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We investigate the consequences of replacing the global flavor symmetry of minimal flavor violation (MFV) $SU(3)_Q \times SU(3)_U \times SU(3)_D \times \dots$ by a discrete $\mathcal{D}_Q \times \mathcal{D}_U \times \mathcal{D}_D \times \dots$ symmetry. Goldstone bosons resulting from the breaking of the flavor symmetry generically lead to bounds on new flavor structure many orders of magnitude above the TeV scale. The absence of Goldstone bosons for discrete symmetries constitute the *primary* motivation of our work. Less symmetry implies further invariants and renders the mass-flavor basis transformation observable in principle and calls for a hierarchy in the Yukawa matrix expansion. We show, through the dimension of the representations, that the (discrete) symmetry in principle does allow for additional $\Delta F = 2$ operators. If though the $\Delta F = 2$ transitions are generated by two subsequent $\Delta F = 1$ processes, as, for example, in the standard model, then the four crystal-like groups $\Sigma(168) \approx \text{PSL}(2, \mathbb{F}_7)$, $\Sigma(72\varphi)$, $\Sigma(216\varphi)$ and especially $\Sigma(360\varphi)$ do provide enough protection for a TeV-scale discrete MFV scenario. Models where this is not the case have to be investigated case by case. Interestingly $\Sigma(216\varphi)$ has a (nonfaithful) representation corresponding to an A_4 symmetry. Moreover we argue that the, apparently often omitted, (D) groups are subgroups of an appropriate $\Delta(6g^2)$. We would like to stress that we do not provide an actual model that realizes the MFV scenario nor any other theory of flavor.

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I. INTRODUCTION

In the absence of Yukawa interaction, $G_F = U(3)^5 = U(3)_Q \times U(3)_{U_R} \times U(3)_{D_R} \times U(3)_L \times U(3)_{E_R}$ is the maximal global symmetry that commutes with the gauge groups of the standard model (SM) [1]. The Yukawa matrices $Y_{U,D,E}$ break this symmetry down to¹

$$G_F = U(3)^5 \xrightarrow{Y_{U,D,E}} U(1)_B \times U(1)_L. \quad (1)$$

It was realized a long time ago [3] that these sort of flavor symmetries forbid flavor changing neutral currents (FCNCs) at tree level. Most models of new physics do not possess a rigid flavor structure and large FCNC effects should be expected in general. On the other hand experiments in the quark flavor sector CLEO, BABAR, Belle, NA48, KTeV, KLOE, Tevatron, etc., do not, yet, show any significant deviations of FCNC or CP violation. This motivated the *effective field theory* approach, called minimal flavor violation (MFV) [4], where it is postulated that the sole sources of flavor violation are the Yukawa matrices. We shall be more precise later on. New physics contributions of the MFV type compared to the SM in K_0 and B_d oscillations can still be as large as $C_{SM}/C_{MFV} \approx$

$(0.5 \text{ TeV})^2/m_W^2$ [4].² It is the size of the Wilson coefficients C_{MFV} , including its implicit loop suppression factor, what we refer to as the “TeV-scale MFV scenario.” The extension of the concept from the quark sector to the lepton sector depends on the nature of the neutrino masses [6]. For notational simplicity we shall focus in this work on the quark sector. The results can easily be transferred to the lepton sector.

In the MFV approach one immediate question raises itself: How is the G_F symmetry broken? If the symmetry is broken spontaneously, which is what has been proposed and investigated so far (cf. [7]), this would then give rise to $3 \cdot 8 + 2 = 26$ CP -odd massless Goldstone bosons³ associated with the breaking of $U(3)^3 \rightarrow U(1)_B$. In connection with non-Abelian family symmetries such Goldstone bosons are known as familons [8]; they are formally similar to an axion arising from the breaking of an axial $U(1)$ symmetry. Because of the fact that they are Goldstone bosons their interactions with SM currents take the universal form $\sim (\partial_\mu \phi_F/f_F) j_\mu^{SM}$. The breaking scale f_F (decay constant) is bounded for instance from the processes $K^+ \rightarrow \pi^+ \bar{\nu} \nu$.

²We have to keep in mind that by quoting the scale 0.5 TeV we implicitly assume a loop suppression factor as in the SM. Besides loop suppression factors the actual scale of new physics is further masked by renormalization group effects as well as possible mixing angles of the underlying theory. Somewhat stronger bounds can be found in a more recent investigation [5].

³Bearing in mind mass contributions from explicit breaking and anomalous $U(1)$ factors.

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¹The further breaking of this group down to $U(1)_{B-L}$ due to the chiral anomaly [2] is not central to this work.

The latter competes with $K^+ \rightarrow \pi^+ \phi_F$ since the familon ϕ_F escapes detection due to its weak coupling to matter. In the case of a global family symmetry with $J_\mu^{\text{SM}} = \bar{s} \gamma_\mu d_L$ the bound $f_{F_{ds}} > 10^8$ TeV is rather high [9]. In MFV it might be a bit lower because of a $|V_{td} V_{ts}^*|$ suppression factor. The relation of f_F to the scale in the Wilson coefficient C_{MFV} is a model-dependent question, but we would expect it to be rather higher than lower. Summarizing, whereas direct bounds on (continuous) MFV allow for traces of TeV-scale operators, the dynamics of spontaneous symmetry breaking suggests the bounds are many orders of magnitude higher and would imply that MFV flavor physics is outside (current) experimental reach.

In this paper we aim to ameliorate this situation by replacing G_F by a discrete symmetry. Spontaneous breaking of discrete symmetries does *not* lead to Goldstone bosons (e.g. [11]). The absence of the latter in this framework constitutes the *primary* motivation of our work.⁴ The main remaining issue is then to investigate whether the reduced symmetries (or what discrete groups) do provide enough protection for a discrete TeV-scale MFV scenario. Are the bounds on the coefficient $C_{d\text{MFV}}$ in the TeV-scale range?

On the technical side this endeavor amounts to studying the invariants of discrete subgroups. The reduced symmetry unavoidably leads to further invariants as compared to G_F . This bears as a consequence that the flavor-mass basis transformation will become observable. The crucial question for discrete MFV is then at what level these new invariants are coming in. In this respect we find it useful to distinguish models where $\Delta F = 2$ operators arise from two $\Delta F = 1$ processes, as in the SM, and those where this is not the case.

Finally we would like to stress two points. First, since we are following the effective field theory approach there is no obvious connection to the scheme of constrained MFV [13], which assumes no new operators beyond those present in the SM. Second, there is no attempt made in this paper to explain valuable textures of the Yukawa matrices, i.e. the question of what is flavor. The symmetry is solely used to ensure that the Yukawa structure governs all flavor transition. Our work is complementary to the field of flavor models with family symmetries, revived by recent experimental information on neutrino masses and mixing angles [Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix]. Those models often involve discrete symmetries and extended Higgs sectors attempting to explain the origin of flavor hierarchies. For a recent review and references on the subject, with emphasis on the neutrino sector, we refer the reader to the write-up [14].

⁴Another alternative is to gauge the flavor symmetries, i.e. use the Higgs-Englert-Brout mechanism. A dedicated analysis [10] has been announced in Ref. [7]. This was investigated in connection with family symmetries some time ago [12].

The paper is organized as follows: In Sec. II we state the problem in a more precise form. Section III summarizes some useful facts about groups and gives an overview of the discrete $SU(3)$ subgroups. In Sec. IV it is shown at what level new invariants necessarily arise and which groups have the least invariants. Section V deals with the physical consequences of the previous findings and proposes to distinguish flavor models according to the generation mechanism of $\Delta F = 2$ operators. In the epilogue we summarize our findings and reflect on the framework and its possible extensions.

II. FORMULATION

A. Minimal flavor violation

In the SM the quark masses and the Cabibbo-Kobayashi-Maskawa (CKM) structure originate from the Yukawa Lagrangian

$$\mathcal{L}_Y = \bar{Q}_L H Y_D D_R + \bar{Q}_L H_c Y_U U_R + \text{H.c.}, \quad (2)$$

which breaks the flavor symmetry down to $U(1)_B$; cf. Eq. (1). The symbols U_R , D_R and $Q_L = (U_L, D_L)^T$ denote right- and left-handed $SU(2)_L$ singlets and doublets, respectively, of up $U = (u, c, t)$ and down $D = (d, s, b)$ quarks. The conjugate Higgs field is defined as $H_c = i\tau_2 H^*$.

It is observed that the quark flavor symmetry

$$G_q = SU(3)_{Q_L} \times SU(3)_{U_R} \times SU(3)_{D_R} \quad (3)$$

can be formally restored by associating to the Yukawa matrices the following transformation properties:

$$Y_U \sim (\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1})_{G_q}, \quad Y_D \sim (\mathbf{3}, \mathbf{1}, \bar{\mathbf{3}})_{G_q}. \quad (4)$$

In fact the flavor symmetry can even be further enhanced by two $U(1)$ factors by appropriately assigning $U(1)$ charges to the quark fields and the Yukawa matrices. In our opinion there is some freedom in choosing them.⁵

An effective theory constructed from the SM fields and the Yukawa matrices is then said to obey the principle of *minimal flavor violation* [4], if the higher dimensional operators are invariant under G_q and CP .⁶ Operators with $\Delta F = 1, 2$ then assume the following form [4]:

⁵Any pair of $U(1)$ charges for the fields Q_L , U_R , D_R , Y_U , and Y_D which leaves (1) invariant is in principle an option. N.B. in Ref. [7] $U(1)_{U_R} \times U(1)_{D_R}$ was chosen.

⁶The latter condition might be relaxed by allowing for arbitrary CP -odd phases in the coefficients C_{MFV} of the effective operators. This has been done, for example, for the minimal supersymmetric standard model (MSSM) in Ref. [15]. One could go even further and assume strong phases as well, which could be due to low energy degrees of freedom, such as the ones studied in the unparticle scenario [16]. Working in the MFV scenario we though implicitly assume that the new structure does not involve new light degrees of freedom.

$$\begin{aligned}
O^{\Delta F=1'} &= (\bar{D}_L Y_U Y_U^\dagger Y_D \sigma \cdot F D_R), \\
O^{\Delta F=1} &= (\bar{D}_L Y_U Y_U^\dagger \gamma_\mu D_L) \bar{D}_L \gamma^\mu D_L, \\
O^{\Delta F=2} &= (\bar{D}_L Y_U Y_U^\dagger \gamma_\mu D_L)^2,
\end{aligned} \tag{5}$$

in the left-handed D sector. The symbol $\sigma \cdot F = \sigma_{\mu\nu} F^{\mu\nu}$ denotes the contracted electromagnetic tensor, $2D_{L/R} \equiv (1 \pm \gamma_5)D$ and we would like to add that notation $\Delta F = 1'$, used here, is rather nonstandard. In the remainder of this paper we shall omit the explicit indication of the γ matrices. Transitions to right-handed quarks demand substitutions of the form $D_L \rightarrow Y_D D_R$, etc., by virtue of G_q invariance (3) and (4), leading to the well-known phenomenon of chiral suppression. The operators in the U sector are simply obtained by interchanging the role of the U and D families. Let us parenthetically note that the predictivity of MFV [4] in the D sector is in large part due to the fact that the top is much heavier than the other U quarks

$$(Y_U Y_U^\dagger)_{ij} = (V^\dagger \text{diag}(y_w, y_c, y_t)^2 V)_{ij} \simeq y_t^2 V_{3i}^* V_{3j}, \tag{6}$$

where $V = \mathcal{U}_L^\dagger \mathcal{D}_L$ is the CKM matrix, resulting from the biunitary diagonalization of the

$$\begin{aligned}
U_L &\rightarrow \mathcal{U}_L U_L, & U_R &\rightarrow \mathcal{U}_R U_R, \\
D_L &\rightarrow \mathcal{D}_L D_L, & D_R &\rightarrow \mathcal{D}_R D_R,
\end{aligned} \tag{7}$$

Yukawa matrices. The masses are related to the Yukawa couplings as $\sqrt{2}m_i = v y_i$ (with $v \simeq 246$ GeV) and it is worth noting that in the limit of degenerate masses the Glashow-Iliopoulos-Maiani mechanism reveals itself through $(Y_U^\dagger Y_U)_{ij} \rightarrow \delta_{ij}$.

B. Discrete minimal flavor violation

Replacing the continuous flavor symmetry with a discrete flavor symmetry requires the following additional information or assumptions:

- (a) the group $D_q = \mathcal{D}_{Q_L} \times \mathcal{D}_{U_R} \times \mathcal{D}_{D_R} \subset G_q$,
 $\mathcal{D} \subset SU(3)$;
- (b) the representation $R_3(\mathcal{D}_{Q_L})$ (3D irrep of families);
- (c) (possibly) Yukawa expansion $Y_{U/D} \rightarrow \kappa Y_{U/D}$,
 $\kappa \in \mathbf{R}$.

Since the three families have to transform in a 3D irreducible representation (irrep) this leads us naturally to study the discrete $SU(3)$ subgroups, which we denote by the symbol \mathcal{D} . The irrep has to be specified since some groups have more than one of them. The reduced symmetry (a) leads to new invariants and therefore renders the transition matrices (7) observable. We will argue in Sec. VA that this gives rise to a rather anarchic pattern of flavor transitions. This can be controlled by assuming a hierarchy in the

Yukawa expansion.⁷ In a perturbative-type model, for example, the operators with several Yukawa insertions might originate from higher dimensional operators suppressed by some high scale Λ and could mean $\kappa \simeq v/\Lambda$ if the Yukawa assume a vacuum expectation value around the electroweak scale. A rough but conservative estimate in Sec. VB indicates that for $\kappa \simeq 0.2$ $C_{d\text{MFV}}$ has the same bounds as C_{MFV} . In this paper we shall not discuss the $U(1)$ factors, e.g. (1), any further. We can think of them as being replaced by a discrete \mathbb{Z}_n symmetry, $D_q \rightarrow D_q \times \mathbb{Z}_n \dots$ and they do not play a role in the type of invariants we are considering.⁸ Generally the embedding could play a more subtle role. First there is freedom of embedding \mathcal{D} into $SU(3)$. We will discuss this issue in Sec. VA where it is argued that the observability of the rotation matrices (7) can be suppressed, modulo V_{CKM} , for certain groups by a suitable embedding. Second, we would like to draw the reader's attention to the assumption of D_q being embedded as a direct product of the discrete $SU(3)$ subgroups (a) into G_q . We shall comment on it in the epilogue.

III. ON (DISCRETE) GROUPS

In this section we shall first state a few useful facts about invariants, groups and representations, which is at the heart of this paper. Then we shall say a few things about the classification of discrete $SU(3)$ subgroups.

A. Useful facts

Consider irreps $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots\}$ of some group, continuous or discrete, and denote the explicit vectors of the irreps by $\mathcal{V} = \{a_r, b_s, c_t, \dots\}$. By the orthogonality theorem the number of times that the identity appears in the Kronecker product, denoted by n_1 ,

$$\mathbf{A} \times \mathbf{B} \times \mathbf{C} \times \dots = n_1 \mathbf{1} + \dots, \tag{9}$$

is *equal* to the number of independent invariants

$$I_n = I_n^{rst\dots} a_r b_s c_t \dots, \quad n = 1 \dots n_1, \tag{10}$$

that can be built out of the set \mathcal{V} specified above. Throughout this paper repeated indices are thought to be summed over. It is this statement that we shall use shortly for our main result. Before going on we would like to mention another fact, peculiar to discrete groups: The number of elements of the group is equal,

$$|\mathcal{D}| = \sum_{i \in \text{irreps}} \dim(R_i(\mathcal{D}))^2, \tag{11}$$

to the sum of the squared dimensions of all irreps.

⁷Such a notion has, for example, been introduced in Ref. [17]. The authors use the notation $\epsilon_{u,d} Y_{u,d}$ and distinguish the case $\epsilon_{u,d} \ll 1$ linear MFV and $\epsilon_{u,d} \sim \mathcal{O}(1)$ nonlinear MFV. In the latter case nonlinear σ -model techniques are imperative [17,18].

⁸In principle though, one could think of embeddings where they play a more subtle role; cf. the epilogue.

B. Discrete $SU(3)$ subgroups

The discrete subgroups of $SU(3)$ were classified a long time ago [19] and further analyzed as alternatives to $SU(3)_F$ in the context of the eightfold way [20]. Explicit representations and Clebsch-Gordan coefficients were systematically worked out in a series of papers around 1980 [21,22], partly motivated by the usage of discrete $SU(3)$ subgroups as an approximation to $SU(3)_{\text{color}}$ in lattice QCD, and further elaborated very recently [23–25] in the context of family symmetries. That the catalog of [20] as compared to [19] is not complete already surfaced in the 1980s [22,26] and has recently been discussed more systematically in a remarkable diploma thesis [27].

The discrete subgroups of $SU(3)$ are of two kinds. The first type consist of the analogue of crystal groups of which we list here the maximal subgroups: $\Sigma(168)$, $\Sigma(360\varphi)$ and $\Sigma(216\varphi)$. The factor φ can in general either be one or three depending on whether the center of $SU(3)$ is divided out or not. For the maximal subgroups it is three.⁹ The second kind are the infinite sequence of groups, sometimes called “dihedral-like” or “trihedral,” $\Delta(3n^2) \approx (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes \mathbb{Z}_3$ and $\Delta(6n^2) \approx (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes S_3$ for $n \in \mathbb{Z}$. The symbol “ \approx ” stands for isomorphic and “ \rtimes ” denotes a semidirect product. The largest irreps of $\Delta(3n^2)$ and $\Delta(6n^2)$ are three- and, respectively, six-dimensional, independent of n . Note, for all groups the number denotes the order of the group, i.e. the number of elements. In Appendix A 2 a we argue that the (D) groups recently emphasized in [27], or more precisely the six-parameter $D(n, a, b; d, r, s)$ matrix groups, are subgroups of $\Delta(6g^2)$, where g is the lowest common multiple of n , d and 2. We therefore do not need to discuss them separately. Further aspects of some groups and some of their invariants are discussed in Appendix A.

IV. INVARIANTS

The study of invariants I is the main ingredient of this paper since the effective Lagrangian approach. In the absence of the knowledge of the dynamics of the underlying model the effective Lagrangian assumes the following form:

$$\mathcal{L}^{\text{eff}} = \sum_n C_{d\text{MFV}}^n (I_n(\text{quarks, Yukawas}) + \text{H.c.}),$$

$$C_{d\text{MFV}}^n = \frac{c_n}{\Lambda^{\dim(I_n)-4}}, \quad (12)$$

where the dimension of the operator (invariant) brings in a certain hierarchy in the infinite sum above. As previously mentioned the association of C^n with the scale of new

⁹The further groups that are listed in [20] are all subgroups: $\Sigma(60) \subset \Sigma(360\varphi)$, $\Sigma(36\varphi) \subset \Sigma(72\varphi) \subset \Sigma(216\varphi)$ for $\varphi = 1, 3$ and of course generally $\Sigma(n) \subset \Sigma(n\varphi)$ for $n = 36, 72, 216, 360$ [20,27]. Once we have established an interesting candidates we shall proceed to have a look at its maximal subgroups.

structure is generally obscured by loop factors, mixing angles and renormalization group effects. Finding the invariants is equivalent to finding the constant tensors of the symmetry group. All objects are either in a $\mathbf{3}$ or the corresponding $\bar{\mathbf{3}}$ representation for which we write lower and upper indices, e.g.¹⁰

$$I^{(m,n)} \sim I_{a_1 \dots a_m}^{b_1 \dots b_n}, \quad (13)$$

as is common practice in the literature e.g. [28]. Nonconstant tensors will be denoted by $\mathcal{T}^{(m,n)}$. In principle this tensor classification is not sufficient for our general problem since there are three different group factors (8). It will though prove sufficient here to contract the other indices.¹¹ We therefore contract the \mathcal{D}_{U_r} index and directly write

$$(\Delta_U)_a{}^r \equiv (Y_U Y_U^\dagger)_a{}^r \quad (\Delta \in \mathcal{T}^{(1,1)}). \quad (14)$$

We shall often drop the subscript U when there is no reason for confusion. In the remainder we shall use the following notation for left-handed down quarks:

$$D_L = (d_L, s_L, b_L) \rightarrow D_i = (D_1, D_2, D_3) \quad (D_i \in \mathcal{T}^{(1,0)}). \quad (15)$$

The operators in Eq. (5) are associated with invariants of the form

$$\begin{aligned} I_n^{(2,2)} &= (I_n)_{rs}^{ab} (\bar{D}^r \Delta_a^s D_b), & \Delta F &= 1', \\ I_n^{(3,3)} &= (I_n)_{rst}^{abc} (\bar{D}^r \Delta_a^s D_b) \bar{D}^t D_c, & \Delta F &= 1, \\ I_n^{(4,4)} &= (I_n)_{rstu}^{abcd} (\bar{D}^r \Delta_a^s D_b) \bar{D}^t D_c^u D_d, & \Delta F &= 2. \end{aligned} \quad (16)$$

Note that there are groups where $\Delta F = 2$ operators are possible with $I^{(2,2)}$ but those structures are definitely too far away from MFV to be of any interest to us. The group $\Sigma(60) \approx A_5$ is an example which is discussed in Appendix A 1.

A. No. 27.—On the necessity of new invariants at the $I^{(4,4)}$ level

In this subsection we will present a general argument that there are necessarily further invariants for $I^{(4,4)}$, corresponding to the generic $\Delta F = 2$ transition (16), as compared to the $SU(3)$ case. Note the $\Delta F = 2$ operator (5) of G_q is obtained for $(I_1)_{rstu}^{abcd} \equiv \delta_r^a \delta_s^b \delta_t^c \delta_u^d$. The problem

¹⁰In this notation the basic constant tensors of $SU(3)$ are $I^{(1,1)} \sim \delta_a^b$, $I^{(0,3)} \sim \epsilon^{abc}$ and $I^{(3,0)} \sim \epsilon_{abc}$; cf. [28].

¹¹A refined treatment is only necessary when there are new $I^{(2,2)}$ invariants and those groups are not of interest to us anyway as explained in the text.

in Eq. (16) reduces to finding the invariants of the following Kronecker product:

$$\begin{aligned} K^{\mathcal{D}_{Q_L}} &= (\bar{\mathbf{3}} \times \mathbf{3} \times \bar{\mathbf{3}} \times \mathbf{3})_s \times (\bar{\mathbf{3}} \times \mathbf{3} \times \bar{\mathbf{3}} \times \mathbf{3})_s \\ &= n_1 \mathbf{1} + \dots, \end{aligned} \quad (17)$$

where s stands for the symmetric part and can be justified as follows: Since the Kronecker product is associative we may choose an ordering adapted to the symmetries in Eq. (16). The most economic way is to tensor the quarks and the Yukawas separately for which only the symmetric part, indicated above, is needed. Let us first look at the invariants that can be generated in the case where $\mathcal{D}_{Q_L} \rightarrow SU(3)$:

$$K^{SU(3)} = ((\mathbf{1} + \mathbf{8}) \times (\mathbf{1} + \mathbf{8}))_s \times ((\mathbf{1} + \mathbf{8}) \times (\mathbf{1} + \mathbf{8}))_s. \quad (18)$$

We shall focus on the product of the four $\mathbf{8}$'s:

$$\begin{aligned} K^{SU(3)} &= (\mathbf{8} \times \mathbf{8})_s \times (\mathbf{8} \times \mathbf{8})_s + \dots \sim (\bar{D}T^A D \times \bar{D}T^B D)_s \\ &\quad \times (\text{tr}[\Delta T^C] \times \text{tr}[\Delta T^D])_s + \dots. \end{aligned} \quad (19)$$

The Clebsch-Gordan coefficients of the $\mathbf{8}$ are, of course, just the generators T^A , $A = 1 \dots 8$, of the $SU(3)$ Lie algebra. The Kronecker product of the 8D irrep in $SU(3)$ reads (e.g. [29])

$$(\mathbf{8} \times \mathbf{8})_{SU(3)} = (\mathbf{1} + \mathbf{8} + \mathbf{27})_s + (\mathbf{8} + \mathbf{10} + \overline{\mathbf{10}})_a. \quad (20)$$

As argued above only the symmetric part is needed. Equation (20) is telling us on the one hand that the decomposition in (19) will lead to three different invariants but more importantly it tells us how the discrete subgroup \mathcal{D}_{Q_L} has to decompose in order not to generate further invariants. A necessary condition for an identical decomposition is that the discrete group contains a $\mathbf{27}$. The trihedral groups $\Delta(3n^2)$ and $\Delta(6n^2)$ are not in this category since their largest irreps are at most three- and, respectively, six-dimensional. Going through the character tables in [20] and the more recent work [27] we realize that there is *no* discrete subgroup of $SU(3)$ which has a 27D irrep. Note on even more general grounds that $\dim(\mathbf{27})^2 = 729$ almost saturates the relation in Eq. (11) and leaves $|\Sigma(360\varphi)| = 1080$ as the only hypothetical candidate among the crystal-like groups.

B. Groups with no new invariants at the $I^{(2,2)}$ ($I^{(3,3)}$) level

In this subsection we investigate the structure of the invariants $I^{(2,2)}$, which correspond to $\Delta F = 1'$ transitions (16). With the same reasoning as above this reduces to analyzing

$$K^{\mathcal{D}_{Q_L}} = (\bar{\mathbf{3}} \times \mathbf{3} \times \bar{\mathbf{3}} \times \mathbf{3}) = n_1 [\mathcal{D}_{Q_L}] \mathbf{1} + \dots, \quad (21)$$

the number of invariants that can be formed from the

Kronecker product (21). According to Sec. III A the number of invariants in $SU(3)$ equals the number of invariants of the discrete subgroup, i.e. $n_1[\mathcal{D}_{Q_L}] = n_1[SU(3)]$ if we can choose a $R_3(\mathcal{D}_{Q_L})$ such that

$$(\mathbf{3} \times \bar{\mathbf{3}})_{\mathcal{D}_{Q_L}} = \mathbf{1} + \mathbf{8}. \quad (22)$$

Once more the trihedral groups $\Delta(3n^2)$ and $\Delta(6n^2)$ can be excluded immediately since they possess maximally three- and, respectively, six-dimensional irreps. It turns out that all three maximal crystal-like subgroups $\Sigma(168)$, $\Sigma(216\varphi)$ and $\Sigma(360\varphi)$ (recall $\varphi = 3$) are in the category of (22) and do therefore not generate any further $I^{(2,2)}$ invariants. Below we shall briefly discuss the representations and some other relevant aspects of the candidates.

- (I) $\Sigma(168) \approx \text{PSL}(2, \mathbb{F}_7) \approx \text{GL}(3, \mathbb{F}_2)$ [23] is discussed in some more detail in Appendix A 1 as well. In what follows we shall permit ourselves to present the irreps in a compact way, though not unambiguous in a strict sense, via the relation (11):

$$\begin{aligned} |\Sigma(168)| &= 168 \\ &= |\mathbf{1}|^2 + 2|\mathbf{3}|^2 + |\mathbf{6}|^2 + |\mathbf{7}|^2 + |\mathbf{8}|^2. \end{aligned} \quad (23)$$

Explicit representation matrices can be found in [23] and the one 3D irrep satisfies (22).

- (II) $\Sigma(216\varphi)$.—Equation (11) reads [27]

$$\begin{aligned} |\Sigma(216\varphi)| &= 3 \cdot 216 \\ &= 3|\mathbf{1}|^2 + 3|\mathbf{2}|^2 + 7|\mathbf{3}|^2 + 6|\mathbf{6}|^2 \\ &\quad + 3|\mathbf{8}|^2 + 2|\mathbf{9}|^2. \end{aligned} \quad (24)$$

Out of the seven 3D irreps one is not faithful¹² and out of the three eight-dimensional irreps two are complex and therefore not of interest. The other six 3D irreps come in complex conjugate pairs. In the notation of [27]

$$\begin{aligned} \Sigma(216\varphi): \mathbf{3}_2 \times \mathbf{3}_6 &= \mathbf{3}_3 \times \mathbf{3}_5 = \mathbf{3}_4 \times \mathbf{3}_7 \\ &= \mathbf{1}_1 + \mathbf{8}_1 \end{aligned} \quad (25)$$

are the pairings à la Eq. (22).

- (III) $\Sigma(360\varphi)$.—Equation (11) reads [27]

$$\begin{aligned} |\Sigma(360\varphi)| &= 3 \cdot 360 \\ &= |\mathbf{1}|^2 + 4|\mathbf{3}|^2 + 2|\mathbf{5}|^2 + 2|\mathbf{6}|^2 \\ &\quad + 3|\mathbf{8}|^2 + 3|\mathbf{9}|^2 + |\mathbf{10}|^2 + 2|\mathbf{15}|^2. \end{aligned} \quad (26)$$

In the notation of [27] the interesting Kronecker products are

¹²It is isomorphic to A_4 [27], which is a popular group in attempts to explain tribimaximal mixing (e.g. [14]).

$$\begin{aligned}\Sigma(360\varphi): \mathbf{3}_1 \times \mathbf{3}_4 &= \mathbf{1}_1 + \mathbf{8}_1, \\ \mathbf{3}_2 \times \mathbf{3}_3 &= \mathbf{1}_1 + \mathbf{8}_2,\end{aligned}\quad (27)$$

which implies that both 8D irreps are real and both are valid candidates for our problem.

Of course the question of whether any subgroups of $\Sigma(168)$, $\Sigma(360\varphi)$, and $\Sigma(216\varphi)$ are in the category (22) is a relevant question here. Some, of the smaller groups, can be excluded on grounds of their order since by virtue of (11); Eq. (22) demands

$$\text{order} \geq 74 = 8^2 + 3^2 + 1^2. \quad (28)$$

The group $\Sigma(168)$ has the permutation group S_4 and the Frobenius group $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$ as maximal subgroups of which both can be discarded since their order, $|\mathcal{S}_4| = 24$ and $|\mathbb{Z}_7 \rtimes \mathbb{Z}_3| = 21$, does not satisfy (28). In the case of $\Sigma(216\varphi)$ we are aware of two maximal subgroups, $\Sigma(216) = \Sigma(216\varphi)/\mathbb{Z}_3$ and $\Sigma(72\varphi)$. The first one has a single 3D irrep which decomposes as $\mathbf{3} \times \mathbf{3} = \mathbf{1} + \mathbf{1}' + \bar{\mathbf{1}}' + 2 \cdot \mathbf{3}$ and is therefore not suitable. The second one:

(IV) $\Sigma(72\varphi)$ —Equation (11) reads [27]

$$\begin{aligned}|\Sigma(72\varphi)| &= 3 \cdot 72 \\ &= 4|\mathbf{1}|^2 + |\mathbf{2}|^2 + 8|\mathbf{3}|^2 + 2|\mathbf{5}|^2 \\ &\quad + 2|\mathbf{6}|^2 + |\mathbf{8}|^2,\end{aligned}\quad (29)$$

and the eight 3D irreps fall into four complex conjugate pairs as follows [27]:

$$\begin{aligned}\Sigma(72\varphi): \mathbf{3}_1 \times \mathbf{3}_2 &= \mathbf{3}_3 \times \mathbf{3}_4 = \mathbf{3}_5 \times \mathbf{3}_6 \\ &= \mathbf{3}_7 \times \mathbf{3}_8 = \mathbf{1}_1 + \mathbf{8}_1,\end{aligned}\quad (30)$$

and we therefore have four choices for a pair of irreps.

In the case of $\Sigma(360\varphi)$ we are aware of the two maximal subgroups $\Sigma(60) \approx A_5$ and $\Sigma(360) \approx \Sigma(360\varphi)/\mathbb{Z}_3$ of which the former can be excluded by virtue of (28) and the latter does not admit a 3D irrep¹³ as can be inferred from the character table (e.g. [20]) and is therefore not suitable.

In order to count the number of invariants up to $I^{(4,4)}$ it is sufficient to know the $\mathbf{8} \times \mathbf{8}$ Kronecker products. We shall list those for $SU(3)$ (20), $\Sigma(168)$ [23] $\Sigma(72\varphi)$, $\Sigma(216\varphi)$ and $\Sigma(360\varphi)$. The latter three have been computed from the character tables given in Ref. [27]:

¹³The same thing happens in the continuum; the $SU(2)/\mathbb{Z}_2 \approx SO(3)$ does not admit 2D representations.

TABLE I. Number of complex conjugate pairs and number of invariants for tensors of type $I^{(n,n)}$, whose definition can be inferred from Eq. (16). The subscripts x and y indicate symmetrizations of x and y pairs of $\mathbf{3}$ and $\bar{\mathbf{3}}$ indices. $I_{2,1}^{(3,3)}$ and $I_{2,2}^{(4,4)}$ correspond to the (symmetric) contractions of $\Delta F = 1$ and $\Delta F = 2$ in (16).

Group	Order	Pairs						
		$(\mathbf{3}, \bar{\mathbf{3}})$	$I^{(2,2)}$	$I^{(3,3)}$	$I_{2,1}^{(3,3)}$	$I^{(4,4)}$	$I_{3,1}^{(4,4)}$	$I_{2,2}^{(4,4)}$
$SU(3)$	∞	1	2	6	5	23	15	14
$\Sigma(360\varphi)$	1080	2	2	6	5	28	18	17
$\Sigma(216\varphi)$	648	3	2	7	6	40	27	23
$\Sigma(168)$	168	1	2	7	6	44	29	25
$\Sigma(72\varphi)$	216	4	2	11	8	92	55	43

$$(\mathbf{8} \times \mathbf{8})_{SU(3)} = (\mathbf{1} + \mathbf{8} + \mathbf{27})_s + (\mathbf{8} + \mathbf{10} + \bar{\mathbf{10}})_a,$$

$$\begin{aligned}(\mathbf{8} \times \mathbf{8})_{\Sigma(168)} &= (\mathbf{1} + \mathbf{8} + (2 \cdot \mathbf{6} + \mathbf{7} + \mathbf{8}))_s \\ &\quad + (\mathbf{8} + (\mathbf{3} + \mathbf{7}) + (\bar{\mathbf{3}} + \mathbf{7}))_a,\end{aligned}$$

$$\begin{aligned}(\mathbf{8} \times \mathbf{8})_{\Sigma(72\varphi)} &= (\mathbf{1}_1 + \mathbf{8} + (\mathbf{1}_2 + \mathbf{1}_3 + \mathbf{1}_4 + 3 \cdot \mathbf{8}))_s \\ &\quad + (\mathbf{8} + (\mathbf{8} + \mathbf{2}) + (\mathbf{8} + \mathbf{2}))_a,\end{aligned}$$

$$\begin{aligned}(\mathbf{8} \times \mathbf{8})_{\Sigma(216\varphi)} &= (\mathbf{1} + \mathbf{8}_1 + (\mathbf{3}_1 + \mathbf{8}_1 + \mathbf{8}_2 + \mathbf{8}_3))_s \\ &\quad + (\mathbf{8}_1 + (\mathbf{8}_2 + \mathbf{2}_2) + (\mathbf{8}_3 + \mathbf{2}_3))_a,\end{aligned}$$

$$\begin{aligned}(\mathbf{8}_1 \times \mathbf{8}_1)_{\Sigma(360\varphi)} &= (\mathbf{1}_1 + \mathbf{8}_1 + (\mathbf{5}_1 + \mathbf{5}_2 + \mathbf{8}_2 + \mathbf{9}_1))_s \\ &\quad + (\mathbf{8}_1 + \mathbf{10} + \mathbf{10})_a.\end{aligned}\quad (31)$$

The brackets indicate the branching rules. The *a priori* unclear pairings $(\mathbf{8}_2 + \mathbf{2}_2)_{\Sigma(216\varphi)}$ and $(\mathbf{3} + \mathbf{7})_{\Sigma(168)}$ can be inferred from Ref. [30]. The product $(\mathbf{8}_2 \times \mathbf{8}_2)_{\Sigma(360\varphi)}$ is obtained from the one in (31) by interchanging the subscripts $1 \leftrightarrow 2$. To this end we shall give an overview of the number of invariants for $I^{(2,2)}$, $I^{(3,3)}$ and $I^{(4,4)}$ in Table I. The number of 3D complex conjugate pairs are also listed. The symmetrized tensors $I_{x,y}$ are explained in the caption.

V. BACK TO PHYSICS

A. Flavor to mass basis—new invariants lead to flavor anarchy

By switching from the flavor basis to the physical (mass) basis we employ biunitary transformations (7) in a $U(3)_{U_L} \times U(3)_{D_L} \times U(3)_{U_R} \times U(3)_{D_R}$ space. In the SM this group is broken down to $U(3)^3$ [cf. Eq. (3)]¹⁴ by the $SU(2)_L$ gauge group, rendering $V_{\text{CKM}} = \mathcal{U}_L^\dagger \mathcal{D}_L$ observable. By choosing a discrete symmetry D_q (8) the group is further broken down and this will *in general* render the

¹⁴In this section we will take a cavalier attitude towards the question of the proper $U(1)$ factors.

transformation matrices (7) observable¹⁵ at the order in the Yukawa expansion where the invariants of the groups \mathcal{D} and $SU(3)$ differ.

One can take different points of view here. From a certain perspective the misalignment of the flavor and mass basis is simply observable and the result has to be accepted. We can though push the bar and take a more active viewpoint and ask the question: Given arbitrary Yukawa matrices is there an embedding of the discrete group \mathcal{D} into $SU(3)$ which allows us to choose $\mathcal{U}_R \simeq 1$, $\mathcal{D}_R \simeq 1$ and either $\mathcal{U}_L \simeq 1$ or $\mathcal{D}_L \simeq 1$? Note that \mathcal{U}_L and \mathcal{D}_L can be interchanged via the CKM matrix. This question is investigated in Appendix B and the answer is no for the crystal groups but appears to be affirmative for the trihedral group $\Delta(3n^2)$ and $\Delta(6n^2)$ for adjusted n and choice of 3D irrep. Although this in principle allows one to make the new angles and phases arbitrarily small it is a fact that non- $SU(3)$ invariants by themselves lead to new flavor patterns. Let assume that $\mathcal{U}_R \simeq 1$, $\mathcal{D}_R \simeq 1$ and $\mathcal{U}_L \simeq 1$, implying $\mathcal{D}_L \simeq V_{\text{CKM}}$, and then the $\Delta(3n^2)/\Delta(6n^2)$ invariant (A14) and (A20) becomes

$$\begin{aligned} I_2^{(2,2)} &\sim (\bar{D}^1 D_1 \bar{D}^2 D_2 + \bar{D}^2 D_2 \bar{D}^3 D_3 + \bar{D}^1 D_1 \bar{D}^3 D_3) \\ &\rightarrow (\bar{d}(V_{ud}^* V_{us})_{S_L})(\bar{d}(V_{cd}^* V_{cs})_{S_L}) + \dots \\ &\leftrightarrow \mathcal{L}_{\text{eff}}^{\Delta S=2} \sim \mathcal{O}(\lambda^2), \end{aligned} \quad (32)$$

and becomes a $\Delta S = 2$ operator, at second order in the Wolfenstein parameter [$\lambda \simeq \cos(\theta_C) \simeq 0.22$], in the mass basis, whereas in the MFV scenario this transition is governed by $|V_{is}^* V_{id}|^2 \sim \mathcal{O}(\lambda^{10})$. This seems rather anarchic. New invariants therefore spoil predictivity and it seems desirable to track or control them in some way.

**B. If $(\Delta F = 2) \approx (\Delta F = 1) \times (\Delta F = 1)$,
then $I^{(4,4)} \rightarrow I^{(2,2)} I^{(2,2)}$**

The view that in a quantum field theory any term, not forbidden by symmetry, emerges dynamically is deeply rooted (e.g. [31]). The crucial pragmatic question for our approach is what are the bounds on the coefficients $C_{d\text{MFV}}$. We will find it useful to divide models in certain classes and reflect on a few specific examples.

In the case where there is no suppression of higher order terms, other than the Yukawa expansion itself, the results of Sec. IVA indicate that at the $\Delta F = 2$ level (16) new invariants of the type $I^{(4,4)}$ (could) emerge. This seems rather dangerous at first sight because the results of the previous subsection imply that new invariants upset the flavor structure *and* predictivity since the mass-flavor basis

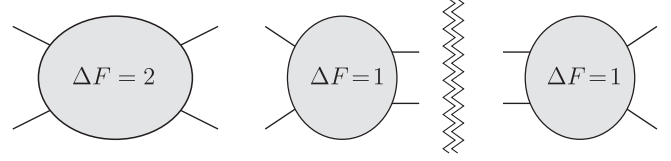


FIG. 1. (Left) Family irreducible: generic $\Delta F = 2$ process. (Right) Family reducible: $\Delta F = 2$ process can be disconnected into two $\Delta F = 1$ parts. There is no horizontal or family charge flowing from the left to the right. The family reducible property is a *sufficient* property for a TeV-scale discrete MFV scenario for any of the groups Eq. (33).

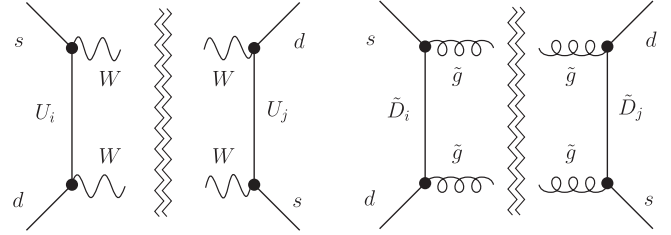


FIG. 2. *Family reducible*: The double wiggly lines indicate cuts through the diagrams where there is no (horizontal) family charge flowing. (Left) SM box diagram. (Right) An example of a gluino contribution in the R -parity conserving MSSM.

transformation becomes observable. As hinted at above it would appear too hasty to conclude that no discrete flavor group is suitable. The generation mechanism of $\Delta F = 2$ operators has to be reflected upon. In rather general terms we may want to distinguish the two cases where the $\Delta F = 2$ process is generated via two subsequent $\Delta F = 1$ parts and where this is not the case.¹⁶ We shall call the former “family irreducible” and the latter “family reducible”; cf. Fig. 1. The SM or the R -parity conserving MSSM (cf. Fig. 2), as presumably many perturbative models, are of the family reducible type. The composite technicolor model of Ref. [1] cannot be claimed, in the absence of the understanding of the nonperturbative dynamics of the preon confinement, to be in the family reducible class.

The family irreducible property (cf. Fig. 1) is a *sufficient* condition for a TeV-scale discrete MFV scenario if the global flavor group D_q is built from the following crystal-like groups:

$$\Sigma(168), \quad \Sigma(72\varphi), \quad \Sigma(216\varphi) \text{ and } \Sigma(360\varphi). \quad (33)$$

Essentially in this case the potentially dangerous invariants factorize $I^{(4,4)} \rightarrow I^{(2,2)} I^{(2,2)}$ and the latter have the same invariants as the groups in Eq. (33).

Below we would like to reflect upon this rather general statement via examples and argue that even family-irre-

¹⁵From another viewpoint it is the absence of the Goldstone bosons, in the approach followed here, that leads to further observable parameters. The latter are in one to one correspondence with the reduction of observable parameters. Counting in the quark sector, $18 \cdot 2$ Yukawa parameters minus 26 Goldstones gives 4 CKM parameters and 6 quark masses [7].

¹⁶One is tempted to say somewhat in the spirit of the phenomenological superweak model for CP violation [32], bearing in mind though that not all features such as, for example, the reality of the CKM matrix are relevant here.

ducible-type cases may be suitable in (many) perturbative-type models.

- (i) $\Sigma(360\varphi)$ model independent.—The most suitable candidate is $\Sigma(360\varphi)$ since the first new invariants appear only at the $I^{(4,4)}$ level; cf. Table I. The discussion of the previous subsection [cf. Eq. (32)] suggests that the most severe constraints could come from $s \rightarrow d$ transitions. We shall attempt a rough estimate of the real part of $\Delta S = 2$ transitions. In the notation of Eq. (12) the effective Lagrangian assumes the following form:

$$\delta \mathcal{L}_{\text{eff}}^{\Delta S=2} = \frac{\kappa^4}{\Lambda^2} \sum_n c_n I_n(\Sigma(360\varphi))_{1stu}^{abc2} \times (\bar{d}\Delta_a^s \Delta_b^t \Delta_c^u s_L)(\bar{d}\Delta_1^2 s_L), \quad (34)$$

where the transition matrices could be either Δ_U or Δ_D (14). The symbol κ denotes the Yukawa expansion parameter (8). It appears to the fourth power because of the four additional Yukawa matrices. We can now ask the following question: *How small does κ need to be in order for $C_{d\text{MFV}}$ (12) to satisfy the same kind of experimental bounds as for C_{MFV} found in Ref. [4]?* The discussion of the subsection [cf. Eq. (32)] suggests that $s \rightarrow d$ could be induced at first order in λ as compared to order $|V_{ts}V_{td}^*| \sim \lambda^5$ in MFV. The total $\Delta S = 2$ transition could therefore be $\mathcal{O}(\lambda^6)$ as compared to $\mathcal{O}(\lambda^{10})$ in MFV. According to our reflection above the condition is $\kappa^4/\lambda^4 \simeq 1$ and therefore $\kappa \simeq \lambda \simeq 0.2$.

- (ii) *MSSM soft terms and $\Sigma(360\varphi)$.*—In the MSSM some additional flavor structure enters through the soft terms, e.g. the squark mass matrix \tilde{m}_Q^2 , which can be considered to be a $\mathcal{T}^{(1,1)}$ tensor. It was suggested a long time ago in the spirit of MFV [33] that the nine parameters of the Hermitian \tilde{m}_Q^2 could be organized into a Yukawa matrix expansion:

$$(\tilde{m}_Q^2)_r^c = \tilde{m}^2(a_1 \delta_r^c + b_1^{(1)}(\Delta_U)_r^c + b_1^{(2)}(\Delta_D)_r^c + b_2^{(1)}(\Delta_U \Delta_D + \text{H.c.})_r^c + \dots).$$

The first correction of the type given in Eq. (34), in this expansion, would be given by

$$(\delta \tilde{m}_Q^2)_r^d = b_3^{(1)} I_n(\Sigma(360\varphi))_{rstu}^{abcd} \Delta_a^s \Delta_b^t \Delta_c^u. \quad (35)$$

Note the assumption of the Yukawa hierarchy translates into $\mathcal{O}(b_1) \simeq \mathcal{O}(b_2 \kappa^{-2}) \simeq \mathcal{O}(b_3 \kappa^{-4})$ for the coefficients.¹⁷

¹⁷The assumption of a Yukawa hierarchy is not always imposed in the literature (e.g. [34]). The finite dimension of the matrices makes the general series collapse at order b_5 . Although the expansion contains more parameters than unknowns predictivity results from the assumption that the b_i are of the same order. Moreover CKM and mass hierarchies also help in this respect.

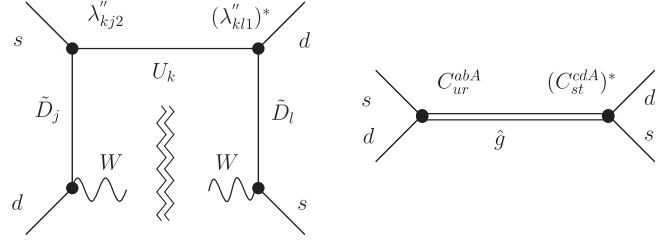


FIG. 3. *Family irreducible*: The double wiggles indicate cuts through the diagrams where there is no (horizontal) family charge flowing. (Left) An example of a squark contribution in the R -parity violating MSSM. Yet, this diagram does not generate new flavor structure in discrete MFV. (Right) Engineered example which is family irreducible and does lead to new invariants in discrete MFV.

- (iii) *Family irreducible examples (cf. Fig. 3).*—

- (1) In the R -parity violating MSSM, which is family irreducible [Fig. 3(left)], the fact that each vertex has to be D_q -invariant prevents the generation of nonfactorizable $I^{(4,4)}$ invariants. This effectively happens because the R -parity violating vertices $\lambda_{ijk}'' U^i D^j D^k$ (U and D are superfields) do not allow for new structures because $\mathbf{3} \times \mathbf{3} \times \mathbf{3}$ contains the trivial representation only once for the groups in Eq. (33).¹⁸
- (2) Lacking a concrete example, let us imagine an effective theory with an interaction vertex $\mathcal{L}_{\text{eff}} \sim C_{rs}^{abA} (\bar{D}^r \Delta_a^s D_b) \hat{g}_A$. The variable A is an index of a nontrivial representation \mathbf{A} appearing in $\mathbf{3} \times \mathbf{3} \times \mathbf{3} = \mathbf{A} + \dots$. The symbol C_{rs}^{abA} denotes a generalized Clebsch-Gordan coefficient that makes the interaction \mathcal{D}_{Q_L} invariant. The scalar particle \hat{g} is supposed to be heavy and, when integrated out, leads to $\Delta S = 2$ structure with a (new) nonfactorizable $I^{(4,4)}$ structure; cf. Fig. 3(right).

VI. EPILOGUE

Before contemplating the scenario we shall briefly summarize our main results. The reduced symmetry leads generally to further invariants and renders the mass-flavor basis transformation matrices observable, which can also be seen as a direct consequence of the absence of the Goldstone bosons themselves. Moreover non- $SU(3)$ invariants do upset MFV hierarchies in a rather anarchic way; cf. Sec. VA. In Sec. IV we have established, through an argument based on the absence of a $\mathbf{27}$ for discrete $SU(3)$ subgroups, that there necessarily are new invariants for $I^{(4,4)}$. The latter enter generic $\Delta F = 2$ transitions (16). In

¹⁸To be even more concrete, in MFV the structure is given by $\lambda_{ijk}''^{\text{MFV}} = a_1 \epsilon_{ajk} (Y_U Y_D^1)_i^a + a_2 \epsilon_{abc} (Y_U)_i^a (Y_D)_j^b (Y_D)_k^c + \dots$ provided the $U(1)$ structure does not forbid them from the start [34].

cases where the $\Delta F = 2$ process is generated via two $\Delta F = 1$ subprocesses, which we called family reducible, the invariant factorizes: $I^{(4,4)} \rightarrow I^{(2,2)} I^{(2,2)}$. For the latter the groups $\Sigma(168)$, $\Sigma(72\varphi)$, $\Sigma(216\varphi)$ and $\Sigma(360\varphi)$ do provide enough symmetry to imitate $SU(3)$ at this level and thus are valid candidates for a TeV-scale discrete MFV scenario with Yukawa expansion parameters of the order of $\kappa_{\Sigma(360\varphi)} \simeq 0.2$. Models which are not of the family reducible type may still be viable candidates, especially if they are perturbative. An overview of the number of invariants is given in Table I. Below we shall add a few not necessarily connected thoughts on MFV and the framework proposed here.

- (i) *Origin of discrete symmetry.*—One might of course ask the question about the origin of such discrete symmetries. They might originate from compactifications in string theory, where it was found that the trihedral group $\Delta(54)$ can appear [35] or they could appear from the breaking of a continuous symmetry; cf. [36] for a recent investigation. The latter has to happen, presumably, at some high energy in order not to make the so far unseen familons too visible. We would like to add that whether global symmetries originate from local ones or not can have subtle physical consequences [37].
- (ii) *This and that.*—Surely it is possible that the groups $\mathcal{D}(8)$ are of different types. We have focused on \mathcal{D}_{Q_L} , which governs the $D(U)_L \rightarrow D(U)'_L$ transitions. The groups \mathcal{D}_{D_R} would matter once we consider $D_R \rightarrow D'_{R(L)}$ -type operators, where again the groups (33) would provide the most protection. Needless to say that if the question of flavor is not linked to a scale close to the TeV scale, and the breaking of D_q (or G_q) happens at a high scale, then experimental bounds do not favor any particular groups. Though in the MSSM, for example, the implementation of MFV is related to supersymmetry breaking through the soft terms [4,33] (35) and this in turn suggests a link of flavor to the hierarchy problem.
- (iii) *Embeddings.*—The formulation (8) could be refined by constructing a discrete subgroups of G_q (3) which does not factor into direct products of $SU(3)$ subgroups, much in the same way as the discrete subgroup $\Sigma(81) \subset U(3) \approx U(1) \times SU(3)$ ¹⁹ cannot be written as a direct product of a discrete $SU(3)$ and $U(1)$ subgroup; $\Sigma(81) \notin \mathcal{D}_{U(1)} \times \mathcal{D}_{SU(3)}$. One might wonder what the consequences for the invariants are.
- (iv) *Model of (d)MFV.*—MFV is an empirically motivated effective field theory approach. Up to now no

explicit model of MFV has been constructed,²⁰ though the seeds of a scenario were put forward in [7,18]. As hinted at in the introduction the relation between the MFV scale $\sqrt{16\pi^2 C_{\text{MFV}}}$ and the breaking scale(s) f_F can only be answered model by model. It has to be added that a model of MFV without the input of the CKM and mass structure seems to be at the same level of difficulty as constructing a theory of flavor which has proven to be a hard problem. One should not forget that besides being predictive and testable MFV has other appealing properties: In an R -parity violating MSSM, MFV provides enough protection to evade bounds on the proton decay [40]. MFV also serves as a reference point for any model with flavor structure and facilitates comparison of different models.

Our aim, in this work, was to point out general issues of implementing MFV via a discrete group. We would hope that this work would be of some help for further investigations towards more specific models, where for one reason or another one or the other invariant does not turn out to be as menacing as in the essentially model-independent approach followed here.

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APPENDIX A: EXAMPLES OF INVARIANTS IN THE FLAVOR BASIS

In this appendix we discuss characteristic invariants of specific groups (in the flavor basis). From Sec. IVA we already know that new invariants are present at the level of the effective theory. The aim of this appendix is to present a few instructive (concrete) examples. We shall use the notation (15) for the left-handed D quarks.

1. Crystal groups Σ

We shall discuss $\Sigma(60)$ and $\Sigma(168)$ which are both instructive.

The group $\Sigma(60)$.—The representations of $\Sigma(60) \approx A_5 \approx I_{\text{cosahedral}} \subset SO(3)$ have for instance been studied in [41]. There are two real 3D irreps which we shall denote by

¹⁹ $\Sigma(81) \supset \Delta(27)$ was first introduced in Ref. [38] and discussed in further detail in Appendix B of Ref. [39].

²⁰Yet, in practice anomaly mediated supersymmetry breaking, with CKM structure, is close to MFV [4].

$\mathbf{3}_1$ and $\mathbf{3}_2$. Their product representation takes the following form:

$$\mathbf{3}_i \times \mathbf{3}_i = \mathbf{1} + \mathbf{3}_i + \mathbf{5}, \quad i = 1 \dots 2. \quad (\text{A1})$$

The $\mathbf{3}_i$ on the right-hand side (RHS) of (A1), with (15), reads

$$\mathbf{3}_i \sim (\bar{D}_3 D_2 - \bar{D}_2 D_3, \bar{D}_1 D_3 - \bar{D}_3 D_1, \bar{D}_2 D_1 - \bar{D}_1 D_2)^T. \quad (\text{A2})$$

Since $\bar{\mathbf{3}}_i = \mathbf{3}_i$ the singlet is obtained by simply taking the scalar product of the vector above

$$I_{\Sigma(60)_2}^{(2,2)} = \bar{D}_3 D_2 \bar{D}_3 D_2 + \dots. \quad (\text{A3})$$

The group $\Sigma(60)$ does allow for a $\Delta F = 2$ structure even in the absence of any Yukawa matrices. The symmetry is

$$d_{abcd} = d^{abcd} = \sum_{\alpha\beta} K_{ab}^\alpha K_{cd}^\beta C_{\alpha\beta} = \begin{cases} 1/\sqrt{2} \\ 0 \end{cases}$$

where cyclic refers to $1113 \rightarrow 3111 \rightarrow 1311 \rightarrow 1131$, etc. The first equality sign above is to be understood on the level of indices and not at the level of tensors. Note that the invariant

$$f = d^{abcd} D_a D_b D_c D_d \sim D_3 (D_1)^3 + D_1 (D_2)^3 + D_2 (D_3)^3 \quad (\text{A5})$$

just corresponds to Klein's famous quartic invariant [42]. From (A4) we can build an invariant of the form

$$(I_{\Sigma(168)_3})_{rstu}^{abcd} = d^{abcd} d_{rstu}. \quad (\text{A6})$$

This invariant tensor, with field content (16), leads to terms of the form

$$I_{\Sigma(168)_3}^{(4,4)} \sim \bar{D}_1 \bar{D}_1 \Delta_1^1 \Delta_2^3 D_2 D_2 + \text{permutations}, \quad (\text{A7})$$

revealing a rather anarchic structure of flavor transitions even in the flavor basis. Moreover the seventh tensor $I_{\Sigma(168)_7}^{(3,3)}$ (cf. Table I) can easily be constructed:

$$(I_{\Sigma(168)_7}^{(3,3)})_{rst}^{abc} = d_{abci} d^{irst}. \quad (\text{A8})$$

At last we would like to remark that the tensor $d_{ab}^{cd} = d_{abij} d^{ijcd}$

$$d_{ab}^{cd} = \begin{cases} 1 & a = b = c = d, \\ 1/2 & a = c, b = d \text{ or } a = d, b = c, \\ 0 & \text{otherwise,} \end{cases} \quad (\text{A9})$$

acts, as expected, like a Kronecker symbol in the $\mathbf{6}$ space.

2. The trihedral groups $\Delta(3n^2)$ and $\Delta(6n^2)$

The main purpose of this subsection is to give some explicit non- $SU(3)$ $I^{(2,2)}$ invariants. In passing we would

simply not strong enough to constrain flavor transitions in any way.

The group $\Sigma(168)$.—The isomorphisms of this group are $\Sigma(168) \approx \text{PSL}(2, \mathbb{F}_7) \approx \text{GL}(3, \mathbb{F}_2)$ [23]. The irreps can be read off from (23) and since the first nontrivial representations have the same dimension this implies that they are identical [23]: $\bar{\mathbf{3}} \times \mathbf{3} = \mathbf{1} + \mathbf{8}$. The $\mathbf{8}$ is therefore real but the difference appears at the level of product of two $\mathbf{8}$'s; cf. Eq. (31). As a consequence of the general discussion in Sec. IV A the absence of a $\mathbf{27}$ implies further invariants. We may construct one of these invariants with the results given in [23] as follows: Consider the product $\mathbf{3} \times \mathbf{3} = \bar{\mathbf{3}} + \mathbf{6}$ and the information that the 6D is real, i.e. $\bar{\mathbf{6}} = \mathbf{6}$, and we may infer that the following product $\mathbf{3} \times \mathbf{3} \times \mathbf{3} \times \mathbf{3} = \mathbf{1} \cdot \mathbf{1} + \dots$ contains the trivial representation once. The invariant tensor may be read off from [23]

$$abcd = \{1113, 2221, 3332\} \text{ and cyclic,} \quad (\text{A4})$$

otherwise,

like to mention that as long as no real representations are generated (we have checked that for specific n , k , and l this is the case), all flavor transitions are governed by the Yukawa matrices in the flavor basis. This fact is interesting but irrelevant to our work because the passage to the mass basis changes everything; cf. Sec. VA. At last we argue that the D groups are subgroups of an appropriate $\Delta(6g^2)$ which is not known to the authors from any other source.

The groups $\Delta(3n^2)$.—The group admits the following isomorphism [22]: $\Delta(3n^2) \approx (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes \mathbb{Z}_3$. As previously mentioned this group has only one and 3D irreps. They are labeled by the pair (k, l) , where $k, l = 0 \dots n-1$ but $(k, l) \neq (0, 0)$ [and additionally $(k, l) \neq s(n/3, n/3)$ with $s = 1 \dots 2$ for $n = 3\mathbb{Z}$] and the following pairs $(k, l) \approx (-k-l, k) \approx (l, -k-l)$ describe equivalent irreps. The complex conjugate representation is obtained by reversing the sign of k and l , i.e. $\bar{\mathbf{3}}_{(k,l)} = \mathbf{3}_{(-k,-l)}$. Anything relevant to us can be gained from the following Kronecker product [22,24]:

$$\bar{\mathbf{3}}_{(k,l)} \times \mathbf{3}_{(k,l)} = \mathbf{3}_{(0,0)} + \mathbf{3}_{(-2k-l,k-l)} + \mathbf{3}_{(2k+l,l-k)}, \quad (\text{A10})$$

and the branching rules for the RHS of (A10) are

$$\mathbf{3}_{(0,0)} \rightarrow \mathbf{1}_{0,0} + \mathbf{1}_{1,0} + \mathbf{1}_{2,0}, \quad (\text{A11})$$

and for $n = 3\mathbb{Z}$ with k and $l, l \in \mathbb{Z}n/3$, which reduces to $(k, l) = (0, \pm n/3)$ under equivalences

$$\begin{aligned} &\mathbf{3}_{(n/3,n/3)} + \mathbf{3}_{(-n/3,-n/3)} \\ &\rightarrow \mathbf{1}_{0,1} + \mathbf{1}_{1,1} + \mathbf{1}_{2,1} + \mathbf{1}_{0,2} + \mathbf{1}_{1,2} + \mathbf{1}_{2,2}, \end{aligned} \quad (\text{A12})$$

there are nine one-dimensional irreps. With (15) and for $(k, l) = (0, n/3)$ they take on the form

$$\begin{aligned}
\mathbf{1}_{r,0} &\sim \bar{D}^1 D_1 + \omega^{-r} \bar{D}^2 D_2 + \omega^r \bar{D}^3 D_3, \\
\mathbf{1}_{r,1} &\sim \bar{D}^2 D_1 + \omega^{-r} \bar{D}^3 D_2 + \omega^r \bar{D}^1 D_3, \\
\mathbf{1}_{r,2} &\sim \bar{D}^3 D_1 + \omega^{-r} \bar{D}^1 D_2 + \omega^r \bar{D}^2 D_3,
\end{aligned} \tag{A13}$$

and for $(k, l) = (0, -n/3)$ the roles of $\mathbf{1}_{r,1/2}$ are reversed [24]. Two of the generators, a and c , act in a nontrivial manner [24]: $a \circ \mathbf{1}_{r,s} = \omega^r \mathbf{1}_{r,s}$ and $c \circ \mathbf{1}_{r,s} = \omega^s \mathbf{1}_{r,s}$ implying $\mathbf{1}_{r,s} \times \mathbf{1}_{r',s'} = \mathbf{1}_{r+r',s+s'}$. Note $\mathbf{1}_{0,0}$ is therefore the only singlet. For any n and k, l there are at least five invariant tensor at the level of $I^{(2,2)}$, as compared to two for $SU(3)$; cf. Table I. For the symmetric contraction $\bar{D}^a \bar{D}^b D_c D_d$ the $\mathbf{1}_{1,0} \times \mathbf{1}_{2,0}$ invariant reads

$$\begin{aligned}
I_{\Delta(3n^2)_2}^{(2,2)} &= I_{\Delta(3n^2)_1}^{(2,2)} \\
&\quad - 3(\bar{D}^1 D_1 \bar{D}^2 D_2 + \bar{D}^2 D_2 \bar{D}^3 D_3 + \bar{D}^1 D_1 \bar{D}^3 D_3), \\
I_{\Delta(3n^2)_1}^{(2,2)} &= (\bar{D}^1 D_1 + \bar{D}^2 D_2 + \bar{D}^3 D_3)^2,
\end{aligned} \tag{A14}$$

where we have indicated the $SU(3)$ invariant I_1 for notational convenience. For the sake of completeness we shall indicate the explicit 3D irreps, which can be obtained from Appendix D of Ref. [24], up to a single transformation of the generator a ,

$$\begin{aligned}
\mathbf{3}_{(-2k-l,k-l)} &\sim (\bar{D}_1 D_2, \bar{D}_2 D_3, \bar{D}_3 D_1)^T, \\
\bar{\mathbf{3}}_{(-2k-l,k-l)} &\sim (\mathbf{3}_{(2k+l,l-k)})^*.
\end{aligned} \tag{A15}$$

From the explicit forms (A13) and (A15) it is a simple matter to obtain the $I^{(2,2)}$ invariants and even beyond. To this end we shall briefly discuss two cases of $\Delta(3n^2)$ which are popular in the literature.

(a) $n = 2$.— $\Delta(12) \approx A_4$. In fact this group was brought into particle physics as early as 1979 [43]. There is only one 3D irrep with $(k, l) = (0, 1)$, which is real. The latter fact can either be checked explicitly, asserted from there being only one 3D irrep or inferred from the fact that $A_4 \subset SO(3)$. The number of $I^{(2,2)}$ invariants is seven and the reality of the $\mathbf{3}$ allows to form an invariant

$$\begin{aligned}
I_{\Delta(3n^2)_4}^{(2,2)} &= (\bar{D}_1 D_2)^2 + (\bar{D}_2 D_3)^2 + (\bar{D}_3 D_1)^2, \\
\text{and } I_5 &= I_4^*.
\end{aligned} \tag{A16}$$

(b) $n = 3$.— $\Delta(27)$ is the first group that admits nine one-dimensional irreps. The remaining one completing the relation (11) is a complex conjugate pair of 3D irreps. There are nine $I^{(2,2)}$ invariants, which can easily be obtained from (A13).

The groups $\Delta(6n^2)$.—The group admits the following isomorphism [22]: $\Delta(6n^2) \approx (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes S_3$. The irreps are 6, 3, 2 and 1D. The 6D representations $\mathbf{6}_{(k,l)}$ are labeled by a pair (k, l) , where $k = 0 \dots (n-1)$ and neither $k = 0, l = 0$ nor $k + l = 0 \bmod n$ [and additionally $(k, l) \neq s(n/3, n/3)$ with $s = 1 \dots 2$ for $n = 3\mathbb{Z}$]. Moreover the

following six pairs $(k, l) \approx (-k-l, k) \approx (l, -k-l) \approx (-l, -k) \approx (k+l, -l) \approx (-k, k+l)$ describe equivalent irreps. There are two types of 3D irreps originating from $\mathbf{6}_{k,l}$ when $k+l = 0 \bmod n$ and $(k, l) \neq (0, 0)$. The two types of representations can therefore be labeled by $\mathbf{3}_1^{(l)}$ and $\mathbf{3}_2^{(l)}$. Complex conjugate irreps are obtained by reversing the sign of (k, l) and (l) , respectively. For $n \in 3\mathbb{Z}$ there are three further 2D irreps denoted by $\mathbf{2}_2, \mathbf{2}_3$, and $\mathbf{2}_4$ [25], which are not relevant for $I^{(2,2)}$ invariants. The Kronecker product for the latter reads [25]

$$(\bar{\mathbf{3}}_1^{(l)} \times \mathbf{3}_1^{(l)}) \times (\bar{\mathbf{3}}_1^{(l)} \times \mathbf{3}_1^{(l)}) = (\mathbf{3}_1^{(0)} + \mathbf{6}_{(l,l)}) \times (\mathbf{3}_1^{(0)} + \mathbf{6}_{(l,l)}), \tag{A17}$$

where the explicit vectors on the RHS, using the parametrization (15), are

$$\begin{aligned}
\mathbf{3}_1^{(0)} &= (\bar{D}^1 D_1, \bar{D}^2 D_2, \bar{D}^3 D_3)^T, \\
\mathbf{6}_{(l,l)} &= (\bar{D}^1 D_3, \bar{D}^3 D_2, \bar{D}^2 D_1, \bar{D}^1 D_2, \bar{D}^2 D_3, \bar{D}^3 D_1)^T.
\end{aligned}$$

Our form looks slightly more symmetric than the one in Ref. [25] because we have chosen the (l, l) rather than the $(-l, 2l)$ representative. The remaining relevant Kronecker products are

$$\begin{aligned}
\mathbf{3}_1^{(0)} \times \mathbf{3}_1^{(0)} &= \mathbf{3}_1^{(0)} + \mathbf{6}_{(0,0)}, \\
\mathbf{3}_1^{(0)} \times \mathbf{6}_{(l,l)} &= 3 \cdot \mathbf{6}_{(l,l)}, \\
\mathbf{6}_{(l,l)} \times \mathbf{6}_{(l,l)} &= \mathbf{6}_{(0,0)} + \mathbf{6}_{(2l,2l)} + \mathbf{6}_{(3l,0)} + \mathbf{6}_{(0,3l)} \\
&\quad + \mathbf{6}_{(-l,2l)} + \mathbf{6}_{(2l,-l)}.
\end{aligned} \tag{A18}$$

The RHS remains the same when $\mathbf{3}_1$ is exchanged with $\mathbf{3}_2$ on the left-hand side [25]. The relevant branching rules are

$$\mathbf{3}_1^{(0)} \rightarrow \mathbf{1}_1 + \mathbf{2}_1, \quad \mathbf{6}_{(0,0)} \rightarrow \mathbf{1}_1 + \mathbf{2}_1 + \mathbf{1}_2 + \mathbf{2}_2. \tag{A19}$$

The Clebsch-Gordan coefficient of $\mathbf{3}_1^{(0)}$ and $\mathbf{6}_{(0,0)}$ on the RHS of the top Eq. (A18) are immediate from the ones of (A17) and the ones for $\mathbf{6}_{(0,0)}$ are [25]

$$\mathbf{6}_{(0,0)} \sim (\bar{D}^1 D_6, \bar{D}^2 D_5, \bar{D}^3 D_4, \bar{D}^4 D_3, \bar{D}^5 D_2, \bar{D}^6 D_1).$$

We have used an obvious generalization of (15). It can be said that at the $I^{(2,2)}$ level there are at least three invariants to be compared to two for $SU(3)$; cf. Table I. The Clebsch-Gordan coefficients allow us to obtain them explicitly. We leave it to the reader to figure out the precise association of (n, l) and the number of invariants. The question is then how the singlets in Eq. (A18) can be obtained from $\mathbf{3}_1^{(0)}$ and $\mathbf{6}_{(0,0)}$. In both cases the generators c and d act trivially and then it remains to work out which combination remains invariant under the two remaining generators a and b . Not surprisingly they are obtained by summing all the entries of the vectors. The correspondences are

$$\begin{aligned}
 \mathfrak{3}_1^{(0)} &\leftrightarrow I_{\Delta(6n^2)_1}^{(2,2)} = \bar{D}^1 D_1 \bar{D}^1 D_1 + \bar{D}^2 D_2 \bar{D}^2 D_2 + \bar{D}^3 D_3 \bar{D}^3 D_3, \\
 \mathfrak{6}_{(0,0)} &\leftrightarrow I_{\Delta(6n^2)_2}^{(2,2)} = \bar{D}^1 D_2 \bar{D}^2 D_1 + \bar{D}^1 D_3 \bar{D}^3 D_1 + \bar{D}^3 D_2 \bar{D}^2 D_3.
 \end{aligned}
 \tag{A20}$$

It worth noting that both $\mathfrak{6}_{(0,0)}$ give rise to the same invariant $I_2^{(2,2)}$ under the symmetric contraction $\bar{D}^a \bar{D}^b D_c D_d$. Note that $I_{\Delta(6n^2)_{1,2}}$ are different from $I_{\Delta(3n^2)_{1,2}}$ but since the two pairs are linearly dependent they are effectively the same.

The (D) groups are subgroups of $\Delta(6n^2)$

In the classic work of Miller, Blichfeldt, and Dickson [19] the so-called (C) and (D) subgroups of $SU(3)$ are defined as matrix groups. In [27] it is shown that the (C) groups are nothing but a special case of $\Delta(3n^2)$. We shall argue here that the (D) groups are nothing but subgroups of an appropriate $\Delta(6g^2)$.

In [27], the generators of the (D) groups have been worked out:

$$\begin{aligned}
 H &= \begin{pmatrix} \eta^a & 0 & 0 \\ 0 & \eta^b & 0 \\ 0 & 0 & \eta^{-a-b} \end{pmatrix}, & T &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \\
 R &= \begin{pmatrix} \delta^r & 0 & 0 \\ 0 & 0 & \delta^s \\ 0 & -\delta^{-r-s} & 0 \end{pmatrix}, & & \tag{A21}
 \end{aligned}$$

where $\eta \equiv \exp(2\pi i/n)$ and $\delta \equiv \exp(2\pi i/d)$. They give rise to a collection of (not necessarily simple) six-parameter subgroups $D(n, a, b; d, r, s)$ of $SU(3)$. When viewed as a matrix subgroup of $SU(3)$ in its fundamental $\mathfrak{3}$ representation, the matrices belonging to these (D) groups have exactly one nonzero entry in every row and column. Furthermore, the nonzero entries are powers of the g th root of unity, with $g = \text{lcm}(n, d, 2)$, where lcm stands for the lowest common multiple. The collection of all such matrices evidently forms a group with $6 \times g^2$ elements (the nonzero entries in the first and second rows determine the nonzero entry in the third, and there are six ways to place the elements). As this group must be just the $\Delta(6n^2)$ group with $n = g$, these (D) groups are subgroups (proper or not) of the $\Delta(6n^2)$ groups. Hence, we can obtain their irreps by decomposing the $\Delta(6n^2)$ irreps. As such, they cannot possess an irreducible representation whose dimension exceeds six. An immediate consequence is that the group $D(n, a, b; d, r, s)$ shares the two invariants Eq. (A20) with $\Delta(6g^2)$. The latter assertion can also be checked explicitly from the generators given in (A21). At last we would like to mention that from the viewpoint of the (algebraically

defined) $\Delta(6g^2)$ groups the (D) groups correspond to non-faithful representations.

APPENDIX B: EMBEDDING OF DISCRETE GROUPS INTO $SU(3)$

We would like to settle the question of whether it possible to approximate an arbitrary $SU(3)$ element (a basis transformation) by an element of a discrete group $\mathcal{D} \subset SU(3)$ suitably embedded into $SU(3)$. The embedding of \mathcal{D} into $SU(3)$ can be varied by conjugation with an arbitrary $SU(3)$ matrix. The problem therefore reduces to the question of whether there exist a $D \in \mathcal{D}$ and $B \in SU(3)$ for a specific $A \in SU(3)$ such that

$$A \simeq BDB^\dagger. \tag{A22}$$

We shall argue below that for $\Delta(3n^2)$ this is possible. Equation (A22) is true if A and D have (approximately) the same invariants. The invariants are given by the coefficients of the characteristic polynomial which are just the trace of the matrix and the trace of the square of the matrix. A sufficient condition for the traces to be (approximately) the same is that the eigenvalues are (approximately) the same. This immediately eliminates the crystal groups since there traces, i.e. characters, only assume very specific values. This can be inferred from the character tables. We shall proceed our argument via the eigenvalues. An $SU(3)$ matrix has in general three eigenvalues of the form $\lambda_i = e^{i\phi_i}$ with the determinant condition $\phi_1 + \phi_2 + \phi_3 = 0 \text{ mod } 2\pi$. The generators of $\Delta(3n^2)$ in the representation $\mathfrak{3}_{(k,l)}$ read [24]

$$\begin{aligned}
 a &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, & c &= \begin{pmatrix} \eta^l & 0 & 0 \\ 0 & \eta^k & 0 \\ 0 & 0 & \eta^{-k-l} \end{pmatrix}, \\
 d &= \begin{pmatrix} \eta^{-k-l} & 0 & 0 \\ 0 & \eta^l & 0 \\ 0 & 0 & \eta^k \end{pmatrix}, & & \tag{A23}
 \end{aligned}$$

with $\eta = \exp(2\pi i/n)$ and it is readily seen that the parameters k, l , and n can be adjusted such that the eigenvalues, of for example c , are arbitrarily close to any pair of unitary complex numbers. The third one is fixed in both cases by the determinant condition. For $\Delta(6n^2)$ this is also possible: The elements $c_{3_1}^a d_{3_1}^b$, with generators c_{3_1} and d_{3_1} as given in [25], approximate any two eigenvalues with arbitrary precision for suitable $a, b, n, l \in \mathbb{Z}$.

We conclude that $\Delta(3n^2)$ and $\Delta(6n^2)$ contrary to the crystal groups can be embedded into $SU(3)$ such that one of its elements is arbitrarily close to any $SU(3)$ element.

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