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Census taking in the hat: FRW/CFT duality

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In this paper a holographic description of eternal inflation is developed. We focus on the description of an open Friedmann-Robertson-Walker (FRW) universe that results from a tunneling event in which a false vacuum with positive vacuum energy decays to a supersymmetric vacuum with vanishing cosmological constant. The observations of a "census taker" in the final vacuum can be organized into a holographic dual conformal field theory that lives on the asymptotic boundary of space. We refer to this bulk-boundary correspondence as FRW/CFT duality. The dual conformal field theory (CFT) is a Euclidean two-dimensional theory that includes a Liouville 2D gravity sector describing geometric fluctuations of the boundary. The renormalization-group flow of the theory is richer than in the AdS/CFT correspondence, and generates two space-time dimensions—one spacelike and one timelike. We discuss a number of phenomena such as bubble collisions, and the Garriga, Guth Vilenkin "persistence of memory," from the dual viewpoint.

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I INTRODUCTION

There are two views of eternal inflation [1]. According to the "global" point of view the entire multiverse is an infinite system of pocket universes populating a landscape [2–5]. The global view raises some difficult questions of principle: for example, in saying the multiverse is a *system*, do we mean a quantum system described by a wave function Ψ? If so, what variables does Ψ depend on? How local is the description? And what relevance does the existence of cosmic event horizons have? If the description is (approximately) local, then what operational meaning can be attached to correlation functions between variables in regions that are out of causal contact?

According to the second local or causal patch viewpoint, [6] cosmology should be formulated without any reference to unobservable events beyond the observer's horizon. In such a formulation the wave function Ψ depends only on the degrees of freedom that have operational meaning to the observer. The biggest question raised by the local viewpoint is what meaning to attach to events beyond the observer's horizon.

In this paper we will focus on the local description of an observer located in a so-called terminal vacuum with an exactly vanishing cosmological constant. Following Shenker, we will call such an observer a census taker (CT). However, we will also give reasons to believe that the global and the local descriptions are both correct, being related by a complementarity principle similar to black hole complementarity.

Obviously the causal past of the census taker contains the most information in the asymptotic late-time limit. Thus, to define the CT's space of states we need to specify the late-time limit of the CT's trajectory. All trajectories in an eternally inflating universe eventually end in some kind of terminal state. The terminal states are easiest to describe if we assume that the mechanism for de Sitter decay is bubble nucleation of the type considered by Coleman and De Luccia [7]. The result of such a decay is always an open, negatively curved, Friedmann-Robertson-Walker (FRW) universe. This paper will expand on the proposal of [8], in which a holographic duality was introduced between an FRW cosmology and a two-dimensional Euclidean conformal field theory (CFT). This duality will be called the FRW/CFT correspondence. The regulator that cuts off the census taker's observations is time: the longer the CT observes, the more he counts. Using the dictionary provided by FRW/CFT, the cutoff may be identified with the ultraviolet regulator of the two-dimensional CFT.

II. THE CENSUS BUREAU

A. Causal patch

Let us begin with a precise definition of a causal patch. Start with a cosmological space-time and assume that a future causal boundary exists. For example, in flat Minkowski space the future causal boundary consists of $\mathcal{I}^+$ (future lightlike infinity) and a single point: timelike infinity. For a Schwarzschild black hole, the future causal boundary has an additional component: the singularity.

A causal patch is defined in terms of a point $\underline{\mathbf{a}}$ on the future causal boundary. We call that point the "census bureau.[1]" By definition, the causal patch is the causal past of the census bureau, bounded by its past light cone. For Minkowski space, one usually picks the census bureau to be timelike infinity. In that case the causal patch is all of the Minkowski space as seen in Fig. 1.

[1] This term originated during a discussion with Steve Shenker.

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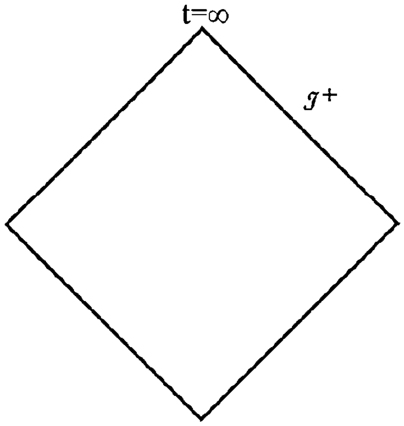

FIG. 1. Conformal diagram for ordinary flat Minkowski space. The causal patch associated with the ‘‘census bureau’’ at $t = \infty$ is the entire space-time.

In the case of the Schwarzschild geometry, **a** can again be chosen to be timelike infinity, in which case the causal patch is everything outside the horizon of the black hole. There is no clear reason why one cannot choose **a** to be on the singularity [9], but it would lead to obvious difficulties.

de Sitter space has the causal structure shown in Fig. 2. In this case, all points at future infinity are equivalent: the census bureau can be located at any of them. However, string theory and other considerations [10] suggest that de Sitter minima are never stable. After a series of tunneling events they eventually end in terminal vacua with exactly zero or a negative cosmological constant. The entire distant future of de Sitter space is replaced by a fractal of terminal bubbles.

Transitions to vacua with negative cosmological constant always lead to singular crunches in which the energy density blows up, or approaches the Planck scale. As in the case of the black hole, we will not consider census bureaus located on singularities. That leaves only the supersymmetric bubbles with zero cosmological constant.[2] Such a bubble evolves to an open, negatively curved, FRW universe bounded by a ‘‘hat’’ [8]. The census bureau is at the tip of the hat.

The term ‘‘census taker’’ denotes an observer in such an FRW universe who looks back into the past and collects data. He can count galaxies, other observers, hydrogen atoms, colliding bubble universes, civilizations, or anything else within his own causal past. As time elapses the census taker sees more and more of the causal patch. Eventually all census takers within the same causal patch arrive at the census bureau where they can compare data.

There are two possible connections between hatted terminal geometries and observational cosmology with a nonzero cosmological constant. First, for many purposes, the current cosmological constant is so small that it can be set to zero. Later we will argue that the conformal field theory description of the approximate hat which results from a nonzero cosmological constant is an ultraviolet incomplete version of the type of field theory that describes a hat.

[2]We will assume that nonsupersymmetric vacua never have exactly zero cosmological constant.

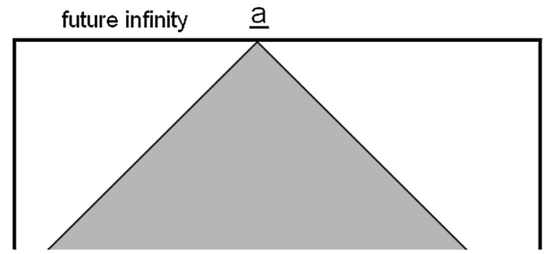

FIG. 2. Conformal diagrams for eternal de Sitter space. The causal past of the census bureau at **a** is shown in grey.

This paper is mainly concerned with the second connection in which hatted geometries are used as probes of eternal inflation. It was emphasized by Maloney, Shenker, and Susskind [11] that because any de Sitter vacuum will eventually decay, a census taker can look back into it from a point at or near the tip of a hat and gather information. In principle the census taker can look back, not only into the ancestor vacuum (our vacuum in this case), but also into bubble collisions with other vacua of the landscape. Much of this paper is about the gathering of information as the census taker’s time progresses, and how it is encoded in the renormalization-group (RG) flow of a holographic field theory.

B. Asymptotic coldness

String theory is a powerful tool in the study of quantum gravity, but only in special backgrounds such as flat space and anti-de Sitter (AdS) space. Effective as it is in describing scattering amplitudes in asymptotically-flat (supersymmetric) space-time, and black holes in anti-de Sitter space, it is an inflexible tool which at present is not useful as a mathematical framework for cosmology. What is it that is so special about flat and anti-de Sitter space backgrounds that allows a rigorous formulation of quantum gravity, and why are cosmological backgrounds so difficult?

The problem is frequently blamed on time dependence, but time-dependent deformations of anti-de Sitter space or matrix theory [12] are easy to describe. Time dependence in itself does not seem to be the problem. There is one important difference between the usual string theory backgrounds and more interesting cosmological backgrounds. Asymptotically-flat and anti-de Sitter backgrounds have a property that we will call *asymptotic coldness*. Asymptotic coldness means that the boundary conditions require the energy density to go to zero at the asymptotic boundary of space-time. Similarly, the fluctuations in geometry tend to zero. This condition is embodied in the statement that all physical disturbances are composed of normalizable modes. Asymptotic coldness is obviously important to defining an S matrix in flat space-time, and plays an equally important role in defining the observables of anti-de Sitter space.

But in cosmology, asymptotic coldness is never the case. Closed universes have no asymptotic boundary, and homogeneous infinite universes have matter, energy, and geometric variation out to spatial infinity. Under this circumstance an S matrix cannot be formulated. String theory at present is ill equipped to deal with asymptotically warm geometries. To put it another way, there is a conflict between a homogeneous cosmology and the holographic principle [13–16] which requires an isolated, cold boundary.

Consider the three kinds of decay products that can occur in eternally inflating space-time: de Sitter bubbles with positive cosmological constant; crunching bubbles with negative cosmological constant; and supersymmetric hatted geometries with zero cosmological constant. de Sitter geometries are never asymptotically cold; the thermal fluctuations continue forever or at least until the de Sitter space decays. Crunches are obviously not good candidates for asymptotic coldness. That leaves vacua with a vanishing cosmological constant bounded by hats.

The geometry under a hat is not asymptotically cold. A negatively curved FRW universe is spatially homogeneous on large scales and not empty. Thus if we fix the time and go out to spatial infinity, conditions do not become cold. However, if the cosmological constant is zero, the temperature and density of matter do tend to zero at asymptotically late times. Although we may not be able to define an S matrix, the late-time universe can be described in terms of a supersymmetric spectrum of free particles described by string theory. This partial asymptotic coldness makes hatted geometries the best candidates for a precise mathematical formulation of eternally inflating cosmology.

III. THE FRW/CFT DUALITY

A. Open FRW universe

The classical space-time in the interior of a Coleman-De Luccia bubble [7] has the form of an open infinite FRW universe. Let $\mathcal{H}_3$ represent a hyperbolic geometry with constant negative curvature,

$$d\mathcal{H}_3^2 = dR^2 + \sinh^2 R d\Omega_2^2. \tag{3.1}$$

The metric of open FRW is

$$ds^2 = -dt^2 + a(t)^2 d\mathcal{H}_3^2, \tag{3.2}$$

or in terms of conformal time T (defined by $dT = dt/a(t)$)

$$ds^2 = a(T)^2(-dT^2 + d\mathcal{H}_3^2). \tag{3.3}$$

Note that in (3.1) the radial coordinate R is a dimensionless hyperbolic angle and that the symmetry of the spatial sections is the noncompact group $O(3, 1)$. This symmetry plays a central role in what follows.

If the vacuum energy in the bubble is zero, i.e., no cosmological constant, then the future boundary of the FRW region is a hat. The scale factor $a(t)$ then has the early and late-time behaviors

$$a(T) \sim t \sim H^{-1} e^{(T+T_0)}, \tag{3.4}$$

where H is the Hubble constant of the ancestor vacuum. For early time when $T \to -\infty$ the constant T_0 is zero,

$$a(T) = H^{-1} e^T \qquad (T \to -\infty). \tag{3.5}$$

In the simplest thin-wall case, T_0 is zero for all time (within the FRW region). In general the sign of T_0 at late time depends on the equation of state at the intermediate stage. If there is an accelerating (decelerating) phase between the early and the late-time phases (3.4), T_0 is positive (negative) at late time.

In Fig. 3, a conformal diagram of FRW is illustrated, with surfaces of constant T and R shown. The colored region represents the de Sitter ancestor vacuum. Figure 4 shows the census taker, as he approaches the tip of the hat, looking back along his past light cone.

The geometry of a spatial slice of constant T is a three-dimensional, negatively curved, hyperbolic plane. It is identical to 3D Euclidean anti-de Sitter space. The two-dimensional analog is well illustrated in Fig. 5 by Escher's drawing "Limit Circle IV." It is both a drawing of Euclidean AdS and also a fixed-time slice of open FRW.

In Fig. 5, the green circle is the intersection of the census taker's past light cone with the time slice. As the census taker advances in time, the green circle moves out toward the boundary.

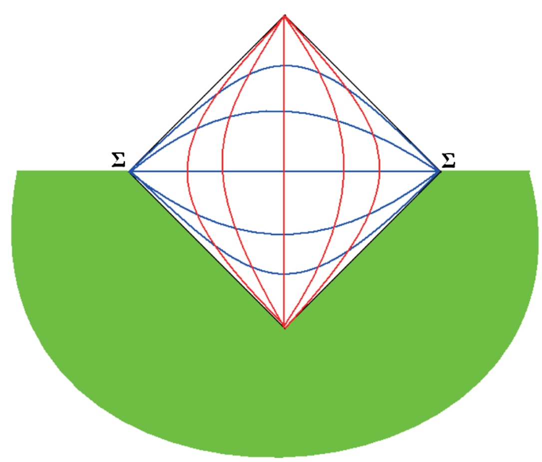

FIG. 3 (color online). A conformal diagram for the FRW universe created by bubble nucleation from an "ancestor" metastable vacuum. The colored region is the ancestor vacuum. The timelike and spacelike curves are surfaces of constant T and R. The two-sphere at spatial infinity is indicated by Σ.

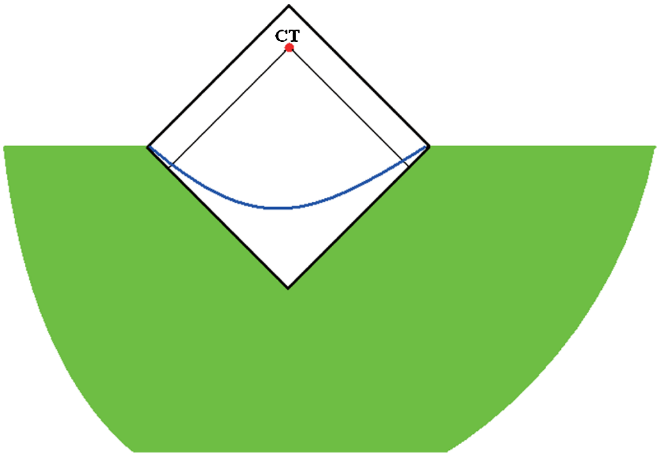

FIG. 4 (color online). The census taker is indicated by the red dot. The thin black lines represent his past light cone and the blue curve is a spacelike surface of constant T.

A fact (to be explained later) which will play a leading role in what follows, concerns the census taker's angular resolution, i.e., his ability to discern small angular variation. If the time at which the CT looks back is called T_{CT}, then the smallest angle he can resolve is of order $\exp(-T_{\mathrm{CT}})$. It is as if the CT were looking deeper and deeper into the ultraviolet structure of a quantum field theory on Σ. This observation motivates an FRW/CFT correspondence.

FIG. 5 (color online). Escher's drawing of the Hyperbolic Plane, which represents Euclidean anti-de Sitter space or a spatial slice of open FRW. The green circle shows the intersection of the census taker's past light cone, which moves toward the boundary with census taker time.

The boundary of anti-de Sitter space plays a key role in the AdS/CFT correspondence, where it represents the extreme ultraviolet degrees of freedom of the boundary theory. The corresponding boundary in the FRW geometry is labeled Σ and consists of the intersection of the hat $\mathcal{I}^+$, with the spacelike future boundary of de Sitter space. From within the interior of the bubble, Σ represents spacelike infinity. It is the obvious surface for a holographic description. As one might expect, the $O(3, 1)$ symmetry which acts on the time slices, also has the action of two-dimensional conformal transformations on Σ. Whatever the census taker sees, it is very natural for him to classify his observations under the conformal group. Thus, the apparatus of (Euclidean) conformal field theory, such as operator dimensions, and correlation functions, should play a leading role in organizing his data. The conjecture of [8] is that there exists an exact duality between the bulk description of the hatted FRW universe and a conformal field theory living on Σ.

In complicated situations, such as multiple bubble collisions, Σ requires a precise definition. The asymptotic light cone $\mathcal{I}^+$ (which is, of course, the limit of the census taker's past light cone), can be thought of as being formed from a collection of lightlike generators. Each generator, at one end, runs into the tip of the hat, while the other end eventually enters the bulk space-time. The set of points where the generators enter the bulk define Σ.

B. Observer complementarity and the census taker

This paper is about a duality between the FRW patch under a hat and two-dimensional conformal field theory. However, it is possible that there is a larger point at stake—a possible complementarity between the census taker's patch and the entire multiverse. The question is whether or not degrees of freedom beyond the census taker's horizon have meaning (we believe the answer is yes) and whether they are independent of the degrees of freedom on the observer's side (in our opinion, no). In the context of black holes the situation is fairly clear by now and is encapsulated in the principle of black hole complementarity. A generalization of black hole complementarity is sometimes called observer complementarity. In this section we will review the complementarity principle [17–19] and then discuss its possible application to the relation between the census taker and the multiverse.

Consider any ordinary quantum system with a Hilbert space $\mathcal{H}$ and a collection of observables (we work in the Heisenberg picture). Pick two times, one in the past and one in the future. Call them t_{in} and t_{out}. For every observable $A(t_{\mathrm{out}})$ (to be measured at t_{out}), there is another observable that can be measured at t_{in} with exactly the same spectrum, and the same probability distribution as $A(t_{\mathrm{out}})$ in any state. To find that operator all we have to do is solve the Heisenberg equations of motion, and express $A(t_{\mathrm{out}})$ as a functional of the operators at t_{in}. An example from elementary quantum mechanics is the free particle on a line. Choose $A(t_{out})$ to be the position of the particle $X(t_{out})$. The corresponding operator at time t_{in} is

$$X(t_{in}) + \frac{P(t_{in})}{M}(t_{out} - t_{in}).$$

The point is not that measuring $X(t_{in}) + \frac{P(t_{in})}{M}(t_{out} - t_{in})$ at time t_{in} is the same thing as measuring $X(t_{out})$ at the later time—it is clearly not—but that in any Heisenberg state, the probability distribution for the two measurements are the same.

To say it another way, imagine preparing a particle in the remote past in some state. We may measure X at time t_{out} or we may measure $X + \frac{P}{M}(t_{out} - t_{in})$ at time t_{in}. The probability distributions for the two experiments are identical.

Thus, by solving the equations of motion one can express $A(t_{out})$ as a functional $A_*(t_{in})$ of operators at time t_{in}. [$A_*(t_{in})$ has the same probability distribution as $A(t_{out})$ and is the same Heisenberg operator as $A(t_{out})$]. A formal expression for $A_*(t_{in})$ is given by

$$A_*(t_{in}) = U^\dagger(t_{out}, t_{in})A(t_{in})U(t_{out}, t_{in}) \qquad (3.6)$$

where $U(t_{out}, t_{in})$ is the usual time development operator from t_{in} to t_{out}.

Now let us consider a process in which a black hole forms and evaporates. We begin by following an in-falling system in "free floating coordinates," such as Eddington-Finkelstein or Painleve-Gullstand coordinates. In such coordinates the in-falling system can be described by ordinary low energy physics as it crosses the horizon, at least until it approaches the singularity. Consider a low energy observable $A(p)$ at a point p, behind the horizon of the black hole (see Fig. 6). One can always find an observable $A_*(t_{in})$ in the remote past, outside the black hole, with the

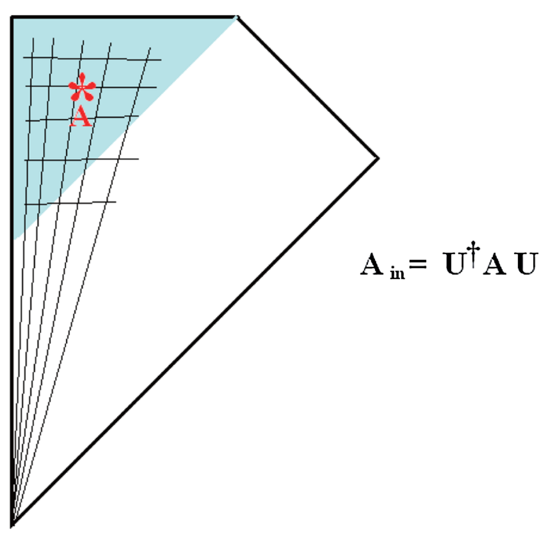

FIG. 6 (color online). A low energy operator A behind the horizon can be evolved into the past. The useful coordinates are freely falling.

same expectation value and probability distribution as $A(p)$. All we need is low energy physics to run the operator backward to t_{in}. Then we get $A_*(t_{in})$ by running $A(t_{in})$ forward with U,

$$A_*(t_{in}) = U^\dagger A(t_{in})U. \qquad (3.7)$$

We emphasize again, that throughout this operation the description in the in-falling frame is governed by conventional low energy quantum field theory. Note that $A_*(t_{in})$ is a function of operators at timelike and lightlike past infinity. In other words it is a function of asymptotic in operators.

Next, assume that in the exterior frame of reference—the frame of an observer who remains outside the horizon—the process is governed by an S matrix, connecting in states to out states. The Hilbert space for this observer should be isomorphic to the one for the in-falling observer, but the operator that describes the same physics will depend on which observer we are considering. In the exterior frame, the operator $A(t_{in})$ evolves into a late-time operator $A(q) = S^\dagger A(t_{in})S$, which consists of the outside degrees of freedom (Hawking radiations). The operator $A_{**}(t_{in})$ which has the same probability distribution as $A(q)$ is given by

$$A_{**}(t_{in}) = S^\dagger A(t_{in})S = S^\dagger U A_*(t_{in})U^\dagger S. \qquad (3.8)$$

In other words there is a mapping from an operator behind the horizon to an operator composed of the outgoing degrees of freedom. The mapping is simply the conjugation by $S^\dagger U$. Of course the mapping is not an easy one to decode. Black holes are very efficient "scramblers" of information [20].

Consider a laboratory falling into a black hole. How much information can that laboratory contain? The answer is obvious: if every operator in the laboratory can also be represented in the Hilbert space of the outgoing evaporation products, then the information in the laboratory cannot exceed the entropy of the black hole [9]. Let us define a concept of information capacity for a system or subsystem.[3] The information capacity is the maximum amount of information that the system can contain. Equivalently it is the maximum entropy of the system. For a quantum system it is the logarithm of the dimensionality of the Hilbert space of states needed to describe the system. What we have argued in the previous paragraph is that the information capacity of any subsystem behind the horizon is bounded by the information capacity of the black hole, i.e., the Bekenstein-Hawking entropy.

The census taker's causal patch is bounded by a horizon. Part of that horizon is the hat itself but it also extends into the bulk geometry as in Fig. 7. It should be clear that the census taker's region is the analog of the exterior of a black hole, and the portion of the multiverse beyond the horizon is the analog of the interior of the black hole. Most of the

[3]Not to be confused with channel capacity.

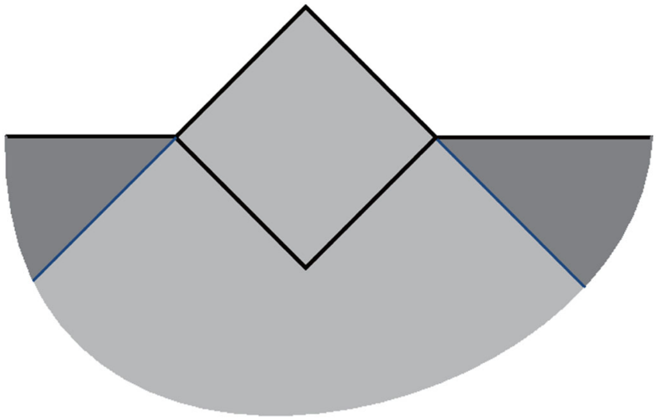

FIG. 7 (color online). The horizon of the CT consists of the hat and its continuation into the bulk.

multiverse is behind the horizon and naively cannot be directly detected by the CT. But there is radiation coming into the CT's patch that is analogous to the Hawking radiation.

The situation is similar to that of information behind the horizon of a black hole and its evaporation products. In the practical sense, the information becomes so scrambled that it is lost to an outside observer. In what follows, we will conjecture a version of observer complementarity that applies to the CT's horizon (even though the logical foundation for the existence of unitary evolution might be weaker, since there is no analog of the formation and evaporation of a black hole).

We take the following assertions as given:

(i) The degrees of freedom accessible to the CT are a complete description of the multiverse.

(ii) The Hilbert space of the CT's patch, i.e., the FRW region under the hat, is isomorphic to the Hilbert space of the multiverse. Operators outside the CT's horizon are complementary to operators within the horizon in the sense that they have the same statistics.

At first sight these assumptions appear to be impossible since the FRW patch of the CT is a proper subset of the multiverse. It seems clear that the CT can never have enough information to decode the multiverse.

However, here is where the current setup differs from a black hole. The black hole can only store a finite amount of information—the entropy of the black hole—and an observer on the outside cannot collect more information than that. But in the case of the census taker, if he waits long enough there is no bound to the amount of information that he can collect. In other words, the information capacity in the CT's patch is infinite. (It is worth noting that the information capacity of a de Sitter space if it were stable would not be infinite.)

The ''bulk'' theory of the multiverse also requires an unbounded number of degrees of freedom to describe it. The phenomenon of eternal inflation will eventually populate the multiverse with an infinite number of events of every kind. For these reasons, counting and comparing the information capacity of the CT's patch and the information capacity of the multiverse can only make sense for regularized versions of the theories.

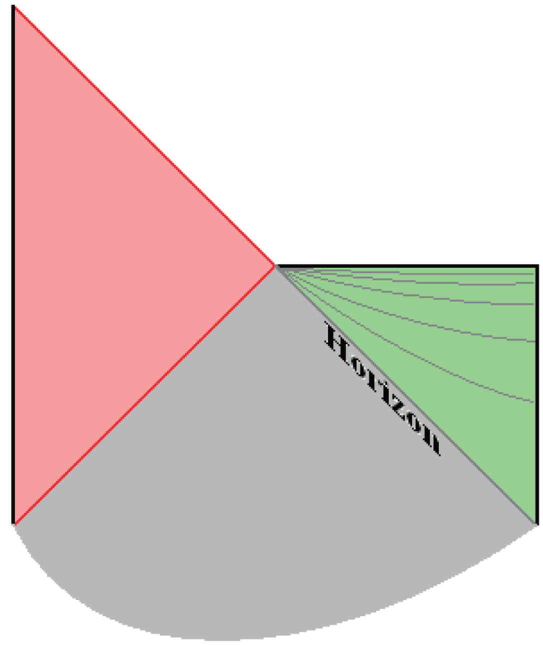

FIG. 8 (color online). The FRW patch and the portion of the ancestor behind the horizon share a common boundary Σ.

There is one more point that is worth mentioning. Consider Fig. 8. The triangular region on the left is the FRW patch and the triangle on the right is the portion of the ancestor vacuum which lies beyond the CT's horizon. Note that the two regions share a common boundary, namely Σ. This suggests that the boundary-holographic theory describing the census taker's patch may also be the holographic description of the rest of the multiverse.

The rest of this paper is about the duality of the boundary CFT and the FRW patch. It is independent of the conjectured observer complementarity relating it to the global description of the multiverse.

IV. THE HOLOGRAPHIC WHEELER-DEWITT EQUATION AND THE FSSY CONJECTURE

The traditional approach to quantum cosmology, the Wheeler-DeWitt equation, is the opposite of string theory: it is very flexible from the point of view of background dependence—it does not require an an asymptotically cold boundary condition, it can be formulated for a closed universe, a flat or open FRW universe, de Sitter space, or for that matter, flat and anti-de Sitter space space-time—but it is not a consistent quantum theory of gravity. It is based on the existence of local bulk degrees of freedom and therefore fails to address the problems that string theory and the holographic principle were designed to solve, namely, the huge overcounting of degrees of freedom implicit in a local field theory.

Freivogel, Sekino, Susskind, and Yeh (FSSY) [8] suggested a way out of the dilemma: synthesize the Wheeler-DeWitt philosophy with the holographic principle to construct a holographic Wheeler-DeWitt (WDW) theory. We will begin with a review of the basics of conventional WDW theory; for a more complete treatment, especially of infinite cosmologies, see [21].

The ten equations of general relativity take the form

$$\frac{\delta}{\delta g_{\mu\nu}} I = 0 \tag{4.1}$$

where I is the Einstein action for gravity coupled to matter. The canonical formulation of general relativity makes use of a time-space split [22]. The six space-space components are more or less conventional equations of motion, but the four equations involving the time index have the form of constraints. These four equations are written,

$$H^{\mu}(x) = 0. \tag{4.2}$$

They involve the space-space components of the metric g_{nm}, the matter fields Φ, and their conjugate momenta. The time component $H^0(x)$ is a local Hamiltonian which "pushes time forward" at the spatial point x. More generally, if integrated with a test function,

$$\int d^3x f(x) H^0(x) \tag{4.3}$$

it generates infinitesimal transformations of the form

$$t \rightarrow t + f(x). \tag{4.4}$$

Under certain conditions H^0 can be integrated over space in order to give a global Hamiltonian description. Since H^0 involves second space derivatives of g_{nm}, it is necessary to integrate by parts in order to bring the Hamiltonian to the conventional form containing only first derivatives. In that case the Arnowitt-Deser-Misner equations can be written as

$$\int d^3x H = E. \tag{4.5}$$

The Hamiltonian density H has a conventional structure, quadratic in canonical momenta, and the energy E is given by a Gaussian surface integral over spatial infinity. The conditions which allow us to go from (4.2) to (4.5) are satisfied in asymptotically cold flat-space time, as well as in anti-de Sitter space; in both cases global Hamiltonian formulations exist. Indeed, in anti-de Sitter space the Hamiltonian of the holographic boundary description is identified with the Arnowitt-Deser-Misner energy, but, as we noted, cosmology, at least in its usual forms, is never asymptotically cold. The only recourse for a canonical description is the local form of Eqs. (4.2).

When we pass from classical gravity to its quantum counterpart, the usual generalization of the canonical Eqs. (4.2) becomes the Wheeler-DeWitt equations,

$$H^{\mu}|\Psi\rangle = 0 \tag{4.6}$$

where the state vector $|\Psi\rangle$ is represented by a wave functional that depends only on the space components of the metric g_{mn}, and the matter fields Φ.

The first three equations

$$H^{m}|\Psi\rangle = 0 \qquad (m = 1, 2, 3) \tag{4.7}$$

have the interpretation that the wave function is invariant under spatial diffeomorphisms,

$$x^n \rightarrow x^n + f^n(x^m). \tag{4.8}$$

In other words $\Psi(g_{mn}, \Phi)$ is a function of spatial invariants. These equations are usually deemed to be the easy Wheeler-DeWitt equations.

The difficult equation is the time component

$$H^{0}|\Psi\rangle = 0. \tag{4.9}$$

It represents invariance under local, spatially varying, time translations. Not only is Eq. (4.9) difficult to solve; it is difficult to even formulate: the expression for H^0 is riddled with factor ordering ambiguities. Nevertheless, as long as the equations are not pushed into extreme quantum environments, they can be useful.

A. Wheeler-DeWitt and the emergence of time

Asymptotically cold backgrounds come equipped with a global concept of time. But in the more interesting asymptotically warm case, time is an approximately derived concept [21,23], which emerges from the solutions to the Wheeler-DeWitt equation. The perturbative method for solving (4.9) that was outlined in [21] can be adapted to the case of negative spatial curvature. We begin by decomposing the spatial metric into a constant curvature background and fluctuations. Since we will focus on open FRW cosmology, the spatial curvature is negative, the space metric having the form

$$ds^2 = a^2(dR^2 + \sinh^2 R(d\theta^2 + \sin^2\theta d\phi^2)) + a^2 h_{mn} dx^m dx^n. \tag{4.10}$$

In (4.10) a is the usual FRW scale factor and the x's are (R, θ, ϕ).

The first step in a semiclassical expansion is the so-called mini-superspace approximation in which all fluctuations are ignored. In lowest order, the Wheeler-DeWitt wave function depends only on the scale factor a. To carry out the leading approximation in open FRW, it is necessary to introduce an infrared regulator which can be done by bounding the value of R,

$$R < R_0 \qquad (R_0 \gg 1). \tag{4.11}$$

Let us also define the total dimensionless coordinate volume within the cutoff region, to be V_0,

$$V_0 = 4\pi \int dR \sinh^2 R \approx \frac{1}{2}\pi e^{2R_0}. \tag{4.12}$$

The first (mini-superspace) approximation is described by the action

$$L = \frac{-aV_0\dot{a}^2 - V_0 a}{2}. \tag{4.13}$$

Defining P to be the momentum conjugate to the scale factor a,

$$P = -aV_0\dot{a} \tag{4.14}$$

the Hamiltonian H is given by [4]

$$H = \frac{1}{2V_0} P \frac{1}{a} P - \frac{1}{2} V_0 a. \tag{4.15}$$

Finally, using $P = -i\partial_a$, the first approximation to the Wheeler-DeWitt equation becomes

$$-\partial_a \frac{1}{a} \partial_a \Psi - V_0^2 a \Psi = 0. \tag{4.16}$$

The equation has two solutions,

$$\Psi = \exp(\pm i V_0 a^2/2), \tag{4.17}$$

corresponding to expanding and contraction universes; to see which is which we use (4.14). The expanding solution, labeled Ψ_0 is

$$\Psi_0 = \exp(-iV_0 a^2/2). \tag{4.18}$$

From now on we will only consider this branch.

At first sight there is something peculiar about (4.18). Multiplying V_0 by a^2 seems like an odd operation. $V_0 a^3$ is the proper volume, but what is $V_0 a^2$? In flat space it has no invariant significance, but in hyperbolic space its geometric meaning is simple: it is just the proper area of the boundary at R_0. One sees from the metric (3.3) that the coordinate volume V_0 and the coordinate area A_0, of the boundary at R_0, are (asymptotically) equal to one another, to within a factor of 2,

$$A_0 = 2V_0. \tag{4.19}$$

Thus the expression in the exponent[5] in (4.18) is $-iA_0/4$, where A_0 is the proper area of the boundary at R_0,

$$\Psi_0 = \exp(-iA_0/4). \tag{4.20}$$

This is very suggestive. It is the first indication of a holographic version of the WDW theory for an open FRW universe.

To go beyond the mini-superspace approximation, one writes the wave function as a product of Ψ_0, and a second factor $\psi(a, h, \Phi)$ that depends on the fluctuations,

$$\Psi(a, h, \Phi) = \Psi_0 \psi(a, h, \Phi) = \exp(-iA_0 a^2/4)\psi(a, h, \Phi). \tag{4.21}$$

By integrating the Wheeler-DeWitt equation over space, and substituting (4.21), an equation for ψ can be obtained,

$$i\partial_a \psi - \frac{1}{A_0}\partial_a \frac{1}{a}\partial_a \psi = H_m \psi. \tag{4.22}$$

In this equation H_m has the form of a conventional Hamiltonian (quadratic in the momenta) for both matter and metric fluctuations.

In the limit of a large scale factor, the term $\frac{1}{A_0}\partial_a \frac{1}{a}\partial_a \psi$ becomes negligible and (4.22) takes the form of a Schrodinger equation,

$$i\partial_a \psi = H_m \psi. \tag{4.23}$$

Evidently the role of a is not as a conventional observable, but a parameter representing the unfolding of cosmic time. One does not calculate its probability, but instead constrains it—perhaps with a delta function or a Lagrange multiplier. As Banks has emphasized [23], in this limit, and maybe only in this limit, the wave function ψ has a conventional interpretation as a probability amplitude.

As we have described it, the Wheeler-DeWitt theory is a throwback to an older view of quantum gravity based on the existence of bulk, space-filling degrees of freedom. It has become clear that this is a serious overestimate of the capacity of space to contain quantum information. The correct (holographic) counting of degrees of freedom is in terms of the area of the boundary of space [13]. The question addressed by FSSY [8] is how to combine the flexibility of the Wheeler-DeWitt theory with the requirements of the holographic principle.

B. The FSSY conjecture

The conjecture of [8] is as follows:

The holographic description of the FRW region under a hat is a Wheeler-DeWitt theory in which the ordinary bulk degrees of freedom are replaced by degrees of freedom that reside on the asymptotic boundary of space, i.e., on Σ. The Wheeler-DeWitt wave function is a functional of those boundary degrees of freedom. Semiclassical bulk degrees of freedom are approximate concepts reconstructed from the precise boundary quantities.

Unlike the AdS case in which the boundary is asymptotically cold, the holographic degrees of freedom include a dynamical boundary metric.

As in the AdS/CFT correspondence [16], it is useful to define a regulated boundary, Σ_0, at $R = R_0$. In the regulated theory the number of degrees of freedom is proportional to the area of the regulated boundary in Planck units.

[4]The factor ordering in the first term is ambiguous. We have chosen the simplest Hermitian factor ordering.

[5]In this subsection we are setting $4\pi G/3$ equal to 1. Including this factor, the wave function is $\Psi_0 = \exp(-i(\frac{3}{4\pi})(\frac{A_0}{4G}))$.

In principle R_0 can depend on the angular location on Ω_2,

$$R_0 = R_0(\Omega_2).$$

In fact, later we will discuss invariance under gauge transformations of the form

$$R \rightarrow R + f(\Omega_2). \tag{4.24}$$

[The notation $f(\Omega_2)$ indicating that f is also a function of location on Σ_0.]

Let us now consider the boundary degrees of freedom. In AdS/CFT the boundary theory is typically a gauge theory and does not include gravitational degrees of freedom. Asymptotic coldness is the statement that the bulk fields are frozen at the boundary and do not fluctuate. But in an asymptotically warm geometry, the boundary geometry will fluctuate. For this reason the boundary degrees of freedom must include a two-dimensional spatial metric on Σ_0. The induced spatial geometry of the boundary can always be described in the conformal gauge in terms of a Liouville field $U(\Omega_2)$,

$$ds^2 = e^{2U(\Omega_2)} e^{2R_0(\Omega_2)} d\Omega_2^2. \tag{4.25}$$

The Liouville field U may be decomposed into a homogeneous term U_0, and a fluctuation. Obviously the homogeneous term can be identified with the FRW scale factor,

$$e^{U_0} = a. \tag{4.26}$$

In Sec. VI, we will give a more detailed definition of the Liouville degree of freedom.

We also postulate a collection of boundary "matter" fields. The boundary matter fields, y, are not the limits of the usual bulk fields Φ, but are analogous to the boundary gauge fields in the AdS/CFT correspondence. In this paper, we will not speculate on the detailed form of these boundary matter fields.

C. The wave function

In addition to U and y, we assume a local Hamiltonian $H(x^i)$ that depends only on the boundary degrees of freedom (the notation x^i refers to coordinates of the boundary Σ), and a wave function $\Psi(U, y)$,

$$\Psi(U, y) = e^{-(1/2)S + iW}. \tag{4.27}$$

At every point of Σ, Ψ satisfies

$$H(x^i)\Psi(U, y) = 0. \tag{4.28}$$

In Eq. (4.27), $S(U, y)$ and $W(U, y)$ are real functionals of the boundary fields. For reasons that will become clear, we will call S the action. However, S should not in any way be confused with the four-dimensional Einstein action.

We make the following three preliminary assumptions about S and W:

(i) Both S and W are invariant under conformal transformations of Σ. This follows from the $O(3, 1)$ symmetry of the background geometry.

(ii) The leading (nonderivative) term in the regulated form of W is $-A/4$ where A is the proper area of Σ_0,

$$W = -\frac{1}{4}\int_\Sigma e^{2R_0} e^{2U} + \ldots \tag{4.29}$$

This follows from (4.20). This term plays a role in determining the expectation values of momentum conjugate to U, but will play secondary roles in this paper.

(iii) S has the form of a local two-dimensional Euclidean action on Σ. In other words it is an integral over Σ, of densities that involve U, y, and their derivatives with respect to x^i.

The first of these conditions is just a restatement of the symmetry of the Coleman-De Luccia (CDL) instanton. Later we will see that this symmetry is broken by a number of effects, including the extremely interesting "persistence of memory" discovered by Garriga, Guth, and Vilenkin (GGV) [24].

The second condition follows from the bulk analysis described earlier in Eq. (4.20). It allows us to make an educated guess about the dependence of the local Hamiltonian $H(x^i)$ on U. A simple form that reproduces (4.20) is

$$H(x) = \tfrac{1}{2} e^{-2U}\pi_U^2 - \frac{e^{4R_0}}{8} e^{2U} + \ldots \tag{4.30}$$

where π_U is the momentum conjugate to U. It is easily seen that the solution to the equation $H\Psi = 0$ has the form (4.20).

The highly nontrivial assumption is the third item—the locality of the action. That the action S is local is far from obvious; as a rule quantum field theory wave functions are not local in this sense. In our opinion item three is the strongest of our assumptions and the one most in need of confirmation. At present our best evidence for the locality is the discrete tower of boundary correlators, including a two-dimensional transverse, traceless, dimension-2 tensor-correlation function, discovered in [8]. This is consistent with a dimension-2 energy-momentum tensor—a necessary condition for a local field theory.

In principle, much more information can be obtained from bulk multipoint functions, continued to Σ. For example, correlation functions of h_{ij} would allow us to study the operator product expansion of the energy-momentum tensor.

The assumption that S is local is a very strong one, but we mean it in a rather weak sense. One of the main points of this paper is that there is a natural RG flow in cosmology (see Sec. VI). By locality we mean only that S is in the basin of attraction of a local field theory. If it is true, locality would imply that the measure

$$\Psi^*\Psi = e^{-S} \tag{4.31}$$

has the form of a local two-dimensional Euclidean field theory with action S, and that the census taker's observations could be organized not only by conformal invariance but by conformal field theory.

V. DATA

The conjectured locality of the action S is based on data calculated by FSSY. The background geometry studied in [8] was the Minkowski continuation of a thin-wall CDL instanton, describing transitions from the ancestor vacuum to a hatted supersymmetric vacuum with zero cosmological constant. In this section, the data of [8] will be reviewed.

We begin with some facts about three-dimensional hyperbolic space and the solutions of its massless Laplace equation. An important distinction is between normalizable modes and non-normalizable modes. A minimally coupled scalar field χ is sufficient to illustrate the important points.

The norm in hyperbolic space is defined in the obvious way:

$$\langle\chi|\chi\rangle = \int dR d\Omega_2 \chi^2 \sinh^2 R. \tag{5.1}$$

In flat space, fields that tend to a constant at infinity are on the edge on normalizability. With the help of the delta function, the concept of normalizability can be generalized to continuum normalizability, and the constant "zero mode" is included in the spectrum of the wave operator, but in hyperbolic space the normalization integral (5.1) is exponentially divergent for constant χ. The condition for normalizability is that $\chi \to 0$ at least as fast as e^{-R}. The constant mode is therefore non-normalizable.

Normalizable and non-normalizable modes have very different roles in the conventional AdS/CFT correspondence. Normalizable modes are dynamical excitations with finite energy and can be produced by events internal to the anti-de Sitter space. By contrast non-normalizable modes cannot be excited dynamically. Shifting the value of a non-normalizable modes is equivalent to changing the boundary conditions from the bulk point of view, or changing the Lagrangian from the boundary perspective. But, as we will see, in the cosmological framework of FSSY, asymptotic warmness blurs this distinction.

The massless scalar correlator in the CDL background stays finite when the points approach the boundary [8]. In other words, non-normalizable modes (which have arbitrary angular dependence, and stay finite near the boundary) are excited. We will call this geometry asymptotic warm, in the sense that we cannot turn off perturbations at spatial infinity.

The reason for this asymptotic warmness is the fact that the Euclidean version of the CDL instanton is compact (even though the spatial slice of the Lorentzian geometry is noncompact). The Euclidean metric is of the form

$$ds^2 = a^2(X)(dX^2 + d\theta^2 + \sin^2\theta\Omega_2^2). \tag{5.2}$$

Note that the Euclidean version of de Sitter space is a four-dimensional sphere (for which $a(X) = H^{-1}/\cosh X$, with $-\infty \le X \le \infty$, where H is the Hubble constant of the ancestor vacuum). The Euclidean CDL geometry in the thin-wall limit is a sphere cut at certain value of X and patched with a flat disc [for which $a(X) \propto H^{-1}e^X$] on the negative X side. The open FRW universe (3.3) is given by the analytic continuation

$$X \to T + \pi i/2, \qquad \theta \to iR, \tag{5.3}$$

from (5.2). (Note $e^X \to ie^T$, and that S^3 goes to $\mathcal{H}_3$).

Correlators are calculated in the Euclidean background (5.2) and analytically continued to the FRW universe, as we will review below.

A. Scalars

Correlation functions of massless (minimally coupled) scalars, χ, depend on time and on the dimensionless geodesic distance between points on $\mathcal{H}_3$. In the limit in which the points tend to the holographic boundary Σ at $R \to \infty$, the geodesic distance between points 1 and 2 is given by

$$l_{1,2} = R_1 + R_2 + \log(1 - \cos\alpha) \tag{5.4}$$

where α is the angular distance on S^2 between 1 and 2. It follows on $O(3, 1)$ symmetry grounds that the correlation function $\langle\chi(1)\chi(2)\rangle$ has the form

$$\begin{aligned}\langle\chi(1)\chi(2)\rangle &= G\{l_{1,2}, T_1, T_2,\} \\ &= G\{R_1 + R_2 + \log(1 - \cos\alpha), T_1, T_2\}.\end{aligned} \tag{5.5}$$

Before discussing the data on the CDL background, let us consider the form of correlation functions for scalar fields in anti-de Sitter space. We work in units in which the radius of the anti-de Sitter space is 1. By symmetry, the correlation function can only depend on l, the proper distance between points. The two-point function has the form

$$\langle\chi(1)\chi(2)\rangle \sim \frac{e^{-(\Delta-1)l}}{\sinh l}. \tag{5.6}$$

In anti-de Sitter space the dimension Δ is related to the mass of χ by

$$\Delta(\Delta - 2) = m^2. \tag{5.7}$$

We will be interested in the limit in which the two points 1 and 2 approach the boundary at $R \to \infty$. Using (5.4) gives

$$\langle\chi(1)\chi(2)\rangle \sim e^{-\Delta R_1} e^{-\Delta R_2}(1 - \cos\alpha)^{-\Delta}. \tag{5.8}$$

It is well known that the "infrared cutoff" R, in anti-de Sitter space, is equivalent to an ultraviolet cutoff in the boundary-Holographic description [16]. The exponential factors, $\exp(-\Delta R)$ in (5.8) have an important quantum field theoretic meaning: they exactly correspond to cutoff dependent wave function renormalization constants. These factors are normally stripped off when defining field theory correlators. However, in this paper we will not remove them.

The remaining factor, $(1-\cos\alpha)^{-\Delta}$ is the conformally covariant correlation function of a boundary field of dimension Δ.

Now let us briefly explain how to calculate the correlator on the CDL background. Equation of motion for massless scalar χ in the Euclidean CDL geometry is

$$\left[-\partial_X^2 + \frac{a''(X)}{a(X)} - \nabla_S^2\right](a(X)\chi) = 0, \tag{5.9}$$

where ∇_S^2 is the Laplacian on S^3. The Euclidean correlator is expressed as

$$\langle\chi(X_1,0)\chi(X_2,\theta)\rangle = \frac{1}{a(X_1)a(X_2)} \times \int_{C_1} \frac{dk}{2\pi} u_k^*(X_1) u_k(X_2) G_k(\theta), \tag{5.10}$$

where $u_k(X)$ are the eigenfunctions of the Schrödinger operator: $[-\partial_X^2 + a''(X)/a(X)]u_k(X) = (k^2+1)u_k(X)$, and $G_k(\theta) = \sinh k(\pi-\theta)/(\sin\theta \sinh k\pi)$ is the Green's function on S^3 with mass (k^2+1). The correlator in the thin-wall limit can be written in terms of the reflection coefficient $\mathcal{R}(k)$ in the potential $a''(X)/a(X)$. $\mathcal{R}(k)$ has a pole at $k=i$, corresponding to a bound state [we can see that $u_B(X) \propto a(X)$ is a bound state with eigenvalue $k=i$]. The integration contour C_1 indicates that the integration is done along the real axis and the contribution from the residue of the pole at $k=i$ is added [which is equivalent to the integration along the contour (a) in Fig. 9].[6] We can easily show that (5.10) indeed satisfies (5.9) with a delta function on the right-hand side.

Performing the analytic continuation (5.3), we get the correlator on the Lorentzian CDL background,

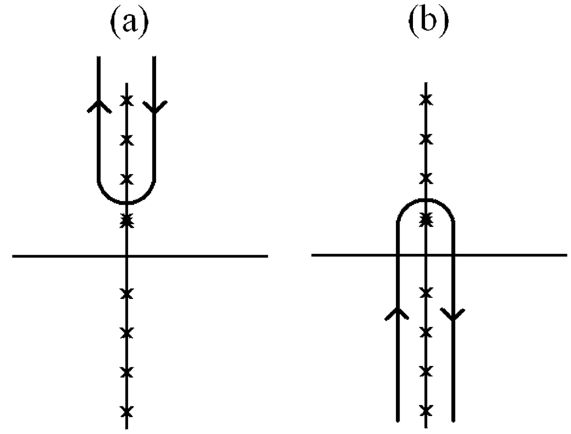

FIG. 9. Contours of integration for the two contributions G_1, G_2.

$$\langle\chi(T_1,0)\chi(T_2,l)\rangle = e^{-(T_1+T_2)} \int_{C_1} \frac{dk}{2\pi} (e^{ik(T_1-T_2+\pi)} + \mathcal{R}(k) e^{-ik(T_1+T_2)}) \frac{\sin kl}{\sinh l \sinh k\pi} \tag{5.11}$$

where l is the geodesic distance on $\mathcal{H}_3$.

The first term, which has nontrivial dependence on $(T_1 - T_2)$, exists even when there is no bubble nucleation. Namely, the FRW region in the thin-wall limit can be thought of as a part of flat Minkowski space (called the Milne universe); the first term gives the correlator in Minkowski space written in hyperbolic coordinates.[7] So we will call this term the flat space piece. The second term, which involves $\mathcal{R}(k)$ depends on the details of bubble nucleation, such as the Hubble constant of the false vacuum and the tension of the domain wall.

In FSSY [8], it was shown that the correlator on the CDL background can be written as a sum of AdS correlators with definite masses (dimensions). By deforming the contour for the k integration, we get a discrete sum over residues at the poles. The integrand has poles at integer multiples of $k=i$, and there is a double pole at $k=i$. In addition $\mathcal{R}(k)$ may have other singularities in the lower half plane.[8]

Let us study the second term of (5.11) which involves $\mathcal{R}(k)$. In the region of our interest ($l\to\infty$), the contour can be closed in the following way:

[6]We take this contour since we have to include the bound state mode in the complete set in the X space. Otherwise, the correlator becomes singular at $X\to-\infty$ (or $T\to-\infty$) even though the background is smooth there. The presence of a bound state in the Euclidean problem corresponds to the presence of non-normalizable modes in the FRW universe as we will see below.

[7]We can do the integral for the first term in (5.11) by contour deformation, and see that it agrees with the massless correlator in flat space, $\langle\chi(1)\chi(2)\rangle = 1/\{(\hat{T}_1-\hat{T}_2)^2 - |\hat{X}_1^a - \hat{X}_2^a|^2\}$, where $\hat{T}$, $\hat{X}$ are the Minkowski coordinates, related to the hyperbolic coordinates by $\hat{T} = e^T \cosh R$, $\hat{X}^a = e^T \sinh R\,\Omega_a$.

[8]Poles in the lower half plane correspond to "virtual states" and resonances, which blow up at $X\to\pm\infty$.

$$e^{-(T_1+T_2)}\left\{\oint_a \frac{dk}{2\pi}\mathcal{R}(k)\frac{e^{-ik(T_1+T_2-l)}}{2\sinh l\sinh k\pi} + \oint_b \frac{dk}{2\pi}\mathcal{R}(k)\frac{e^{-ik(T_1+T_2+l)}}{2\sinh l\sinh k\pi}\right\} \quad (5.12)$$

where the contours $\oint_a$ and $\oint_b$ are shown in Fig. 9. FSSY ignored the contribution from poles in the lower half plane on the basis that they are negligible at late time. In fact those terms have significance that we will come back to, but first we will review the terms studied in FSSY.

First, there is the normalizable contribution, G_1. This is an infinite sum, each term having the form (5.8) with T-dependent coefficients,

$$G_1 = \sum_{\Delta=2}^{\infty} G_\Delta e^{(\Delta-2)(T_1+T_2)}\frac{e^{-(\Delta-1)l}}{\sinh l} \to \sum_{\Delta=2}^{\infty} G_\Delta e^{(\Delta-2)(T_1+T_2)} e^{-\Delta R_1} e^{-\Delta R_2}(1-\cos\alpha)^{-\Delta} \quad (5.13)$$

where Δ takes on integer values from 2 to ∞, and G_Δ are a series of constants which depend on the detailed CDL solution.

The connection with conformal field theory correlators is obvious; Eq. (5.13) is a sum of correlation functions for fields of definite dimension Δ, but with coefficients which depend on the time T. (It should be emphasized that the dimensions Δ in the present context are not related to bulk four-dimensional masses.) Note that the sum in (5.13) begins at $\Delta = 2$, implying that every term falls at least as fast as $\exp(-2R)$ with respect to either argument. Thus every term is normalizable.

Let us now extrapolate (5.13) to the surface Σ. Σ can be reached in two ways—the first being to go out along a constant T surface to $R = \infty$. Each term in the correlator has a definite R dependence which identifies its dimension.

Another way to get to Σ is to first pass to lightlike infinity, $\mathcal{I}^+$, and then slide down the hat, along a lightlike generator, until reaching Σ. For this purpose it is useful to define light cone coordinates, $T^\pm = T \pm R$,

$$G_1 = e^{-(T_1^+ + T_2^+)}\sum_\Delta G_\Delta e^{(\Delta-1)(T_1^- + T_2^-)}(1-\cos\alpha)^{-\Delta}. \quad (5.14)$$

We note that apart from the overall factor $e^{-(T_1^+ + T_2^+)}$, G_1 depends only on T^-, and therefore tends to a finite limit on $\mathcal{I}^+$. If we strip that factor off, then the remaining expression consists of a sum over CFT correlators, each proportional to a fixed power of e^{T^-}. In the limit ($T^- \to -\infty$) in which we pass to Σ, each term of fixed dimension tends to zero as $e^{(\Delta-1)(T_1^- + T_2^-)}$ with the dimension-2 term dominating the others.

The second term in the scalar correlation function discussed by FSSY consists of a single term, which comes from the double pole at $k = i$,

$$G_2 = \frac{e^l}{\sinh l}(T_1 + T_2 + l) \to \{T_1^+ + T_2^+ + \log(1-\cos\alpha)\}. \quad (5.15)$$

The contribution (5.15) does not have the form of a correlator of a conformal field of definite dimension. To understand its significance, consider a canonical massless scalar field in two dimensions. On a two-sphere the correlation function is ultraviolet divergent and has the form

$$\log\{\kappa^2(1-\cos\alpha)\} \quad (5.16)$$

where κ is the ultraviolet regulator momentum. If the regulator momentum varies with location on the sphere—for example in the case of a lattice regulator with a variable lattice spacing—formula (5.16) is replaced by

$$\log\{(1-\cos\alpha)\} + \log\kappa_1 + \log\kappa_2. \quad (5.17)$$

Evidently if we identify the UV cutoff κ with T^+,

$$\log\kappa = T^+, \quad (5.18)$$

the expressions in (5.15) and (5.17) are identical. The relation (5.18) is one of the central themes of this paper, that as we will see, relates RG flow to the observations of the census taker.

That the UV cutoff of the 2D boundary theory depends on R is very familiar from the UV/IR connection [16] in anti-de Sitter space. In that case the T coordinate is absent and the log of the cutoff momentum in the conformal field theory would just be R. The additional time-dependent contribution in (5.18) will become clear later when we discuss the Liouville field.

The logarithmic ultraviolet divergence in the correlator is a signal that massless 2D scalars are ill defined; the well-defined quantities being derivatives of the field.[9] When calculating correlators of derivatives, the cutoff dependence disappears. Thus for practical purposes, the only relevant term in (5.17) is $\log(1-\cos\alpha)$.

The existence of a dimension-zero scalar field on Σ is a surprise. It is obviously associated with bulk field modes which do not go to zero for large R. Such modes are non-normalizable on the hyperbolic plane, and are usually not included among the dynamical variables in anti-de Sitter space.

In string theory the only massless scalars in the supersymmetric hatted vacua would be moduli, which are expected to be "fixed" in the ancestor. For that reason FSSY considered the effect of adding a four-dimensional mass

[9]Exponentials of the field can also have well-defined dimensions. The dependence on the cutoff κ in (5.18) becomes exponentiated. The resulting power law dependence on κ is recognized as the wave function renormalization factor.

term, $m^2\chi^2$, in the ancestor vacuum. The mass term is assumed to vanish in the supersymmetric hatted vacuum.

The result on the boundary scalar was to shift its dimension from $\Delta = 0$ to $\Delta = \mu$ (when mass is small, $\mu \propto m^2$). If m is sufficiently small relative to the ancestor Hubble constant the corresponding mode stays non-normalizable. However the correlation function was not similar to those in G_1, each term of which had a dependence on T^-. The dimension μ term depends only on T^+:

$$G_2 \to e^{-\mu T_1^+} e^{-\mu T_2^+} (1 - \cos\alpha)^{-\mu}. \tag{5.19}$$

The two terms, (5.14) and (5.19) depend on different combinations of the coordinates, T^+ and T^-. It seems odd that there is one and only one term that depends solely on T^+ and all the rest depend on T^-. In fact the only reason for this was that FSSY ignored an entire tower of higher dimension terms, which, like (5.19), depend only on T^+. These terms all come from the contour **b** in Fig. 9.

From now on we will group all terms independent of T^- into the single expression G_2:

$$G_2 = \sum_{\Delta'} \tilde{G}_{\Delta'} e^{-\Delta'(T_1^+ + T_2^+)} (1 - \cos\alpha)^{-\Delta'}. \tag{5.20}$$

The Δ' include μ, the positive integers, and contributions from whatever other poles in the lower half plane. In the case $\mu = 0$, the leading term in G_2 is (5.15).

Finally there is the flat space piece coming from the first term of (5.11),

$$G_{\rm flat} = \frac{e^{-(T_1+T_2)}}{2\sinh l} \sum_{n=1}^{\infty} (e^{-nT_1^+} e^{nT_2^-} + e^{nT_1^-} e^{-nT_2^+}). \tag{5.21}$$

Here the T^+ dependence at one point is combined with the T^- dependence at the other point.

We will return to the two terms G_1 and G_2 in Sec. VI B.

B. Metric fluctuations

To prove that there is a local field theory on Σ, the most important test is the existence of an energy-momentum tensor. In the AdS/CFT correspondence, the boundary energy-momentum tensor is intimately related to the bulk metric fluctuations. We assume a similar connection between bulk and boundary fields in the present context. In FSSY, metrical fluctuations were studied in a particular gauge which we will call the *spatially transverse-traceless* (STT) gauge. The coordinates of region I can be divided into FRW time, T, and space x^m where $m = 1, 2, 3$. The STT gauge for metric fluctuations is defined by

$$\nabla^m h_{mn} = 0, \qquad h^m_m = 0. \tag{5.22}$$

In these equations, the index is raised with the aid of the background metric (3.3). The main benefit of the STT gauge is that metric fluctuations satisfy minimally coupled massless scalar equations, and the correlation functions are similar to G_1 and G_2. However the index structure is rather involved. We define the correlator,

$$\langle h^\mu_\nu h^\sigma_\tau \rangle = G\{^{\mu\sigma}_{\nu\tau}\} = G_1\{^{\mu\sigma}_{\nu\tau}\} + G_2\{^{\mu\sigma}_{\nu\tau}\}. \tag{5.23}$$

The complicated index structure of G was worked out in detail in FSSY. In this paper we quote only the results of interest—in particular, those involving elements of $G\{^{\mu\sigma}_{\nu\tau}\}$ in which all indices lie in the two-sphere Ω_2. We consider the part $G_1\{^{ik}_{jl}\}$, which contains a term with dimension 2. This is the dimension that energy-momentum tensor in a two-dimensional conformal field theory should have.[10]

As in the scalar case, G_1 consists of an infinite sum of correlators, each corresponding to a field of dimension $\Delta = 2, 3, 4, \ldots$. The asymptotic T and R dependence of the terms is identical to the scalar case, and the first term has $\Delta = 2$. Once again this term is also time independent.

After isolating the dimension-2 term and stripping off the factors $\exp(-2R)$, the resulting correlator is called $G_1\{^{ik}_{jl}\}|_{\Delta=2}$. The calculations of FSSY revealed that this term is two-dimensionally traceless, and transverse,

$$G_1\begin{Bmatrix} ik \\ il \end{Bmatrix}\Bigg|_{\Delta=2} = G_1\begin{Bmatrix} ik \\ jk \end{Bmatrix}\Bigg|_{\Delta=2} = 0 \qquad \nabla_i G_1\begin{Bmatrix} ik \\ jl \end{Bmatrix}\Bigg|_{\Delta=2} = 0. \tag{5.24}$$

Equation (5.24) is the clue that, when combined with the dimension-2 behavior of $G_1\{^{ik}_{il}\}|_{\Delta=2}$, hints at a local theory on Σ. It insures that it has the precise form of a two-point function for an energy-momentum tensor in a conformal field theory. The only ambiguity is the numerical coefficient connecting $G_1\{^{ik}_{jl}\}|_{\Delta=2}$ with $\langle T^i_j T^k_l \rangle$. We will return to this coefficient momentarily.

The existence of a transverse, traceless, dimension-2 operator is a necessary condition for the boundary theory on Σ to be local: at the moment it is our main evidence. But there is certainly more that can be learned by computing multipoint functions. For example, from the three-point function $\langle hhh \rangle$ it should be possible to verify the operator product expansion and the Virasoro algebra for the energy-momentum tensor.

Dimensional analysis allows us to estimate the missing coefficient connecting the metric fluctuations with T^i_j, and at the same time determine the central charge c. In [8] we found c to be of order of the horizon entropy of the ancestor vacuum. We repeat the argument here:

Assume that the (bulk) metric fluctuation h has canonical normalization, i.e., it has bulk mass dimension 1 and a canonical kinetic term. Either dimensional analysis or explicit calculation of the two-point function $\langle hh \rangle$ shows that it is proportional to the square of the ancestor Hubble constant,

$$\langle hh \rangle \sim H^2. \tag{5.25}$$

[10] G_2 does not contain the $\Delta = 2$ term because $\mathcal{R}(k)$ is zero at $k = -i$ which would correspond to $\Delta = 2$ [8].

Knowing that the three-point function $\langle hhh \rangle$ must contain a factor of the gravitational coupling (Planck length) l_p, it can also be estimated by dimensional analysis,

$$\langle hhh \rangle \sim l_p H^4. \tag{5.26}$$

Now assume that the 2D energy-momentum tensor is proportional to the boundary dimension-2 part of h, i.e., the part that varies like e^{-2R}. Schematically, we write

$$T = qh \tag{5.27}$$

with q being a numerical constant. It follows that

$$\langle TT \rangle \sim q^2 H^2 \qquad \langle TTT \rangle \sim q^3 l_p H^4. \tag{5.28}$$

Lastly, we use the fact that the ratio of the two- and three-point functions is parametrically independent of l_p and H because it is controlled by the classical algebra of diffeomorphisms: $[T, T] = T$. Putting these elements together we find

$$\langle TT \rangle \sim \frac{1}{l_p^2 H^2}. \tag{5.29}$$

Since we already know that the correlation function has the correct form, including the short distance singularity, we can assume that the right-hand side of (5.29) also gives the central charge. It can be written in the rather suggestive form:

$$c \sim \text{area}/G \qquad (G = l_p^2) \tag{5.30}$$

where area refers to the horizon of the ancestor vacuum. In other words, the central charge of the hypothetical CFT is proportional to the horizon entropy of the ancestor.

C. Dimension-zero term

The term $G_2\{^{ik}_{jl}\}$ begins with a term, which like its scalar counterpart, has a nonvanishing limit on Σ. It is expressed in terms of a standard 2D bitensor $t\{^{ik}_{jl}\}$ which is traceless and transverse in the two-dimensional sense. If the correlation function were given just by $t\{^{ik}_{jl}\}$, it would be a pure gauge artifact. One can see this by considering the linearized expression for the 2D curvature-scalar C,

$$C = \nabla_i \nabla_j h^{ij} - 2\nabla^i \nabla_i \, \mathrm{Tr} h. \tag{5.31}$$

The 2D curvature associated with a traceless-transverse fluctuation vanishes, and since $t\{^{ik}_{jl}\}$ by itself is traceless transverse with respect to both points, it would be pure gauge if it appeared by itself.

However, the actual correlation function $G_2\{^{ik}_{jl}\}$ is given by

$$G_2\begin{Bmatrix} ik \\ jl \end{Bmatrix} = t\begin{Bmatrix} ik \\ jl \end{Bmatrix}\{R_1 + T_1 + R_2 + T_2 + \log(1 - \cos\alpha)\}. \tag{5.32}$$

The linear terms in $R + T$, being proportional to $t\{^{ik}_{jl}\}$ are pure gauge, but the finite term

$$t\begin{Bmatrix} ik \\ jl \end{Bmatrix} \log(1 - \cos\alpha) \tag{5.33}$$

gives rise to a nontrivial 2D curvature-curvature correlation function of the form

$$\langle CC \rangle = (1 - \cos\alpha)^{-2}. \tag{5.34}$$

One difference between the metric fluctuation h, and the scalar field χ, is that we cannot add a mass term for h in the ancestor vacuum to shift its dimension.

Finally, as in the scalar case, there is a tower of higher dimension terms in the tensor correlator, $G_2\{^{ik}_{jl}\}$ that only depend on T^+.

The existence of a zero dimensional term in $G_2\{^{ik}_{jl}\}$, which remains finite in the limit $R \to \infty$ indicates that fluctuations in the boundary geometry—fluctuations which are due to the asymptotic warmness—cannot be ignored. One might expect that in some way these fluctuations are connected with the field U that we encountered in the holographic version of the Wheeler-DeWitt equation. In the next section we will elaborate on this connection.

D. Three-point functions

Three-point functions in the CDL geometry can also be calculated by analytic continuation from the Euclidean space. They take the form of AdS three-point functions with mass of the propagators summed over. Let us illustrate this by taking the tree-level three-point function for a massless field χ with an interaction term $\int d^4x \sqrt{g} \chi^3$, as an example.

The Euclidean three-point function is given by integrating the vertex over the whole Euclidean space,

$$\langle \chi(x_1)\chi(x_2)\chi(x_3) \rangle = \int d^4x_0 \langle \chi(x_1)\chi(x_0) \rangle \langle \chi(x_2)\chi(x_0) \rangle \times \langle \chi(x_3)\chi(x_0) \rangle, \tag{5.35}$$

where the subscript 0 denotes the coordinates of the vertex. The propagators $\langle \chi\chi \rangle$ are given by the two-point functions (5.10). In the thin-wall background, $\langle \chi\chi \rangle$ is written in terms of the reflection or the transmission coefficients $\mathcal{R}(k)$, $\mathcal{T}(k)$. The external points are put on the flat side, since we want three-point functions in the flat FRW universe. When the vertex is on the flat side,

$$\langle \chi(X_1, \Omega_1)\chi(X_0, \Omega_0) \rangle = \frac{1}{a(X_1)a(X_0)} \int_{C_1} \frac{dk}{2\pi} (e^{ik(X_1 - X_0)} + \mathcal{R}(k) e^{-ik(X_1 + X_0)}) G_k(\Omega_{10}), \tag{5.36}$$

and when the vertex is on the de Sitter side,

$$\langle\chi(X_1,\Omega_1)\chi(X_0,\Omega_0)\rangle = \frac{1}{a(X_1)a(X_0)}\int_{C_1}\frac{dk}{2\pi}\frac{k+i\tanh X_0}{k+i} \times \mathcal{T}(k)e^{-ik(X_1-X_0)}G_k(\Omega_{10}), \tag{5.37}$$

where Ω_1, Ω_0 denote the positions on S^3, and Ω_{10} is the geodesic distance on S^3.

Performing the integration over the Euclidean time (X_0) of the vertex, we get

$$\begin{aligned}&\langle\chi(X_1,\Omega_1)\chi(X_2,\Omega_2)\chi(X_3,\Omega_3)\rangle\\ &= \frac{1}{a(X_1)a(X_2)a(X_3)}\int_{C_1}\frac{dk_1}{2\pi}\int_{C_1}\frac{dk_2}{2\pi}\int_{C_1}\frac{dk_3}{2\pi}\\ &\quad\times G_{k_1k_2k_3}(\Omega_1,\Omega_2,\Omega_3)\cdot[(e^{ik_1X_1}+\mathcal{R}(k_1)e^{-k_1X_1})\\ &\quad\cdot(2)\cdot(3)\cdot c^{(\text{flat})}_{k_1k_2k_3}+\mathcal{T}(k_1)e^{-ik_1X_1}\cdot(2)\cdot(3)\cdot c^{(\text{dS})}_{k_1k_2k_3}]\end{aligned} \tag{5.38}$$

where (2) and (3) denote the factors given by replacing the subscripts in the previous factors by 2 and 3. $c^{(\text{flat})}_{k_1k_2k_3}$ and $c^{(\text{dS})}_{k_1k_2k_3}$ are the coefficients we get when the vertex is on the flat side and de Sitter side, respectively. $G_{k_1k_2k_3}(\Omega_1,\Omega_2,\Omega_3)$ is a three-point function for massive fields on S^3,

$$G_{k_1k_2k_3}(\Omega_1,\Omega_2,\Omega_3) = \int d^3\Omega_0 G_{k_1}(\Omega_{10})G_{k_2}(\Omega_{20})G_{k_3}(\Omega_{30}). \tag{5.39}$$

To get the three-point function in the FRW universe from (5.38), we analytically continue the coordinates of the external points by (5.3), do the k integrals by closing the contour in the appropriate directions, and rotate the integration contour for $\int d^3\Omega_0$ to get an integral over $\mathcal{H}_3$. The final result is a sum of massive three-point functions on $\mathcal{H}_3$ (three-dimensional Euclidean AdS). The mass (dimension) for each AdS propagator is summed over; as in the two-point function, the dimension takes integer values starting from 2, and other possible values coming from the poles of $\mathcal{R}(k)$ or $\mathcal{T}(k)$.

The detailed structure of three-point functions is under study. Three-point functions involving a graviton will tell us the precise value of the central charge, as mentioned in Sec. V B. Also, the study of operator algebra will provide nontrivial consistency checks for our proposal of identifying a bulk field with a tower of CFT operators.

That is the data about correlation functions on the boundary sphere Σ that form the basis for our conjecture that there exists a local holographic boundary description of the open FRW universe. There are a number of related puzzles that this data raises: First, how does time emerge from a Euclidean QFT? The bulk coordinate R can be identified with scale size just as in AdS/CFT[11] but the origin of time requires a new mechanism.

The second puzzle concerns the number of degrees of freedom in the boundary theory. The fact that the central charge is the entropy of the ancestor suggests that there are only enough degrees of freedom to describe the false vacuum and not the much large number needed for the open FRW universe at late time. The resolution of both puzzles involves the Liouville field.

VI. LIOUVILLE THEORY

A. Breaking free of the STT gauge

The existence of a Liouville sector describing metrical fluctuations on Σ seems dictated by both the holographic Wheeler-DeWitt theory and from the data of the previous section. It is clear that the Liouville field is somehow connected with the non-normalizable metric fluctuations whose correlations are contained in (5.32), although the connection is somewhat obscured by the choice of gauge in [8]. In the STT gauge, the fluctuations h are traceless, but not transverse (in the 2D sense). From the viewpoint of 2D geometry they are not pure gauge as can be seen from the fact that the 2D curvature correlation does not vanish. One might be tempted to identify the Liouville mode with the zero-dimension piece of (5.32). To do so would of course require a coordinate transformation on Ω_2 in order to bring the fluctuation h_i^j to the "conformal" form $\tilde{h}\delta_i^j$.

This identification may be useful but it is not consistent with the Wheeler-DeWitt philosophy. The Liouville field U that appears in the Wheeler-DeWitt wave function is not tied to any specific spatial gauge. Indeed, the wave function is required to be invariant under gauge transformations,

$$x^\mu \rightarrow x^\mu + f^\mu(x) \tag{6.1}$$

under which the metric transforms:

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + \nabla_\mu f_\nu + \nabla_\nu f_\mu. \tag{6.2}$$

Let us consider the effect of such transformations on the boundary limit of h_{ij}. The components of f along the directions in Σ induce 2D coordinate transformation under which h transforms conventionally. Invariance under these transformations merely mean that the action S must be a function of 2D invariants.

Invariance under the shifts f^R and f^T are more interesting. In particular, the combination $f^+ = f^R + f^T$ generates nontrivial transformations of the boundary metric h_{ij}. An easy calculation shows that

$$h_i^j \rightarrow h_i^j + f^+(\Omega_2)\delta_i^j. \tag{6.3}$$

In other words, shift transformations f^+, induce Weyl rescalings of the boundary metric. This prompts us to

[11] This is often described by saying that R is related to the renormalization-group flow parameter.

modify the definition of the Liouville field from

$$U = T + \tilde{h} \tag{6.4}$$

to

$$U = T + \tilde{h} + f^{+}. \tag{6.5}$$

One might wonder about the meaning of an equation such as (6.5). The left side of the equation is supposed to be a dynamical field on Σ, but the right side contains an arbitrary function f^{+}. The point is that in the Wheeler-DeWitt formalism the wave function must be invariant under shifts, but in the original analysis of FSSY a specific gauge was chosen. Thus, in order to render the wave function gauge invariant, one must allow the shift f^{+} to be an integration variable, giving it the status of a dynamical field.

A similar example is familiar from ordinary gauge theories. The analog of the Wheeler-DeWitt gauge-free formalism would be the unfixed theory in which one integrates over the time component of the vector potential. The analog of the STT gauge would be the Coulomb gauge. To go from one to the other we would perform the gauge transformation

$$A_0 \rightarrow A_0 + \partial_0 \phi. \tag{6.6}$$

Integrating over the gauge function ϕ in the path integral would restore the gauge invariance that was given up by fixing the Coulomb gauge.

Returning to the Liouville field, since both $\tilde{h}$ and f are linearized fluctuation variables, we see that the classical part of U is still the FRW conformal time.

One important point: because the effect of the shift f^{+} is restricted to the trace of h, it does not influence the traceless-transverse (dimension-2) part of the metric fluctuation, and the original identification of the 2D energy-momentum tensor is unaffected.

Finally, invariance under the shift f^{-} is trivial in this order, at least for the thin-wall geometry. The reason is that in the background geometry, the area does not vary along the T^{-} direction.

Given that the boundary theory is local, and includes a boundary metric, it is constrained by the rules of two-dimensional quantum gravity laid down long ago by Polyakov [25]. Let us review those rules for the case of a conformal matter field theory coupled to a Liouville field. Two-dimensional coordinate invariance implies that the central charge of the Liouville sector cancels the central charge of all other fields. We have argued in [8] (and in Sec. V) that the central charge of the matter sector is of order of the horizon area of the ancestor vacuum, measured in Planck units. It is obvious from the four-dimensional bulk viewpoint that the semiclassical analysis that we have relied on only makes sense when the Hubble radius is much larger than the Planck scale. Thus we take the central charge of matter to satisfy $c \gg 1$. As a consequence, the central charge of the Liouville sector, c_L, must be large and negative. Unsurprisingly, the negative value of c is the origin of the emergence of time.

The formal development of Liouville theory begins by defining two metrics on Ω_2. The first is what we will call the reference metric $\hat{g}_{ij}$. Apart from an appropriate degree smoothness, and the assumption of Euclidean signature, the reference metric is arbitrary but fixed. In particular it is not integrated over in the path integral. Moreover, physical observables must be independent of $\hat{g}_{ij}$.

The other metric is the "real" metric denoted by g_{ij}. The purpose of the reference metric is merely to implement a degree of gauge fixing. Thus one assumes that the real metric has the form

$$g_{ij} = e^{2U} \hat{g}_{ij}. \tag{6.7}$$

The real metric—that is to say U—is a dynamical variable to be integrated over.

For positive c_L the Liouville Lagrangian is

$$L_L = \frac{Q^2 \sqrt{\hat{g}}}{8\pi} \{\hat{\nabla} U \hat{\nabla} U + \hat{R} U\} \tag{6.8}$$

where $\hat{R}$, $\hat{\nabla}$, all refer to the sphere Ω_2, with metric $\hat{g}$. The constant Q is related to the central charge c_L by

$$Q^2 = \frac{c_L}{3}. \tag{6.9}$$

The two-dimensional cosmological constant has been set to zero for the moment, but it will return to play a surprising role. For future reference we note that the cosmological term, had we included it, would have had the form

$$L_{cc} = \sqrt{\hat{g}} \lambda e^{2U}. \tag{6.10}$$

It is useful to define a field $\phi = QU$ in order to bring the kinetic term to the canonical form. One finds

$$L_L = \frac{\sqrt{\hat{g}}}{8\pi} \{\hat{\nabla} \phi \hat{\nabla} \phi + Q \hat{R} \phi\} \tag{6.11}$$

and, had we included a cosmological term, it would be

$$L_{cc} = \sqrt{\hat{g}} \lambda \exp \frac{2\phi}{Q}. \tag{6.12}$$

By comparison with the case of positive c_L, very little is rigorously understood about Liouville theory with negative central charge. In this paper we will make a leap of faith: we assume that the theory can be defined by analytic continuation from positive c_L. To that end we note that the only place that the central charge enters (6.11) and (6.12) is through the constants Q and γ, both of which become imaginary when c_L becomes negative. Let us define

$$Q = i\mathcal{Q}. \tag{6.13}$$

Equations (6.11) and (6.12) become

$$L_L = \frac{\sqrt{\hat{g}}}{8\pi}\{\hat{\nabla}\phi\hat{\nabla}\phi + iQ\hat{R}\phi\} = \frac{\sqrt{\hat{g}}}{8\pi}\{\hat{\nabla}\phi\hat{\nabla}\phi + 2iQ\phi\} \tag{6.14}$$

(where we have used $\hat{R} = 2$), and

$$L_{cc} = \sqrt{\hat{g}}\lambda \exp\frac{-2i\phi}{Q}. \tag{6.15}$$

Let us come now to the role of λ. First of all λ has nothing to do with the four-dimensional cosmological constant, either in the FRW patch or the ancestor vacuum. Furthermore it is not a constant in the action of the boundary theory. Its proper role is as a *Lagrange multiplier* that serves to specify the time T, or more exactly, the global scale factor. The procedure is motivated by the Wheeler-DeWitt procedure of identifying the scale factor with time. In the present case of the thin-wall limit, we identify $\exp 2U$ with $\exp 2T$. Thus we insert a δ function in the path integral,

$$\delta\left(\int \sqrt{\hat{g}}(e^{2U} - e^{2T})\right) = \int dz \exp iz\left(\int \sqrt{\hat{g}}(e^{2U} - e^{2T})\right). \tag{6.16}$$

The path integral (which now includes an integration over the imaginary 2D cosmological constant z) involves the action

$$L_L + L_{cc} = \frac{\sqrt{\hat{g}}}{8\pi}\left\{\hat{\nabla}\phi\hat{\nabla}\phi + 2iQ\phi + 8\pi iz \exp\frac{-2i\phi}{Q} - 8\pi ize^{2T}\right\}. \tag{6.17}$$

The action (6.17) has a saddle point[12] when the potential

$$V = 2iQ\phi + 8\pi iz \exp\frac{-2i\phi}{Q} - 8\pi ize^{2T} \tag{6.18}$$

is stationary; this occurs at

$$\exp\frac{-2i\phi}{Q} = e^{2T} \qquad z = i\frac{Q^2}{8\pi}e^{-2T} \tag{6.19}$$

or in terms of the original variables

$$e^{2U} = e^{2T} \qquad \lambda = \frac{Q^2}{8\pi}e^{-2T}. \tag{6.20}$$

Once λ has been determined in terms of T by (6.20), the Liouville theory with that value of λ determines expectation values of the remaining variables as functions of the time.

Thus, as we mentioned earlier, the cosmological constant is not a constant of the theory but rather a parameter that we scan in order to vary the cosmic time.

[12]It should be noted that the saddle point only occurs for negative central charge.

B. Liouville, renormalization, and correlation functions

1. Preliminaries

There are two preliminary discussions that will help us understand the application of Liouville theory to cosmic holography. The first is about the AdS/CFT connection between the bulk coordinate R, and renormalization-group running of the boundary field theory. There are three important length scales in every quantum field theory. The first is the ''low energy scale''; in the present case the low energy scale is the radius of the sphere which we will call L.

The second is the ''bare'' cutoff scale—where the underlying theory is prescribed. Call the bare scale[13] a.

The bare input is a collection of degrees of freedom, and an action coupling them. In a lattice gauge theory the degrees of freedom are site and link variables, and the couplings are nearest neighbor to insure locality[14] In a ferromagnet they are spins situated on the sites of a crystal lattice.

The previous two scales have obvious physical meaning but the third scale is arbitrary: a sliding scale called the renormalization or reference scale. We denote it by δ. The reference scale is assumed to be much smaller than L and much larger than a, but otherwise it is arbitrary. It helps to keep a concrete model in mind. Instead of a regular lattice, introduce a ''dust'' of points with average spacing a. It is not essential that a be uniform on the sphere. Thus the spacing of dust points is a function of position, $a(\Omega_2)$. The degrees of freedom on the dust points, and their nearest-neighbor couplings, will be left implicit.

Next we introduce a second dust at larger spacing, δ. The δ dust provides the reference scale. It is well known that for length scales greater than δ, the bare theory on the a dust can be replaced by a renormalized theory defined on the δ dust. The renormalized theory will typically be more complicated, containing second, third, and nth neighbor couplings.

Generally, the dimensionless form of the renormalized theory will depend on δ in just such a way that physics at longer scales is exactly the same as it was in the original theory. The dimensionless parameters will flow as the reference scale is changed.

If there is an infrared fixed point, and if the bare theory is in the basin of attraction of the fixed point, then as δ becomes much larger than a, the dimensionless parameters will run to their fixed-point values. In that case the continuum limit ($a \to 0$) will be a conformal field theory with $SO(3, 1)$ invariance.

These things hold in the holographic theory of anti-de Sitter space. The infrared scale is provided by the

[13]The lattice spacing a should not be confused with the FRW scale factor, also called a.

[14]Nearest neighbor is common but not absolutely essential. However this subtlety is not important for us.

geometry of the boundary, namely, a sphere. The compact nature of the boundary gives the theory an energy gap.

The bare scale is not so important in AdS/CFT. One might as well take the continuum limit from the start. On the other hand, the running renormalization scale is all important. It defines the holographically generated radial dimension of space. A useful slogan is ‘‘motion along the R direction is the same as renormalization-group flow.'' The reference scale δ is related to R_0 by $\delta = e^{-R_0}$.

More generally, we can allow the renormalization scale to depend on position along the boundary:

$$\delta(\Omega) = e^{-R_0(\Omega)}. \tag{6.21}$$

Now we move to the second preliminary—some observations about the Liouville theory. Again, a concrete model is helpful. Liouville theory is closely connected with the theory of dense, planar, ‘‘fishnet'' diagrams [26] such as those which appear in large N gauge theories and matrix models [27–29].

We assume the fishnet has the topology of a sphere, i.e., it can be drawn on a sphere with no crossing of lines, and that it has a very large number of vertices. Following [26], we draw the diagram so that it is locally isotropic. If it is not locally isotropic it may be made so by shifting the points around. Once the diagram is isotropic, the remaining freedom in drawing the fishnet is conformal transformations.

The fishnet plays the role of the bare lattice in the previous discussion, but now it is dynamical—we sum over all fishnet diagrams, assuming only that the spacing (on the two-sphere) is everywhere much smaller than the sphere size, L. As before, we call the angular spacing between neighboring vertices on the sphere, $a(\Omega)$.

Each fishnet defines a metric on the sphere. Let $d\alpha$ be a small angular interval (measured in radians). The fishnet metric is defined by

$$ds^2 = \frac{d\alpha^2}{a(\Omega)^2}. \tag{6.22}$$

We again introduce a reference scale δ. It can also be a fishnet, but now it is fixed, its vertices nailed down, not to be integrated over. We continue to assume that δ satisfies the inequalities, $a(\Omega) \ll \delta(\Omega) \ll L$, but otherwise it is arbitrary. The δ metric is defined by

$$ds_\delta^2 = \frac{d\alpha^2}{\delta(\Omega)^2}. \tag{6.23}$$

It should be clear that the metrics in (6.22) and (6.23) are the same as the real and reference metrics in (6.7). We can now define the Liouville field U. All it is is the ratio of the reference and fishnet scales:

$$e^U \equiv \delta/a. \tag{6.24}$$

Using (6.24) together with $\delta = e^{-R}$ and $ds = \frac{d\alpha}{a}$, we see that U is also given by the relation

$$ds = d\alpha e^{(R_0+U)}. \tag{6.25}$$

In (6.25) both R_0 and U are functions of location on Ω_2, but only U is dynamical, i.e., to be integrated over.

2. Liouville in the hat

With that in mind, we return to cosmic holography and consider the metric on the regulated spatial boundary of FRW, Σ_0. In the absence of fluctuations it is

$$ds^2 = e^{2R_0} e^{2T} d^2\Omega_2.$$

In general relativity it is natural to allow both R_0 and T to vary over the sphere, so that

$$ds^2 = e^{2R_0(\Omega_2)} e^{2T(\Omega_2)} d^2\Omega_2. \tag{6.26}$$

The parallel between (6.25) and (6.26) is obvious. Exactly as we might have expected from the Wheeler-DeWitt interpretation, the Liouville field, U, may be identified with time T, when both are large,

$$U \approx T. \tag{6.27}$$

The connection between T and U is ambiguous when U becomes small. In that limit the fishnet diagrams become very sparse and any detailed identification with the continuous variable T breaks down.

To summarize, let us list a number of correspondences:

$$\delta \leftrightarrow e^{-R}, \qquad \lambda \leftrightarrow e^{-T}, \qquad a \leftrightarrow e^{-T^+} = e^{-(T+R)}. \tag{6.28}$$

We also recall the correspondences with the real and reference metric:

$$ds^2 = \frac{d\alpha^2}{a(\Omega)^2}, \qquad ds_\delta^2 = \frac{d\alpha^2}{\delta(\Omega)^2}. \tag{6.29}$$

One other point about the Liouville theory: the density of vertices of a fishnet is normally varied by changing the weight assigned to vertices. When the fishnet is a Feynman diagram, the weight is a coupling constant g. It is well known that the coupling constant and Liouville cosmological constant are alternate descriptions of the same thing. Either can be used to vary the average vertex density—increasing it either by increasing g or decreasing λ. The very dense fishnets correspond to large U and therefore large FRW time, whereas very sparse diagrams dominate the early Planckian era.

3. RG-covariant and RG-invariant objects in quantum field theory

There are two kinds of objects[15] in Wilsonian renormalization [30] that correspond quite closely to the terms G_1 and G_2 that we have found in Sec. V. We call them *‘‘RG*

[15] In an earlier version of this paper the terms ‘‘proactive'' and ‘‘reactive'' were used instead of RG-covariant and RG-invariant.

covariant" and *"RG invariant"*. RG-covariant quantities depend on the arbitrary reference scale. They are not directly measurable quantities. The best example is the exact Wilsonian action, defined at a specific reference scale. The form of RG-covariant quantities depends on that reference scale, and so does the value of their matrix elements; indeed their form varies with δ in such a way as to keep the physics fixed at longer distances.

By contrast, RG-invariant objects are observables whose value does not depend on the reference scale. They do depend on the bare cutoff scale a through wave function renormalization constants, which typically tend to zero as $a \to 0$. The wave function renormalization constants are usually stripped off when defining quantum fields, but we will find it more illuminating to keep them.

The distinction between these two kinds of objects is subtle and is perhaps best expressed in Polchinski's version of the exact Wilsonian renormalization group [31]. In that scheme, at every scale there is a renormalized description in terms of local defining fields $\phi(x)$, but the RG-covariant action grows increasingly complicated as the reference scale is lowered.

Consider the exact effective action defined at reference scale δ. It is given by an infinite expansion of the form

$$L_W(\delta) = \sum_{\Delta=2}^{\infty} g_\Delta \mathcal{O}_\Delta, \tag{6.30}$$

where $\mathcal{O}_\Delta$ are a set of operators of dimension Δ, and g_Δ are dimensional coupling constants. The renormalization flow is expressed in terms of the dimensionless coupling constants,

$$\tilde{g}_\Delta = g_\Delta \delta^{(2-\Delta)}. \tag{6.31}$$

The $\tilde{g}$ satisfy RG equations,

$$\frac{d\tilde{g}}{d\log\delta} = -\beta(\tilde{g}). \tag{6.32}$$

If the theory flows to a fixed point, in that limit the $\tilde{g}$ become constant. Thus the dimensional constants g_Δ in the Lagrangian will grow with δ. Normalizing them at the bare scale a, in the fixed-point case we get

$$g_\Delta(\delta) = g_\Delta(a)\left\{\frac{\delta}{a}\right\}^{(\Delta-2)}, \tag{6.33}$$

$$L_W(\delta) = \sum_{\Delta=2}^{\infty} \mathcal{O}_\Delta \left\{\frac{\delta}{a}\right\}^{(\Delta-2)}. \tag{6.34}$$

Now consider the two-point function of the effective action, $\langle L_W(\delta)L_W(\delta)\rangle$, evaluated at distance scale $L \gg \delta$

$$\langle L_W(\delta)L_W(\delta)\rangle = \sum_{\Delta=2}^{\infty} \langle \mathcal{O}_\Delta \mathcal{O}_\Delta\rangle \left(\frac{\delta}{a}\right)^{2(\Delta-2)}. \tag{6.35}$$

Suppose the theory is defined on a sphere of radius L and we are interested in the correlator $\langle L_W(\delta)L_W(\delta)\rangle$ between points separated by angle α. The factor $\langle \mathcal{O}_\Delta \mathcal{O}_\Delta\rangle$ is the two-point function of a field of dimension Δ, in a theory on the sphere of size L with an ultraviolet cutoff at the reference scale δ. Accordingly it has the form

$$\langle \mathcal{O}_\Delta \mathcal{O}_\Delta\rangle = \left(\frac{\delta}{L}\right)^{2\Delta}(1-\cos\alpha)^{-\Delta} \tag{6.36}$$

where the two factors of $(\frac{\delta}{L})^{\Delta}$ are the ultraviolet-sensitive wave function renormalization constants. The final result is

$$\langle L_W(\delta)L_W(\delta)\rangle = \sum_{\Delta=2}^{\infty} C_\Delta\left(\frac{\delta}{a}\right)^{2(\Delta-2)}\left(\frac{\delta}{L}\right)^{2\Delta}(1-\cos\alpha)^{-\Delta}. \tag{6.37}$$

Note the dependence of (6.37) on the arbitrary reference scale δ. That dependence is typical of RG-covariant quantities.

Now consider a RG-invariant quantity such as a fundamental field, a derivative of such a field, or a local product of fields and derivatives. Their matrix elements at distance scale L will be independent of the reference scale (although it will depend on the bare cutoff a) and be of order

$$\langle \phi\phi\rangle \sim \left(\frac{a}{L}\right)^{2\Delta_\phi}(1-\cos\alpha)^{-\Delta_\phi} \tag{6.38}$$

where Δ_ϕ is the operator dimension of ϕ. Thus we see two distinct behaviors for the scaling of correlation functions:

$$\left(\frac{\delta}{a}\right)^{2(\Delta-2)}\left(\frac{\delta}{L}\right)^{2\Delta} \qquad \text{RG covariant} \tag{6.39}$$

and

$$\left(\frac{a}{L}\right)^{2\Delta_\phi} \qquad \text{RG invariant.} \tag{6.40}$$

The formulas are more complicated away from a fixed point but the principles are the same.

We note that the effective action is not the only RG-covariant object. The energy-momentum tensor and various currents computed from the effective action will also be RG covariant. As we will see, these two behaviors—RG covariant and RG invariant—exactly correspond to the dependence in (5.14) and (5.15).

Now we are finally ready to complete the discussion about the relation between the correlators of Sec. V and RG-covariant/invariant operators. Begin by noting that in AdS/CFT, the minimally coupled massless (bulk) scalar is the dilaton, and its associated boundary field is the Lagrangian density. It may seem puzzling that in the present case, an entire infinite tower of operators seems to replace what in AdS/CFT is a single operator. In the case of the metric fluctuations, a similar tower replaces the energy-momentum tensor. The puzzle may be stated another way. The FRW geometry consists of an infinite number of Euclidean AdS time slices. At what time (or what 2D cosmological constant) should we evaluate the boundary limits of the metric fluctuations in order to define the energy-momentum tensor? As we will see, a parallel ambiguity exists in Liouville theory.

We return now to the three scales of Liouville theory: the infrared scale L, the reference scale δ, and the fishnet scale a, with $L \gg \delta \gg a$. It is natural to assume that the basic theory is defined at the bare fishnet scale a by some collection of degrees of freedom at each lattice site, and also specific nearest-neighbor couplings—the latter insuring locality. Now imagine a Wilsonian integration of all degrees of freedom on scales between the fishnet scale and the reference scale, including the fishnet structure itself. The result will be a RG-covariant effective action of the type we described in Eq. (6.30). Moreover the correlation function of $L_{\rm eff}$ will have the form (6.37). But now, making the identifications (6.21) and

$$\frac{\delta}{a} = e^U = e^T, \tag{6.41}$$

we see that Eq. (6.39) for RG-covariant scaling becomes (for each operator in the product)

$$e^{(\Delta-2)T} e^{-\Delta R}. \tag{6.42}$$

This is in precise agreement with the coefficients in the expansion (5.13).

Similarly, the RG-invariant scaling (6.40), $e^{-\Delta T^+}$, is in agreement with the properties of G_2.

What happens to the RG-covariant objects if we approach Σ by sending $T^+ \to \infty$ and $T^- \to -\infty$? In this limit only the dimension-2 term survives: exactly what we would expect if the matter action ran toward a fixed point. All of the same things hold true for the tensor fluctuations. Before the limit $T^- \to -\infty$, the energy-momentum tensor consists of an infinite number of higher dimension operators but in the limit, all tend to zero except for the dimension-2 term.

It should be observed that the higher dimension contributions to $G_1\{^{ik}_{jl}\}$ are not transverse in the two-dimensional sense. This is to be expected: before the limit is taken, the Liouville field does not decouple from the matter field, and the matter energy momentum is not separately conserved. But if the matter theory is at a fixed point, i.e., scale invariant, the Liouville and matter do decouple and the matter energy momentum should be conserved. Thus, in the limit in which the dimension-2 term dominates, it should be (and is) transverse traceless.

The physical reason why the fluctuations of the Liouville field decouples at late time deserves some comment. As we have emphasized, the reason that the boundary geometry is dynamical is the asymptotic warmness of the FRW background as $R \to \infty$ at fixed time. But unlike de Sitter space, FRW cosmology becomes cold at late time.

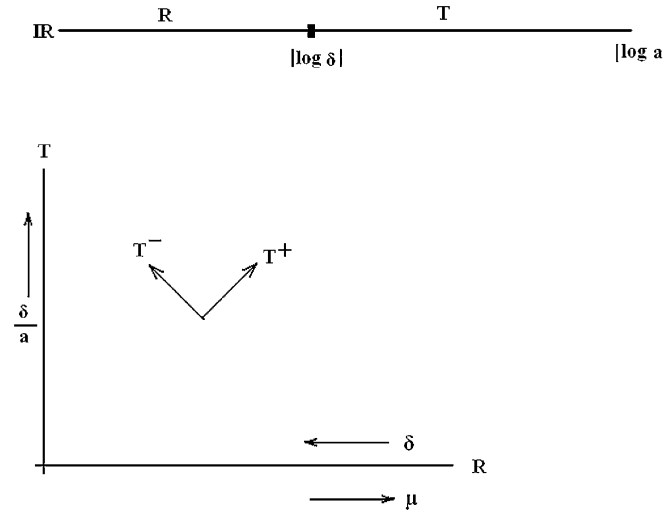

FIG. 10. The Wilson line of scales and the two-dimensional R, T plane.

Thus it makes sense that the Liouville field decouples as $T \to \infty$.

RG flow is usually thought of in terms of a single independent flow parameter. In some versions it is the logarithm of the bare cutoff scale, and in other formulations it is the log of the renormalization scale. In the conventional AdS/CFT framework, R can play either role. One can imagine a bare cutoff at some large R_0 or one can push the bare cutoff to infinity and think of R as a running renormalization scale.

However, for our purposes, it is better to keep track of both scales. One can either think of a one-dimensional (logarithmic) axis—we can call it the "Wilson line"—extending from the infrared scale to the fishnet scale a, or a two-dimensional R, T plane. In either case the effective action is a function of two independent variables. Figure 10 shows a sketch of the Wilson line and the two-dimensional plane representing the two directions R and T.

The two independent parameters can be chosen to be a and δ, or equivalently R and T. Yet another choice is to work in momentum space. The reference energy scale is usually called μ,

$$\mu = e^R. \tag{6.43}$$

And in the case of negative central charge, the two-dimensional cosmological constant λ can replace T.

In this light, it is extremely interesting that the distinction between RG-covariant and RG-invariant scaling corresponds to motion along the two lightlike directions T^- and T^+ as depicted in Fig. 11.

It may not be obvious why the bulk fields should correspond to RG-covariant and RG-invariant boundary fields in the way that they do. The solutions to the wave equation in the bulk are generally sums of two types of modes,

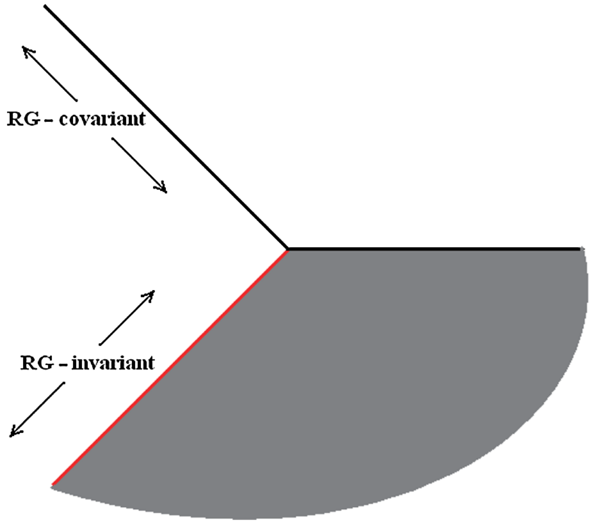

FIG. 11 (color online). RG-covariant and RG-invariant quantities scale with the two lightlike directions T^- and T^+.

$$\chi_- \to g_-(\Omega_2)F_-(T^-)e^{-2T}, \qquad \chi_+ \to g_+(\Omega_2)F_+(T^+). \tag{6.44}$$

Some insight can be obtained from Fig. 8. The lower (red) lightlike line represents the initial condition of the FRW universe at $T = -\infty$. In that limit the RG-covariant contributions to the correlation functions tend to zero, and only the RG-invariant terms are present. Thus it makes sense to think of the RG-invariant quantities as inputs to the RG flow—in other words as the bare action. Given the input on the red initial surface, the bulk field equations can be solved to determine the output on the hat. Note that on the hat, the RG-invariant quantities tend to zero, leaving only the RG covariant.

This duality between the renormalization properties of Liouville theory and the cosmological coordinates R and T is very remarkable and clearly deserves more attention. It is important to understand that the duality between FRW cosmology and Liouville 2D gravity does not only involve the continuum fixed-point theory. As long as T is finite the theory has some memory of the bare theory. Cosmology in the hat consists of an entire RG flow from some bare theory to a final fixed point that governs late times. It is only in the limit $T \to \infty$ that the theory flows to the fixed point and loses memory of the bare details. We will come back to this point in Sec. VIII when we discuss the GGV [24] ''persistence of memory'' phenomenon.

VII. SCALING AND THE CENSUS TAKER

A. Moments

Now we come to the connection between the scaling behavior of two-dimensional quantum field theory and the observations of a census taker as he moves toward the census bureau. In order to better understand the connection between the cutoff scale and T^+, let us return to the similar connection between cutoff and the coordinate R in anti-de Sitter space. We normalize the AdS radius of curvature to be 1; with that normalization, the Planck area is given by $1/c$ where c is the central charge.

Consider the proper distance between points 1 and 2 given by (5.4). The relation $l = R_1 + R_2 + \log(1 - \cos\alpha)$ is approximate, valid when l and $R_{1,2}$ are all large. When $l \sim 1$ or equivalently, when

$$\alpha^2 \sim e^{-(R_1+R_2)} \tag{7.1}$$

Eq. (5.4) breaks down. For angles smaller than (7.1) the distance in anti-de Sitter space behaves like

$$l \sim e^R \alpha. \tag{7.2}$$

Thus a typical correlation function will behave as a power of $(1 - \cos\alpha)$ down to angular distances of order (7.1) and then fall quickly to zero.

The angular cutoff in anti-de Sitter space has a simple meaning. The solid angle corresponding to the cutoff is of order e^{-2R} while the area of the regulated boundary is e^{+2R}. Thus, metrically, the cutoff area is of order unity. This means that in Planck units, the cutoff area is the central charge c of the boundary conformal field theory.

Now consider the cutoff angle in the hat. Equations (5.17) and (5.18) imply the UV cutoff is of order

$$\alpha^2 \sim e^{-(T_1^+ + T_2^+)}. \tag{7.3}$$

Once again this corresponds to a small patch of proper area (on Σ_0, the regulated boundary) which is time independent, and in Planck units, of order the central charge. It is also equal to the area on the horizon in the ancestor vacuum. The degrees of freedom which describe this patch will possibly be matrices as in the usual AdS/CFT correspondence.

Consider the census taker looking back from some late time $T_{\rm CT}$. For convenience we place the CT at $R = 0$. His backward light cone is the surface

$$T + R = T_{\rm CT}. \tag{7.4}$$

The CT can never quite see Σ. Instead he sees the regulated surfaces corresponding to a fixed proper cutoff (Fig. 4). The later the CT observes, the smaller the angular structure that he can resolve on the boundary. This is another example of the UV/IR connection, this time in a cosmological setting.

Let us consider a specific example of a possible observation. The massless scalar field χ of Sec. V has an asymptotic limit on Σ that defines the dimension-zero field $\chi(\Omega)$. Moments of χ can be defined by integrating it with spherical harmonics,[16]

[16]We are using l for geodesic distance on $\mathcal{H}_3$, and ℓ for the angular momentum quantum number.

$$\chi_{\ell m} = \int \chi(\Omega) Y_{\ell m}(\Omega) d^2\Omega. \tag{7.5}$$

It is worth recalling that in anti-de Sitter space the corresponding moments would all vanish because the normalizable modes of χ all vanish exponentially as $R \to \infty$. The possibility of nonvanishing moments is due entirely to the asymptotic warmness of open FRW.

We can easily calculate[17] the mean square value of $\chi_{\ell m}$ (it is independent of m).

$$\langle \chi_\ell^2 \rangle = H^2 \int d(\cos\alpha) \log(1 - \cos\alpha) P_\ell(\cos\alpha) \sim H^2 \frac{1}{\ell(\ell+1)}. \tag{7.6}$$

It is evident that at a fixed census taker time $T_{\rm CT}$, the angular resolution is limited by (7.3). Correspondingly, the largest moment that the CT can resolve corresponds to

$$\ell_{\max} \sim e^{T_{\rm CT}}. \tag{7.7}$$

Thus we arrive at the following picture: the census taker can look back toward Σ but at any given time his angular resolution is limited by (7.3) and (7.7). As time goes on more and more moments come into view. Once they are measured they are frozen and cannot change. In other words the moments evolve from being unknown quantum variables, with a Gaussian probability distribution, to classical boundary conditions that explicitly break rotation symmetry (and therefore conformal symmetry). One sees from (7.6) that the symmetry breaking is dominated by the low moments.

This phenomenon never occurs in an undiluted form. Realistically speaking, we do not expect massless scalars in the nonsupersymmetric ancestor. In Sec. V we discussed the effect of a small mass term, in the ancestor vacuum, on the correlation functions of χ. The result of such a mass term is a shift of the leading dimension from 0 to μ. This has an effect on the moments. The correlation function becomes

$$H^2 e^{-\mu T_1^+} e^{-\mu T_2^+} (1 - \cos\alpha)^{-\mu}, \tag{7.8}$$

and the moments take the form

$$\langle \chi_\ell^2 \rangle = H^2 e^{-2\mu T_{CT}} \int d(\cos\alpha)(1 - \cos\alpha)^{\mu} P_\ell(\cos\alpha). \tag{7.9}$$

The functional form of the ℓ dependence changes a bit, favoring higher ℓ, but more importantly, the observable effects decrease like $e^{-2\mu T_{\rm CT}}$. Thus as $T_{\rm CT}$ advances, the asymmetry on the census taker's sky decreases exponentially with conformal time. Equivalently it decreases as a power of proper time along the CT's worldline.

[17] L. S. is indebted to Ben Freivogel for explaining Eq. (7.6).

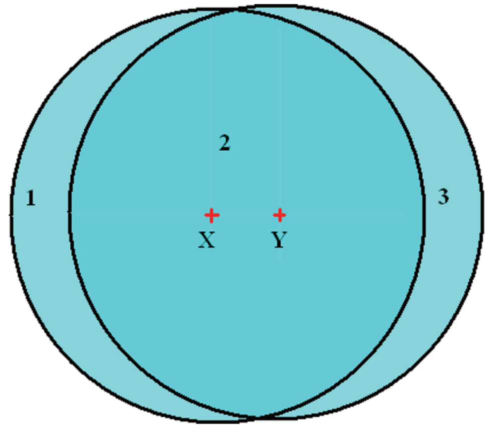

FIG. 12 (color online). Two large spheres centered at X and Y.

B. Homogeneity breakdown

Homogeneity in an infinite FRW universe is generally taken for granted, but before questioning homogeneity we should know exactly what it means. Consider some three-dimensional scalar quantity such as energy density, temperature, or the scalar field χ. Obviously the Universe is not uniform on small scales, so in order to define homogeneity in a useful way we need to average χ over some suitable volume. Thus at each point X of space, we integrate χ over a sphere of radius r and then divide by the volume of the sphere. For a mathematically exact notion of homogeneity, the size of the sphere must tend to infinity. The definition of the average of χ at the point X is

$$\overline{\chi(X)} = \lim_{r\to\infty} \frac{\int \chi d^3x}{V_r}. \tag{7.10}$$

Now pick a second point Y and construct $\overline{\chi(Y)}$. The difference $\overline{\chi(X)} - \overline{\chi(Y)}$ should go to zero as $r \to \infty$ if space is homogeneous. But as the spheres grow larger than the distance between X and Y, they eventually almost completely overlap. In Fig. 12 we see that the difference between $\overline{\chi(X)}$ and $\overline{\chi(Y)}$ is due to the two thin crescent-shaped regions, 1 and 3. It seems evident that the overwhelming bulk of the contributions to $\overline{\chi(X)}$, $\overline{\chi(Y)}$ come from the central region 3, which occupies almost the whole figure. The conclusion seems to be that the averages, if they exist at all, must be independent of position. Homogeneity while true, is trivial.

This is correct in flat space, but surprisingly it can break down in hyperbolic space.[18] The reason is quite simple: despite appearances the volume of regions 1 and 3 grow just as rapidly as the volume of 2. The ratio of the volumes is of order

[18] L. S. is grateful to Larry Guth for explaining this phenomenon and to Alan Guth for emphasizing its importance in cosmology.

$$\frac{V_1}{V_2} = \frac{V_3}{V_2} \sim \frac{l}{R_{\text{curvature}}} \tag{7.11}$$

(where l is the distance between X and Y) and remains finite as $r \to \infty$.

To be more precise, we observe that

$$\overline{\chi(X)} = \frac{\int_1 \chi + \int_2 \chi}{V_1 + V_2} \qquad \overline{\chi(Y)} = \frac{\int_3 \chi + \int_2 \chi}{V_3 + V_2} \tag{7.12}$$

and that the difference $\overline{\chi(X)} - \overline{\chi(Y)}$ is given by

$$\overline{\chi(X)} - \overline{\chi(Y)} = \frac{\int_1 \chi}{V_1 + V_2} - \frac{\int_3 \chi}{V_3 + V_2} \tag{7.13}$$

which, in the limit $r \to \infty$ is easily seen to be proportional to the dipole moment of the boundary theory,

$$\overline{\chi(X)} - \overline{\chi(Y)} = l \int \chi(\Omega) \cos\theta d^2\Omega = l\chi_{1,0}. \tag{7.14}$$

Since, as we have already seen for the case $\mu = 0$, the mean square fluctuation in the moments does not go to zero with distance, it is also true that the average value of $|\overline{\chi(X)} - \overline{\chi(Y)}|^2$ will be nonzero. In fact it grows with separation.

However there is no reason to believe that a dimension-zero scalar exists. Moduli, for example, are expected to be massive in the ancestor, and this shifts the dimension of the corresponding boundary field. In the case in which the field χ has dimension μ, the effect (nonzero rms average of moments) persists in a somewhat diluted form. If a renormalized field is defined by stripping off the wave function normalization constants, $\exp(-\mu T^+)$, the squared moments still have finite expectation values and break the symmetry. However, from an observational point of view there is no reason to remove these factors. Thus it seems that as the census taker time tends to infinity, the observable asymmetry will decrease like $\exp(-2\mu T_{CT})$.

VIII. BUBBLE COLLISIONS AND OTHER MATTERS

By looking back toward Σ, the census taker can see into bubbles of other vacua—bubbles that in the past collided with his hatted vacuum.

As Guth and Weinberg recognized long ago [32], a single isolated bubble is infinitely unlikely. A typical "pocket universe" will consist of a cluster of an unbounded number of colliding bubbles, although if the nucleation rate is small the collisions will in some sense be rare. To see why such bubble clusters form, it is sufficient to recognize why a single bubble is infinitely improbable. In Fig. 13 the main point is illustrated by drawing a timelike trajectory that approaches Σ from within the ancestor vacuum. The trajectory has infinite proper length, and assuming that there is a uniform nucleation rate, a second bubble will eventually swallow the trajectory and collide with the original bubble. Repeating this process will produce an infinite bubble cluster.

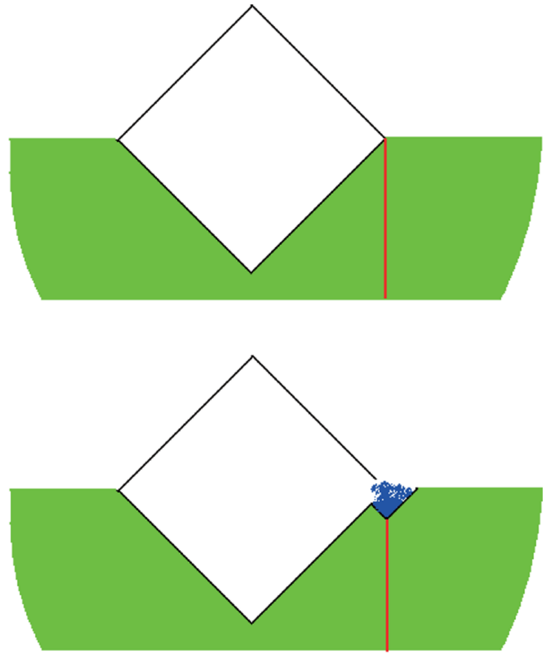

FIG. 13 (color online). The top figure represents a single nucleated bubble. The red trajectory is a timelike curve of infinite length approaching Σ. Because there is a constant nucleation rate along the curve, it is inevitable that a second bubble will nucleate as in the lower figure. The two bubbles will collide.

More recently, GGV [24] have argued that the multiple bubble collisions must spontaneously break the $SO(3, 1)$ symmetry of a single bubble, and in the process render the (pocket) universe inhomogeneous and anisotropic. The breaking of symmetry in [24] was described, not as spontaneous breaking, but as explicit breaking due to initial conditions. However, spontaneous symmetry breaking *is* the persistent memory of a temporary explicit symmetry breaking, if the memory does not fade with time. For example, a small magnetic field in the very remote past will determine the direction of an infinite ferromagnet for all future time.

The actual observability of bubble collisions depends on the amount of slow-roll inflation that took place after tunneling. Much more than 60 e-foldings [33] would probably wipe out any signal, but the interest of this paper is conceptual. We will take the viewpoint that anything within the past light cone of the census taker is in principle observable.

In the last section we saw that perturbative infrared effects are capable of breaking the $SO(3, 1)$ symmetry, and it is an interesting question what the relation between these two mechanisms is. The production of a new bubble would seem to be a nonperturbative effect that adds to the perturbative symmetry breaking effects of the previous section. Whether it adds distinctly new effects that are

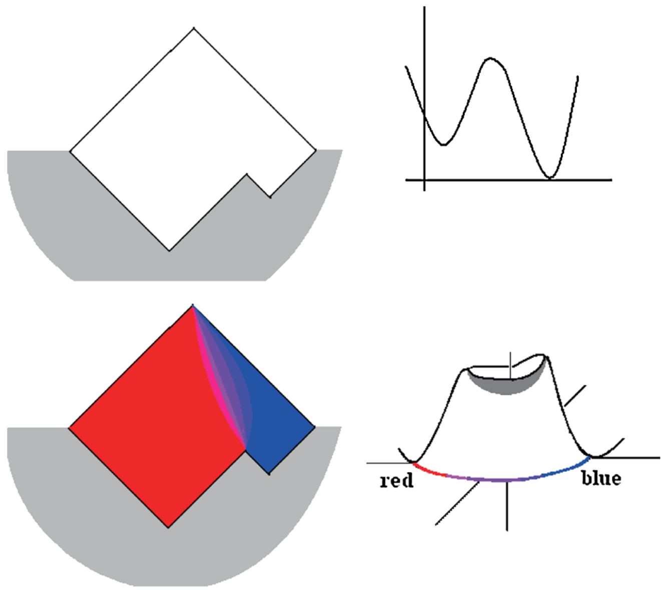

FIG. 14 (color online). In the top figure two identical bubbles collide. This would be the only type of collision in a simple landscape with two discrete minima—one of positive energy and one of zero energy. In the lower figure a more complicated situation is depicted. In this case, the false vacuum F can decay to two different true vacua, ''red'' and ''blue,'' each with vanishing energy. The two true vacua are connected by a flat direction, but CDL instantons only lead to the red and blue points.

absent in perturbation theory is not obvious and may depend on the specific nature of the collision. Let us classify some possibilities.

A. Collisions with identical vacua

The simplest situation is if the true-vacuum bubble collides with another identical bubble, the two bubbles coalescing to form a single bubble, as in the top of Fig. 14.

The surface Σ is defined by starting at the tip of the hat and tracking back along lightlike trajectories until they end—in this case at a false vacuum labeled **F**. The collision is parametrized by the spacelike separation between nucleation points. Particles produced at the collision of the bubbles just add to the particles that were produced by ordinary FRW evolution. The main effect of such a collision is to create a very distorted boundary geometry, if the nucleation points are far apart. When they are close, the double nucleation blends in smoothly with the single bubble. These kinds of collisions seem to be no different than the perturbative disturbances caused by the non-normalizable mode of the metric fluctuation. GGV compute that the typical observer will see multipole moments on the sky, but as we have seen, similar multipole moments can also occur perturbatively.

In the bottom half of Fig. 14 we see another type of collision in which the colliding bubbles correspond to two different true vacua: red (r) and blue (b). But in this case red and blue are on the same moduli space, so that they are connected by a flat direction.[19] Both vacua are included within the hat. In the bulk space-time they bleed into each other, so that as one traverses a spacelike surface, blue gradually blends into purple and then red.

On the other hand, the surface Σ is sharply divided into blue and red regions, as if by a one-dimensional domain wall. This seems to be a new phenomenon that does not occur in perturbation theory about either vacuum.

As an example, consider a case in which a red vacuum nucleation occurs first, and then much later a blue vacuum bubble nucleates. In that case the blue patch on the boundary will be very small and the census taker will see it occupying a tiny angle on the sky. How does the boundary field theorist interpret it? The best description is probably as a small blue ''instanton'' in a red vacuum.[20] In both the bulk and boundary theory this is an exponentially suppressed, nonperturbative effect.

However, in a conformal field theory the size of an instanton is a modulus that must be integrated over. As the instanton grows the blue region engulfs more and more of the boundary. Eventually the configuration evolves to a blue 2D vacuum, with a tiny red instanton. One can also think of the two configurations as the observations of two different census takers at a large separation from one another. Which one of them is at the center is obviously ambiguous.

The same ambiguous separation into dominant vacuum, and small instanton, can be seen another way. The nucleation sites of the two bubbles are separated by a spacelike interval. There is no invariant meaning to say that one occurs before the other. A element of the de Sitter symmetry group can interchange which bubble nucleates early and which nucleates later.

Nevertheless, a given census taker will see a definite pattern on the sky. One can always define the CT to be at the center of things, and integrate over the relative size of the blue and red regions. Or one can keep the size of the regions fixed—equal for example—and integrate over the location of the CT.

From both the boundary field theory, and the bubble nucleation viewpoints, the probability for any finite number of red-blue patches is zero. Small red instantons will be sprinkled on every blue patch and vice versa, until the boundary becomes a fractal. The fractal dimensions are closely connected to operator dimensions in the boundary theory. Moreover, exactly the same pattern is expected from multiple bubble collisions.

[19] We assume that there is no symmetry along the flat direction, and that there are only two tunneling paths from the false vacuum: one to red, and one to blue.

[20] We are using the term instanton very loosely. On the one hand, the occurrence of such blue patches is nonperturbative and exponentially suppressed by the decay rate in the bulk. But there does not seem to be any topological stability to the object.

But the census taker has a finite angular resolution. He cannot see angular features smaller than $\delta\alpha \sim \exp(-T_{\rm CT})$. Thus he will see a finite sprinkling of red and blue dust on the sky. As $T_{\rm CT}$ increases, the UV cutoff scale tends to zero and the CT sees a homogeneous ''purple'' fixed-point theory.

The red and blue patches are reminiscent of the Ising spin system (coupled to a Liouville field). As in that case, it makes sense to average over small patches and define a continuous ''color field'' ranging from intense blue to intense red. It is interesting to ask whether Σ would look isotropic, or whether there will be finite multipole moments of the renormalized color field (as in the case of the χ field). The calculations of GGV suggest that multipole moments would be seen. But unless for some reason there is a field of exactly zero dimension, the observational signal should fade with census taker time.

There are other types of collisions that seem to be fundamentally different from the previous. Let us consider a model landscape with three vacua—two false, B and W (black and white); and one true vacuum T. Let the vacuum energy of B be bigger than that of W, and also assume that the decays $B \to W$, $B \to T$, and $W \to T$ are all possible. Let us also start in the black vacuum and consider a transition to the true vacuum. The result will be a hat bounded by Σ.

However, if a bubble of W forms, it may collide with the T bubble as in Fig. 15. The W bubble does not end in a hat but rather, on a spacelike surface. By contrast, the true-vacuum bubble does end in a hat. The surface Σ is defined as always, by following the lightlike generators of the hat backward until they enter the bulk—either Black or White—as in Fig. 15.

In this case a portion of the boundary Σ butts up against B, while another portion abuts W. In some ways this situation is similar to the previous case where the boundary was separated into red and blue regions, but there is no analog of the gradual bleeding of vacua in the bulk. In the previous case the census taker could smoothly pass from red to blue. But in the current example, the CT would have to crash through a domain wall in order to pass from T to W. Typically this happens extremely fast, long before the CT could do any observation. In fact if we define census takers by the condition that they eventually reach the census bureau, then they simply never enter W.

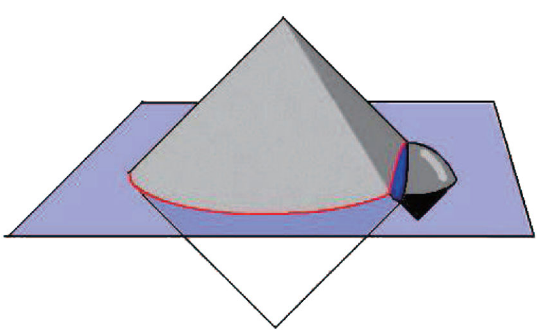

FIG. 15 (color online). A bubble of true vacuum forms in the black false vacuum and then collides with a bubble of white vacuum. The true vacuum is bounded by a hat but the white vacuum terminates in a spacelike surface. Some generators of the hat intersect the black vacuum and some intersect the white. Thus Σ, shown as the red curve, is composed of two regions.

From the field theory point of view this example leads to a paradox. Naively, it seems that once a W patch forms on Σ, a B region cannot form inside it. A constraint of this type on field configurations would obviously violate the rules of quantum field theory; topologically (on a sphere) there is no difference between a small W patch in a B background, and a small B patch in a W background. Thus field configurations must exist in which a W region has smaller black spots inside it. There is no way consistent with locality to forbid bits of B in regions of W.

Fortunately the same conclusion is reached from the bulk point of view. The rules of tunneling transitions require that if the transition $B \to W$ is possible, so must be the transition $W \to B$, although the probability for the latter would be smaller (by a large density of states ratio). Thus one must expect B to invade regions of W.

As the census taker time advances he will see smaller and smaller spots of each type. If one assumes that there are no operators of dimension zero, then the pattern should fade into a homogeneous average grey, although under the conditions we described it will be almost white.

The natural interpretation is that the boundary field theory has two phases of different free energy, the B free energy being larger than that of W. The dominant configuration would be the ones of lower free energy with occasional fluctuations to higher free energy.

B. The persistence of memory

Returning to Fig. 13, one might ask why no bubble formed along the red trajectory in the infinitely remote past. The authors of [24] argue that eternal inflation does not make sense without an initial condition specifying a past surface on which no bubbles had yet formed. That surface invariably breaks the $O(3,1)$ symmetry and distinguishes a ''preferred census taker'' who is at rest in the frame of the initial surface. He alone sees an isotropic sky whereas all the other census takers see nonzero anisotropy. The effect persists, no matter how late the nucleation takes place.

The ''persistence of memory'' reported in [24] had nothing to do with whether or not the census taker sees a fading signal: GGV were not speaking about census taker time at all. They were referring to the fact that no matter how long after the start of eternal inflation a bubble nucleates, it will remember the symmetry breaking imposed by the initial conditions; not whether the signal fades with $T_{\rm CT}$.

To be clear about this, consider two (proper) times, t_N and $t_{\rm CT}$. The first, t_N, is the time after the initial condition at which the census taker's bubble nucleates. The second, $t_{\rm CT}$, is the census taker time measured from the nucleation event. The persistence of memory refers to t_N. No matter how late the bubble nucleates, the census taker will see some memory of the initial conditions at finite $t_{\rm CT}$.

An entirely separate question is whether the symmetry breaking effects of the initial condition fade away with $t_{\rm CT}$ and if so how fast. The answer to this is determined by the spectrum of dimensions in the conformal field theory. If there are no dimension-zero operators the effects will dilute with increasing census taker time.

Let us consider how the persistence of memory fits together with the RG flow discussed earlier.[21] Begin by considering the behavior for finite δ in the limit of small a. It is reasonable to suppose that in integrating out the many scales between a and δ, the theory will run to a fixed point. Now recall that this is the limit of very large T. If in fact the theory has run to a fixed point it will be conformally invariant. Thus we expect that the symmetry $O(3, 1)$ will be unbroken at a very late time.

On the other hand, consider the situation of δ/a of order 1. The reference and bare scales are very close and there are few degrees of freedom to be integrated out. There is no reason why the effective action should be near a fixed point. The implication is that at a very early time (recall, $\delta/a = e^T$) the physics on a fixed time slice will not be conformally invariant. Near the beginning of an RG flow the effective action is strongly dependent on the bare theory. The implication of a breakdown of conformal symmetry is that there is no symmetry between census takers at different locations in space. In such situations the center of the (deformed) anti-de Sitter space is indeed special. The GGV boundary condition at the onset of eternal inflation is the same thing as the initial condition on the RG flow. In other words, varying the GGV boundary condition is no different from varying the bare fishnet theory.

Is it possible to tune the bare action so that the theory starts out at the fixed point? If this were so, it would be an initial condition that allowed exact conformal invariance for all time. Of course it would involve an infinite amount of fine tuning and is probably not reasonable. But there may be reasons to doubt that it is possible altogether, even though in a conventional lattice theory it is possible.

The difficulty is that the bare and renormalized theories are fundamentally different. The bare theory is defined on a variable fishnet whose connectivity is part of the dynamical degrees of freedom. The renormalized theory is defined on the fixed reference lattice. The average properties of the underlying dynamical fishnet are replaced by conventional fields on the reference lattice. Under these circumstances it is hard to imagine what it would mean to tune the bare theory to an exact fixed point.

The example of the previous subsection involving two false vacua, B and W, raises some interesting questions. First imagine starting with GGV boundary conditions such that, on some past spacelike surface, the vacuum is pure black and that a bubble of true vacuum nucleates in that environment. Naively the boundary is mostly black. That means that in the boundary theory the free energy of black must be lower than that of white.

But we argued earlier that white instantons will eventually fill Σ with an almost white, very light grey color, exactly as if the initial GGV condition were white. That means that white must have the lower potential energy. What then is the meaning of the early dominance of B from the 2D field theory viewpoint?

The point is that it is possible for two rather different bare actions to be in the same broad basin of attraction and flow to the same fixed point. The case of black GGV conditions corresponds to a bare starting point (in the space of couplings) where the potential of B is lower than W. During the course of the flow to the fixed point the potential changes so that at the fixed point W has the lower energy.

On the other hand, white GGV initial conditions correspond to starting the flow at a different bare point—perhaps closer to the fixed point—where the potential of W is lower.

This picture suggests a powerful principle. Start with the space of two-dimensional actions, which is broad enough to contain a very large landscape of 2D theories. With enough fields and couplings the space could probably contain everything. As Wilson explained [30], the space divides itself into basins of attraction. Each initial state of the Universe is described either as a GGV initial condition, or as a bare starting point for an RG flow. The endpoints of these flows correspond to the possible final states—the hats—that the census taker can end up in.

We have not exhausted all the kinds of collisions that can occur—in particular collisions with singular, negative cosmological constant vacua. A particularly thorny situation results if there is a Bogomol'nyi-Prasad-Sommerfield domain wall between the negative and zero cosmological-constant bubbles, then as shown by Freivogel, Horowitz, and Shenker [34] the entire hat may disappear in a catastrophic crunch. The meaning of this is unclear.

C. Flattened hats

In a broad sense this paper is about phenomenology. The census taker could be us: if we lived in an ideal thin-wall hat we would see, spread across the sky, correlation functions of a holographic quantum field theory; we could measure the dimensions of operators both by the time dependence of the received signals and their angular dependence; bubble collisions would appear as patches resembling instantons.

[21] These observations are based on the work with Steve Shenker.

Unfortunately (or perhaps fortunately) we are insulated from these effects by two forms of inflation—the slow-roll inflation that took place shortly after bubble nucleation and the current accelerated expansion of the Universe. The latter means that we do not live in a true hatted geometry. Rather we live in a flattened hat, at least if we ignore the final decay to a terminal vacuum.

The Penrose diagram in Fig. 16 shows an ancestor, with large vacuum energy, decaying to a vacuum with a very small cosmological constant. The important new feature is that the hat is replaced by a spacelike future infinity. Consider the census taker's final observations as he arrives at the flattened hat. It is obvious from Fig. 16 that he cannot look back to Σ. His past light cone is at a finite value of T^+. Thus for each time slice T, there is a maximum radial variable $R = R_0(T)$ within his ken, no matter how long he waits [35]. In other words, there is an unavoidable ultraviolet cutoff. It is evident that a final de Sitter bubble must be described by a theory with no continuum limit; in other words not only a nonlocal theory, but one with no ultraviolet completion.[22]

Another, perhaps more serious limitation, is that all of the memory of a past bubble nucleation may, for observational purposes, be erased by the slow-roll inflation that took place shortly after the CDL tunneling event—unless it lasted for the minimum permitted number of e-foldings [33]. In principle the effects are imprinted on the sky, but in an exponentially diluted form.

Nevertheless, it may be interesting to explore the phenomenology of a limiting case in which the amount of slow-roll inflation is very near the observational lower bound [33] and in which the cosmological constant is nonzero but arbitrarily small. The point of this exercise would be to get some idea of the possible effects of eternal inflation and how they are encoded in the FRW/CFT correspondence. In this paper, we will only make a few simple observations. In order to keep track of the ancestor cosmological constant and the present cosmological constant, we will use the notation H for the ancestor Hubble constant and h for the current value.

[22]This suggests that de Sitter space may have an intrinsic imprecision. Indeed, as Seiberg has emphasized, the idea of a metastable vacuum is imprecise, even in condensed matter physics where they are common. L. S. is grateful to Nathan Seiberg for discussions on this point.

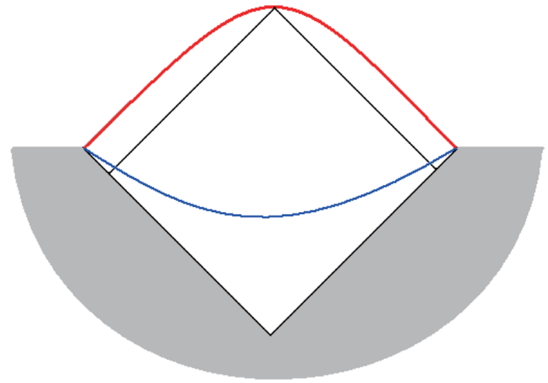

FIG. 16 (color online). If a CDL bubble leads to a vacuum with a small positive cosmological constant, the hat is replaced by a rounded spacelike surface. The result is that no census taker can look back to Σ.

The simplest way to compare the census taker with an observer in the later stages of a universe with a nonzero cosmological constant is to simply cut off the RG flow when the area $H^{-2}\exp(2T^+)$ is equal to the horizon area in the late-time de Sitter vacuum, namely h^{-2}. Thus we simulate the effect of the event horizon by imposing a final value of T^+—call it T_f^+,

$$e^{T_F^+} \sim \frac{H}{h}. \tag{8.1}$$

As an example, consider the situation described in Sec. VII.

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[1] J. R. Gott, Nature (London) **295**, 304 (1982); A. H. Guth and E. Weinberg, Nucl. Phys. **B212**, 321 (1983); P. J. Steinhardt, in *The Very Early Universe*, edited by G. W. Gibbons, S. W. Hawking, and S. T. C. Siklos (Cambridge University Press, Cambridge, 1983); J. R. Gott and T. S. Statler, Phys. Lett. **136B**, 157 (1984); A. D. Linde, Mod. Phys. Lett. A **1**, 81 (1986); A. D. Linde, Phys. Lett. B **175**, 395 (1986).

[2] R. Bousso and J. Polchinski, J. High Energy Phys. 06 (2000) 006.

[3] S. Kachru, R. Kallosh, A. Linde, and S. P. Trivedi, Phys. Rev. D **68**, 046005 (2003).

[4] L. Susskind, arXiv:hep-th/0302219.

[5] M. R. Douglas, J. High Energy Phys. 05 (2003) 046.

[6] See for example: R. Bousso, R. Harnik, G. D. Kribs, and G. Perez, Phys. Rev. D **76**, 043513 (2007); R. Bousso, B.

Freivogel, and I.-S. Yang, Phys. Rev. D **74**, 103516 (2006).
[7] S. R. Coleman and F. De Luccia, Phys. Rev. D **21**, 3305 (1980).
[8] B. Freivogel, Y. Sekino, L. Susskind, and C.-P. Yeh, Phys. Rev. D **74**, 086003 (2006); see also, D. S. Park, J. High Energy Phys. 06 (2009) 023.
[9] G. Polhemus, A. J. S. Hamilton, and C. S. Wallace, J. High Energy Phys. 09 (2009) 016.
[10] N. Goheer, M. Kleban, and L. Susskind, J. High Energy Phys. 07 (2003) 056.
[11] A. Maloney, S. Shenker, and L. Susskind (unpublished).
[12] T. Banks, W. Fischler, S. Shenker, and L. Susskind, Phys. Rev. D **55**, 5112 (1997).
[13] G. 't Hooft, arXiv:gr-qc/9310026; L. Susskind, J. Math. Phys. (N.Y.) **36**, 6377 (1995).
[14] J. Maldacena, Int. J. Theor. Phys. 38 (**1999**) 1113.
[15] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).
[16] L. Susskind and E. Witten, arXiv:hep-th/9805114.
[17] L. Susskind, L. Thorlacius, and J. Uglum, Phys. Rev. D **48**, 3743 (1993).
[18] Y. Kiem, H. L. Verlinde, and E. P. Verlinde, Phys. Rev. D **52**, 7053 (1995).
[19] T. Banks and W. Fischler, arXiv:hep-th/0102077.
[20] P. Hayden and J. Preskill, J. High Energy Phys. 09 (2007) 120; Y. Sekino and L. Susskind, J. High Energy Phys. 10 (2008) 065.
[21] T. Banks, W. Fischler, and L. Susskind, Nucl. Phys. **B262**, 159 (1985).
[22] R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: An Introduction to Current Research*, edited by Louis Witten (Wiley, New York, 1962), Chap. 7, pp. 227–265.
[23] T. Banks, Nucl. Phys. **B249**, 332 (1985).
[24] J. Garriga, A. H. Guth, and A. Vilenkin, Phys. Rev. D **76**, 123512 (2007).
[25] A. M. Polyakov, Phys. Lett. **103B**, 207 (1981).
[26] A. Kraemmer, H. B. Nielsen, and L. Susskind, Nucl. Phys. **B28**, 34 (1971).
[27] G. 't Hooft, Nucl. Phys. **B72**, 461 (1974).
[28] M. R. Douglas and S. H. Shenker, Nucl. Phys. **B335**, 635 (1990).
[29] D. J. Gross and A. A. Migdal, Phys. Rev. Lett. **64**, 127 (1990).
[30] K. G. Wilson, Rev. Mod. Phys. **47**, 773 (1975).
[31] J. Polchinski, Nucl. Phys. **B231**, 269 (1984).
[32] A. H. Guth and E. J. Weinberg, Nucl. Phys. **B212**, 321 (1983).
[33] B. Freivogel, M. Kleban, M. Rodriguez Martinez, and L. Susskind, J. High Energy Phys. 03 (2006) 039.
[34] B. Freivogel, G. T. Horowitz, and S. Shenker, J. High Energy Phys. 05 (2007) 090.
[35] N. Kaloper, M. Kleban, and L. Sorbo, Phys. Lett. B **600**, 7 (2004); M. Kleban, arXiv:hep-th/0412055.