case of [3], we use techniques similar to those in [35] and assume that near the center of the dust cloud we can expand $R(t, x)$ as

$$R(t, x) = R_1(t)x + R_2(t)x^2 + \cdots. \tag{92}$$

For the classical collapse the mass function can be expanded as

$$F(x) = F_3x^3 + F_4x^4 + \cdots. \tag{93}$$

Substituting the expansion for $R(t, x)$ and for $F(x)$ in the classical first order equation $\dot{R}^2R = \kappa R + F$, we find that the lowest order term on the left (as also the second term on the right) of the above equation goes as x^3. This suggests that the series expansion for the energy function $\kappa(x)$ should be

$$\kappa(x) = \kappa_2x^2 + \kappa_3x^3 + \cdots. \tag{94}$$

It also implies that at $x = 0$, the center of the cloud, $\kappa(x) = 0$ and therefore for the case where $\kappa(x) > 0$ outside $x = 0$, κ_2 should be greater than zero. On the other hand, for $-1 < \kappa(x) < 0$, κ_2 should be less than zero. However, if we consider our effective κ as in (54) then we can have lower order terms in κ. Since the lowest order term in α is of order x^3 and that in f is of order x we can have the lowest order term in κ behave as x^{-3}. This would then imply that at the center κ blows up whereas classically for negative κ we have the condition $-1 < \kappa < 0$.

With this caveat we now consider (89) to lowest order after using various series expansions

$$c_1R_1^3x^3\left(1 - c_2^2R_1^2x^2 - \frac{c_2^2\kappa_{-3}R_1^2}{x}\right)R_1 + 3c_2c_3\kappa_{-3}R_1^6x^2 = 2c_2F_2R_1x^2. \tag{95}$$

Here c_1, c_2, c_3, and κ_{-3} are the coefficients of the lowest order terms in the expansion of α, f, l, and κ, respectively. Here we assume the orbital vertex number $\mathcal{N}$ to be nearly constant around the center. (This gives rise to the strongest effect from inverse triad corrections and allows direct comparisons with the matching results from [13].) The lowest order term in F is F_2 instead of F_3 because of the effective density correction. For $x \approx 0$ the first two terms in the parenthesis on the left can be ignored compared to the third term and thus the above equation implies that $\dot{R}_1 = 0$ for

$$\kappa_{-3} = \frac{2c_2F_2}{(3c_2c_3 - c_1c_2^2)R_1^5}. \tag{96}$$

It turns out that $3c_2c_3 - c_1c_2^2 < 0$ implying that $\dot{R}_1$ can be zero only for $\kappa_{-3} < 0$ as in the classical case. If we now look at (90) near the center we find that the condition for a bounce is

$$\kappa_{-3} > \frac{2F_2}{3c_3R_1^5} \tag{97}$$

which means that, as in the classical case, κ should be positive implying that we do not have a bounce for the central shell.

For version two the expression for the effective density (53) and the effective energy function (55), respectively, imply that to lowest order the mass function can behave as x^{-3} and the energy function as x^{-10}. To look at the possibility of a bounce at the center of the cloud consider (85) to lowest order

$$R_1 - c_1^2R_1^7x^6 - \frac{\kappa_{-10}c_1^2R_1^7}{x^4} + \frac{10\kappa_{-10}c_1c_4R_1^7}{x^4} = -\frac{3F_{-3}}{x^4}. \tag{98}$$

Here c_4, F_{-3}, and κ_{-10} are the coefficients in the expansion of h, F, and κ, respectively. Ignoring the first two terms for $x \approx 0$, the above equation implies

$$\kappa_{-10} = -\frac{3F_{-3}}{(10c_1c_4 - c_1^2)R_1^7} \tag{99}$$

as the condition for $\dot{R}_1 = 0$. We note that the denominator here is positive implying that $\kappa_{-10} > 0$ if $F_{-3} < 0$ (negative mass function near the center) and $\kappa_{-10} < 0$ if $F_{-3} > 0$ (positive but decreasing mass function near the center). None of these behaviors could occur classically; either negative total energy ($F < 0$) or a negative density ($F' < 0$) would be required. To see if the above condition implies a bounce we use (86) to lowest order and find

$$\kappa_{-10} > -\frac{3F_{-3}}{10c_1c_4R_1^7} \tag{100}$$

as the condition for getting a bounce. This means that for $F_{-3} < 0$, κ_{-10} has to be positive (in agreement with the condition for $\dot{R}_1 = 0$ found above) implying that a bounce for the central shell is possible if the above inequality is satisfied. For $F_{-3} > 0$, κ_{-10} has to be greater than a negative number and thus if it is positive then a bounce again seems possible.

VI. COLLAPSE BEHAVIOR NEAR THE CENTER

Using (92)–(94) we see that to lowest order in x (which is x^3) the classical equation $\dot{R}^2R = \kappa R + F$ implies

$$dt = \pm\frac{dR_1}{\sqrt{\kappa_2 + \frac{F_3}{R_1}}}. \tag{101}$$

This has the solution (choosing the minus sign which corresponds to collapse)

$$t = -\frac{R_1\sqrt{\kappa_2 + \frac{F_3}{R_1}}}{\kappa_2} + \frac{F_3}{2\kappa_2^{3/2}} \times \log\left(F_3 + 2\kappa_2R_1 + 2\sqrt{\kappa_2}\sqrt{\kappa_2 + \frac{F_3}{R_1}}R_1\right) \tag{102}$$

for $\kappa_2 > 0$ and

$$t = \frac{R_1\sqrt{\frac{F_3}{R_1} - |\kappa_2|}}{\kappa_2} + \frac{F_3}{2|\kappa_2|^{3/2}} \times \arctan\left[\frac{\sqrt{\frac{F_3}{R_1} - |\kappa_2|}(2|\kappa_2|R_1 - F_3)}{2\sqrt{|\kappa_2|}(\kappa_2R_1 - F_3)}\right] \tag{103}$$

for $\kappa_2 < 0$.

We now proceed with a similar analysis for the first version of the inverse triad correction. Near the center the various quantities (α, f, l) behave as

$$\alpha = \left(\frac{2}{\gamma l_{\mathrm{P}}^2}\right)^{3/2}R_1^3x^3, \qquad f = \sqrt{\frac{8e^{1-\pi/2}}{\gamma l_{\mathrm{P}}^2}}R_1x, \qquad l = \frac{1}{5}\left(\frac{2}{\gamma l_{\mathrm{P}}^2}\right)^{3/2}\sqrt{\frac{8e^{1-\pi/2}}{\gamma l_{\mathrm{P}}^2}}R_1^4x^4 \tag{104}$$

where the way the near center behavior for l has been determined is described around Eq. (32). In what follows we will denote the constants appearing in the expansion of (α, f, l) by (c_1, c_2, c_3), respectively. Thus substituting the series expansions in (43) we get

$$c_1(1 - c_2^2R_1^2x^2 - \kappa_2c_2^2R_1^2x^4)R_1^4x^3 + c_1\dot{R}_1^2R_1^4x^5 + 2\dot{R}_1^2R_1x^2 - 2\kappa_2c_2c_3R_1^6x^7 = 3c_2F_3R_1x^3. \tag{105}$$

We now consider three different possibilities.

A. Case I: No correction to the expansion of F and κ

If we work with (105) directly then we find that the lowest order term on the left-hand side is $2\dot{R}_1^2R_1x^2$ and the lowest order term on the right-hand side goes as x^3 implying $\dot{R}_1 = 0$.

B. Case II: Modification to the expansion of F and no modification to κ

However because of the presence of an extra factor of x on the right-hand side we can start the expansion of F with the leading term behaving as x^2. In this case the lowest order term on both the left-hand side and right-hand side are of order x^2 and we get

$$2\dot{R}_1^2R_1x^2 = 2c_2F_2R_1x^2 \tag{106}$$

which has the solution (for collapsing dust cloud)

$$R_1(t) = 1 - \sqrt{c_2F_2}(t - t_0) \tag{107}$$

where we choose the initial condition $R_1(t = t_0) = 1$. We see that the central singularity $R_1 = 0$ forms in a finite time $t_s = (1 + \sqrt{c_2F_2}t_0)/\sqrt{c_2F_2}$.

C. Case III: Modifications to the series expansion of F and κ

There is a third option which is suggested by the possibility of a corrected energy function as discussed earlier. If we consider this correction then the lowest order term in the expansion of κ goes as κ_{-3}/x^3. With this included the matching of lowest order terms in (105) gives

$$-\kappa_{-3}c_1c_2^2R_1^5 + 2\dot{R}_1^2 + 3\kappa_{-3}c_2c_3R_1^5 = 2c_2F_2 \tag{108}$$

and thus

$$dt = -\frac{\sqrt{2}dR_1}{\sqrt{2c_2F_2 + (\kappa_{-3}c_1c_2^2 - 3\kappa_{-3}c_2c_3)R_1^5}} \tag{109}$$

with the solution

$$t = -\frac{\sqrt{2}R_1\sqrt{1+\frac{(\kappa_{-3}c_1c_2^2-3\kappa_{-3}c_2c_3)R_1^5}{2c_2F_2}}F_1(\frac{1}{5},\frac{1}{2},\frac{6}{5},-\frac{(\kappa_{-3}c_1c_2^2-3\kappa_{-3}c_2c_3)R_1^5}{2c_2F_2})}{\sqrt{2c_2F_2+(\kappa_{-3}c_1c_2^2-3\kappa_{-3}c_2c_3)R_1^5}} + c_0 \tag{110}$$

where $2F_1(a, b; c; x)$ is the hypergeometric function and where c_0 is constant of integration.

VII. HOMOGENEOUS LIMIT AND MINKOWSKI SPACE

In the classical case, the first order equation is $\dot{R}^2R = \kappa(x)R + F(x)$ with corresponding expression for the mass function $F' = 8\pi GR^2R'\epsilon$. This allows isotropic space-times as special solutions. We use the ansatz $R(t, x) = a(t)x$ with the condition that at time $t = 0$, $a(0) = a_0$ and assume that the density $\epsilon = \epsilon_0$ is a constant and that the energy function goes as $\kappa = -kx^2$ where k is a constant. When used in the first order equation, this gives

$$\dot{a}^2a = -ka + \frac{8\pi G\epsilon_0}{3} \tag{111}$$

which is the Friedmann equation.

We would now like to see if we can get a Friedmann-like solution within the LTB class with inverse triad corrections included. This would indicate whether there can be an effective geometry of the classical homogeneous form, although the notion of homogeneity itself might change on a quantum space-time. Using the ansatz for R and the assumed form for the mass function and the energy function we find that for the first version of the inverse triad correction (43) we get

$$\frac{\alpha(1-f^2)}{f}a + k\alpha f^2ax^2 + \frac{\alpha\dot{a}^2ax^2}{f} + \frac{2a\dot{a}^2x^2}{f} + 2klax^2 = 8\pi G\epsilon_0x^2. \tag{112}$$

Although the resulting expression is not as simple as in the marginal case, we can see that Friedmann-like solutions are not possible since x^2 does not cancel from the first term while a is allowed to depend only on t.

The second version of the inverse triad correction (39) gives

$$(1-\alpha^2+k\alpha^2x^2)a + \frac{3\dot{a}^2ax^2}{\alpha^2} - \frac{2\dot{a}^2a^2}{\alpha^3}\frac{d\alpha}{d(ax)} + 2k\alpha hax^2 = 8\pi G\epsilon_0x^2. \tag{113}$$

Again the Friedmann solution is prohibited. Since the first term, which spoils the homogeneous limit, is the same for an analysis based on (72), there is no homogeneous limit in that case, either. One can also see from the first term that no other x-dependent κ, which might implement quantum corrections to the notion of homogeneity, can resolve the nonexistence of homogeneous effective geometries subject to our equations.

The homogeneous limit, as a special case, would also include Minkowski space as the vacuum solution. Classically the first order equation for $F = 0$ and $\kappa = 0$ implies that $R \equiv R(x)$ and we recover the Minkowski space-time. However from (43) we see that after choosing the mass function and the energy function equal to zero the equation becomes

$$\alpha(1-f^2)R' + \alpha\dot{R}^2R' + 2\dot{R}\dot{R}'R = 0 \tag{114}$$

which, due to the presence of the first term, implies that R will be dependent on both (t, x). Even though the equation has the correct classical limit, it is not straightforward to see how the time dependence of R should disappear in the Minkowski limit.

For the second version (39) gives

$$(1-\alpha^2)R' + \frac{\dot{R}^2R'}{\alpha^2} + \frac{2R\dot{R}\dot{R}'}{\alpha^2} - \frac{2R\dot{R}^2R'}{\alpha^3}\frac{d\alpha}{dR} = 0 \tag{115}$$

which again implies that even though in the classical limit we do have a Minkowski solution, there is still time dependence in R in corrected solutions.

Strong corrections are suggested at small values of the argument of α, which, given that R determines that value, may seem unacceptable because the center in Minkowski space is not physically distinguished. However, the radius R and thus the center is directly relevant for the size of corrections only if there is no lattice refinement in which case the only parameter which α depends on is R. The primary argument of α is, however, the size Δ of discrete patches rather than R^2 of whole spherical orbits. With a nontrivial refinement scheme, $\alpha(R^2/\mathcal{N})$ will also depend on the number of vertices per orbit, which for a good semiclassical state must provide a more uniform distribution of quantum corrections not distinguishing a center: $\mathcal{N}$ must be small when R^2 is small. If discrete patch sizes on all orbits are nearly similar, quantum corrections are uniform and do not distinguish a center. A detailed discussion would go beyond the scope of this paper, but one can already see the crucial role played by lattice refinement for the correct semiclassical limit.

VIII. MATCHING OF THE LOOP CORRECTED LTB INTERIOR WITH GENERALIZED VAIDYA EXTERIOR

To verify the corrected mass formulas from a different perspective, it is instructive to match our corrected models of dust collapse to radiative generalized Vaidya solutions. The interior metric for an inverse triad corrected nonmarginal LTB model can be written in the form

$$ds^2 = -dt^2 + \frac{(R')^2}{(1+\kappa)q^2}dx^2 + R^2 d\Omega^2 \qquad (116)$$

where q takes the appropriate form depending on whether the correction function corresponds to first version or the second version. Following [13], we match this interior solution with the generalized Vaidya metric

$$ds^2 = -\left(1 - \frac{2M}{\chi}\right)dv^2 + 2dvd\chi + \chi^2 d\Omega^2 \qquad (117)$$

where M is allowed to be a function of v as well as χ.

Let the boundary of the dust cloud be at $x = x_b$. Here, the exterior coordinates will then be functions $v \equiv v_b(t)$, $\chi \equiv \chi_b(t)$ of the interior coordinate t such that

$$dv = \dot{v}_b dt, \qquad d\chi = \dot{\chi}_b dt. \qquad (118)$$

With this, the metric induced from the exterior on the matching surface becomes

$$ds^2 = -\left(\left(1 - \frac{2M}{\chi}\right)\dot{v}_b^2 - 2\dot{v}_b\dot{\chi}_b\right)dt^2 + \chi_b^2 d\Omega^2. \qquad (119)$$

Matching the first fundamental form at the boundary (where $dx = 0$ in the interior coordinates) we get

$$-dt^2 + R_b^2 d\Omega^2 = -\left(\left(1 - \frac{2M}{\chi_b}\right)\dot{v}_b^2 - 2\dot{v}_b\dot{\chi}_b\right)dt^2 + \chi_b^2 d\Omega^2 \qquad (120)$$

where R_b is the value of the area radius at the boundary. Equating the coefficients of dt^2 and $d\Omega^2$, respectively, on the two sides implies

$$\chi_b = R_b \qquad \left(1 - \frac{2M}{\chi_b}\right)\dot{v}_b^2 - 2\dot{v}_b\dot{\chi}_b = 1. \qquad (121)$$

We now match the second fundamental forms (extrinsic curvature) at the boundary. For this we need the normal to the boundary in the interior as well as the exterior coordinates. In the interior we have $dx = 0$ which implies that the normal $n^{\rm i}_\mu = (0, a, 0, 0)$ where a is a constant fixed by the normalization $g^{\mu\nu}_{\rm i} n^{\rm i}_\mu n^{\rm i}_\nu = 1$ (the label i standing for "interior"). This gives $a = R'_b/q_b\sqrt{1+\kappa}$ and the normal in the interior is $n^{\rm i}_\mu = (0, R'_b/q_b\sqrt{1+\kappa}, 0, 0)$. Evaluating the $\theta\theta$ component of extrinsic curvature $K^{\rm i}_{\mu\nu} = n^{\rm i}_{\mu;\nu}$ we find

$$K^{\rm i}_{\theta\theta} = q_b R_b\sqrt{1+\kappa}. \qquad (122)$$

From (118), at the boundary, we have $dv - \dot{v}_b dt = 0$, $dt = \frac{d\chi}{\dot{\chi}_b}$, and thus

$$-\dot{\chi}_b dv + \dot{v}_b d\chi = 0. \qquad (123)$$

In terms of the Vaidya coordinates for the exterior, the normal to the boundary is given by $n^{\rm e}_\mu = (-c\dot{\chi}_b, c\dot{v}_b, 0, 0)$ where c is a constant fixed again by the requirement $g^{\mu\nu}_{\rm e} n^{\rm e}_\mu n^{\rm e}_\nu = 1$. Using the inverse of the metric in the exterior we find that $c = 1$. The $\theta\theta$ component of the extrinsic curvature in the exterior is

$$K^{\rm e}_{\theta\theta} = -\chi_b\dot{\chi}_b + \chi_b\left(1 - \frac{2M}{\chi_b}\right)\dot{v}_b. \qquad (124)$$

Equating (122) and (124) and using (121) to simplify the resulting expression we obtain

$$q_b\sqrt{1+\kappa} = -\dot{R}_b + \left(1 - \frac{2M}{R_b}\right)\dot{v}_b \qquad (125)$$

implying

$$\dot{v}_b = \frac{q_b\sqrt{1+\kappa} + \dot{R}_b}{1 - \frac{2M}{R_b}}. \qquad (126)$$

Using this and its square, in the second line of (121) we have

$$2M = (1 - q_b^2(1+\kappa) + \dot{R}_b^2)R_b. \qquad (127)$$

At this stage, we can note that the expression for the Vadiya mass resulting from matching is the same as the one obtained for the effective Misner-Sharp mass in (50) and (52).

The exterior region can contain trapped surfaces when the condition $2M = \chi$ is satisfied. Using the first expression in (121) along with (127) this implies that the boundary will be trapped when $(1 - q_b^2(1+\kappa) + \dot{R}_b^2)R_b = R_b$, or

$$\dot{R}_b = -q_b\sqrt{1+\kappa} \qquad (128)$$

where the negative sign has been chosen since we have a collapsing scenario. This formula shows that the horizon condition is corrected by q_b simply by multiplying $1+\kappa^2$ as it is suggested by the effective metric (116).

IX. CONCLUSIONS

We have extended the treatment of [3] to nonmarginal models, where additional corrections from spin connection terms arise. With these additional terms the original derivation appears more arbitrary, which led us to provide an independent derivation of equations corrected by the treatment of inverse triads in loop quantum gravity. In this new derivation, anomaly freedom is implemented first and LTB conditions are imposed afterwards to select a special class of solutions. The structure of the resulting equations is very similar in both derivations, showing the robustness. By the alternative method, which is much less arbitrary than the one extended from [3] to nonmarginal models, we thus show that the more phenomenological treatment of corrections used in [3] is reliable. In details, however, the resulting equations do differ which is always possible due to quantization ambiguities. The effects analyzed in this paper do not appear to depend sensitively on the method, but further analysis may well provide restrictions on acceptable equations, and thus on quantization ambiguities.

Our analysis in this paper has been done for inverse triad corrections, while holonomy corrections, which to some degree were treated in the marginal case of [3], are technically more involved. Already for inverse triad corrections, the extension provided here is an interesting step in the analysis of inhomogeneous collapse and singularities. Comparing with homogeneous models and matching results of [13] would suggest easy resolutions of singularities by bounces. Marginal models were not entirely conclusive in this regard since their homogeneous analog is that of a spatially flat Friedmann-Robertson-Walker model which under inverse triad corrections gives rise to bounces only with a negative matter potential [36]. In the collapse analysis, however, we have used positivity conditions for the mass function which indicate that bounces in marginal LTB models with inverse triad corrections should not be expected. For nonmarginal models, on the other hand, homogeneous special solutions with positive spatial curvature exist, which do show bounces with inverse triad corrections and positive matter terms [12]. One would thus expect nonmarginal models to result in bounces much more easily than marginal ones do.

This, however, is not the case: we mostly confirm the results found in marginal models where (i) bounces are not obvious and (ii) a homogeneous limit of quantum corrected solutions may not even exist. As for the first property, bounces seem somewhat easier to achieve than in marginal models, but in contrast to the expectation turn out to be hard to realize generically. Moreover, a complete analysis would have to involve an investigation of shell-crossing singularities which can be involved even classically. (See [3] for more discussions on this in marginal corrected models.) As for property (ii) about the homogeneous limit, one can evade ruling out a homogeneous limit at the dynamical level only if one assumes a quantum notion of symmetry which would imply effective isotropic space-time metrics different from classical Friedmann-Robertson-Walker models. This may well be expected, as indeed the deformed constraint algebra (60) shows that there is a corrected quantum space-time structure. It would be interesting to see how this influences space-time symmetries.

Finally, several issues discussed here involved the role of lattice refinement for the semiclassical limit. As treatable models between homogeneous ones and the full theory, LTB models turn out to be quite instructive. This should also be expected for an implementation at the state (rather than phenomenological) level which was started in [3] for marginal models but which we have not attempted here for nonmarginal models.

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