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Can cosmic acceleration be caused by exotic massless particles?

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To describe dark energy we introduce a fluid model with no free parameter on the microscopic level. The constituents of this fluid are massless particles which are a dynamical realization of the unextended $D = (3 + 1)$ Galilei algebra. These particles are exotic as they live in an enlarged phase space. Their only interaction is with gravity. A minimal coupling to the gravitational field, satisfying Einstein's equivalence principle, leads to a dynamically active gravitational mass density of either sign. A two-component model containing matter (baryonic and dark) and dark energy leads, through the cosmological principle, to Friedmann-like equations. Their solutions show a deceleration phase for the early universe and an acceleration phase for the late universe. We predict the Hubble parameter $H(z)/H_0$ and the deceleration parameter $q(z)$ and compare them with available experimental data. We also discuss a reduced model (one-component dark sector) and the inclusion of radiation. Our model shows no stationary modification of Newton's gravitational potential.

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I. INTRODUCTION

Astrophysical observations (supernovae data [1,2]) suggest that the Universe is undergoing an accelerated expansion. This conclusion was drawn by interpreting these data in the framework of the cosmological Friedmann equations which describe the Universe as being homogeneous and isotropic on the largest scales (cf. [3]). Within this framework the origin of the cosmic acceleration is attributed to an exotic component, called dark energy, which is the source of repulsive gravitation (due to its negative pressure—according to the present interpretation).

But there exist other interpretations of the astrophysical data which do not invoke dark energy:

(i) Cosmic acceleration could be an apparent effect due to the averaging of large scale inhomogeneities in the Universe. (See [4] and the literature quoted therein. For a nonexpert explanation see [5].) However, it is an open question as to whether this interpretation is in agreement with all available cosmological data (see [6], Sec. 5.3).

(ii) The geometric part of the Einstein-Hilbert action could be modified by replacing the Ricci scalar R by an arbitrary function $f(R)$ or by introducing higher-order derivative terms (see [7] and the literature quoted therein). Some models are based on modified teleparallel gravity [8]. All such models, however, suffer from having to rely on an arbitrary function which cannot be derived from more fundamental assumptions.

Hence we assume that some sort of dark energy is the cause of the cosmic acceleration. Before we present our model let us give a very brief critical overview of the presently available dark energy models (see also [9]).

The simplest model, also called the ΛCDM model (see any review of dark energy, e.g., [6]), involves the use of a positive cosmological constant Λ whose value has to be determined from experimental data. Its small value (as determined by such considerations) causes some problems when we interpret Λ as the energy density of the vacuum (cf. [10,11]). The most popular dynamical dark energy models use instead a scalar field (see the reviews in [10]). However, such models have less predictive power as one can always construct a scalar field potential that gives rise to a given cosmic evolution [9].

Another class of models unifies dark matter and dark energy into a one-component dark sector. Then the acceleration comes from a new kind of interaction within the dark sector. In the case of a Chaplygin gas this interaction is given by an *ad hoc* assumed equation of state with negative pressure [11]. Other models use a complex scalar field [12] or a phenomenological antifriction force which can be understood as a nonminimal coupling of the cosmic gas to the curvature [13].

In summary, so far we do not have any dark energy model which has been derived from fundamental physics [6]. All known models contain at least one new parameter in the microscopic action [14].

In this paper we introduce our dark energy model which, on the microscopic level, contains no new parameters. To do this we start with the well-known fact that cosmology can be discussed without using general relativity as the basic Friedmann equations can be derived from the Newtonian gravity [15]. If we now want to consider some new nonrelativistic particles as the cause of cosmic acceleration they must necessarily be massless as the usual massive particles always lead to attractive gravitation. The

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possibility of having nonrelativistic massless particles as a dynamical realization of an extended Galilei algebra has already been discussed in some of our recent papers [16]. In the present paper we show that massless particles can exist also as a dynamical realization of an unextended Galilei algebra (a related realization for massless particles moving with infinite velocity has quite recently been found by Duval and Horvathy [17]).

The existence of nonrelativistic massless particles may appear strange at first sight; however, as we show in Sec. II, such particles possess a modified relation between energy and momentum (or velocity) and so they live in an enlarged phase space. For this reason we will call these particles ''exotic.'' Because of the enlarged phase space we have some freedom on how to introduce the gravitational coupling for our particles. Here we will do this in a minimal way which satisfies the general form of the Einstein equivalence principle but which does not use the concept of a rest mass of the gravitating particle. This can be stated in the form of the requirement that ''a freely falling observer does not feel any effect of gravitation'' [18]. This minimal coupling has the important property that, in a many exotic particle system, it leads to a dynamically generated active gravitational mass density of either sign which can then be a source of the gravitational field. Then a fluid mechanical generalization of this model can serve as a new model for dark energy.

A further extension, to a two-fluid model, including baryonic and dark matter besides dark energy, then leads, using the cosmological principle, to Friedmann-like equations for the cosmological scale factor $a(t)$. The solutions show a deceleration phase for the early universe and an acceleration phase for the late universe.

Furthermore, we show that our model allows for a one-component description of the dark sector on large (i.e., cosmological) scales. The choice between these two possibilities has to be made by comparison with the observational data on galactic scales.

We have also looked at the influence of our dark energy sector on local systems. We show that, in particular, it does not lead to the modification of Newton's gravitational potential.

The paper is organized as follows. In Sec. II we present our nonrelativistic massless particle model coupled minimally to gravitation. In Sec. III this model is generalized further and extended to a two-component fluid model for matter (baryonic and dark) and dark energy. In Sec. IV we show that we do not run into instabilities with our model. In Sec. V we describe some solutions, which satisfy the cosmological principle, of the corresponding fluid dynamical equations. These solutions show a decelerating universe at early times and an accelerating one at late times. We will see that on the cosmological level our model is an effective one-component model for the dark sector. In Sec. VI we include radiation. Observational consequences discussed in Sec. VII are as follows:

(i) The prediction of the Hubble parameter $H(z)/H_0$ and of the deceleration parameter $q(z)$, having fixed two integration constants by using measured cosmological parameters.

(ii) The proof that Newton's gravitational potential requires no stationary modification.

We conclude with some final remarks (Sec. VIII). Some technical details are given in Appendix A. In Appendix B we speculate on a relativistic generalization of our nonrelativistic particles describing tachyons.

II. NONRELATIVISTIC MASSLESS PARTICLES AND THEIR GRAVITATIONAL INTERACTION

In our second paper in [16] we have introduced the Lagrangian

$$L = p_i(\dot{x}_i - y_i) + q_i\dot{y}_i - \frac{1}{2\kappa}q_i^2, \tag{1}$$

where, in the three-dimensional case, x_i (y_i) are the components of spatial position (velocity) of a particle and p_i (q_i) are the components of the corresponding momenta. We use a Euclidean metric and Einstein's summation convention with $i = 1, 2, 3$. An overdot represents a time derivative.

The Lagrangian (1) leads to a dynamical realization of the acceleration-extended Galilei group in any dimension with one central charge (κ) for a noninteractive massless particle. Without the last term in (1) we have a dynamical realization of the Galilei group without any central charge (i.e., without any free parameter).

To show that we note that when $\kappa = \infty$, the equations of motion that follow from (1) are

$$\dot{x}_i = y_i, \qquad \dot{p}_i = 0, \qquad \dot{q}_i = -p_i, \qquad \dot{y}_i = 0. \tag{2}$$

These equations correspond to the canonical Poisson brackets

$$\{x_i, p_j\} = \delta_{ij}, \qquad \{y_i, q_j\} = \delta_{ij}, \tag{3}$$

which can be derived from the Hamiltonian

$$H = p_i y_i. \tag{4}$$

If we now introduce the conserved Galilean boost generator K_i which is given by

$$K_i = p_i t + q_i, \tag{5}$$

we find that

$$\{p_i, K_j\} = 0, \tag{6}$$

which clearly shows that we are dealing with a massless particle.

Going the other way, i.e., by starting with $m = 0$, with (6) as a requirement, it can be shown that the Lagrangian

$$L = p_i(\dot{x}_i - y_i) + q_i\dot{y}_i, \tag{7}$$

defined in a 12-dimensional phase space, is the minimal one [19].

Furthermore, we note that the conserved angular momentum is given by

$$\vec{J} = \vec{x} \times \vec{p} + \vec{y} \times \vec{q}, \tag{8}$$

and that the Poisson brackets of $\vec{p}$, $\vec{K}$, $\vec{J}$, and H build the unextended Galilei algebra. The presence of the second term in (8) shows that our particles possess a nontrivial spin.

To couple this particle to gravity we start with the general form of Einstein's equivalence principle. In a nonrelativistic context this can be stated as follows: locally, i.e., at each fixed space point $\vec{x}$, a gravitational force $-\vec{\nabla}\phi(\vec{x}, t)$ is equivalent to a time-dependent acceleration $\vec{b}(t)$. The only known equation of motion for the particle trajectory $\vec{x}(t)$ satisfying this form of the equivalence principle is given by the Newton law:

$$\ddot{x}_i(t) = -\partial_i \phi(\vec{x}(t), t), \tag{9}$$

because (9) is invariant with respect to arbitrary time-dependent translations [20]

$$x_i \longrightarrow x_i' = x_i + a_i(t), \tag{10}$$

if $\phi(\vec{x}, t)$ transforms to

$$\phi'(\vec{x}', t) = \phi(\vec{x}, t) - \ddot{a}_i(t)x_i + h(t). \tag{11}$$

Hence considering $\phi(\vec{x}, t)$ as an external gravitational field we can take its interaction term with our particle $L_{\rm int}$ in the form:

$$L_{\rm int} = q_i \partial_i \phi(\vec{x}(t), t). \tag{12}$$

Clearly, with this term, the equation of motion for x_i is given by (9) and the second equation in (2) becomes $\dot{p}_i = q_k \partial_k \partial_i \phi$. Then our system is invariant under arbitrary time-dependent translations (10) where ϕ transforms according to (11) with q_i and p_i being invariant.

III. TWO-FLUID DYNAMICS

In this section we consider a two-fluid cosmological model where one fluid component M consists of massive matter (baryonic and dark) and the other fluid D consists of the exotic massless particles, introduced in the previous section and representing dark energy. The only interaction considered within the fluids and between them is gravitational.

A. Lagrange picture

First we generalize the dark energy model introduced in the previous section to the continuum case by introducing comoving coordinates $\vec{\xi} \in R^3$ [21], add continuous massive matter with its standard gravitational interaction, and use the usual Lagrangian for the gravitational field.

Then our Lagrangian is given by

$$L = L_M + L_D + L_\phi, \tag{13}$$

where

$$L_M = m \int d^3\xi \Big(y_i^M \Big(\dot{x}_i^M - \frac{1}{2} y_i^M \Big) - \phi(\vec{x}^M, t) \Big), \tag{14}$$

where m is a mass parameter giving (14) the correct dimension,

$$L_D = \int d^3\xi (p_i(\dot{x}_i^D - y_i^D) + q_i^D \dot{y}_i^D + q_i \partial_i \phi(\vec{x}^D, t)), \tag{15}$$

and

$$L_\phi = -\frac{1}{8\pi G} \int d^3x (\vec{\nabla}\phi(\vec{x}, t))^2. \tag{16}$$

In these expressions all phase space variables are functions of $\vec{\xi}$ and t, i.e., $\vec{x}^M = \vec{x}^M(\vec{\xi}, t)$, etc.

Note that both L_D and L_M are invariant, up to a total time derivative, under the transformations (10) and (11).

The equations of motion (EOM) corresponding to L are given by

(i) M sector

$$\dot{x}_i^M = y_i^M, \qquad \dot{y}_i^M = -\partial_i \phi(\vec{x}^M, t). \tag{17}$$

(ii) D sector

$$\begin{aligned} &\dot{x}_i^D = y_i^D, \qquad \dot{q}_i^D = -p_i^D, \\ &\dot{y}_i^D = -\partial_i \phi(\vec{x}^D, t), \qquad \dot{p}_i^D = q_k \partial_k \partial_i \phi(\vec{x}^D, t). \end{aligned} \tag{18}$$

(iii) ϕ sector

$$\begin{aligned} \Delta\phi(\vec{x}, t) = 4\pi G \int d^3\xi (& m\delta(\vec{x} - \vec{x}^M(\vec{\xi}, t)) \\ &+ q_i(\vec{\xi}, t)\partial_i \delta(\vec{x} - \vec{x}^D(\vec{\xi}, t))). \end{aligned} \tag{19}$$

The last term in (19) represents a dynamically generated active gravitational mass density.

B. Eulerian picture

In the Eulerian picture the dynamics of the fluid is described in terms of $\vec{x}$ and t dependent fields: particle number density $n(\vec{x}, t)$, velocity $u_i(\vec{x}, t)$, momentum $p_i(\vec{x}, t)$, and pseudomomentum $q_i(\vec{x}, t)$.

Assuming uniform distribution in $\vec{\xi}$ the Lagrangian phase space variables are transformed to the Eulerian fields by

$$n(\vec{x}, t) = \int d^3\xi \delta^3(\vec{x} - \vec{x}(\vec{\xi}, t)), \tag{20}$$

and

$$n(\vec{x},t)p_i(\vec{x},t) = \int d^3\xi\, p_i(\vec{\xi},t)\delta^3(\vec{x}-\vec{x}(\vec{\xi},t)), \tag{21}$$

and an analogous expression for $u_i(\vec{x},t)$ [in the expression above replace $p_i(\vec{x},t)$ by $u_i(\vec{x},t)$ and $p_i(\vec{\xi},t)$ by $y_i(\vec{\xi},t)$], and similarly for $q_i(\vec{x},t)$. In fact, (21) holds for any function of relevant variables.

To derive the EOM in the Eulerian picture we follow the standard procedure (cf. [21]) and obtain from (17)–(19) by using (20) and (21) the corresponding equations in the Eulerian picture:

$$\partial_t n^A(\vec{x},t) + \partial_k(n^A u_k^A)(\vec{x},t) = 0, \tag{22}$$

where $A = (M, D)$, i.e., the continuity equations for the particle number densities n^M and n^D, and from (19) the Poisson equation for the gravitational field

$$\triangle\phi(\vec{x},t) = 4\pi G(\rho^M + \partial_i(n^D q_i)), \tag{23}$$

where the mass density ρ^M is defined by $\rho^M := mn^M$.

Note that the last term in (23) represents the dynamically generated active gravitational mass density of the dark energy fluid.

We have, in addition, the following Euler equations:

$$D_t^M u_i^M = -\partial_i\phi \tag{24}$$

from the second equation in (17) and from the third equation in (18)

$$D_t^D u_i^D = -\partial_i\phi,$$

where we have defined $D_t^A = \partial_t + u_i^A\partial_i$.

Suppose now that u_i^M and u_i^D obey the same initial conditions. Then (24) shows that $u_i^D = u_i^M = u_i$, i.e., (24) becomes one universal Euler equation valid for all fluid components.

$$D_t u_i = -\partial_i\phi. \tag{25}$$

Finally, the second and fourth equations in (18) give

$$D_t q_i = -p_i, \qquad D_t p_i = q_k\partial_i\partial_k\phi. \tag{26}$$

Looking at (25) and (26) we note that, in contrast to standard fluid mechanics, the two vector fields $\vec{p}(\vec{x},t)$ and $\vec{u}(\vec{x},t)$ are not parallel to each other.

C. Symmetries

First we note that all of our EOM (22)–(26) are obviously rotationally symmetric.

To consider other symmetries we observe that if we perform an infinitesimal time-dependent translation $\delta x_i = a_i(t)$ we see that

$$\begin{aligned}\delta u_i(\vec{x},t) &= \dot{a}_i(t) - a_k(t)\partial_k u_i(\vec{x},t),\\ \delta\phi(\vec{x},t) &= -\ddot{a}_i(t)x_i + h(t) - a_k(t)\partial_k\phi(\vec{x},t),\end{aligned} \tag{27}$$

and

$$\delta\zeta(\vec{x},t) = -a_k(t)\partial_k\zeta(\vec{x},t),$$

where $\zeta \in (n^A, p_i, q_i)$. Thus the EOM are invariant under such translations and so, locally, the general form of Einstein's principle of equivalence is satisfied as in general relativity. Moreover, as shown recently by one of us (P. C. S.), we obtain, when neglecting the massive matter part, as symmetry algebra, the expansionless conformal Galilei algebra with dynamical exponent $z = \frac{5}{3}$ [19].

D. Stress tensor and pressure

To see that our massless particle fluid may, indeed, represent dark energy we show now that the pressure can be negative. To do this we consider the local momentum conservation

$$\partial_t P_i(\vec{x},t) + \partial_j T_{ij}(\vec{x},t) = 0, \tag{28}$$

where, in our case, the momentum density P_i and stress tensor T_{ij} are given by

$$P_i(\vec{x},t) = (np_i)(\vec{x},t), \qquad T_{ij}(\vec{x},t) = (P_i u_j)(\vec{x},t) + \mathcal{P}\delta_{ij}, \tag{29}$$

and $\mathcal{P}$ is the pressure.

Note that the stress tensor is not symmetric. The deeper reason for that is the presence of a spin part in the conserved angular momentum [see (8) and [22]].

To have the system as simple as possible we consider first a one-dimensional self-gravitating massless particle fluid. The EOM for the momentum field $p(x,t)$ then, due to the Poisson equation (19), becomes

$$D_t p = 4\pi G q\partial_x(nq). \tag{30}$$

Using the continuity equation (22), together with (30) we obtain for (28)

$$\partial_t P + \partial_x(Pu - 2\pi G(nq^2)) = 0, \tag{31}$$

i.e., we get the negative pressure

$$\mathcal{P} = -2\pi G(nq)^2. \tag{32}$$

For a three-dimensional case we obtain a result similar to (32) if we assume that the vector field $n\vec{q}$ is irrotational (see Sec. V), i.e.,

$$(nq_k)(\vec{x},t) = \partial_k h(\vec{x},t). \tag{33}$$

Then, from (19), we get

$$\partial_i\phi = 4\pi G n q_i, \tag{34}$$

leading to the EOM for the momentum field p_i

$$D_t p_i = 4\pi G q_k\partial_i(nq_k), \tag{35}$$

and finally, analogously to (31), to the pressure

$$\mathcal{P} = -2\pi G(nq_k)^2. \tag{36}$$

If $n\vec{q}$ is not irrotational then the stress tensor T_{ij} is more complicated than that given in (29).

E. Nature of the gravitational mass

According to the Poisson equation (23) the active gravitational mass density of the dark energy fluid $\hat{\rho}^D$ is given by

$$\hat{\rho}^D = \partial_i(nq_i). \tag{37}$$

This result should be compared with the expression, from general relativity, for the perfect fluid

$$\hat{\rho}^D = \left(\rho^D + \frac{3\mathcal{P}}{c^2}\right), \tag{38}$$

where $c^2\rho^D$ is the energy density in the rest frame of the fluid and $\mathcal{P}$, the corresponding pressure. Neglecting the massive matter component and considering the particular case of an irrotational field $n\vec{q}$ [see (33)] we find from the Lagrangian (15) and (16) and the transformations (20) and (21) that the energy density $\mathcal{E}^D(\vec{x}, t)$ is given by

$$\mathcal{E}^D = np_iu_i - 2\pi G(nq_i)^2, \tag{39}$$

and so

$$\rho^D = -\frac{2\pi G(nq_i)^2}{c^2}. \tag{40}$$

Then, with $\mathcal{P}$ given by (36), we obtain for (38)

$$\hat{\rho}^D = -\frac{8\pi G(nq_i)^2}{c^2}. \tag{41}$$

The reason for the difference between (37) and (41) is obvious: in our nonrelativistic massless particle model there is no place for the velocity of light c. Furthermore, we note that (37) is not built from the components of the energy-stress tensor and, in fact, (37) is the simplest Galilei invariant expression in our model with the dimension of mass density.

IV. STABILITY CONSIDERATIONS

The Hamiltonian corresponding to the Lagrangian (15) of the dark energy fluid component is linear in the momenta p_i^D and thus not bounded from below. This property is well known for any higher-derivative Lagrangian (theorem of Ostrogradsky [23]) and it arises in our case because our one-particle Lagrangian considered in Sec. II may be understood as the limiting case of the higher-order Lagrangian (1), which in the configuration space takes the form $L = \frac{\kappa}{2}\ddot{x}_i^2$ (see the second part of [16]). However, for a free particle, this does not concern us too much, as the Hamiltonian (4) can always be transformed to a positive quadratic form by a complex-valued canonical transformation[1]:

$$p_i = iap_i' - bq_i', \qquad y_i = -bq_i' - iap_i',$$
$$q_i = \frac{1}{2b}y_i' - \frac{i}{2a}x_i', \qquad x_i = -\frac{i}{2a}x_i' - \frac{1}{2b}y_i', \tag{42}$$

where a, b are arbitrary real numbers, leading to

$$H = a^2p_i'^2 + b^2q_i'^2. \tag{43}$$

Thus all serious problems (like, e.g., the collapse in the classical case or a nonunitary time development in the quantum case) may arise only in the presence of interactions [25,26].

Another possible instability of interacting theories containing negative energy involves the spontaneous decay of any state into a collection of positive and negative energy particles [26]. This instability is excluded in our model due to the particle number conservation (22).

To find out what happens in our fluid model it is sufficient to consider only a self-gravitating massless particle system, i.e., to neglect its dark matter component. To treat things as simple as possible, in the following we consider only a one-dimensional system. In Sec. IV A we show that a two-particle system at zero energy can, indeed, collapse. When, in the next section, we generalize these considerations to the continuum, i.e., a fluid dynamical case, we find that in this case the collapse does not take place. This reassures us in our belief that the three-dimensional case is also collapse free.

A. Two-particle case

Specializing the EOM (18), for $d = 1$, to the two-particle case we obtain for the relative motion:

$$\ddot{x}(t) = 4\pi Gq(t)\delta(x(t)), \tag{44}$$

and

$$\ddot{q}(t) = 2\pi Gq^2(t)\delta'(x(t)), \tag{45}$$

where we have defined the relative variables (the indices 1, 2 label the two particles)

$$x := x_1 - x_2, \qquad q := q_1 - q_2.$$

The variables of the two-particle center, $R := \frac{1}{2}(x_1 + x_2)$ and $Q := \frac{1}{2}(q_1 + q_2)$ satisfy

$$\ddot{Q} = 0, \quad \text{and} \quad \ddot{R} = -4\pi GQ\delta(x). \tag{46}$$

To obtain these equations we have used the fact that, for a generic $x \in R^1$, $\partial_x\phi(x, t)$ is given, due to (19), by

$$\partial_x\phi(x, t) = 4\pi G\sum_{\alpha=1}^{2}\delta(x - x_\alpha(t))q_\alpha(t). \tag{47}$$

Here we have taken the particular solution $Q(t) = 0$ of (46).

Then the energy of the two-particle system is given by

$$E = -\frac{\dot{q}\,\dot{x}}{2} + \pi Gq^2\delta(x). \tag{48}$$

[1]Similar ideas can be used to demonstrate the absence of ghosts in the Pais-Uhlenbeck model. This was recently shown, in a different way, by Bender and Mannheim [24].

A solution of the EOM (44) and (45), for vanishing energy E, is clearly given by

$$\dot{x} = 2\pi G\lambda q^2, \qquad \dot{q} = \lambda^{-1}\delta(x), \tag{49}$$

where λ is arbitrary and needed for dimensional reasons.

If we now take $\lambda > 0$ so that

$$\dot{x}(t) \geq 0, \quad \text{for all } t \in R^1, \tag{50}$$

and choose

$$x(t) < 0, \dot{x}(t) > 0 \quad \text{for } t < t_0 \quad \text{and} \quad x(t_0) = 0, \tag{51}$$

we obtain from (49) that

$$q(t) = q_0(1 - \theta(t - t_0)), \tag{52}$$

and

$$\dot{x}(t) = \begin{cases} 2\pi G\lambda q_0^2 & \text{for } t < t_0, \\ 0 & \text{for } t > t_0, \end{cases} \tag{53}$$

i.e., the two particles collide at $t = t_0$ and stay together for all later times. This is a collapse situation.

B. Hydrodynamic case

In the Eulerian picture for $d = 1$ the analog of (47) is now

$$\partial_x \phi = 4\pi G n q, \tag{54}$$

and so the hydrodynamic EOM obtained from (22), (25), and (26) take the form

$$\partial_t n + \partial_x(un) = 0, \tag{55}$$

$$D_t u = -4\pi G n q, \tag{56}$$

and

$$D_t^2 q = -4\pi G q \partial_x(nq). \tag{57}$$

In analogy to (39) the energy density $\mathcal{E}(x, t)$ is given by

$$\mathcal{E} = n(-(D_t q)u - 2\pi G n q^2). \tag{58}$$

Next we proceed as in the two-particle case. We make the ansatz

$$u = 2\pi G\lambda q^2. \tag{59}$$

Then from (56) we find that

$$D_t q = -\lambda^{-1} n, \tag{60}$$

which, together with (59), demonstrates the vanishing of the energy density $\mathcal{E}$.

Then, as can be easily checked, the unique solution of the remaining EOM (55), (57), and (59) is given by

$$n(x, t) = n_0 (= \text{const}), \qquad q(x, t) = \frac{-n_0 t + c}{\lambda}, \tag{61}$$

demonstrating a collapse-free situation.

V. COSMOLOGICAL SOLUTIONS OF FLUID DYNAMICS EQUATIONS

In order for the Universe to be homogeneous and isotropic on large scales we require, as usual, that

$$n^A = n^A(t), \tag{62}$$

and

$$u_i = \frac{\dot{a}(t)}{a(t)} x_i, \tag{63}$$

where $a(t)$ is the cosmic scale factor.

Then (25) tells us that

$$\partial_i \phi = x_i \varphi(t), \tag{64}$$

with

$$\varphi(t) = -\frac{\ddot{a}}{a}.$$

Putting (63) and (64) into the second equation in (26) gives us

$$D_t p_i = -q_i \frac{\ddot{a}}{a}. \tag{65}$$

To solve the first equation in (26) and (65) we make an ansatz

$$q_i = f_q(t) x_i, \quad \text{and} \quad p_i = f_p(t) x_i. \tag{66}$$

Then, using (63)–(65) we eliminate f_p and get

$$\ddot{f}_q + 2\frac{\dot{a}}{a}\dot{f}_q = 0, \tag{67}$$

which can be integrated once giving us

$$\dot{f}_q(t) = \frac{\beta}{a^2(t)}, \quad \text{with } \beta = \text{const.} \tag{68}$$

Furthermore, with (62) and (63) the continuity equations (22) can be integrated as usual giving us for ρ^M and n^D

$$\rho^M = \frac{M}{\frac{4\pi}{3} a^3(t)}, \tag{69}$$

and

$$n^D = \frac{D}{\frac{4\pi}{3} a^3(t)}, \tag{70}$$

where M and D are positive constants.

Inserting (64) and (69) into the Poisson equation (23) we get from (66)

$$-\ddot{a} = \frac{G}{a^2}(M + 3Df_q), \tag{71}$$

where f_q should be taken as a solution of (68). Equation (71) is one of our Friedmann-like equations.

We should now distinguish two cases:

(i) $\beta = 0$, which implies $f_q = \text{const}$. Then Eq. (71) gives us that

$$\ddot{a} > 0 \quad \text{for any } t, \quad \text{if } f_q < -\frac{M}{3D}, \tag{72}$$

i.e., we obtain an accelerated expansion for all times (this contradicts the known cosmological facts).

(ii) $\beta \neq 0$. Then putting (68) into (71) we get

$$-\ddot{a} = \frac{\dot{f}_q G}{\beta}(M + 3Df_q). \tag{73}$$

Integrating once we find

$$-\dot{a} = \frac{f_q G}{\beta}\left(M + \frac{3}{2}Df_q\right) + c_1. \tag{74}$$

Multiplying (74) by $\dot{f}_q$ and using (68) on the left-hand side we obtain

$$-\frac{\dot{a}\beta}{a^2} = \frac{\dot{f}_q f_q G}{\beta}\left(M + \frac{3}{2}Df_q\right) + c_1\dot{f}_q, \tag{75}$$

which after integration gives us

$$\frac{\beta}{a} = \frac{G}{2\beta}f_q^2(M + Df_q) + c_1 f_q + c_0, \tag{76}$$

where c_0 and c_1 are integration constants.

Let us now discuss, given (76), the behavior of f_q as a function of the scale factor a.

Performing the transformation:

$$f_q \rightarrow g(a) := f_q + \frac{M}{3D}, \tag{77}$$

we arrive at (redefining c_0 and c_1)

$$g^3(a) + c_1 g(a) + c_0\left(1 - \frac{a_t}{a}\right) = 0, \tag{78}$$

where we have defined

$$a_t := \frac{2\beta^2}{GDc_0}. \tag{79}$$

Let us look now at the solution of (78) with the constants c_0 and c_1 being positive, $c_{0,1} > 0$. First of all we note that the scale factor a may serve as a measure of time due to $\dot{a} > 0$ (expanding universe).

For $a < a_t$ we have $g(a) > 0$ and so, due to (71) $\ddot{a} < 0$. So, for $a < a_t$, we are in the deceleration phase of the early universe. On the other hand, clearly, for $a > a_t$ we have $g(a) < 0$ and then, due to (71), $\ddot{a} > 0$. So, for $a > a_t$ we are in the acceleration phase of the late universe and we see that a_t defines the transitional scale factor at which the deceleration stops and the acceleration takes over. It can easily be seen that the condition $c_{0,1} > 0$ is also necessary to obtain these results.

Next we observe that by differentiating (78) with respect to a we have

$$g'(a) = -\frac{a_t c_0}{a^2(c_1 + 3g^2(a))}, \tag{80}$$

where the prime denotes the derivative with respect to a. If we now put (80) into (68) we get

$$\dot{a} = -\frac{\beta(c_1 + 3g^2)(a))}{c_0 a_t}, \tag{81}$$

thus showing that, for $\dot{a} > 0$, we need $\beta < 0$.

Equation (81) is our second Friedmann-like equation. Note that the first Friedman-like equation (71) is a consequence of the second one (81) if $g(a)$ is a solution of the cubic equation (78).

To integrate (81) we need the explicit form of $g(a)$. To obtain $g(a)$ we note that $g(a)$ is the real valued solution of the cubic equation (78). This solution is given by

$$g(a) = u_+(a) + u_-(a), \tag{82}$$

with

$$u_\pm(a) = \left(-\frac{q}{2} \pm \left[\left(\frac{c_1}{3}\right)^3 + \left(\frac{q}{2}\right)^2\right]^{1/2}\right)^{1/3}, \tag{83}$$

where

$$q := c_0\left(1 - \frac{a_t}{a}\right).$$

Then, from (81) we find that

$$t - t_0 = \frac{c_0 a_t}{|\beta|}\int da \frac{1}{c_1 + 3g^2(a)}, \tag{84}$$

with $g(a)$ given by (82).

In Appendix A we present a detailed discussion of the evaluation of (84) in terms of the roots of (78). As our final results are not very transparent let us mention here some asymptotic results:

(i) At large a, i.e., $a \gg a_t$, a grows linearly with t. This follows from the observation that at large a, q goes to c_0 and so the integrand in (84) becomes independent of a.

(ii) At a very close to a_t we get from (78) that

$$g(a) \simeq -\frac{c_0}{c_1 a_t}(a - a_t). \tag{85}$$

Then, by choosing t_0 as the time at which $a = a_t$ we obtain from (84)

$$t - t_0 \simeq -\frac{-i}{\delta}\log\frac{1 + i\gamma(a(t) - a_t)}{1 - i\gamma(a(t) - a_t)}, \tag{86}$$

where

$$\gamma := \frac{\sqrt{3}c_0}{c_1^{3/2}a_t},$$

and

$$\delta := \frac{2\sqrt{3}|\beta|}{c_1^{1/2} a_t^2}.$$

Inverting (86) and taking the first terms of the power series expansion in $t - t_0$ we obtain

$$a(t) - a_t = \frac{c_1|\beta|}{c_0 a_t}(t - t_0) + \frac{|\beta|^3}{c_0 a_t^5}(t - t_0)^3 + O((t - t_0)^5). \tag{87}$$

Considering (85) with (71) and (80) it is easy to see that higher-order corrections to (85) do not change the first two terms in the expansion (87).

(iii) For small a, i.e., for $a \ll a_t$, we obtain from (78)

$$g(a) \simeq \left(\frac{c_0 a_t}{a}\right)^{1/3}, \tag{88}$$

leading to, due to (84) with $t_0 = 0$ and $a(0) = 0$,

$$a(t) \sim t^{3/5}, \tag{89}$$

thus showing that the combined effect of matter and dark energy at the early times differs from the behavior of the matter dominated universe for which $a(t) \sim t^{2/3}$.

Note that this result (89) is exactly the scale invariant solution for $a(t)$ corresponding to the dynamical exponent $z = 5/3$ [19].

A. Dark sector with one or two components?

In Sec. III we made the usual assumption that the dark sector possesses a two-component structure. However, our results for the Friedmann-like equations (71) and (81) and for Eq. (78) determining $g(a)$ are all independent of the constant M, defined by (69). Thus, as long as we do not compare (71) and (81) with the original Friedmann equations which would be physically senseless due to the different nature of the gravitational mass in our model (see Sec. III E), it is sufficient to keep the dark energy fluid (now to be called "dark fluid") as the only component within the dark sector. In our case, this dark fluid takes over the role of dark matter and dark energy, at least on large scales, like in the cases of the Chaplygin gas [11] or the complex scalar field [12]. The baryonic component, which corresponds to about 4% of the energy of the Universe, is negligible on these scales.

To describe the Universe correctly, at the scale of galaxies, the dark fluid must behave like dark matter, i.e., exhibit attractive gravitation at local scales (see [12] and the literature cited therein). This point still has to be examined in more detail.

VI. COSMOLOGY INCLUDING RADIATION

Including radiation (photons), and also massless neutrinos, within our framework, would require a full relativistic treatment. However, what we really need here is somewhat less ambitious. For the cosmology as outlined above we need a description of radiation as a nonrelativistic fluid[2] component R with an equation of state parameter (defined as the ratio of pressure and energy density) [6]

$$\omega^R = \tfrac{1}{3}. \tag{90}$$

To get the required result we follow McCrea [27] and Harrison [28] who extended Newtonian cosmology by taking pressure into account. For a homogeneous and isotropic universe we have therefore to add to our hydrodynamic equations the continuity equation for the radiation energy density $c^2\rho^R$

$$\dot{\rho}^R + 4\frac{\dot{a}}{a}\rho^R = 0, \tag{91}$$

whose solution is given by

$$\rho^R = \frac{R}{\frac{4\pi}{3}a^4(t)}, \tag{92}$$

where R is a positive constant. Furthermore we must change the Poisson equation (23) by adding to its right-hand side the active gravitational radiation mass density $2\rho^R(t)$ leading to

$$\triangle \phi = 4\pi G(\rho^M + \partial_i(n^D q_i) + 2\rho^R). \tag{93}$$

From (93) we conclude that the first Friedmann-like equation (71) now becomes

$$-\ddot{a} = \frac{G}{a^2}\left(M + 3Df_q + \frac{2R}{a}\right). \tag{94}$$

Unfortunately, when $R \neq 0$, it is not possible to integrate analytically the coupled system of differential equations (68) and (94). Nevertheless, we can conclude, as usual, that at very early times the last term in (94) dominates, i.e., the Universe is radiation dominated. In the following, we will consider, as we have already done in Sec. V, the Universe only for the later times, i.e., when the last term in (94) is negligible.

VII. OBSERVATIONAL CONSEQUENCES

Our exotic massless particles possess no nongravitational interaction, neither with the particles of the standard model nor with the dark matter particles. Thus their existence can only lead to observational consequences at cos-

[2]Note that within a hydrodynamic description of radiation the velocity field at a point $\vec{x}$ is an average over all directions of radiation velocities whose modulus is therefore less than c (it might even be small when compared to c).

mological scales (see Secs. VII A and VII B) and, perhaps, also at local scales (see Sec. VII D).

A. Predicting the Hubble and the deceleration parameters from our model

To calculate the Hubble parameter H in our model we introduce the redshift z by

$$a = \frac{1}{1+z}, \tag{95}$$

and then consider

$$H(z) := \frac{\dot{a}}{a}(z). \tag{96}$$

Then from (71) we find that $H(z)$ is given by

$$H(z) = \frac{|\beta| c_1}{a c_0 a_t}\left(1 + 3\frac{g^2(a)}{c_1}\right)\bigg|_{a=(1/1+z)}. \tag{97}$$

Next we define H_0 as $H_0 := H(z=0)$ and so find that

$$h(z) := \frac{H(z)}{H_0} = \frac{1 + 3\frac{g^2(1/1+z)}{c_1}}{1 + 3\frac{g^2(1)}{c_1}}(1+z). \tag{98}$$

In a similar way we see that the deceleration parameter $q(z)$ defined as

$$q(z) = -\frac{\ddot{a}}{aH^2(z)}$$

is given by

$$q(z) = \frac{6a_t}{a}\frac{c_0}{c_1^{3/2}}\frac{g(a)}{c_1^{1/2}}\left(1 + 3\frac{g^2(a)}{c_1}\right)^{-2}\bigg|_{a=(1/1+z)}. \tag{99}$$

Note that both are functions of only a_t and of $\kappa = c_0/c_1^{3/2}$. Hence to determine them we need two experimental data.

Before we try to determine $q(z)$ and $H(z)$, let us observe that there are a few things we can say about their behavior for any values of the two parameters. First of all, we easily see from (88) that at large z, i.e., for $z \gg z_t = \frac{1-a_t}{a_t}$,

$$\frac{g(\frac{1}{1+z})}{\sqrt{c_1}} \simeq \left(\frac{z}{1+z_t}\right)^{1/3}\left(\frac{c_0}{c_1^{3/2}}\right)^{1/3}, \tag{100}$$

and so from (99) we get that, for $z \gg z_t$,

$$q(z) \simeq \tfrac{2}{3}. \tag{101}$$

Moreover, $h(z)$ is monotonically increasing. To see this we take (97) and note that

$$h(z) = k(1 + 3\tilde{g}^2(z))(1+z), \tag{102}$$

where we have defined

$$\tilde{g}(z) := c_1^{-1/2} g\left(\frac{1}{1+z}\right),$$

and, similarly, the overall positive constant k. Then (80) is equivalent to

$$\tilde{g}\,'(z) = \frac{\kappa}{(1+z_t)(1+3\tilde{g}^2)} > 0. \tag{103}$$

Then

$$h'(z) = k\left(1 + 3\tilde{g}^2 + \frac{6\kappa(1+z)\tilde{g}}{(1+z_t)(1+3\tilde{g}^2)}\right) > 0. \tag{104}$$

In addition, from (100) and (102) we see that, for $z \gg z_t$,

$$h(z) \sim z^{5/3}. \tag{105}$$

B. Estimation of $H(z)$ and of $q(z)$

To obtain our "predictions" for $h(z)$ and $q(z)$, we use the data from the first reference in [29]. They give us $q(0) = -0.57$ and $z_t = 0.71$, both with small errors which we do not mention here as the curves we will show here depend very little on the exact values of these parameters. These values are obtained by fitting the matter part Ω_m of the ΛCDM model to observational data. $q(0)$ then serves to determine the constant κ in our model. We will give the curves obtained with these values subscript S.

We can also use the model independent values from the other two references in [29]. The data from the paper by Cunha are $q(0) = -0.73$ and $z_t = 0.49$ and from the paper by Lu *et al.* $q(0) = -0.788$ and $z_t = 0.632$. The curves corresponding to them will carry the indices C and L, respectively. Note that the values of κ for the three cases are $\kappa = 0.8667$ (S), 0.977 (C), and 1.166 (L).

In Figs. 1 and 2 we plot our predictions for $H(z)$ and $q(z)$, respectively. We have normalized $H(z)$ to its value at $z = 0$, so in fact, our plots are of $h(z)$. In Fig. 3 we present the corresponding values for $\tilde{g}(z)$. We note that all 3 cases are quite similar.

We have also attempted to compare our results to the experimental data given in Table 1 in [30]. The results given there have large experimental errors and are given

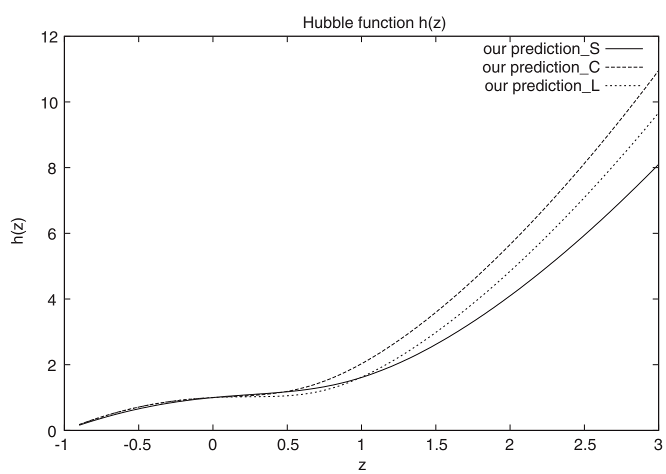

FIG. 1. Our prediction for $h(z)$.

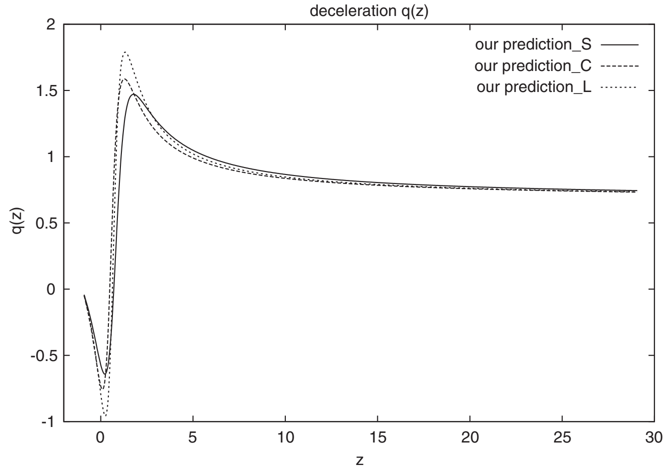

FIG. 2. Our prediction for $q(z)$.

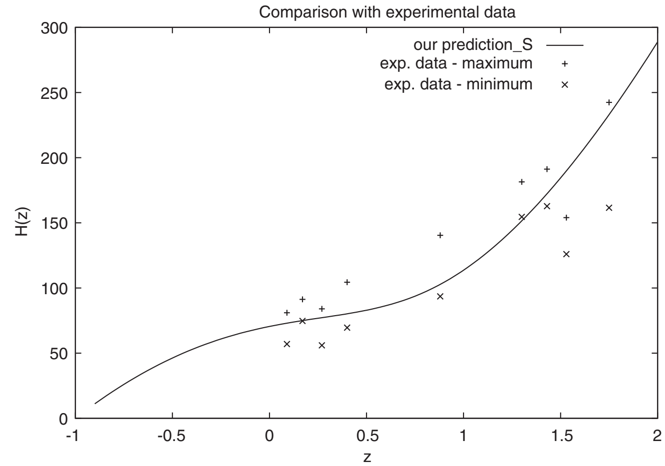

FIG. 4. Comparison with experimental data, $H(z)$ in units of $\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

only for a few values of z. Hence they will not be too conclusive or reliable. However, to perform any comparison we need the value of H_0. We can, of course, take this value from the second part of [2]. There we find $H_0 = 70.5$. Hence in Fig. 4 we present our data (with the normalization fixed by $H_0 = 70.5$) and compare them with the experimental data (constructed from the data in [30]) corresponding to the experimental data $+1$ standard deviation error (called "maximum") and -1 standard deviation error (called "minimum"). We note the general agreement and so we are heartened by this result.

C. How to compare our predictions with the ΛCDM model and the CMB-shift parameter?

For the ΛCDM model $h(z)$ is given, for a flat universe, by the equation (cf. the Appendix of [29])

$$h(z) = [\Omega_m(1+z)^3 + (1-\Omega_m)]^{1/2}, \tag{106}$$

where Ω_m is the present value of the matter content of the Universe. From the combined supernovae (SN) data in [2] we find that $\Omega_m \simeq 0.29$, whereas the combination of the SN data and the WMAP observations gives $\Omega_m \simeq 0.274$ (second part of [2]) so that it is sufficient to compare our predictions given in Fig. 1 with the plot of $h(z)$) for $\Omega_m = 0.3$ in [31], Fig. 2. We note a good consistency in the range $0 < z < 1$ for all three predictions but only a reasonable agreement with the S prediction for $1 < z < 2$. Note the comparison with the experimental data given in Fig. 4.

For larger values of z, all of our predictions, given the assumed input data, are systematically higher than the predictions of the ΛCDM model. However, for these values of z no (SN) or other direct data of $h(z)$ are really available. One source of information, for larger values of z, is based on the CMB-shift parameter R defined, for a flat universe, by (cf. the second part of [2])

$$R = \Omega_m^{1/2}\int_0^{1090}\frac{dz}{h(z)}, \tag{107}$$

where $h(z)$, within the integral range, has to be determined by some model estimate—containing the same value of Ω_m as used in front on the integral in (107) [32]. But as we have discussed in Sec. VA, on cosmological scales, our model does not allow the separation of the dark sector into the dark matter and dark energy parts. Hence, in agreement with the arguments given in [33] we have to ignore R in this paper.

To analyze the CMB data in terms of our model is a challenge for future research.

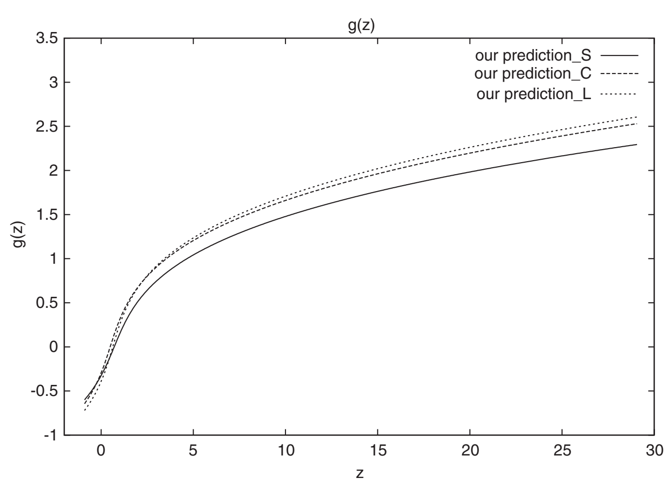

FIG. 3. Our prediction for $\tilde{g}(z)$.

D. Influence on local systems

Here we look at the problem of how a two-body system, bound by the standard Newtonian potential, may be affected by the dark sector proposed in this paper. To study this we consider two different mechanisms:

(i) The effect of the dark fluid at cosmological scales giving rise to an additional time-dependent term for the two-body potential,

$$\delta\phi(r,t) = -\frac{r^2}{2}\frac{\ddot{a}}{a}. \tag{108}$$

The equations for the two-body relative motion then take the form:

$$\ddot{\vec{r}} = \frac{\ddot{a}}{a}\vec{r} - \frac{G\mu}{r^3}\vec{r}, \tag{109}$$

where μ is the reduced mass.

As we do not have the explicit form of the time dependence of the scale factor $a(t)$, we use instead a as a measure of time. Then (109) leads to the following differential equation for $\vec{r}(a)$:

$$\vec{r}''\dot{a}^2 + \vec{r}'\ddot{a} = \frac{\ddot{a}}{a}\vec{r} - \frac{G\mu}{r^3}\vec{r} \tag{110}$$

or using the Friedman-like equations (71) and (81), we obtain

$$\vec{r}''(a)\beta^2\frac{(c_1 + 3g^2(a))^2}{c_0^2 a_t^2} - \frac{3DG}{a^2}g(a)\vec{r}'(a) = -\frac{3DG}{a^3}g(a)\vec{r}(a) - \frac{G\mu}{r^3(a)}\vec{r}(a), \tag{111}$$

where $g(a)$ is given by (82) and a prime denotes differentiation with respect to a.

To solve (111) numerically we would have to know, besides the constants a_t and $c_0/c_1^{3/2}$ known from Sec. VII B, also the values of constants β and c_0. Recent estimates of the effects caused by $\delta\phi(r,t)$ in the case of a constant $w^D < -1$ [34] have found observable effects on a time scale given by billions of years.[3] We expect similar results for our model.

(ii) The other issue involves a possible modification of Newton's gravitational potential by a local, stationary dark energy fluid. To study this we consider a point mass m located at $\vec{x} = 0$. We will show that the corresponding stationary dark energy flow leads to a vanishing extra gravitational mass density $\partial_i(\rho^D q_i^D)$ and so there is no extra contribution to $\phi(r)$. To see this we consider the D sector of our equations of motion given in Sec. III B for the stationary case. They become

$$\partial_k(n^D u_k) = 0, \tag{112}$$

$$u_k\partial_k u_i = -\partial_i\phi, \tag{113}$$

$$u_k\partial_k q_i^D = -p_i^D, \tag{114}$$

$$u_k\partial_k p_i^D = q_k^D\partial_k\partial_i\phi, \tag{115}$$

together with the Poisson equation

$$\Delta\phi = 4\pi G(m\delta(\vec{x}) + \partial_i(n^D q_i^D)). \tag{116}$$

Then we use (113) and (114) to eliminate p_i^D and $\partial_i\phi$ in (115) and obtain

$$u_k\partial_k u_l\partial_l q_i^D = q_k^D\partial_k u_l\partial_l u_i. \tag{117}$$

Looking at (117) we note that it implies that q_i^D has to be proportional to u_i

$$q_i^D \sim u_i, \tag{118}$$

and so, due to (112), we obtained the desired result, i.e.,

$$\partial_k(n^D q_k^D) = 0. \tag{119}$$

VIII. FINAL REMARKS

Given that there are already many dark energy models, what are the reasons why we have introduced another one? The reasons are twofold:

(i) There are no free parameters in the microscopic formulation of our model.

(ii) Our model introduces new physical ideas in the form of nonrelativistic massless particles whose minimal coupling to gravity leads to the generation of an active gravitational mass density of either sign.

This last point poses the question about the relation of these new physical ideas to Newton's and Einstein's theory of gravity. As our particles are a dynamical realization of the unextended Galilei algebra, they fit into the general scheme of nonrelativistic physics. The gravitational coupling, satisfying Einstein's equivalence principle, leads to the same equation of motion (8) in configuration space as in the massive case. Thus we can consider our model of a gravitationally coupled, nonrelativistic massless particle as an extension of Newton's theory of gravity.

However, it seems not possible to obtain our model as a nonrelativistic limit of a relativistic model. Massless relativistic particle models possess conformal Poincaré symmetry leading, in the nonrelativistic limit, to conformal Galilean symmetry (first part of [16]), i.e., $z = 1$. In our case, for a one-component dark sector, $z = \frac{5}{3}$. In Appendix B we have speculated that the relativistic generalization of our Galilean massless particles are tachyons. However, it may be that we are here in a situation similar to Hořava gravity [36]; i.e., we have nonrelativistic symmetry in the ultraviolet limit (small t) and approach general relativity only in the infared (large t) limit. However, to have such a picture we may have to modify our model. This is a challenge for further research.

As a drawback of our model one can consider the existence of additional dimensions in phase space. However, such a case is already well known from the related case of nonrelativistic massless fields (Galilean electromagnetism) in which the Lagrangian formulation requires the introduction of auxiliary fields [37]. In our

[3]For consideration of more general astronomical structures see [35] and the literature cited therein.

case the additional degrees of freedom lead in the Friedmann-like equations to undetermined constants which are integration constants along the additional phase space dimensions. The question then arises as to whether these constants can be determined *a priori* by some physical arguments. This point is currently under investigation.

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APPENDIX A

Here we demonstrate that the integral (84) can be calculated in a closed form.

First we note that due to (80) we have

$$c_0 a_t \int da (c_1 + 3g^2(a))^{-1} = -\int da a^2 g'(a). \qquad \text{(A1)}$$

Next we change the integration variable $a \to g(a)$ and use (78) to rewrite the right-hand side of (A1) as

$$-(c_0 a_t)^2 \int dg (g^3 + c_1 g + c_0)^{-2}. \qquad \text{(A2)}$$

We define the roots of the cubic equation

$$g^3 + c_1 g + c_0 = 0 \qquad \text{(A3)}$$

as g_i. They are given by

$$g_1 = v_+ + v_-, \qquad g_2 = -\frac{v_+ + v_-}{2} + \frac{v_+ - v_-}{2} i\sqrt{3}, \qquad g_3 = g_2^\star, \qquad \text{(A4)}$$

with

$$v_\pm = \left(-\frac{c_0}{2} \pm \left[\left(\frac{c_1}{3}\right)^3 + \left(\frac{c_0}{2}\right)^2\right]^{1/2}\right)^{1/3}. \qquad \text{(A5)}$$

Next we perform the decomposition

$$(g^3 + c_1 g + c_0)^{-1} = \prod_{i=1}^{3} (g - g_i)^{-1} = \sum_{i=1}^{3} a_i (g - g_i)^{-1}, \qquad \text{(A6)}$$

where a_i are given by

$$a_i = ((g_i - g_{i+1})(g_i - g_{i-1}))^{-1}. \qquad \text{(A7)}$$

Here $i = 1, 2, 3$ and cyclic permutation is assumed.

Putting all this together we perform the integration in (A2) and obtain

$$c_0 a_t \int da \frac{1}{c_1 + 3g^2(a)} = (c_0 a_t)^2 \sum_{i=1}^{3} a_i^2 \frac{1}{g(a) - g_i} - 2(c_0 a_t)^2 \sum_{i<j}^{3} \frac{a_i a_j}{g_i - g_j} \times \log \frac{g(a) - g_i}{g(a) - g_j}. \qquad \text{(A8)}$$

Clearly $a_1 = a_1^\star$ and $a_2^\star = a_3$.

APPENDIX B

Here we discuss a possible relativistic correspondence of the nonrelativistic massless particles introduced in Sec. II. Clearly, they cannot correspond to either massive or massless relativistic particles. However, they could correspond to tachyons which can be seen as follows:

(i) The relativistic generalization of the equations of motion (2) are given by the derivatives of the corresponding four-vectors with respect to the relativistic parameter τ:

$$\dot{x}_\mu = y_\mu, \qquad \dot{p}_\mu = 0, \qquad \dot{q}_\mu = -p_\mu, \qquad \dot{y}_\mu = 0. \qquad \text{(B1)}$$

(ii) From the second and fourth equations in (B1) we see that

$$p_\mu y^\mu = \text{const.} \qquad \text{(B2)}$$

In order to reproduce, in the nonrelativistic limit, the energy relation (4) the constant appearing on the right-hand side of (B2) must vanish; i.e., we must have

$$p_\mu y^\mu = 0. \qquad \text{(B3)}$$

(iii) From (B3) we see that

$$p_\mu p^\mu = \frac{(\vec{p} \cdot \vec{v})^2}{c^2} - \vec{p}^2 = \leq -\left(1 - \frac{v^2}{c^2}\right)\vec{p}^2 < 0. \qquad \text{(B4)}$$

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