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Bogolyubov-Hartree-Fock approach to studying the QCD ground state

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The quark's behavior while influenced by a strong stochastic gluon field is analyzed. An approximate procedure for calculating the effective Hamiltonian is developed and the corresponding ground state within the Hartree-Fock-Bogolyubov approach is found. The comparative analysis of various Hamiltonian models is given and transition to the chiral limit in the Keldysh model is discussed in detail.

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Nowadays we know that the mixing of the zero modes is the microscopic mechanism of the spontaneous breakdown of chiral symmetry in the instanton liquid model [1]. In this approach the quarks are considered in a given gluon background and the spectrum of the respective Dirac operator is calculated in order to be accompanied then by averaging over the gluon ensemble. It is believed that at low energy the zero modes are effectually overlapped and the eigenvalues of the Dirac operator spread over some range of virtualities. However, up to now the behavior of low-lying quark modes in interacting stochastic (instanton) background cannot be analyzed analytically and one is forced to devise the suitable approximations argued by some general theoretical reasonings. Also great care is needed in order to obtain proper thermodynamical limit with nonzero chiral condensate. A lot of that happens to be in striking contrast to the Nambu-Jona-Lasinio (NJL) model [2] which is cognate to the instanton liquid model based actually on the similar multifermion interaction. Superficially, the main distinction consists in the appearance of some nonlocal form factors instead of the corresponding coupling constant. As to the microscopic consideration, the generation of dynamical quark mass in the NJL model is caused by the reconstruction of the Hamiltonian ground state and the quarks manifest themselves already as the quasiparticles [3] although the multifermion attractive force should be strong enough, roughly speaking. In this paper we emphasize that an instanton model and several other models which are based on treating the stochastic ensemble of strong gluon fields become practically identical in many aspects to the NJL model.

Such an approach is motivated by the conceptual idea of an intricate nature of the QCD vacuum [4] having been populated by intensive stochastic gluon fields of nontrivial topological structure. Moreover, studying the corresponding cooled lattice configurations gives evidence of this component presence [5] and using the instantons in the singular gauge to fit the data turns out to be very fruitful [6] and allows one to evaluate the ensemble density (around one topological charge per fm^4) and the characteristic size of a saturating configuration (about 1 GeV^{-1}). Both estimates are in fairly good agreement with the corresponding results of the instanton liquid model [7]. Nevertheless, the keen search of various confining configurations is still going on [8–10] in parallel with collecting the convincing evidences that the construction of a self-consistent ensemble of such configurations is a too complicated problem (see, for example, the estimate for the (anti-)instanton ensemble done in Ref. [11]).

Supposing the high-frequency component of the stochastic ensemble of gluon fields as the dominating contribution, we develop, in fact, an effective theory [12] by applying the procedure of simplified (averaged in time) system description which is widely used for studying the dynamical systems. Developing the effective theories which are discussed here has been launched to a considerable extent by studying the behavior of light quarks in the instanton gas (liquid) [13]. The zero mode approximation has provided for the quantitative picture of spontaneous chiral symmetry breaking [14]. However, an effective Lagrangian of the NJL type was soon received in Ref. [15] by the direct summation of certain leading diagrams and the obtained vertices of multiquark interactions occurred rather differently from those calculated in the zero mode approximation. Analysis of heavy quark systems behavior affected by the stochastic gluon fields [16] has demonstrated that at constructing the respective effective theory the cluster decomposition of generating functional [17] can be a very efficient tool and the specific role of various characteristic correlation times has been clarified to classify the descriptions. These results together with the criticism of zero mode approximation [18] have contributed to widening the cluster decomposition applications. This approach has been used to analyze the light quark behavior [19] and it is interesting to note the effective Lagrangian has agreed with that obtained in [15]. In the context of our interest here the cluster decomposition is

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called upon to describe the correlations in quantum system inspired by an external process. In this situation, as a matter of fact, the description of system behavior is executed by averaging the generating functional. However, as we show in this paper such a procedure applied to the quantum system could be incomplete and it is more appropriate to base an analysis on the corresponding density matrix. Nonetheless, we argue here that in the ''white noise'' limit (when the time intervals of stochastic impulses are very short) the procedure of averaging the generating functional occurs quite adequately.

The form of the effective Hamiltonian obtained urges us to search the system ground state as the Bogolyubov trial function. The corresponding dressing transformation will be analyzed for various form factors of the effective Hamiltonian. In such an approach the quarks are already treated as the quasiparticles and a rather practical way to get beyond the zero mode approximation appears. Actually, this approach grounds on simple iterations of corresponding integral equation solutions for the dressing transformation which is quite stable unlike the results of mean field approximation [15,19]. The different ensembles are examined and their selection is stipulated by the requirement that one of their asymptotic forms would be the NJL model which plays a calibrating role in our calculations. The chiral limit of the Keldysh model with the correlator behaving as a δ function in the momentum space is studied in detail and the singular behavior of the corresponding mean energy functional is demonstrated.

I. THE HARTREE-FOCK-BOGOLYUBOV APPROXIMATION

We consider the quark (antiquark) ensemble in the background of the strong stochastic gluon field and suppose this field is so strong that we could neglect the gluon interchanging processes (quenched approximation). The stochastic gluon field is characterized by a correlation function and its particular form will be discussed and fixed below. The Lagrangian density is the following:

$$\mathcal{L}_E = \bar{q}(i\gamma_\mu D_\mu + im)q. \tag{1}$$

Here q, $\bar{q}$ are the quark and antiquark fields with covariant derivative $D_\mu = \partial_\mu - igA^a_\mu t^a$, where A^a_μ is the gluon field, $t^a = \lambda^a/2$ are the generators of color gauge group $SU(N_c)$, and m is the current quark mass, $\mu = 1, 2, 3, 4$. We work in the context of the Euclidean field theory and γ_μ means the Hermitian Dirac matrices ($\gamma^+_\mu = \gamma_\mu, \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$) in the chiral representation. Then the corresponding Hamiltonian description results from

$$\mathcal{H} = \pi\dot{q} - \mathcal{L}_E, \qquad \pi = \frac{\partial \mathcal{L}_E}{\partial \dot{q}} = iq^+, \tag{2}$$

and, in particular, for the noninteracting fields we have

$$\mathcal{H}_0 = -\bar{q}(i\gamma\nabla + im)q. \tag{3}$$

In the Schrödinger representation the quark field evolution is determined by the equation for the quark probability amplitude Ψ as

$$\dot{\Psi} = -H\Psi, \tag{4}$$

and the creation and annihilation operators of quarks and antiquarks a^+, a, b^+, b have no ''time'' dependence and consequently look like

$$q_{\alpha i}(\mathbf{x}) = \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{(2|p_4|)^{1/2}} [a(\mathbf{p}, s, c)u_{\alpha i}(\mathbf{p}, s, c)e^{i\mathbf{px}} + b^+(\mathbf{p}, s, c)v_{\alpha i}(\mathbf{p}, s, c)e^{-i\mathbf{px}}]. \tag{5}$$

Here the summation over index s which stands to describe two quark spin polarizations and index c which should play the similar role for a color is implied. Further we make concrete the form of the Dirac conjugated spinor. Fixing a spin polarization as it is known can be done by imposing an additional constraint on spinor (see, below). However, there is no direct analogy with the color polarization and the particular state should be fixed by the corresponding complete set of diagonal operators which includes the Casimir operators as well. In fact, this complete definition of the spinor color state is unnecessary for us here. All observables are usually expressed by summing up the polarization states of some bilinear spinor combinations as the singlet and octet states and the singlet component is obviously playing the specific role.

The density of interaction Hamiltonian can be presented as

$$\mathcal{V}_S = \bar{q}(\mathbf{x})t^a\gamma_\mu A^a_\mu(t, \mathbf{x})q(\mathbf{x}). \tag{6}$$

The obvious dependence on time in this Hamiltonian is present in the gluon field only. As it is mentioned above we are planning to work with the stochastic gluon field implying the random process for which one may define only a probability of realizing some gluon configuration. Such a nature of gluon field urges (and allows) us to develop the approximate procedure for describing the quark field treating (4) as a probabilistic process. Then the system states are described by the corresponding averages (over a time or an ensemble according to the ergodic hypothesis). However, in the quantum theory we face one difficulty in this way because Ψ is a probability amplitude and an immediate averaging of $\langle\Psi\rangle$ can be insignificant. Studying a mean probability density $\langle\overset{*}{\Psi}\Psi\rangle$ looks more promising and can be realized by complicating the procedure of continual integration [20]. In Appendix A we analyze a convincing quantum mechanical example to illustrate the difference between two approaches. One of those is based on constructing the corresponding density matrix $\langle\overset{*}{\Psi}\Psi\rangle$, and the second approach does use the relevant averaging of the functional $\langle\Psi\rangle$. We argue the latter

could be practical for application in the white noise limit with the δ-like time correlation function. Adapting these ideas to the gauge theories we should obviously strive to operate with the gauge invariant quantities which include an ordered exponential, at least. Unfortunately, such a program in what concerns the ensemble consideration is still too far to be realized. However, it is clear that applying the averaging procedure would result in putting in an appearance of a set of corresponding correlation functions $\langle A^2\rangle$, $\langle A^4\rangle$, etc. [21]. Below, we will resort to formulating the gauge background in a particular gauge; cf. Equation (13).

In the interaction representation, where $\Psi = e^{H_0 t}\Phi$, Eq. (4) can be rewritten as

$$\dot{\Phi} = -V\Phi, \qquad V = e^{H_0 t} V_S e^{-H_0 t}. \tag{7}$$

Now the time dependence appears in quark operators as well. Now we recall some features of the averaging description as formulated in Ref. [17]. Presenting Eq. (7) in the integral form as

$$\Phi(t) = \Phi(0) - \int_0^t d\tau V(\tau)\Phi(\tau), \tag{8}$$

where $\Phi(0)$ is an arbitrary initial state of ensemble and performing another iteration one receives

$$\Phi(t) = \Phi(0) - \int_0^t d\tau V(\tau)\Phi(0) + \int_0^t d\tau V(\tau) \times \int_0^\tau d\tau' V(\tau')\Phi(\tau').$$

By averaging the fast-changing component and uncoupling the correlators one approximately approaches the long-wavelength component $\langle\Phi\rangle$ in the highest order (also taking into account that $\langle V\rangle = 0$) as follows:

$$\langle\Phi(t)\rangle \approx \Phi(0) + \int_0^t d\tau \int_0^\tau d\tau' \langle V(\tau)V(\tau')\rangle\langle\Phi(\tau')\rangle. \tag{9}$$

Certainly, it is assumed the characteristic correlation time of stochastic process is smaller than the time characteristic for the process $\langle\Phi\rangle$. By differentiating Eq. (9) it is easy to get rid of the initial condition $\Phi(0)$ and to have

$$\langle\dot{\Phi}(t)\rangle = + \int_0^t d\tau' \langle V(t)V(\tau')\rangle\langle\Phi(\tau')\rangle.$$

Actually this equation should describe a steady-state process and at reversing a time the solution, in general case, will not return to the initial magnitude $\Phi(0)$. Changing the integration variable as $\tau' = t - \tau$ one comes to

$$\langle\dot{\Phi}\rangle = + \int_0^t d\tau \langle V(t)V(t-\tau)\rangle\langle\Phi(t-\tau)\rangle. \tag{10}$$

It is usually supposed that because the correlations are quickly decaying then the upper limit of integration might be changed for ∞ and in order to deal with the local process it is well justified (without a precision loss) to change the argument of function $\langle\Phi\rangle$ for t. Eventually, as a result we have

$$\langle\dot{\Phi}(t)\rangle = + \int_0^\infty d\tau \langle V(t)V(t-\tau)\rangle\langle\Phi(t)\rangle. \tag{11}$$

(The requirements to validate the factorization of the long-wavelength component are discussed, for example, in [17].) Implementing the approximation (11) in the quantum field theory models, we run into trouble trying to get the most general form of the correlation function if the characteristic quark and gluon correlation times are comparable. Fortunately, if the quark fields are considered to be practically constant on the gluon background the problem receives essential simplification. The gluon field contribution may be factorized as a corresponding correlation function $\langle A^a_\mu(x)A^b_\nu(y)\rangle$ [22]. Recent lattice measurements of this correlation function provide us with a reasonable arguments to interpret the result as gluon ''mass'' generation ($\sim$ 300–400 MeV) in the momentum region of order 200 MeV [23].

It is curious to notice that the averaging over ensemble (time) in the right-hand side of Eq. (11) is performed in both the correlator and $\langle\Phi(t)\rangle$. It means that by resumming and averaging a certain class of diagrams in the quantum field theory models, one may take into account high order correlator contributions in different ways if the form of function $\langle\Phi(t)\rangle$ is specified. Besides, the correlation functions in models interesting to us should be translation invariant and it implies that the correlator in Eq. (11) has the following form:

$$\langle V(t)V(t-\tau)\rangle = F(\tau),$$

i.e., for example, a one-dimensional process after having done the integration in Eq. (11) will be described by a constant which characterizes the slow process. In quantum field theory for the problem we are interested in, the correlator connecting two space points

$$\langle\dot{\Phi}(t)\rangle = \int d\mathbf{x}\bar{q}(\mathbf{x},t)t^a\gamma_\mu q(\mathbf{x},t)\int_0^\infty d\tau\int d\mathbf{y}\bar{q}(\mathbf{y},t-\tau) \times t^b\gamma_\nu q(\mathbf{y},t-\tau)g^2\langle A^a_\mu(t,\mathbf{x})A^b_\nu(t-\tau,\mathbf{y})\rangle\langle\Phi(t)\rangle$$

appears instead of a constant. Assuming the correlation function is rapidly decreasing with time we change the time $t-\tau$ dependence in the quark fields for t and perform the inverse transformation to the Schrödinger representation. Then, introducing the function $\chi = e^{-H_0 t}\langle\Phi\rangle$, we have [24] the following equation:

$$\dot{\chi} = -H_{\text{ind}}\chi,$$
$$\mathcal{H}_{\text{ind}} = -\bar{q}(i\gamma\nabla + im)q - \bar{q}t^a\gamma_\mu q\int d\mathbf{y}\bar{q}'t^b\gamma_\nu q' \times \int_0^\infty d\tau g^2\langle A^a_\mu A'^b_\nu\rangle, \tag{12}$$

where $q = q(\mathbf{x})$, $\bar{q} = \bar{q}(\mathbf{x})$, $q' = q(\mathbf{y})$, $\bar{q}' = \bar{q}(\mathbf{y})$, $A^a_\mu = A^a_\mu(t, \mathbf{x})$ and $A'^b_\nu = A^b_\nu(t - \tau, \mathbf{y})$.

In order to receive the final result we should fix the form of the correlation function. As a guide, consider a stochastic ensemble of (anti-)instantons in the singular gauge

$$A^a_\mu(x) = \sum_{i=1}^{N} A^a_\mu(x; \gamma_i), \tag{13}$$

and the instanton solution reads as

$$A^a_\mu(x) = \frac{2}{g} 4\pi^2 i\rho^2 \omega^{ab} \bar{\eta}_{\mu b\nu} \int \frac{dq}{(2\pi)^4} q_\nu \phi(q) e^{iq(x-z)},$$
$$\phi(q) = \frac{1}{q^2}\left(K_2(q\rho) - \frac{2}{q^2\rho^2}\right), \tag{14}$$

where K_2 is the modified Bessel function of imaginary argument, ρ is the instanton size, the matrix ω appoints the pseudoparticle orientation in color space, z is the coordinate of instanton center, and $\bar{\eta}$ stands for the 't Hooft symbol. The distribution of the pseudoparticle orientation in color space is supposed to be homogeneous $\sim d\omega$ as well as the probability to find a pseudoparticle in the volume element is proportional $\sim dz/V$, where V is the volume of the system under consideration. Apparently, specifying the saturating configuration in the form of Eq. (13) is, in a direct way, the gauge fixing procedure. Calculating the quantum corrections for every single pseudoparticle in a one-loop approximation (what corresponds to the zeroth order of the N/V expansion), and exploiting the variation principle [7,11] allows one to ascertain the size distribution of pseudoparticles. In this way it is possible to attach clear meaning to the wave functional and to construct in the thermodynamical limit $\lim_{V\to\infty} N/V \to n$ the state possessing a negative energy density and developing a nonzero gluon condensate. (Uncertain interrelation of perturbative and nonperturbative contributions into the path integral [25] makes the computability of the wave functional highly nontrivial for now.) In Eq. (12) we imply the correlation function integrated over the time for which we receive in the highest order in the density n of (anti-)instanton ensemble

$$\int_0^\infty dx_4 \langle A^a_\mu(x) A^b_\nu(y)\rangle$$
$$= \frac{1}{2}\int_{-\infty}^{\infty} dx_4 \langle A^a_\mu(x) A^b_\nu(y)\rangle$$
$$= \frac{4(4\pi^2)^2}{g^2}\frac{\delta_{ab} n\rho^4}{N_c^2 - 1}(\delta_{\mu\nu}\delta_{\alpha\beta} - \delta_{\mu\alpha}\delta_{\nu\beta})$$
$$\times \int \frac{dp}{(2\pi)^4} p_\alpha p_\beta e^{ip(x-y)} \phi(-p)\phi(p)\frac{1}{2} 2\pi\delta(p_4).$$

The first equality is valid due to the symmetry properties of the instanton solution. Then the correlation function can be presented as

$$\langle \widetilde{A^a_\mu A^b_\nu}(\mathbf{p})\rangle = \frac{(4\pi^2)^2 n\rho^4}{g^2}\frac{2\delta_{ab}}{N_c^2 - 1}[I(p)\delta_{\mu\nu} - J_{\mu\nu}(p)],$$
$$I(p) = \mathbf{p}^2\phi(-p)\phi(p),$$
$$J_{ij}(p) = p_i p_j \phi(-p)\phi(p), \tag{15}$$
$$J_{4i} = J_{i4} = J_{44} = 0.$$

We generalize in what follows to a variety of stochastic ensembles of gluon fields characterized by their profile functions $I(p)$, $J_{\mu\nu}(p)$ and analyze the contribution of the quadratic correlator only. The deficiency of fixing the gauge implicitly for the truncated system in the treatment above is compensated, in a sense, by our investigation of a full spectrum of reasonable correlation functions (including opposite limiting correlators when they are extrapolated even into the perturbative region). Recent considerable progress in studying the confining configurations of lattice gauge theories, in particular, revealing the monopole clusters and their role in confinement (see, for review [26]) as well as detecting the specific features of quark behavior in the uncooled configurations and the indications that low-lying Dirac eigenmodes are localized on the objects of dimension inherent in monopoles and vortices [27] looks entirely urging. However, these results are also bringing the perilous tendencies because reveal some features of lattice gauge theories common with the compact electrodynamics, what makes almost inevitable to draw in the singular (in the continual limit) objects and, hence, to provide them with underlying physical meaning. Searching the form factors (the corresponding ensembles of saturating configurations) interesting for applications one should compare to the reasonable results for the four nonets of light mesons obtained in the NJL model. Apparently, the constants of effective four-quark Hamiltonian (scalar, pseudoscalar, vector, and axial-vector channels) and the parameters of integral saturation (cutoff) should be comparable. It might be carefully supposed that those singular objects (at still an unknown scale) have to reproduce the major features of a successful effective NJL Hamiltonian after the corresponding averaging. In any case, the problems of finding the specific features of such singular objects which admit their experimental identification and of analyzing the quark behavior in the ensembles of monopoles or vortices are of really great interest [28].

With such a form of induced four-fermion interaction we are going to search the ground state as the Bogolyubov probe function with vacuum quantum numbers [29–31]

$$|\sigma\rangle = T|0\rangle,$$
$$T = \Pi_{p,s,c} \exp\{\varphi[a^+(\mathbf{p}, s, c)b^+(-\mathbf{p}, s, c) + a(\mathbf{p}, s, c)b(-\mathbf{p}, s, c)]\}, \tag{16}$$

which is defined by minimizing mean energy

$$E = \langle\sigma|H|\sigma\rangle. \tag{17}$$

Here $\varphi = \varphi(\mathbf{p})$ and $|0\rangle$ is the vacuum of free Hamiltonian, i.e. $a(\mathbf{p}, s, c)|0\rangle = 0$, $b(\mathbf{p}, s, c)|0\rangle = 0$. Introducing the creation and annihilation operators of quasiparticles with the dressing T transformation ($T^{-1} = T^{\dagger}$ for fermions)

$$A = TaT^{-1}, \qquad B^{+} = Tb^{+}T^{-1},$$

we present the operator Eq. (5) as, with the Dirac conjugate spinor

$$\begin{aligned} q(\mathbf{x}) &= \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{(2|p_4|)^{1/2}} [A(\mathbf{p}, s, c)U(\mathbf{p}, s, c)e^{i\mathbf{px}} \\ &\quad + B^{+}(\mathbf{p}, s, c)V(\mathbf{p}, s, c)e^{-i\mathbf{px}}], \\ \bar{q}(\mathbf{x}) &= \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{(2|p_4|)^{1/2}} [A^{+}(\mathbf{p}, s, c)\bar{U}(\mathbf{p}, s, c)e^{-i\mathbf{px}} \\ &\quad + B(\mathbf{p}, s, c)\bar{V}(\mathbf{p}, s, c)e^{i\mathbf{px}}], \end{aligned} \tag{18}$$

where the spinors U and V are defined as

$$\begin{aligned} U(\mathbf{p}, s, c) &= \cos(\varphi)u(\mathbf{p}, s, c) - \sin(\varphi)v(-\mathbf{p}, s, c), \\ V(\mathbf{p}, s, c) &= \sin(\varphi)u(-\mathbf{p}, s, c) + \cos(\varphi)v(\mathbf{p}, s, c). \end{aligned} \tag{19}$$

with $\bar{U}(\mathbf{p}, s, c) = U^{+}(\mathbf{p}, s, c)\gamma_4$ and $\bar{V}(\mathbf{p}, s, c) = V^{+}(\mathbf{p}, s, c)\gamma_4$. Now we have to specify the choice of spinors in the Euclidean variables. They obey the Dirac equations

$$(\hat{p} - im)u(p, s) = 0, \qquad (\hat{p} + im)v(p, s) = 0, \tag{20}$$

(with $\hat{p} = p_4\gamma_4 + \mathbf{p}\gamma$) and an additional constraint which fixes the spinor polarization

$$i\gamma_5\hat{s}u(p, s) = u(p, s), \qquad i\gamma_5\hat{s}v(p, s) = v(p, s), \tag{21}$$

where $\gamma_5 = -\gamma_1\gamma_2\gamma_3\gamma_4$, and the four-vector s is normalized to unit and orthogonal to the four-vector p, i.e. $s^2 = 1$, $(ps) = 0$. It could be, for example,

$$s_4 = \frac{(\mathbf{pn})}{im}, \qquad \mathbf{s} = \mathbf{n} + \frac{(\mathbf{pn})\mathbf{p}}{im(p_4 - im)},$$

where $\mathbf{n}$ is an arbitrary unit vector. If the covariant normalization conditions are satisfied

$$\bar{u}u = 2im, \qquad \bar{v}v = -2im, \tag{22}$$

the spinors are defined with precision up to the phase factor. All these conditions allow us to formulate the following matrix representation:

$$\begin{aligned} u(p, s)\bar{u}(p, s) &= \frac{\hat{p} + im}{2}(1 + i\gamma_5\hat{s}), \\ v(p, s)\bar{v}(p, s) &= \frac{\hat{p} - im}{2}(1 + i\gamma_5\hat{s}). \end{aligned} \tag{23}$$

Calculating the mean energy Eq. (17) we meet spinors with opposite moments. We introduce the four-vector $q = (p_4, -\mathbf{p})$ in order to simplify notations. Using the projection operator we can express the spinor $v(q, s)$ through the spinor $u(p, s)$ (see [32])

$$v(q, s) = \alpha \frac{\hat{q} - im}{-2im} \frac{1 + i\gamma_5\hat{s}}{2} u(p, s). \tag{24}$$

The coefficient α is fixed by the covariant normalization Eq. (22) up to the phase factor as

$$\overset{*}{\alpha} * \alpha = -\frac{2m^2}{(pq) + m^2} = \frac{m^2}{\mathbf{p}^2}, \qquad |\alpha| = \frac{m}{|\mathbf{p}|}.$$

Then summing up over the spinor states results in

$$\begin{aligned} \sum_s u(q, s)\bar{v}(p, s) &= \alpha \frac{\hat{q} + im}{2im}(\hat{p} - im), \\ \sum_s v(p, s)\bar{u}(q, s) &= \overset{*}{\alpha}(\hat{p} - im)\frac{\hat{q} + im}{2im}, \\ \sum_s u(p, s)\bar{v}(q, s) &= \overset{*}{\alpha}(\hat{p} + im)\frac{\hat{q} - im}{2im}, \\ \sum_s v(q, s)\bar{u}(p, s) &= \alpha \frac{\hat{q} - im}{2im}(\hat{p} + im). \end{aligned} \tag{25}$$

The polarization in which the momentum $\mathbf{p}$ and unit polarization vector $\mathbf{n}$ are orthogonal $(\mathbf{pn}) = 0$ turns out to be the most convenient for handling. In such a situation both operators $\hat{p}$ and $\hat{q}$ commute with $\gamma_5\hat{s}$ and the polarization directions of quark and antiquark could be taken identical (although in general case they should be two different directions). Then the summation over polarization of quarks and antiquarks is performed separately in the final equations. It allows us not to control the obligatory constraint to have the vacuum quantum numbers of the pairs present in the intermediate calculations.

When calculating the mean energy Eq. (17) nontrivial contribution bilinear in quark operators comes from the terms of type $\langle\sigma|BB^{+}|\sigma\rangle$ (remember $B|\sigma\rangle = 0$, $A|\sigma\rangle = 0$). The contribution from the terms like $\langle\sigma|AA^{+}|\sigma\rangle$ is absent because of the particular representation of bilocal operator we are using as $\bar{q}q$ (then quadratic terms are expressed by the spinors V, $\bar{V}$. Because of similar reasons the four-quark operators develop only two nonzero contributions $\langle\sigma|BB^{+}B'B'^{+}|\sigma\rangle$ and $\langle\sigma|BAA'^{+}B'^{+}|\sigma\rangle$. The first combination corresponds to the contribution of so-called tadpole diagrams and the latter is related to the asterisk [33] diagrams. As a result the four-fermion interaction contribution can be presented in the following form:

$$\langle \widetilde{A^a_\mu A^b_\nu}(0)\rangle \int \frac{d\mathbf{p}d\mathbf{p}'}{(2\pi)^6}\frac{1}{4|p_4||p'_4|}\bar{V}_{\alpha i}(\mathbf{p},s,c)t^a_{ij}\gamma^\mu_{\alpha\beta}V_{\beta j}(\mathbf{p},s,c)\bar{V}_{\gamma k}(\mathbf{p}',s',c')t^b_{kl}\gamma^\mu_{\gamma\delta}V_{\delta l}(\mathbf{p}',s',c')$$
$$+\int \frac{d\mathbf{p}d\mathbf{p}'}{(2\pi)^6}\frac{1}{4|p_4||p'_4|}\bar{V}_{\alpha i}(\mathbf{p},s,c)t^a_{ij}\gamma^\mu_{\alpha\beta}V_{\delta l}(\mathbf{p},s,c)\bar{U}_{\gamma k}(\mathbf{p}',s',c')t^b_{kl}\gamma^\nu_{\gamma\delta}U_{\beta j}(\mathbf{p}',s',c')\langle \widetilde{A^a_\mu A^b_\nu}(\mathbf{p}+\mathbf{p}')\rangle.$$

Here $\langle \widetilde{A^a_\mu A'^b}_\nu\rangle$ is the Fourier transform of the gluon correlator and the summation over spinor and color indices is implied. The contribution of the first tadpole diagram is an identical zero due to completeness of the spinor basis in color space, giving a unit color matrix (in particular it is valid for color singlet quark configuration). In electrodynamics the considered term provides a dominant contribution. But it is interesting to remark that the singular character of the photon propagator in the infrared region makes this Abelian theory even more complicated to research than in the non-Abelian one. In the compact $U(1)$ electrodynamics (on a lattice) the infrared behavior of correlation function is formed by the monopole contributions but nowadays it is still difficult to define a scale where these effects show up themselves. In the octet channel of non-Abelian theory we obtain the quark repulsion $\sim -1/(4N_c)$ and therefore this regime might be omitted when searching the minimum of mean energy Eq. (17). Then, for the spinors with polarizations summed up, we have

$$V\bar{V} = p_4\gamma_4 + \cos(\theta)(\mathbf{p}\boldsymbol{\gamma} - im) - \frac{\overset{*}{\alpha}+\alpha}{2im}\sin(\theta)(\mathbf{p}^2 - im\mathbf{p}\boldsymbol{\gamma}),$$
$$U\bar{U} = p_4\gamma_4 + \cos(\theta)(\mathbf{p}\boldsymbol{\gamma} + im) + \frac{\overset{*}{\alpha}+\alpha}{2im}\sin(\theta)(\mathbf{p}^2 + im\mathbf{p}\boldsymbol{\gamma}),$$

where angle $\theta = 2\varphi$. In the formulas above the phase inherent in the sum $\overset{*}{\alpha}+\alpha$ (a spinor is defined up to such a phase) is still indefinite. The direct analysis of the mean energy functional demonstrates that the most preferable value of the phase factor (responsible for the color interaction of quarks) is the value when the coefficient α appears to be a real number. For definiteness we put $\alpha = +|m|/p$. The curious fact is that the results of summation are not equal $[V\bar{V}(m) = U\bar{U}(-m)]$ and they coincide in the chiral limit $m = 0$ only, i.e. particles and antiparticles formally generate the different contributions.

The direct calculations lead to the following result for the mean energy (17):

$$\langle\sigma|H_{\mathrm{ind}}|\sigma\rangle = -\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{2N_cp_4^2}{|p_4|}(1-\cos\theta) - \tilde{G}\int\frac{d\mathbf{p}d\mathbf{q}}{(2\pi)^6}\Big\{-(3\tilde{I}-\tilde{J})\frac{p_4q_4}{|p_4||q_4|} + (4\tilde{I}-\tilde{J})\frac{pq}{|p_4||q_4|}$$
$$\times\Big(\sin\theta - \frac{m}{p}\cos\theta\Big)\Big(\sin\theta' - \frac{m}{q}\cos\theta'\Big) - (2\tilde{I}\delta_{ij} + 2\tilde{J}_{ij} - \tilde{J}\delta_{ij})\frac{p_iq_j}{|p_4||q_4|}$$
$$\times\Big(\cos\theta + \frac{m}{p}\sin\theta\Big)\Big(\cos\theta' + \frac{m}{q}\sin\theta'\Big)\Big\}. \tag{26}$$

Here we designated $p = |\mathbf{p}|$, $q = |\mathbf{q}|$, $\tilde{I} = \tilde{I}(\mathbf{p}+\mathbf{q})$, $\tilde{J}_{ij} = \tilde{J}_{ij}(\mathbf{p}+\mathbf{q})$, $\tilde{J} = \sum_{i=1}^3 \tilde{J}_{ii}$, $p_4^2 + p^2 = q_4^2 + q^2 = -m^2$, $\theta' = \theta(q)$, $\tilde{G} = (4\pi^2)^2 n\rho^4$, and as a matter of convenience we singled out the color factor $G' = \frac{2}{N_c^2-1}\tilde{G}$. To obtain this result we performed the regularization (subtracting the free Hamiltonian H_0). It results in the presence of a unit (together with $-\cos\theta$) in the parentheses of the first integral. Let us also recall that in the Euclidean space p_4^2 is a negative magnitude. Then with Eq. (26) available one can find the most advantageous value of the angle θ from the condition

$$\frac{d\langle\sigma|H_{\mathrm{ind}}|\sigma\rangle}{d\theta} = 0. \tag{27}$$

Henceforth we characterize the different stochastic ensembles of the gluon fields by their profile functions $I(p)$, $J_{\mu\nu}(p)$.

II. NAMBU-JONA-LASINIO MODEL

Now let us consider the example in which the correlation function behaves in the coordinate space as the δ function

(simply we assume $J_{\mu\nu}(p) = 0$). Actually, it corresponds to the NJL model [2]. As is well known, regularization is required to obtain an intelligent result in this model. We adjust the NJL model with the parameter set given by Ref. [34], and limit the integration interval over momentum in Eq. (26) with the quantity $|\mathbf{p}| < \Lambda$ ($\Lambda = 631$ MeV). Then the functional (26) is written in the following form (unessential terms contributing the constant values are omitted):

$$W = \int^{\Lambda} \frac{d\mathbf{p}}{(2\pi)^3}\Big[|p_4|(1-\cos\theta) - G\frac{p}{|p_4|}\Big(\sin\theta - \frac{m}{p}\cos\theta\Big)\int^{\Lambda}\frac{d\mathbf{q}}{(2\pi)^3}\frac{q}{|q_4|} \times \Big(\sin\theta' - \frac{m}{q}\cos\theta'\Big)\Big]. \quad (28)$$

Here $m = 5.5$ MeV. The equation to calculate the optimal angle θ (27) reads as

$$(p^2 + m^2)\sin\theta - M(p\cos\theta + m\sin\theta) = 0, \quad (29)$$

where

$$M = 2G\int^{\Lambda}\frac{d\mathbf{p}}{(2\pi)^3}\frac{p}{|p_4|}\Big(\sin\theta - \frac{m}{p}\cos\theta\Big). \quad (30)$$

The constant of four-fermion interaction is $G = \frac{4}{2N_c}\tilde{G}$ while expressed in the same units as the mean energy functional in Eq. (26). For the NJL model Eq. (29) makes it possible to contract a functional space in which the minimum of mean energy functional can be realized. This equation parameterizes the function $\theta(p)$ on the whole interval $p \in [0, \Lambda]$ of searching the solution. Moreover, Eq. (29) itself does not impose any restrictions on the parameter M which may be any real number. Then the functional (28) simply becomes the function of parameter $W(M)$. Now if one expresses the trigonometrical functions via parameter M it is possible to make the representation of minimizing function and the result of its integration (30) agree. As a result we receive three extremal points, two of them correspond the minimal points with negative and positive values of M and the negative value conforms to the state of more stability. The point of unstable equilibrium is located in the vicinity of coordinate origin $\sim m$. The induced quark mass for the parameter magnitudes fixed is $M = -335$ MeV and the quark condensate

$$\langle\sigma|\bar{q}q|\sigma\rangle = \frac{iN_c}{\pi^2}\int_0^{\Lambda} dp\,\frac{p^2}{|p_4|}(p\sin\theta - m\cos\theta), \quad (31)$$

develops the magnitude of $\langle\sigma|\bar{q}q|\sigma\rangle = -i(245\ \text{MeV})^3$. The characteristic constant of the four-fermion interaction is equal to $G/(2\pi^2) = 1.34$. In what follows we rely on these quantitative results.

The situation, if the correlator $J_{\mu\nu}(p)$ is not equal to zero and has the same form of the δ function in coordinate space, can be similarly analyzed. The numerical analysis done teaches that its influence can be essential but we do not show these results due to the lack of any phenomenological estimates of the correlation function magnitude. The nonlocal version of the NJL model in which the correlator has the separable form $I(p, q) = K(p)K(q)$ can be similarly analyzed. In fact, it again displays the above mentioned property which replaces the functional analysis for the analysis of function dependence on some parameter. Although one important difference does exist and it shows that the procedure of integral cutting off is unnecessary for the functions $K(p)$. The regularization is naturally performed by the $K(p)$ kernel and so strong regularization is caused by the separable form interaction kernel. Certainly, such a property can manifest itself in much weaker form for more realistic correlators.

III. THE KELDYSH MODEL

Here we are going to analyze the limit in which the correlation function has a δ-function form in the momentum space

$$I(\mathbf{p}) = (2\pi)^3 G\delta(\mathbf{p}).$$

This limit is an analogue of the Keldysh model which is well known in the physics of condensed matter [35], and the mean energy functional (26) develops the following form in this case [36]:

$$W(m) = \int\frac{d\mathbf{p}}{(2\pi)^3}\Big[|p_4|(1-\cos\theta) - G\frac{p^2}{|p_4|^2}\Big(\sin\theta - \frac{m}{p}\cos\theta\Big)^2\Big]. \quad (32)$$

The optimal values of angle θ are determined by the solutions of the following equation:

$$|p_4|^3\sin\theta - 2G(p\cos\theta + m\sin\theta)(p\sin\theta - m\cos\theta) = 0, \quad (33)$$

and we start analyzing these solutions in the chiral limit $m = 0$. One of the solutions corresponds to the zero angle $\theta = 0$ but the nontrivial one takes the form

$$\cos\theta = \frac{p}{2G}. \quad (34)$$

Both the positive and negative angles θ are suitable as the solutions because of the parity (positive) property of the functional (32) and these (real) solutions (in addition to the trivial one) exist on the limited momentum interval $p < 2G$. There is one real solution for the trivial angle and there are two imaginary (complex-conjugate) solutions beyond this interval. Analyzing the NJL model above we noticed its very convenient property when the solution $\theta(p)$ is defined on the whole interval and, in fact, the functional is parameterized by a single number which is the integral M. In the Keldysh model the situation is much more

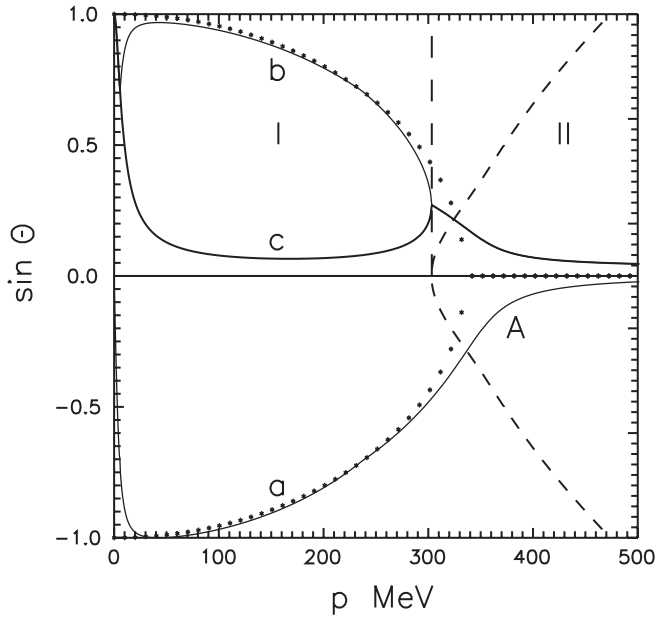

FIG. 1. Phase portrait of the Keldysh model, with $\sin\theta$ as a function of momentum p(MeV). The dotted curves correspond to the solutions in the chiral limit $m = 0$.

sophisticated and the phase portrait of its solutions in the chiral limit (for example, $\sin\theta$ as a function of momentum p) consists of two arches (with positive and negative $\sin\theta$; see Fig. 1) and a straight line corresponding to the trivial solution. Thus, the semiaxis $p \in [0, \infty)$ can be divided into two parts. There are three branches (solutions) at the interval $p \in [0, 2G]$, two of those correspond to the positive and negative angles $\theta = \pm \arccos(p/2G)$ and the trivial one where $\theta = 0$. At the interval $p \in [2G, \infty)$ only one trivial solution $\theta = 0$ exists and in order to construct the solution on the whole semiaxis $p \in [0, \infty)$ one has to add the trivial solution on the interval $p \in [2G, \infty)$ to any detached branch of solutions on the interval $p \in [0, 2G]$. It is easy to see that making use of the imaginary branches of solutions leads to the significant growth of energy and just because of this fact they are uninteresting. The other potentially interesting functions $\theta(p)$ for which it is reasonable to search the functional minima could be received if the interval $p \in [0, 2G]$ is subdivided into smaller intervals and then for each interval when continuing the function to the next interval (for example, to the direction of the momentum p increasing) to use two other branches as well as of the results of continuation on the same branch. Apparently, it results in the piecewise continuous function and, unlike the NJL model, here we have no parameter which restricts the function and watches its integral characteristics. In the chiral limit all the solutions (trajectories) constructed in such a way will acquire strictly fixed (finite) value of the functional $W(0)$ (it will be observed that the functional does not contain the derivatives of angle in momentum). For example, the trajectory which is going along the top arch at the interval $p \in [0, 2G]$ and continuing longer as a trivial solution to the whole semiaxis leads to the magnitude

$$W_{\pm}(0) = -\frac{G^4}{15\pi^2},$$

(similarly for the top negative arch). The chiral condensate (31) turns out then to be

$$\langle\sigma|\bar{q}q|\sigma\rangle(0) = \frac{iN_cG^3}{2\pi},$$

(and for the solution along the negative arch we have the opposite sign). The mean energy and chiral condensate equal to zero for the trivial solution, i.e. ($W_0(0) = 0$, $\langle\sigma|\bar{q}q|\sigma\rangle_0(0) = 0$). Clearly, these piecewise continuous functions will lead to the magnitudes of functional $W(0)$ which fill up the interval $[W_0(0), W_{\pm}(0)]$ densely, a similar pattern takes place for the chiral condensate. With the natural parameterization

$$\sin\theta = \frac{M_\theta}{(p^2 + M_\theta)^{1/2}}, \tag{35}$$

we obtain for the mass M_θ which characterize the angle at the top arch the following result:

$$M_\theta = (4G^2 - p^2)^{1/2}. \tag{36}$$

It is interesting to notice that then the respective energy of nontrivial solutions $E(p) = \sqrt{p^2 + M_\theta^2}$ becomes constant $E(p) = 2G$.

After having done the analysis in the chiral limit which is shown by the dotted lines in Fig. 1 we would like to comment on the situation beyond this limit, i.e. where $m \neq 0$. The evolution of corresponding branches is available on the same plot (Fig. 1), where the behavior of $\theta(p)$ as the function of momentum p in MeV is shown for the solution of Eq. (33). The semiaxis $p \in [0, \infty)$ where we are searching for the solution can be subdivided into two sectors which are demonstrated by the vertical dashed line on the plot. Three solutions denoted by a, b, and c are developing at the first sector denoted in Fig. 1 by I. Besides, there are three solutions at the second sector denoted by II, one real solution designated as A for the negative pairing angle and two complex-conjugate roots with the positive real parts. The imaginary parts of solutions are plotted in Fig. 1 by the dashed lines. The solution A in domain II develops the behavior of $\theta \sim -\frac{2Gm}{p^2}$ with increasing momentum. As in the chiral limit the minimum of the mean energy functional $W(m)$ can be obtained with the piecewise continuous functions which are properly represented by the trajectories aA, bA, cA (for real solutions). The first symbol of this complicated designation implies the branches a, b, or c at the first sector, the second symbol corresponds to the branch at the sector II. Thus, at low momenta we start

with the solution of branches a, b, or c, then the relevant solution passes to the branches interchanging its position in any subinterval. But in any case there is only one way to continue the real solution when momenta goes to infinity and it is related with the branch A when the angle is going to the zero value. Moreover, if the angle θ could take a strictly zero value in sector II then the second term of Eq. (32) leads to the singular contribution coming from the term $\frac{m^2}{p^2}\cos^2\theta$ with the linear divergency at a large momentum. Besides, the other terms develop the logarithmic divergencies as well. It is an amusing fact that the mean energy functional out of chiral limit goes to an infinity at any nonzero value of current quark mass m although in the chiral limit $W(0)$ is well defined. (It is worthwhile to remember here the current mass singularity of zero mode approximation which was discovered in Ref. [18].) The same conclusion is valid for the chiral condensate [see Eq. (31)] in which the first and second terms are developing the linear and quadratic divergencies, respectively. We could conclude here that if the cutoff factor is not used in the integrals when dealing with the solutions on the whole axis the functional $W(m)$ and quark condensate $\langle\sigma|\bar{q}q|\sigma\rangle(m)$ are ill defined.

Let us remember here that by definition the approximation (12) should describe the quark behavior in the background of the stochastic gluon field (which is averaged) at low energies. Then it looks quite natural to introduce an effective cutoff (in momentum) parameter $\tilde{\Lambda}$. The condition for factorization of gluon and quark field contributions gets broken at the momenta above $\tilde{\Lambda}$. In such a situation the dependence of mean energy and quark condensate on the current quark mass is defined not only by the form of integrand but by the value of parameter $\tilde{\Lambda}$ as well. And if this value is pretty large $\tilde{\Lambda} \gg M$ (where M is the dynamical quark mass) the dependence on m of all the observables is mainly defined by the magnitude of cutoff parameter $\tilde{\Lambda}$ because of the singular character of integrals (for example, for the NJL model this magnitude could be estimated as $\sim$1 GeV). Obviously, it means in order to get the dependence of observables on the current quark mass we need to draw essential additional information.

As to the possible interpretation of the singularities available in the mean energy functional we could assume, for example, the mechanism similar to the Cooper pairing which takes place at every scale of the increasing momenta $\tilde{\Lambda}, \tilde{\Lambda}_1, \ldots$. Certainly, we should correct the existing results about the four-quark interaction potential to put the pairing effect on realistic ground. For example, the contribution of the stochastic configurations like the small size instantons which is exponentially suppressed is hardly relevant to provide an efficient pairing mechanism for the momenta above $\tilde{\Lambda}$. Apparently, the hard gluon exchange looks like a more adequate mechanism at small distances. Then the gluon correlation function in Eq. (12) should be transformed in the corresponding gluon propagator. The effective four-quark interaction we are interested in can be derived by the quasiaverage formalism [29] which approximates smoothly the infrared and ultraviolet momentum regions although an alternative scenario could also be quite meaningful (see, for instance, the discussion in [37]). The fact that the Cooper attraction is still large enough despite the coupling constant weakening could signal the dominance of more fundamental fields at very small distances. (Here it is easy to see that all the different models might be classified by the convergence of the integral over momentum with the constant of four-fermion interaction

$$I_G = \int dp G(p),$$

as the integrand. The model falls under the category of a singular one if this integral diverges.)

In Fig. 2 we compare the equilibrium angles θ for the NJL model (solid line) and the aA solution of the Keldysh model (dashed curve) as the functions of momentum p in MeV with the current quark mass $m = 5.5$ MeV. It is interesting to notice that out of the chiral limit the solution (which has a spherical symmetry) passed over zero at $p = 0$ [see, Eq. (37)]. Besides, Fig. 2 demonstrates that out of the chiral limit the pairing process becomes essential not at zero momentum value (as it takes place in the chiral limit) but it is shifted to the magnitude about $p_\theta \sim 40$ MeV for the fitting parameters used. For example, in the NJL model it can be obtained that

$$p_\theta = [m|M - m|]^{1/2}.$$

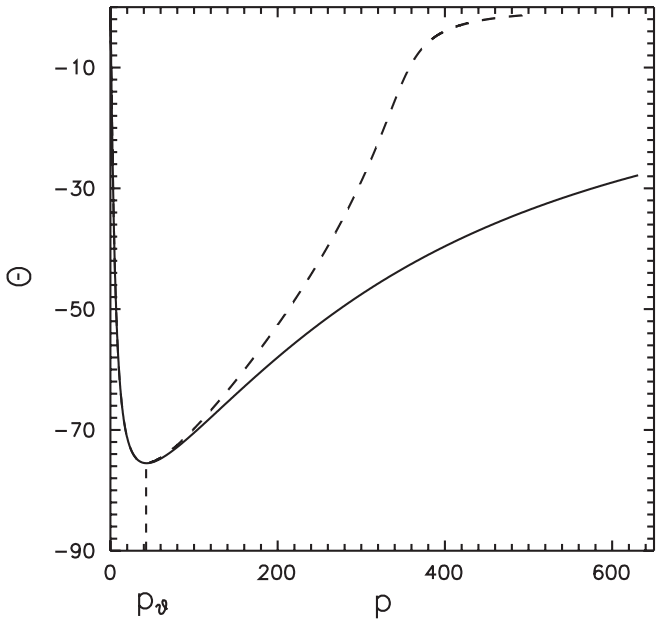

FIG. 2. The equilibrium angle θ (in degrees) as a function of momentum p in MeV. The solid line shows the result of the NJL model and the dashed line corresponds to the most stable branch of the Keldysh model; the current quark mass is taken as $m = 5.5$ MeV.

The quantity $r_\theta = 1/p_\theta$ determines the characteristic size of the region which is efficient for the pairing process. In the chiral limit this region is formally extending to infinity. The curves shown in the plot correspond to the opposite, in a sense, limiting regimes, and it is interesting to evaluate where the model with more realistic correlator could be found out.

One of the important motivations to study the Keldysh model was the question of a natural regularization which presents for the interaction with a separable kernel. We have seen that in the chiral limit for the kernel with the most extensively expressed regularizing property as, for example, the momentum δ function, both the mean energy and chiral condensate are well defined. Out of the chiral limit the unexpected singularity appears. In Ref. [38] the possibility of continuing the mean energy functional and the quark condensate by performing the respective regularization was discussed. As is well known, the meson masses in the NJL model can be presented by the quark condensate which hints that the corresponding expressions in the Keldysh model could be singular as well and one needs to perform another regularization to provide them with clear physical meaning. However, despite the present singularity of the chiral condensate the meson observables are finite and are well matched with the experimental mass scale (see [39]). The reason to have these meson observables as the smooth functions of current quark mass is in the regularizing role of additional vertex form factors which enter the meson mass formulas. Then we may summarize that it does not make sense to debate about an absolute value of quark condensate (in vacuum) for considered mechanism of spontaneous chiral symmetry breakdown because its magnitude depends on the particular observable (characteristic momentum of saturation) which is used for extracting this data. In Appendix B we compare the results obtained in the Hartree-Fock-Bogolyubov approach for the NJL and Keldysh models with the results of mean field approximation.

IV. THE EXPONENTIAL AND GAUSSIAN CORRELATORS

Here we turn to a more realistic situation and analyze the solutions possessing a spherical symmetry in the regime where the correlation function $I(\mathbf{x})$ is rather quickly decreasing with the distance increasing. Performing the integration over the azimuthal angles we can get the equation to derive the optimal angle in the following form:

$$|p_4|^2 \sin\theta - \frac{4G}{\pi}\left(\cos\theta + \frac{m}{p}\sin\theta\right)\iint_0^\infty dq dx \frac{q}{|q_4|} \times (q\sin\theta' - m\cos\theta')I(x)\sin(px)\sin(qx) = 0. \quad (37)$$

Considering the solution behavior at high momentum values p we are interested in analyzing solutions in which the angle θ is going to zero magnitude. Assuming the θ value to be rather small, we expand (37) up to the terms of the θ order and have

$$p^2\theta - \frac{4G}{\pi}\int_0^\infty dq(q\theta' - m)I(p,q) = 0.$$

If the function θ is decreasing faster than $1/q$ the most essential contribution to the integral comes from the term proportional to m and if the kernel $I(p,q)$ is integrable the asymptotic behavior has the following form:

$$\theta = -\frac{4Gm}{\pi p^2}\int_0^\infty dq I(p,q).$$

Let us consider now two concrete examples, with exponential behavior of the correlator $I(\mathbf{x}) = \exp(-a|\mathbf{x}|)$, and with the Gaussian behavior $I(\mathbf{x}) = \exp(-a^2\mathbf{x}^2)$. The integration over x can be performed exactly for both cases and the kernels of integral equations look like

$$\int_0^\infty dx e^{-ax}\sin(px)\sin(qx) = \frac{a}{2}\left(\frac{1}{a^2+(p-q)^2} - \frac{1}{a^2+(p+q)^2}\right),$$

for the exponential correlator and as

$$\int_0^\infty dx e^{-a^2x^2}\sin(px)\sin(qx) = \frac{\sqrt{\pi}}{4a}\left(e^{-((p-q)^2/4a^2)} - e^{-((p+q)^2/4a^2)}\right),$$

for the Gaussian one. Now let us hold the contribution of the first term only at large momentum values p for both examples. Then as a result the corresponding asymptotic behaviors are expressed by the constants which are defined by the integrals with the kernels $I(p,q)$. It allows us to conclude that we have again the singular functional for the mean energy out of the chiral limit. The parameter M_θ [see Eq. (35)] and the quark condensate as functions of the constant G for the Gaussian correlator [both obtained by the numerical computation of Eq. (37)] are depicted in Fig. 3. The solid line demonstrates the solution with the current quark mass $m = 5.5$ MeV and the dashed line is calculated in the chiral limit as the quark condensate presented by the points. The intrinsic change of the parameter M_θ generation out of the chiral limit is easily seen. The similar features are observed for the exponential correlator as well.

Unfortunately, it is a very serious problem to get all the solutions of the nonlinear integral equation (37) and here we are working with only one of its (the most stable) branches. As it was demonstrated above such a situation generates a lot of difficulties for extracting a reliable information on the observables out of the chiral limit. Because of this reason we calculate here the dynamical quark mass ($M = -335$ MeV) and chiral condensate ($|\langle\sigma|\bar{q}q|\sigma\rangle| = (245\ \text{MeV})^3$) in the chiral limit collating

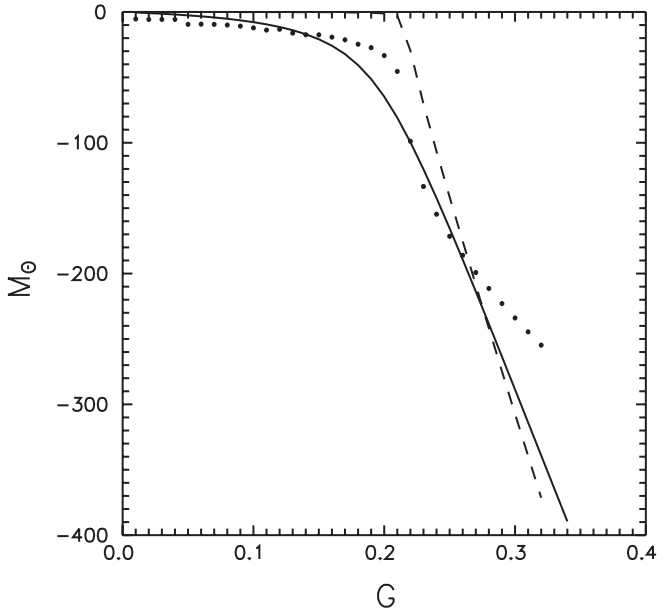

FIG. 3. The parameter M_θ (solid and dashed lines) and quark condensate (without an imaginary unit, in power 1/3) in MeV (shown by points calculated in the chiral limit $m = 0$) as a function of the constant G for the Gaussian correlator. The solid line is calculated with the current quark mass $m = 5.5$ MeV and the dashed line is calculated in the chiral limit.

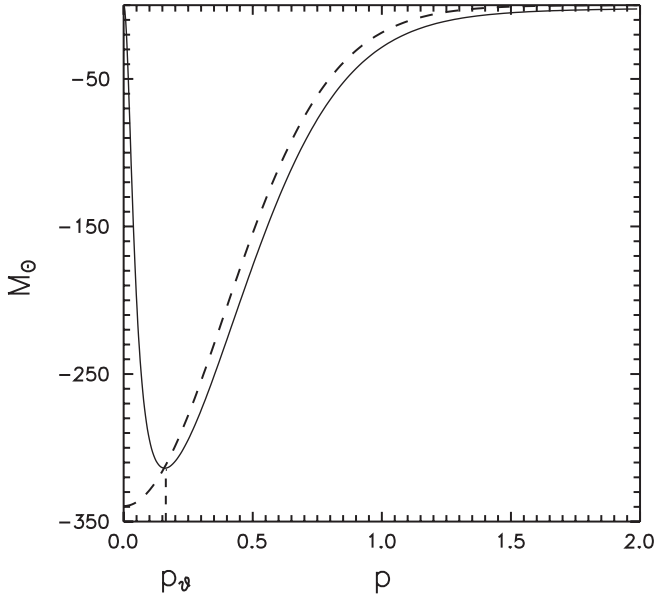

FIG. 4. The parameter M_θ in MeV as a function of momentum p in GeV which corresponds to the best fit of experimental data. The solid curve is calculated for the Gaussian correlator with $m = 5.5$ MeV; the dashed line is calculated for the same correlator but in the chiral limit.

the dynamical quark mass with M_θ and fitting the parameters a and G. The parameter a for the exponential and Gaussian correlators reads as

$$a_{ex} = 0.15 \text{ GeV}, \qquad a_{gs} = 0.16 \text{ GeV}.$$

The most suitable values of G are equal to

$$G_{ex} = 0.35, \qquad G_{gs} = 0.31$$

and $|M_\theta^{ex}| = 338$ MeV, $|\langle\sigma|\bar{q}q|\sigma\rangle_\theta^{ex}| = (228 \text{ MeV})^3$, $|M_\theta^{gs}| = 340$ MeV, $|\langle\sigma|\bar{q}q|\sigma\rangle_\theta^{gs}| = (245 \text{ MeV})^3$. Actually we can collate the obtained value of the four-fermion interaction constant with the packing fraction parameter of which the basic one for the instanton vacuum model is $\tilde{G} = (4\pi^2)^2 n\rho^4$. The result of this exercise $n\rho^4 \sim 10^{-3}$ is quite realistic. However, there is a pretty serious discrepancy in the estimates of characteristic configuration size (we should keep in mind the calculations are done in the chiral limit).

The parameter M_θ as a function of momentum p in GeV calculated with the parameters corresponding the best fit of ''experimental data'' is depicted in Fig. 4. The solid curve is obtained for the Gaussian correlator with $m = 5.5$ MeV and the dashed line is calculated for the same correlator but in the chiral limit. Here we do not mention the results obtained for the behaving exponentially correlator because they practically coincide with the results for the Gaussian correlator. The parameter p_θ is estimated at the current quark mass $m = 5.5$ MeV to be as $p_\theta \sim 150$ MeV, i.e. of the π-meson mass order. The treatment of the correlator with instanton profile together with the detailed analysis of exponential and Gaussian correlators is worthy of special paper and will be studied in the next paper.

V. CONCLUSION

In the present paper we undertake the efforts to systematically study the quark's behavior in various ensembles of stochastic gluon fields developing simple ensemble approximation which is grounded on the circumstantial analysis of the two-particle correlation function. An approximate procedure developed enables us to calculate the effective functional for the mean energy and to estimate the ground state parameters within the Hartree-Fock-Bogolyubov approach. The models with the exponential and Gaussian behaviours of correlators are analyzed in the chiral limit and their parameters are fitted. The results obtained are used to estimate the characteristic region size r_θ in which the possible processes of quark-antiquark pairing might become significant. This size was estimated to be $r_\theta \sim 1/40$ MeV^{-1} for the parameters inherent in the NJL model. For the exponential and Gaussian correlators this estimate looks like $r_\theta \sim 1/150$ MeV^{-1}.

Besides, we clearly demonstrate the presence of singularity in the mean energy functional outside the chiral limit. Finally, let us emphasize the quark ensemble characteristics discussed in the paper are not physically observable and in order to make an intelligent conclusion about the model effectiveness one should explore, for example, the meson correlation function. In fact, it has been done for the

Keldysh model in [39] and the result turned out to be quite encouraging. Despite the singular character of the mean energy of the system the meson observables are finite, quite recognizable, and comparable with the energy scale coming from an experiment.

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APPENDIX A

Here dealing with the simple quantum mechanical example we demonstrate the difference between the description based on the averaging of density matrix and the approach in which the averaging of the wave functional is used (considering the model system in a real time).

(1) Let us suppose the particle described by the stationary Hamiltonian H_0 is also affected by the time dependent force $f(t)$. Then the particle state is circumscribed by the Schrödinger equation

$$i\dot{\Psi} = H\Psi, \qquad H = H_0 + V, \qquad V = fx,$$

where Ψ is the corresponding wave function and we search the perturbative solution as

$$\Psi = \Psi^{(0)} + \Psi^{(1)} + \ldots$$

We expand the wave function of the zeroth order in the eigenfunctions of Hamiltonian H_0 as

$$\Psi^{(0)} = c_n e^{-i\lambda_n t}\psi_n, \qquad H_0\psi_n = \lambda_n\psi_n.$$

The constants c_n are defined by the initial condition here, and the next perturbative orders are calculated in the following form:

$$\Psi^{(j)} = d_n^{(i)} e^{-i\lambda_n t}\psi_n,$$

where the coefficient $d_n^{(i)}$ is determined by the iterations as

$$d_n^{(j)}(t) = -i(\overset{*}{\psi}_n x\psi_m)\int_0^t d\tau f(\tau)e^{i(\lambda_n-\lambda_m)\tau}d_m^{(j-1)}(\tau),$$

and $(\overset{*}{\psi}_n x\psi_m)$ stands here for the matrix element over the eigenfunctions of H_0, $j = 1, 2, \ldots, d_n^{(0)} = c_n$. The energy operator after averaging over the final state Ψ can be presented in the form of trace $\mathrm{Tr}\{H\rho\}$ with the pseudodensity matrix $\rho = \Psi\overset{*}{\Psi}$. In general, an energy being averaged over such a pseudomatrix will be time dependent but at analyzing the quasistationary processes, for example, it might be useful to study its averages in time as reads

$$\overline{\mathrm{Tr}\{H\rho\}} = \int_0^T dt\mathrm{Tr}\{H\rho\}/T.$$

For the sake of clarity we suppose for the force mean value that $\bar{f} = 0$. Then a nontrivial contribution into the interaction mean energy comes from the cross terms of the zeroth and first orders of perturbation expansion $\overset{*}{\Psi}^{(0)}\Psi^{(1)}$, $\Psi^{(0)}\overset{*}{\Psi}^{(1)}$ and we have

$$\mathrm{Tr}\{V\rho\} = ic_k\overset{*}{c}_m(\psi_k x\overset{*}{\psi}_n)(\psi_n x\overset{*}{\psi}_m)e^{i(\lambda_n-\lambda_k)t} \times\int_0^t d\tau f(\tau)f(t)e^{i(\lambda_k-\lambda_n)\tau} + \text{c.c.} \qquad \text{(A1)}$$

At estimating the impact of the stochastic force its contribution may be factorized with a help of the corresponding correlation function $\overline{f(\tau)f(t)} \sim \bar{f}^2F(\tau - t)$ if the characteristic frequencies λ_n are smaller then the stochastic ones. In the particular case of the white noise (when the profile function F has the δ-function shape) the time dependence in the intermediate states $\overset{*}{\psi}_n$, ψ_n in Eq. (A1) disappears (see the corresponding exponentials depending on λ_n). Because of the assumed completeness of eigenvalues basis of H_0, i.e. $\sum_n|\psi_n\rangle\langle\overset{*}{\psi}_n| = 1$, Eq. (A1) may be presented as

$$\mathrm{Tr}\{V\rho^{(2)}\} \simeq ic_k\overset{*}{c}_m(\psi_k x^2\overset{*}{\psi}_m)\bar{f}^2e^{i(\lambda_m-\lambda_k)t} + \text{c.c.}$$

It allows us to conclude that the resulting averaged final state density matrix is weighed with the effective "potential" of form $\bar{f}^2x^2$. The similar results can be received in the next perturbative orders. The cluster decomposition of stochastic exponential e^{ifx} is practical to demonstrate that the same results for the effective potential of interaction take place for a white noise at averaging the wave (generating) functional $\langle\Psi\rangle$ (as it is claimed in the first section of this paper). In general consideration there appears a certain nonlocal potential and its properties are dependent on the system state.

(2) Now let us turn to the description in terms of a density matrix only. It is defined by the following equations:

$$i\dot{\rho} = H\rho - \rho H', \qquad \text{(A2)}$$

and the density matrix is dependent on the coordinates and time $\rho(x, y; t)$. The operator H is acting on the coordinate x and the operator H' is acting on the

coordinate y. In the zeroth order of perturbative expansion we have

$$\rho^{(0)} = c_{nm} e^{i\lambda_n t} \overset{*}{\psi}_n(y) e^{-i\lambda_m t} \psi_m(x),$$

where c_{nm} is a Hermitian matrix which is defined by the initial data. In the first order of the perturbation series we present the solution in the following form:

$$\rho^{(1)} = d^{(1)}_{nm} e^{i\lambda_n t} \overset{*}{\psi}_n(y) e^{-i\lambda_m t} \psi_m(x).$$

It is possible to have for the matrix $d^{(1)}$ the representation as

$$d^{(1)}_{nm} = -ic_{nk}(\overset{*}{\psi}_m x \psi_k) \int_0^t d\tau f(\tau) e^{i(\lambda_n - \lambda_k)\tau} + ic_{km}(\overset{*}{\psi}_k y \psi_n) \int_0^t d\tau f(\tau) e^{i(\lambda_k - \lambda_m)\tau}.$$

Then for the density matrix $\rho^{(1)}$ the following form appears (in order to get it we have to interchange the indices m and n in the second term):

$$\rho^{(1)} = -ic_{nk}(\overset{*}{\psi}_m x \psi_k) e^{i(\lambda_n - \lambda_m)t} \overset{*}{\psi}_n(y) \psi_m(x) \times \int_0^t d\tau f(\tau) e^{i(\lambda_n - \lambda_k)\tau} + ic_{kn}(\overset{*}{\psi}_k y \psi_m) e^{i(\lambda_m - \lambda_n)t} \overset{*}{\psi}_m(y) \psi_n(x) \times \int_0^t d\tau f(\tau) e^{i(\lambda_k - \lambda_n)\tau}.$$

Because of the Hermitian property of density matrix we have $c_{kn} = \overset{*}{c}_{nk}$ the second term is complex conjugate with the first one at the coinciding arguments $x = y$. Then calculating the mean interaction energy $\mathrm{Tr}\{V\rho^{(1)}(x, y; t)|_{y\to x}\}$, we are convinced that the result is identical to what we found out at the beginning of this section.

Let us consider the solution for density matrix in the operator form (and without specifying the basis functions) as

$$\rho = e^{i(H_0' - H_0)t} \tilde{\rho},$$

where the matrix $\tilde{\rho}$ is determined by the solution of following integral equation:

$$\tilde{\rho}(t) = -i \int_0^t d\tau e^{i(H_0 - H_0')\tau} f(\tau)(x - y) e^{i(H_0' - H_0)\tau} \tilde{\rho}(\tau) + \tilde{\rho}(0),$$

and the effective interaction Hamiltonian is given by the operator expression as

$$\mathrm{Tr}\{H\rho\} = \mathrm{Tr}\Big\{[H_0 + f(t)x](-i) e^{i(H_0' - H_0)t} \times \int_0^t d\tau e^{i(H_0 - H_0')\tau} f(\tau)(x - y) \times \rho(x, y; \tau)|_{y\to x}\Big\}.$$

(3) Further we analyze some details of the particular exercise which admits of receiving the overt expressions and consider the forced oscillations defined by the Hamiltonian

$$H_0 = -\frac{1}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2}{2} x^2.$$

The continual integral is exactly calculated for this exercise [20] and the presentation of pseudodensity matrix which we are interested in looks like

$$\overset{*}{\psi}(y_2, t_2)\psi(x_2, t_2) = \iint_{-\infty}^{\infty} dx_1 dy_1 \overset{*}{K}(y_2, t_2; y_1, t_1) \times K(x_2, t_2; x_1, t_1) \overset{*}{\varphi}(y_1, t_1) \times \varphi(x_1, t_1), \quad \text{(A3)}$$

where $\varphi(x_1, t_1)$ is an initial state. The transformation kernel is expressed by the overt formula like

$$K(x_2, t_2; x_1, t_1) = \left(\frac{m\omega}{2\pi i \sin\omega T}\right)^{1/2} e^{iS}, \quad \text{(A4)}$$

where the action is given as

$$S = \frac{m\omega}{2\sin\omega T}[\cos\omega T(x_2^2 + x_1^2) - 2x_2x_1 + 2x_2\phi_2 + 2x_1\phi_1 - F],$$

and the phase factor have the following form:

$$\phi_1 = \frac{1}{m\omega}\int_{t_1}^{t_2} d\tau f(\tau)\sin\omega(t_2 - \tau),$$

$$\phi_2 = \frac{1}{m\omega}\int_{t_1}^{t_2} d\tau f(\tau)\sin\omega(\tau - t_1),$$

$T = t_2 - t_1$. The term F depends on the time parameters only and is immaterial because it is canceled in the exponential exponent of pseudodensity matrix. We introduce the new variable $x_2 = \tilde{x}_2 + \phi_1$ and transform the exponential exponent in the K kernel as

$$\cos\omega T(x_2^2 + x_1^2) - 2x_2x_1 + 2x_2\phi_2 + 2x_1\phi_1 = \cos\omega T(\tilde{x}_2^2 + x_1^2) - 2\tilde{x}_2x_1 + 2(\cos\omega T\phi_1 + \phi_2)\tilde{x}_2 + \cos\omega T\phi_1^2 + 2\phi_1\phi_2.$$

The similar transformations should be done in the kernel $\overset{*}{K}$ with the variable $y_2 = \tilde{y}_2 + \phi_1$. The for-

mulas take a more convenient form if we introduce the auxiliary factor $\bar{\phi}_1 = (\cos\omega T\phi_1 + \phi_2)/\sin\omega T$ which can be written down by the simple transformations in the following form:

$$\bar{\phi}_1 = \frac{1}{m\omega}\int_{t_1}^{t_2} d\tau f(\tau)\cos\omega(t_2 - \tau).$$

Making use of the well-known representation of the kernel K for nonperturbated oscillator ($f = 0$) (see [20]),

$$K_0(x_2, t_2; x_1, t_1) = \sum_{n=0}^{\infty} e^{-i\lambda_n T}\overset{*}{\psi}_n(x_2)\psi_n(x_1).$$

It is easy to understand that the important terms of the pseudodensity matrix kernel $\overset{*}{K}K$ at $y_2 \to x_2$ are represented in the similar form

$$\overset{*}{K}K = \sum_{n=0}^{\infty} e^{i\lambda_n T}\psi_n(\tilde{y}_2)\overset{*}{\psi}_n(y_1)e^{-im\omega\bar{\phi}_1\tilde{y}_2} \times e^{im\omega\bar{\phi}_1\tilde{x}_2}\sum_{m=0}^{\infty} e^{-i\lambda_m T}\overset{*}{\psi}_m(\tilde{x}_2)\psi_m(x_1)|_{y_2=x_2}.$$

The following matrix element,

$$\int_{-\infty}^{\infty} dx_2\psi_n(\tilde{y}_2)e^{-im\omega\bar{\phi}_1\tilde{y}_2}H(x_2)e^{im\omega\bar{\phi}_1\tilde{x}_2}\overset{*}{\psi}_m(\tilde{x}_2)|_{y_2=x_2},$$

will be faced with calculating the mean energy. Now moving the exponential $e^{im\omega\bar{\phi}_1\tilde{x}_2}$ to the left and changing the variable $x_2 = \tilde{x}_2 + \phi_1$ in the Hamiltonian $H(x_2)$ we obtain the representation

$$e^{-im\omega\bar{\phi}_1\tilde{y}_2}H(x_2)e^{im\omega\bar{\phi}_1\tilde{x}_2} = H_0(\tilde{x}_2) - \omega\bar{\phi}_1 i\frac{d}{d\tilde{x}_2} + (m\omega^2\phi_1 + f)\tilde{x}_2 + \frac{m\omega^2}{2}(\phi_1^2 + \bar{\phi}_1^2) + f\phi_1, \quad \text{(A5)}$$

which allows us to see that the mean energy calculated over the final state is expressed by the diagonal elements and matrix elements of coordinate and momentum as well:

$$\int_{-\infty}^{\infty} dx_2\overset{*}{K}H(x_2)K = \sum_{n=0}^{\infty} H_{n,n}\psi_n(x_1)\overset{*}{\psi}_n(y_1) + \sum_{n=0}^{\infty} H_{n,n-1}e^{i\omega T}\psi_n(x_1)\overset{*}{\psi}_{n-1}(y_1) + \sum_{n=0}^{\infty} H_{n-1,n}e^{-i\omega T}\psi_{n-1}(x_1)\overset{*}{\psi}_n(y_1),$$

where

$$H_{n,n} = \left(n + \frac{1}{2}\right)\omega + \frac{m\omega^2}{2}(\phi_1^2 + \bar{\phi}_1^2) + f\phi_1,$$

$$H_{n,n-1} = \overset{*}{H}_{n-1,n} = [m\omega^2(i\bar{\phi}_1 + \phi_1) + f] \times \left(\frac{n}{2m\omega}\right)^{1/2}.$$

(4) Now we would like to analyze the example of oscillations initiated by a periodic perturbation defined as

$$f(t) = F\sin\Omega t.$$

Then the phase factors develop the following form:

$$m\omega^2\phi_1 = F\frac{\omega}{\Omega^2 - \omega^2}(\Omega\sin\omega T - \omega\sin\Omega T),$$

$$m\omega^2\bar{\phi}_1 = F\frac{\Omega\omega}{\Omega^2 - \omega^2}(\cos\Omega T - \cos\omega T), \quad \text{(A6)}$$

(for the sake of simplicity we take the parameter as $t_1 = 0$). As in the limit of classical mechanics these expressions include a resonance behavior at coinciding the external frequency Ω and oscillator frequency ω, and in the resonance vicinity ($\Omega = \omega + \varepsilon$ with the small deviation ε from the oscillator frequency) the motion behaves as the beats, i.e. the small oscillations with the frequency ω and large amplitude. Now we are going to resolve the corresponding classical equation

$$\ddot{x}_c + \omega^2 x_c = -f/m,$$

with the initial conditions as $x_c(0) = 0$, $\dot{x}_c(0) = 0$

$$x_c = \frac{F}{m\omega}\frac{\Omega\sin\omega t - \omega\sin\Omega t}{\Omega^2 - \omega^2}.$$

Comparing Eqs. (A6) and x_c, $\dot{x}_c$ we are able to express the phase factors ϕ_1 and $\bar{\phi}_1$ as the classical coordinates $\phi_1 = x_c$ and velocity $\bar{\phi}_1 = \dot{x}_c/\omega$. In particular, the correction to the diagonal element of effective Hamiltonian (A5) can be presented in the following form:

$$\frac{m\omega^2}{2}(\phi_1^2 + \bar{\phi}_1^2) + f\phi_1 = \frac{m}{2}\dot{x}_c^2 + \frac{m\omega^2}{2}x_c^2 + fx_c.$$

Averaging the mean energy with pseudodensity matrix we get the quadratic form as

$$\mathrm{Tr}\{H\rho\}|_{y_2=x_2} = \sum_{n,m=0}^{\infty}\overset{*}{c}_n H_{n,m}c_m, \quad \text{(A7)}$$

with the coefficients c_n defined by the initial state and normalized as $\sum|c_n|^2 = 1$. In the considered situation of

periodic force acting for a very long (unlimited) time it becomes clear the value of mean energy received is time dependent (analogously to the classical description) and it means a certain asymptotic value for mean energy (as for other observables) simply does not exist [40]. Physical meaning of this fact appears quite transparent. The quantum system is carrying out the repeated transitions to the upper levels of excited state and back (these transitions are controlled by pseudodensity matrix) eventually resulting in some quasistationary process which can be pithily characterized by some observable values averaged in time. Thus, the averaged magnitude of diagonal element of pseudodensity matrix $\bar{H}_{n,n} = \int_0^T dt H_{n,n}/T$ takes the form

$$\bar{H}_{n,n} = \left(n + \frac{1}{2}\right)\omega + \frac{F^2}{m}\frac{\Omega^2 + 3\omega^2}{4(\Omega^2 - \omega^2)^2} + \frac{3F^2}{8m}\frac{1}{\Omega^2 - \omega^2} \times \frac{\sin 2\Omega T}{\Omega T} + \frac{F^2}{2m\omega^2}\frac{\Omega\omega}{\Omega^2 - \omega^2}\left[\frac{\Omega - 2\omega}{\Omega - \omega} \times \frac{\sin(\Omega - \omega)T}{(\Omega - \omega)T} - \frac{\Omega + 2\omega}{\Omega + \omega}\frac{\sin(\Omega + \omega)T}{(\Omega + \omega)T}\right].$$

Then it is not difficult to see that asymptotically a quasistationary regime of quantum ensemble oscillations as the whole can be set in, indeed, and now the question of interest is to determine the minimum of functional (A7) which corresponds to some effective ground state of the system while under the external influence. The effective Hamiltonian (A5)

$$H_{\text{eff}}(x) = e^{-im\omega\bar{\phi}_1\bar{y}} H(x) e^{im\omega\bar{\phi}_1\bar{x}},$$

can be presented by using the classical variables x_c, $\dot{x}_c$ in the following form:

$$H_{\text{eff}}(x) = \frac{(\hat{p} + p_c)^2}{2} + \frac{m\omega^2}{2}(\hat{x} + x_c)^2 + f(\hat{x} + x_c),$$

where $p_c = m\dot{x}_c$. This quantity (at the certain conditions) may be treated in such a way that it is practical to search the ground state with the biased coordinate x_c and momentum p_c. Certainly, the treatment of excited states turns out the nontrivial problem in this situation. At every time moment the pseudodensity matrix is a pure magnitude because the equality $\rho^2 = \rho$ is identically valid. However, it is possible to estimate the purity degree of trial quasistationary state $\varsigma = \text{Tr}\{\overline{\rho^2}\}$, with the time averaged density matrix and to find such states which allow us to develop a description close to 1 in the terms of the Schrödinger equation.

The density matrix formalism is very practical in more general situations, for example, at studying the influence of other quantum ensembles on a particle. It is very actively discussed and developing (being often quite far from our concerns) [41] but our purpose here was to illustrate the difference in describing a system with averaging a density matrix and averaging a wave functional.

APPENDIX B

The standard way to formulate an effective theory is to use path integral formalism. In order to transit to such a description we should construct the corresponding Lagrangian action density from the effective Hamiltonian (12)

$$\mathcal{L} = \bar{q}(i\gamma_\mu\partial_\mu + im)q - G'\bar{q}t^a\gamma_\mu q \times \int d\mathbf{y} I_{\mu\nu}(\mathbf{x} - \mathbf{y})\bar{q}'t^a\gamma_\nu q', \tag{B1}$$

where $q = q(\mathbf{x}, t)$, $\bar{q} = \bar{q}(\mathbf{x}, t)$, $q' = q(\mathbf{y}, t)$, $\bar{q}' = \bar{q}(\mathbf{y}, t)$. For the highest order in N_c the sum of the color group generators looks like $\sum_{a=1}^{N_c^2-1} t_{ij}^a t_{kl}^a \approx \frac{1}{2}\delta_{il}\delta_{kj}$. For the sake of simplicity we consider the correlator of the following form only $I_{\mu\nu}(\mathbf{x} - \mathbf{y}) = \delta_{\mu\nu} I(\mathbf{x} - \mathbf{y})$. Using the Fierz transformation $\gamma_\mu \otimes \gamma_\mu = 1 \otimes 1 + i\gamma_5 \otimes i\gamma_5 - \frac{1}{2}\gamma_\mu \otimes \gamma_\mu - \frac{1}{2}\gamma_\mu\gamma_5 \otimes \gamma_\mu\gamma_5$, and holding only the scalar contribution we receive in the mean field approximation the following effective Lagrangian density:

$$\mathcal{L} = \bar{q}(i\gamma_\mu\partial_\mu + im)q - G'\int d\mathbf{y} I(\mathbf{x} - \mathbf{y})\langle\bar{q}q'\rangle\bar{q}'q. \tag{B2}$$

The brackets in this expression imply the calculation of the corresponding averages. The self-consistency condition of approximation which may be formulated as the following integral equation:

$$-iM(\mathbf{p}) = \int \frac{dq}{(2\pi)^4} G' I(\mathbf{p} - \mathbf{q})\text{Tr}\frac{1_c}{\hat{q} + im + iM(\mathbf{q})}, \tag{B3}$$

allows us to calculate the quark mass. Integrating over the fourth component of momentum

$$\int_{-\infty}^{\infty}\frac{dq_4}{2\pi}\frac{1}{q_4^2 + \mathbf{q}^2 + (m + M(\mathbf{q}))^2} = \frac{1}{2}\frac{1}{[\mathbf{q}^2 + (m + M(\mathbf{q}))^2]^{1/2}}$$

we have

$$M(\mathbf{p}) = 2G'N_c\int\frac{d\mathbf{q}}{(2\pi)^3} I(\mathbf{p} - \mathbf{q})\frac{m + M(\mathbf{q})}{[\mathbf{q}^2 + (m + M(\mathbf{q}))^2]^{1/2}}. \tag{B4}$$

With the correlator corresponding to the NJL model we obtain the well-known gap equation

$$M = 2G'N_c \int^{\Lambda} \frac{d\mathbf{q}}{(2\pi)^3} \frac{m+M}{[\mathbf{q}^2 + (m+M)^2]^{1/2}}.$$

For the Keldysh model we have

$$M(\mathbf{p}) = 2G'N_c \frac{m+M(\mathbf{p})}{[\mathbf{p}^2 + (m+M(\mathbf{p}))^2]^{1/2}},$$

and remember that $I(\mathbf{p}) = (2\pi)^3\delta(\mathbf{p})$. Then it is easy to understand that the solution can be presented as a function of $p(M)$ for convenient handling.

In the Hartree-Fock-Bogolyubov approach the following sum over the color matrices is used, $\sum_{a=1}^{N_c^2-1} t^a_{ij}t^a_{jk} = \frac{N_c^2-1}{2N_c}\delta_{ik}$, and then we have for the quark mass

$$M_\theta(\mathbf{p}) = 2G' \frac{N_c^2-1}{N_c} \int \frac{d\mathbf{q}}{(2\pi)^3} I(\mathbf{p}-\mathbf{q}) \frac{|\mathbf{q}|}{|q_4|} \sin\theta(q). \tag{B5}$$

Comparing this expression to Eq. (B4) it becomes clear that the four-fermion interaction constant acquires the small correction $\sim 1/N_c$ which is rooted in the mean field approximation while the higher order terms in N_c are held. The patent formula for the Keldysh model can be simply received in the chiral limit. In the mean field approximation we have

$$M(p) = [(2G'N_c)^2 - p^2]^{1/2}$$

and in the Hartree-Fock-Bogolyubov we receive

$$M(p) = \left[\left(2G' \frac{N_c^2-1}{N_c}\right)^2 - p^2\right]^{1/2};$$

see also Eq. (36). At $m \neq 0$ the momentum dependencies of masses are quite different. For the Keldysh model in the mean field approximation at zero momentum, for example, we have $M(0) = 2G'N_c$ whereas in the Hartree-Fo approach $M_\theta(0) = 0$. At large momenta the mass in the mean field approximation behaves as $|M(p)| \to 2G'N_c m/p$ and in the Hartree-Fock-Bogolyubov approximation it is $|M(p)| \to (2G'\frac{N_c^2-1}{N_c})^2 m/p^2$; see also Fig. 3. However, generally, if one takes an orientation to the analysis of integral characteristics $M(p)$ the results are not so different. Similar relations could be obtained for the NJL model as well. Apparently it is reasonable to notice here that our analysis of the Hamiltonian equation (12) [Lagrangians equations (B1) and (B2)] is also valid for the Lorentz-invariant formulation when the τ (time) integration is performed for the quark fields as well [for the Lagrangians (B1) and (B2) the integration over time is retained and the form factor becomes a function of four-vector $I(x-y)$].

$$\mathcal{L} = \bar{q}(i\gamma_\mu\partial_\mu + im)q - G\int dy I(x-y)\langle\bar{q}q'\rangle\bar{q}'q. \tag{B6}$$

The self-consistency condition Eq. (B3) acquires the covariant form. In particular, in the Keldysh model in four-dimensional formulation when $I(p) = (2\pi)^4\delta(p)$ the mass gap equation reads as

$$M = 4GN_c \frac{m+M}{p^2 + (m+M)^2}.$$

It allows us to conclude that a quark never comes on the mass shell because

$$p^2 + (m+M)^2 = 4N_cG\frac{m+M}{M} > 0.$$

This feature has already been noticed in Ref. [42]. Similar behavior has been also observed in the analytic models of confinement [43]. Meanwhile, an absence of bound states in the four-dimensional Keldysh model (unlike the Keldysh model with three-dimensional form factors) is its shortage. There appears the additional integration over the fourth component of auxiliary four-momentum l in Eq. (26)

$$\int \frac{d\mathbf{q}}{(2\pi)^3} \to \int \frac{dl_4}{2\pi} \int \frac{d\mathbf{q}}{(2\pi)^3} I(l_4) \frac{1}{|p_4| + |q_4| - il_4},$$

where $I(l_4)$ is the respective part of the form factor. In particular, for the four-dimensional Keldysh model with $I(l_4) = 2\pi\delta(l_4)$ the mean energy functional can be presented in the following way:

$$W(m) = \int \frac{d\mathbf{p}}{(2\pi)^3}\Big[|p_4|(1-\cos\theta) - G\frac{p^2}{|p_4|^2}\frac{1}{2|p_4|} \times \left(\sin\theta - \frac{m}{p}\cos\theta\right)^2\Big]. \tag{51}$$

The singularity revealed in three-dimensional Keldysh model manifests itself as the weaker (logarithmic only) one in the four-dimensional consideration.

[1] T. Schäfer and E. V. Shuryak, Rev. Mod. Phys. **70**, 323 (1998).
[2] M. K. Volkov and A. E. Radzhabov, Phys. Usp. **176**, 569 (2006); D. Ebert, H. Reinhardt, and M. K. Volkov, Prog. Part. Nucl. Phys. **33**, 1 (1994).
[3] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961).
[4] C. G. Callan, R. Dashen, and D. J. Gross, Phys. Lett. **66**,

375 (1977); Phys. Rev. D **17**, 2717 (1978).
[5] A. Di Giacomo, E. Meggiolaro, and H. Panagopoulos, Nucl. Phys. **B483**, 371 (1997); M. D'Elia, A. Di Giacomo, and E. Meggiolaro, Phys. Lett. B **408**, 315 (1997); G. Bali, N. Brambilla, and A. Vairo, Phys. Lett. B **421**, 265 (1998); M. DÉlia, A. Di Giacomo, and E. Meggiolaro, Phys. Rev. D **67**, 114504 (2003).
[6] A. E. Dorokhov, S. V. Esaibegyan, and S. V. Mikhailov, Phys. Rev. D **56**, 4062 (1997); E.-M. Ilgenfritz, B. V. Martemyanov, S. V. Molodtsov, M. Müller-Preussker, and Yu. A. Simonov, Phys. Rev. D **58**, 114508 (1998); E.-M. Ilgenfritz, B. V. Martemyanov, and M. Müller-Preussker, Phys. Rev. D **62**, 096004 (2000).
[7] D. I. Diakonov and V. Yu. Petrov, Nucl. Phys. **B245**, 259 (1984).
[8] L. D. Faddeev, arXiv:hep-th 0805.1624; L. D. Faddeev and A. J. Niemi, Phys. Rev. Lett. **82**, 1624 (1999); Phys. Lett. B **449**, 214 (1999); **464**, 90 (1999).
[9] P. van Baal, Nucl. Phys. B, Proc. Suppl. **108**, 3 (2002); P. van Baal and A. Wipf, Phys. Lett. B **515**, 181 (2001); E. T. Tomboulis, Proc. Sci., LAT2007 (2007) 336; K. R. Ito and E. Seiler, arXiv:hep-lat 0711.4930.
[10] A. P. Protogenov, Phys. Usp. **176**, 689 (2006).
[11] S. V. Molodtsov and G. M. Zinovjev, J. High Energy Phys. 12 (2008) 112.
[12] It is usually encodes the predictions of a quantum field theory at low energies, and all assumptions done in the way to construct it are not of special importance. What is entirely restrictive to fix the effective action at really low energy (i.e. low cutoff) up to a few coupling constants, to develop the approximate procedure to analyze the quark interactions and to introduce the corresponding low energy effective variables is an idea to neglect all the contributions coming from gluon fields generated by the (anti-) quarks. Actually, it means the removal of corresponding cutoff(s) from consideration but by the definition of an effective theory this operation does not pose itself.
[13] G. 't Hooft, Phys. Rev. D **14**, 3432 (1976); Phys. Rev. Lett. **37**, 8 (1976); Phys. Rep. **142**, 357 (1986).
[14] D. I. Diakonov and V. Yu. Petrov, Nucl. Phys. **B272**, 457 (1986); D. I. Diakonov and V. Yu. Petrov, in *Hadronic Matter under Extreme Conditions*, edited by V. Shelest and G. Zinovjev (Naukova Dumka, Kiev, 1986), p. 192.
[15] P. V. Pobylitsa, Phys. Lett. B **226**, 387 (1989).
[16] U. Marguard and H. G. Dosch, Phys. Rev. D **35**, 2238 (1987); H. G. Dosch, Phys. Lett. B **190**, 177 (1987); H. G. Dosch and Yu. A. Simonov, Phys. Lett. B **205**, 339 (1988); Z. Phys. C **45**, 147 (1989).
[17] N. G. Van Kampen, Phys. Rep. **24**, 171 (1976); Physica (Utrecht) **74**, 215 (1974); **74**, 239 (1974).
[18] B. O. Kerbikov, D. S. Kuzmenko, and Yu. A. Simonov, JETP Lett. **65**, 137 (1997); A. G. Zubkov, O. V. Dubasov, and B. O. Kerbikov, Int. J. Mod. Phys. A **14**, 241 (1999).
[19] Yu. A. Simonov, Phys. Lett. B **412**, 371 (1997).
[20] R. P. Feynman and F. L. Vernon, Jr., Ann. Phys. (Leipzig) **24**, 118 (1963); R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).
[21] For example, the spontaneous breaking of chiral symmetryis well understood in the instanton liquid model just due to such a trick [15,19]. It is interesting to notice that the result is expressed by packing fraction parameter $n\bar{\rho}^4$ [where n is the instanton liquid density and $\bar{\rho}$ is the mean size of (anti-)instanton], the covariant derivative in the field of each separate (anti-)instanton and the free Green function.
[22] A. Di Giacomo, H. D. Dosch, V. I. Shevchenko, and Yu. A. Simonov, Phys. Rep. **372**, 319 (2002).
[23] E.-M. Ilgenfritz, M. Müller-Preussker, A. Sternbeck, A. Schiller, and I. L. Bogolubsky, Braz. J. Phys. **37**, 193 (2007); I. L. Bogolubsky, V. G. Bornyakov, G. Burgio, E.-M. Ilgenfritz, M. Müller-Preussker, and V. K. Mitrjushkin, arXiv:hep-lat/0804.1250.
[24] Let us notice that, in general, this form does not coincide with $\langle e^{-H_0 t}\Phi\rangle$.
[25] T. Schäfer, arXiv:hep-lat/0411010.
[26] V. I. Zakharov, Phys. Usp. **174**, 39 (2004).
[27] C. Aubin *et al.* (MILC Collaboration), Nucl. Phys. B, Proc. Suppl. **140**, 626 (2005); V. I. Zakharov, Braz. J. Phys. **37**, 165 (2007).
[28] M. V. Martemyanov and S. V. Molodtsov, JETP Lett. **65**, 142 (1997).
[29] N. N. Bogolyubov, J. Phys. (Moscow) **9**, 23 (1947).
[30] A. Le Yaouanc, L. Oliver, O. Péne, and J.-C. Raynal, Phys. Rev. D **29**, 1233 (1984); A. Le Yaouanc, L. Oliver, S. Ono, O. Péne, and J.-C. Raynal, Phys. Rev. D **31**, 137 (1985).
[31] In order to avoid any misunderstanding we remind here that fixing a form of ground state introduces a primary frame.
[32] M. I. Krivoruchenko, Phys. Usp. **164**, 643 (1994); Yad. Fiz. **47**, 1823 (1987).
[33] If one is interested in stochastic gluon field contribution in the highest one-particle order approximation only all gluon lines in the respective Feynman diagrams are depicted as coming from the single point corresponding to the center coordinate of the gluon configuration. However, working within the Green function method to derive the Schwinger-Dyson equations suppose the resummation of special classes of diagrams. Then taking into account the diagrams with a large number of asterisks (the centers of gluon configurations) looks like an overrange of one-particle approximation. Thus the second star coming to play, for example, signals the contribution of second order in density available.
[34] T. Hatsuda and T. Kunihiro, Phys. Rep. **247**, 221 (1994).
[35] L. V. Keldysh, Doctor thesis, FIAN, 1965 (unpublished); E. V. Kane, Phys. Rev. **131**, 79 (1963); V. L. Bonch-Bruevich, in *Physics of Solid States* (VINITI, Moscow, 1965).
[36] For sake of simplicity we do not consider the contribution of correlator $J_{\mu\nu}(p)$.
[37] B. A. Arbuzov, M. K. Volkov, and I. V. Zaitsev, Int. J. Mod. Phys. A **21**, 5721 (2006); B. A. Arbuzov, Phys. Lett. B **656**, 67 (2007).
[38] S. V. Molodtsov and G. M. Zinovjev, arXiv:0811.4405v2.
[39] S. V. Molodtsov, M. K. Volkov, and G. M. Zinovjev, arXiv:0812.2666.
[40] Apparently, this almost obvious fact was underrated for a rather long time.

[41] B. V. Chirikov, Phys. Rep. **52**, 263 (1979); G. M. Zaslavsky, Phys. Rep. **80**, 157 (1981); *Chaos and Quantum Physics*, UJFG and NATO ASI Proceedings of Les Houches Summer School, Session LII, 1989, edited by M.-J. Jiannony, A. Voros, and J. Zinn-Zustin (Elsevier, New York, 1991); W. H. Zurek, Rev. Mod. Phys. **75**, 715 (2003).

[42] A. E. Dorokhov, G. M. Zinovjev, and S. V. Molodtsov, Phys. At. Nucl. **71**, 759 (2008).

[43] G. V. Efimov and S. N. Nedelko, Eur. Phys. J. C **1**, 343 (1998); A. C. Kalloniatis and S. N. Nedelko, Phys. Rev. D **64**, 114025 (2001).