

Multiple parton interactions in hadron collisions and diffractionPaolo Lipari^{1,*} and Maurizio Lusignoli^{2,†}¹*INFN sezione Roma “La Sapienza,” Piazzale Aldo Moro 2, 00185 Roma, Italy*²*Dipartimento di Fisica and sezione INFN, Università di Roma “La Sapienza,” Piazzale Aldo Moro 2, 00185 Roma, Italy*

(Received 4 August 2009; published 14 October 2009)

Hadrons are composite objects made of quarks and gluons, and during a collision one can have several elementary interactions between the constituents. These elementary interactions, using an appropriate theoretical framework, can be related to the total and elastic cross sections. At high c.m. energy it also becomes possible to identify experimentally a high p_{\perp} subset of the parton interactions and to study their multiplicity distribution. Predictions of the multiple interaction rates are difficult because in principle one needs to have a knowledge of the correlated parton distribution functions that describe the probability to find simultaneously different partons in different elements of phase space. In this work we address this question and suggest a method to describe effectively the fluctuations in the instantaneous configuration of a colliding hadron. This problem is intimately related to the origin of the inelastic diffractive processes. We present a new method to include the diffractive cross section in an eikonal formalism that is equivalent to a multichannel eikonal. We compare with data and present an extrapolation to higher energy.

DOI: [10.1103/PhysRevD.80.074014](https://doi.org/10.1103/PhysRevD.80.074014)

PACS numbers: 13.85.Lg, 13.85.Dz, 96.50.sd

I. INTRODUCTION

The evolution with center of mass energy of the total and elastic cross sections in hadron-hadron collisions, and the properties of multiparticle production in these interactions remain an important open problem in particle physics. This problem is clearly of significant intrinsic interest, but it has also important phenomenological implications: on one hand the estimate of the properties of hadronic interactions is obviously important at the LHC to model the background in the search for Higgs particles and possible forms of “new physics”; on the other hand the detailed properties of hadronic interaction properties are important in cosmic ray studies. The highest energy cosmic ray particles can only be detected indirectly observing the extensive air showers that they produce in the Earth’s atmosphere. The development of air showers is determined by the hadronic cross sections and the properties of particle production in hadronic interactions, and therefore the interpretation of the available (and future) data depends also on theoretical assumptions about hadronic interactions.

Hadrons are composite objects containing quarks and gluons, and it is natural to relate the cross sections for hadron collisions to the more elementary interactions between their parton constituents. The precise method to do this remains however an unsolved problem.

A possible approach is provided by the so-called “minijet” models [1], where the total and elastic cross sections are obtained using an eikonal formalism, in terms of the quantity $\langle n(b, s) \rangle$ that has the physical meaning of the average number of elementary interactions at impact parameter b and c.m. energy \sqrt{s} (for earlier work on the

importance of minijets see [2,3]). An attractive feature of this approach is that it allows one to compute the distribution of the number of elementary interactions that happen in a single hadron collision. This distribution can then be used, with the inclusion of a few additional assumptions, in Monte Carlo codes to predict properties of particle production that can be tested experimentally. For example it is simple to see that events with a large number of parton scatterings must have a larger multiplicity and a more complex structure.

The original version of the minijet approach and several subsequent ones did not include in a consistent way in their formalism the inelastic diffractive processes. Following an approach introduced long ago by Good and Walker [4], several authors [5–9] indicated possible methods to include diffraction using a multichannel eikonal formalism. In this work we rediscuss this problem, and suggest an alternative method to include the diffractive cross section in the eikonal formalism. This method is mathematically equivalent to the multichannel eikonal method but offers additional physical insight.

The fundamental physical idea to explain the existence of inelastic diffraction introduced by Good and Walker [4] is to assume that inelastic diffraction emerges because an interacting hadron can be seen as a superposition of different states that undergo unequal absorptions. It is natural, as originally proposed by Miettinen and Pumplin [10], to identify these “transmission eigenstates” as different “configurations” of the parton constituents inside a hadron. In this theoretical framework the estimate of inelastic diffraction requires some understanding of the ensemble of such parton configurations. This appears as a daunting task.

A possible approach is to make the dramatic approximation of reducing the space of parton configurations to a finite dimensional space (in fact spanned by as few as two

*paolo.lipari@roma1.infn.it

†maurizio.lusignoli@roma1.infn.it

or three base vectors) and to construct explicitly a matrix transition operator. An explicit example (originally constructed in [11]) of this approach is also discussed in this work.

The alternative method we propose here is to construct a mapping from the space of the hadron configurations to the real positive numbers, so that the number of elementary interactions for the configuration \mathbb{C} is $n(\mathbb{C}) = \langle n \rangle \alpha(\mathbb{C})$ (with $\langle n \rangle$ the average over all configurations). The probability distribution of α (together with a model for $\langle n \rangle$) is then sufficient to compute the total cross section and its different components (elastic, absorption, and diffraction).

The consistent inclusion of diffraction in the theory is very important, because it changes dramatically the relation between the inclusive parton cross sections and the directly observable hadron cross sections.

This work is organized as follows. In the next section we start discussing the problem of multiple interactions in a narrower but better defined sense that is limiting our considerations to hard high p_\perp interactions that on one hand can be easily identified experimentally, and on the other hand have an inclusive cross section that is calculable in perturbative QCD using the standard parton distribution functions (PDF's). We will show that the calculation of the multiplicity distribution of these hard interactions requires additional theoretical constructs.

In Sec. III after reviewing some general formalism about total and elastic cross sections we discuss a standard, single channel, version of the eikonal formalism that has been used in the original minijet model of Durand and Pi and in many other works.

In Sec. IV, after a brief introduction to inelastic diffraction we recall the basic ideas of the Good and Walker [4] method, and then we discuss our implementation of this multicomponent ansatz in terms of the function $p(\alpha)$ (where α is a real positive variable). We call the function $p(\alpha)$ the “effective configuration probability distribution.”

In Sec. V we present a simple form for the function $p(\alpha)$ that depends on a single parameter, and use this form, together with a (2 parameters) parametrization of the function $\langle n(b, s) \rangle$ to describe the available data on pp and $\bar{p}p$ scattering at collider energies.

In Sec. VI we discuss the energy dependence of the parameters of our model and discuss extrapolations to higher energy (LHC and ultrahigh energy cosmic rays). Section VII offers some final considerations.

II. MULTIPLE INTERACTIONS

The problem of multiple interactions in a hadron-hadron collision is usually discussed in the context of a calculation of the total (or inelastic) cross section and referring to the *total* number of elementary interactions in a collision. Such a general discussion has serious difficulties. Theoretically the concept of the “total” number of elementary interactions in a collision is not really well defined. For example,

one usually divides the elementary interactions into two classes: “soft” and “hard” choosing a rather arbitrary cutoff in p_\perp , however soft interactions cannot be considered as truly elementary, since they are effective processes, as for instance a “pomeron exchange” that, after decades of efforts, remains a somewhat elusive concept, and the theoretical “counting” of the number of interactions in a collision has significant ambiguities. On the other hand, experimentally it is essentially impossible to measure the number of soft interactions in one collision, and one can at best obtain only some partial and indirect information from the study of particle multiplicities.

To avoid these difficulties, in this section we will discuss a more limited but much better defined problem, namely, the production of parton-parton scatterings with a transverse momentum larger than a chosen threshold. If the threshold p_\perp^{\min} is sufficiently large (above a few Gev), these scatterings are, at least in principle, experimentally identifiable. Multiple parton interactions can be identified and distinguished from multijet production due to higher order corrections (processes of type $2 \rightarrow 3$ or $2 \rightarrow 4$ at the parton level) observing pairs of back to back jets. Such (difficult) studies have already been performed at the Tevatron collider [12].

Moreover, given the colliding hadron PDF's, the inclusive differential cross section for the production of pairs of jets can be estimated in perturbation theory from the well-known expression:

$$\begin{aligned} \frac{d^3\sigma}{dp_\perp dx_1 dx_2} \Big|_{\text{jet pair}}(p_\perp, x_1, x_2; \sqrt{s}) \\ = \sum_{j,k,j',k'} f_j^{h_1}(x_1, \mu^2) f_k^{h_2}(x_2, \mu^2) \frac{d\hat{\sigma}_{jk \rightarrow j'k'}}{dp_\perp}(p_\perp, \hat{s}). \end{aligned} \quad (1)$$

In this expression $f_j^{h_1}(x, \mu^2)$ [$f_k^{h_2}(x, \mu^2)$] is the PDF for parton of type j (k) in the hadron h_1 (h_2) and the scale $\mu^2 \sim p_\perp^2$, and $d\hat{\sigma}_{jk \rightarrow j'k'}/dp_\perp(p_\perp, \hat{s})$ is the differential cross section for the parton-parton scattering of type $j + k \rightarrow j' + k'$ at the squared c.m. energy for the parton-parton interaction $\hat{s} = sx_1x_2$. The fractional momenta $x_{1,2}$ are connected to the rapidities $y_{1,2}$ of the observed jets by the relation:

$$x_{1,2} = \frac{p_\perp}{\sqrt{s}} \exp\left[\pm \frac{(y_1 + y_2)}{2}\right]. \quad (2)$$

Integrating Eq. (1) over the phase space region $p_\perp > p_\perp^{\min}$ and all allowed jet rapidities one obtains the inclusive jet cross section:

$$\begin{aligned} \sigma_{\text{jet}}(p_{\perp}^{\text{min}}, \sqrt{s}) &= \int_{p_{\perp}^{\text{min}}}^{\sqrt{s}/2} dp_{\perp} \int_{4p_{\perp}^2/s}^1 dx_1 \int_{4p_{\perp}^2/(sx_1)}^1 dx_2 \\ &\times \left\{ \sum_{j,k,j',k'} f_j^{h_1}(x_1, \mu^2) f_k^{h_2}(x_2, \mu^2) \frac{d\hat{\sigma}_{jk \rightarrow j'k'}}{dp_{\perp}} \right. \\ &\left. \times (p_{\perp}, \hat{s}) \right\}. \end{aligned} \quad (3)$$

More in general, one could consider the production of jet pairs in a more limited region of phase space selecting, for example, only jets in a certain range of rapidities, with appropriate choice of the limits in the integration over phase space. The discussion below remains valid also in this case, and in the following we will denote with σ_{jet} the inclusive production of jets in a fixed kinematical region determined by p_{\perp}^{min} and appropriate cuts in the jet rapidities, leaving the dependence on these kinematical cuts implicit.

The quantity $\sigma_{\text{jet}}(s)$ is a cross section for parton-parton interactions, and therefore must be interpreted in the appropriate way. Its physical meaning is to give the *inclusive* cross section for the production of jet pairs in the chosen kinematical region. This means that when a detector collects the integrated luminosity L_{int} , the expected number of jet pairs is $L_{\text{int}}\sigma_{\text{jet}}$. Since the total number of inelastic scattering events is $L_{\text{int}}\sigma_{\text{inel}}$, the ratio $\sigma_{\text{jet}}/\sigma_{\text{inel}}$ is the average number of jet pairs produced in one inelastic interaction. In principle it is possible, and in fact it will happen for p_{\perp}^{min} sufficiently small and/or \sqrt{s} sufficiently large, that σ_{jet} exceeds σ_{inel} . This simply implies that some events must contain more than one parton-parton interaction.

In general, an inelastic event can have 0, 1, 2 or more hard interactions. A natural problem is the estimate of the relative frequencies of events that have jet multiplicity k . The probability p_k that an inelastic event contains exactly k pairs of jets can be expressed as the ratio:

$$p_k = \frac{\sigma_k^{\text{jet}}}{\sigma_{\text{inel}}}. \quad (4)$$

The partial cross sections σ_k^{jet} must satisfy the sum rules:

$$\sum_k \sigma_k^{\text{jet}} = \sigma_{\text{inel}}, \quad (5)$$

$$\sum_k k \sigma_k^{\text{jet}} = \sigma_{\text{jet}}, \quad (6)$$

and therefore

$$\langle k \rangle \equiv \sum_k k p_k^{\text{jet}} = \frac{\sigma_{\text{jet}}}{\sigma_{\text{inel}}}. \quad (7)$$

It is important to stress that the set of partial cross sections σ_k^{jet} or equivalently the probabilities p_k are observable quantities. For large p_{\perp}^{min} the identification of the

hard interactions is experimentally straightforward; however the inclusive jet cross section σ_{jet} is much smaller than σ_{inel} and the jet multiplicity distribution becomes ‘‘trivial’’ and only p_0 and p_1 are different from zero. For p_{\perp}^{min} sufficiently small the jet multiplicity distribution becomes broader, and the probability to find more than one hard scattering in a single event becomes significant; however at the same time the experimental identification of the hard scatterings becomes more difficult. At the LHC it should however be possible to identify a value p_{\perp}^{min} sufficiently small to result in an interesting multiplicity distribution of hard interactions and sufficiently large to allow the measurement of such a distribution.

Such an experimental study should be compared with a theoretical prediction. The calculation of the multiplicity distribution of the hard interactions is in fact a very difficult, unsolved problem, that, as we will discuss in the following, requires the introduction of new ideas, beyond the use of the standard PDF’s, that are only sufficient for the calculation of the inclusive jet cross section. One may note that the obvious sum rule (5) actually implies that the calculation of the partial cross sections requires unavoidably a complete theory for the inelastic hadron-hadron cross section.

As a first step toward the calculation of the set of partial cross sections σ_k^{jet} one can note that it is natural to expect that collisions at different impact parameters will result in a different number of hard interactions, with small (large) b corresponding to a larger (smaller) number of interactions. The *average* number of interactions at a fixed impact parameter b can be calculated as

$$\begin{aligned} \langle n_{\text{jet}}(b, s, p_{\perp}^{\text{min}}) \rangle &= \int d^2b_1 \int d^2b_2 P_{\text{int}}(\vec{b} - \vec{b}_1 + \vec{b}_2) \\ &\times \int dp_{\perp} \int dx_1 \int dx_2 \\ &\times \sum_{j,k,j',k'} F_j^{h_1}(x_1, b_1, \mu^2) F_k^{h_2}(x_2, b_2, \mu^2) \\ &\times \frac{d\hat{\sigma}_{jk \rightarrow j'k'}}{dp_{\perp}}(p_{\perp}, \hat{s}) \end{aligned} \quad (8)$$

(where we have left implicit the integration limits over p_{\perp} , x_1 , and x_2). The expression (8) differs from Eq. (3) for three reasons: (i) it replaces the PDF’s $f_j^h(x, \mu^2)$ with the impact parameter dependent PDF’s $F_j^h(x, b, \mu^2)$; (ii) it includes two additional integrations over \vec{b}_1 and \vec{b}_2 that describe the positions in transverse space of the partons inside the two hadrons; and (iii) the function $P_{\text{int}}(\vec{b} - \vec{b}_1 + \vec{b}_2)$ is also included.

The impact parameter dependent PDF’s $F_j^h(x, b, \mu^2)$ describe the probability to find a parton of type j in hadron h with fractional longitudinal momentum x at transverse position b (with respect to the hadron c.m.). These functions are related to the standard PDF’s by the relation:

$$f_j^h(x, \mu^2) = \int d^2b F_j^h(x, b, \mu^2). \quad (9)$$

The function $P_{\text{int}}(\vec{b})$ describe the probability density that two partons (each in a different hadron) separated by the distance b in transverse space interact. The function has the normalization:

$$\int d^2b P_{\text{int}}(\vec{b}) = 1. \quad (10)$$

The simplest choice for $P_{\text{int}}(\vec{b})$ is a delta function [$P_{\text{int}}(\vec{b}) = \delta^2(\vec{b})$], or more in general (but at the cost of introducing a new parameter) a Gaussian of width σ .

Using Eqs. (9) and (10) it is simple to verify that integrating $\langle n_{\text{jet}}(b, s) \rangle$ over all impact parameters one obtains the correct result:

$$\int d^2b \langle n_{\text{jet}}(b, s) \rangle = \sigma_{\text{jet}}(s). \quad (11)$$

The physical interpretation of Eq. (9) is very straightforward. To obtain the expected number of hard scattering in a hadron collision at impact parameter b one must integrate over the distribution of the partons in transverse space around the center of mass of the colliding hadrons. The standard PDF's integrate over these transverse space variables and are therefore insufficient for the calculation, and are only capable of giving the inclusive result.

To proceed in the calculation one must obtain information about the impact parameter dependent PDF's. The simplest hypothesis is to assume that the dependence on the impact parameter of the F_j^h functions can be factorized:

$$F_j^h(x, b, \mu^2) = f_j^h(x, \mu^2) \hat{\rho}_h(b), \quad (12)$$

where the function $\hat{\rho}_h(b)$ satisfies the normalization condition:

$$\int d^2b \hat{\rho}_h(b) = 1. \quad (13)$$

With this factorization assumption the quantity $\langle n(b, s) \rangle$ can be written as

$$\langle n_{\text{jet}}(b, s) \rangle = \sigma_{\text{jet}}(s) A(b), \quad (14)$$

where the ‘‘hadron overlap function’’ $A(b)$ is

$$A(b) = \int d^2b_1 \int d^2b_2 \hat{\rho}_{h_1}(b_1) \hat{\rho}_{h_2}(b_2) P_{\text{int}}(\vec{b} - \vec{b}_1 + \vec{b}_2), \quad (15)$$

and it [using (10) and (13)] satisfies the normalization condition:

$$\int d^2b A(b) = 1. \quad (16)$$

A reasonable first approximation for the overlap function $A(b)$ proposed by Durand and Pi [1] is to estimate it from the electromagnetic form factor of the colliding

hadrons (see Appendix A). Another (more phenomenological) approach has been to assume a Gaussian form of a superposition of Gaussians.

It should however be stressed that the factorization hypothesis (12) has no serious motivation beyond its simplicity, and it is likely to be incorrect. In fact, the impact parameter PDF's can be in principle calculated from the generalized parton distribution functions (GPDF's) [13–15]. Work on this subject is in progress.

The need to consider the dependence on the transverse degrees of freedom introduces a serious complication and uncertainty in the calculation of the partial cross sections σ_k^{jet} , but unfortunately, even having a good theoretical control of the overlap function $A(b)$ (or better a detailed knowledge of the impact parameter dependent PDF's) is not sufficient to estimate the partial cross sections.

To complete the calculation one must make some hypothesis about the fluctuations in the number of hard interactions for collisions at a fixed impact parameter (and c.m. energy). The simplest hypothesis is to assume that the fluctuations are simply Poissonian. The partial jet cross sections are then calculable integrating over all impact parameters the Poisson probability:

$$\sigma_k^{\text{jet}}(s) = \int d^2b \frac{\langle n_{\text{jet}}(b, s) \rangle^k}{k!} e^{-\langle n_{\text{jet}}(b, s) \rangle}. \quad (17)$$

The hypothesis that the multiplicity distribution of the hard interactions at a given impact parameter is Poissonian is however not necessarily correct, and some simple considerations suggest that in fact this distribution is considerably broader than a Poissonian. An argument in this direction can be developed as follows. At the instant t a hadron can be described as an ensemble of partons each having a certain longitudinal momentum x and a transverse position \vec{b} . The set of values $\{q_j, x_j, \vec{b}_j\}_{(j=1,N)}$ (q_j is the complete set of quantum numbers of each parton) is the ‘‘instantaneous configuration’’ of the hadron and will be denoted with the symbol \mathbb{C} . The probability to find hadron h in a certain instantaneous configuration \mathbb{C} can be denoted as $P_h(\mathbb{C})$ with the normalization condition:

$$\int d\mathbb{C} P_h(\mathbb{C}) = 1, \quad (18)$$

where the integration over $d\mathbb{C}$ indicates formally the sum over all possible configurations.

The interaction between (for example) two protons, in the c.m. frame, lasts a crossing time t_{cross} , that can be estimated as

$$t_{\text{cross}} \sim \frac{2R_p}{\gamma_{\text{c.m.}}} \simeq \frac{4R_p m_p}{\sqrt{s}}, \quad (19)$$

where R_p is the linear size of the proton of order 0.5 fm. Because of the Lorentz length contraction the crossing time shrinks $\propto s^{-1/2}$.

The time required for the radiation and absorption of partons, that is for changing the parton configurations, is of order R_p and is therefore much longer than the interaction time. It is therefore a reasonable approximation to assume that during one interaction the hadrons appear as “frozen” each into its own configuration.

The expected number of hard interactions in a collision of impact parameter b is determined by the parton configurations of the two colliding hadrons. If the configuration is composed by a few hard partons, as, for example, is the case when most of the hadron energy is carried by the valence quarks, the number of hard scattering is suppressed, while if the configurations contain many gluons with fractional energy of order $x \sim (p_{\perp}^{\min})^2/s$ the number of hard interactions is enhanced.

We will denote as $n_{\text{jet}}(b, \mathbb{C}_1, \mathbb{C}_2)$ the expected number of hard interactions in a collision at impact parameter b between two hadrons with configurations \mathbb{C}_1 and \mathbb{C}_2 . Integrating over all possible hadron configurations one must recover the expected value of the jet multiplicity, therefore we can write formally:

$$\int d\mathbb{C}_1 \int d\mathbb{C}_2 P_{h_1}(\mathbb{C}_1) P_{h_2}(\mathbb{C}_2) n_{\text{jet}}(b, \mathbb{C}_1, \mathbb{C}_2) = \langle n_{\text{jet}}(b, s) \rangle. \quad (20)$$

If we make the further assumption that the actual number of interactions in a single collision at impact parameter b with the hadrons in the configurations \mathbb{C}_1 and \mathbb{C}_2 has a Poisson distribution around the expected value, we can now write the partial jet cross sections as

$$\sigma_k^{\text{jet}} = \int d^2b \int d\mathbb{C}_1 \int d\mathbb{C}_2 P_{h_1}(\mathbb{C}_1) P_{h_2}(\mathbb{C}_2) \times \left\{ \frac{[n_{\text{jet}}(b, \mathbb{C}_1, \mathbb{C}_2)]^k}{k!} \exp[-n_{\text{jet}}(b, \mathbb{C}_1, \mathbb{C}_2)] \right\}. \quad (21)$$

Of course Eq. (21) is only a formal solution of our problem since we have not yet developed the instruments to estimate the probability of the different configurations and to perform the integration over the configurations space.

In order to make progress we will make the assumption that the expected value of the jet multiplicity for a certain configuration of the hadrons is related to the averaged one by the factorized relation:

$$n_{\text{jet}}(b, \mathbb{C}_1, \mathbb{C}_2) = \langle n_{\text{jet}}(b, s) \rangle \alpha(\mathbb{C}_1, \mathbb{C}_2), \quad (22)$$

where $\alpha(\mathbb{C}_1, \mathbb{C}_2)$ is a real positive number. With this simplifying assumption we can construct a function $p(\alpha)$ that is independent from b :

$$p(\alpha) = \int d\mathbb{C}_1 \int d\mathbb{C}_2 P_{h_1}(\mathbb{C}_1) P_{h_2}(\mathbb{C}_2) \delta[\alpha(\mathbb{C}_1, \mathbb{C}_2) - \alpha]. \quad (23)$$

It is straightforward to see that because of the normaliza-

tion condition (18) one has

$$\int_0^\infty d\alpha p(\alpha) = 1, \quad (24)$$

consistently with the interpretation of $p(\alpha)$ as a probability density; moreover because of Eq. (20) one has also that

$$\int_0^\infty d\alpha \alpha p(\alpha) = 1. \quad (25)$$

Using the definition of (23) one can rewrite (21) in the simpler form:

$$\sigma_k^{\text{jet}} = \int d^2b \int_0^\infty d\alpha p(\alpha) \left\{ \frac{\alpha^k \langle n_{\text{jet}}(b, s) \rangle^k}{k!} \times \exp[-\alpha \langle n_{\text{jet}}(b, s) \rangle] \right\}. \quad (26)$$

In this equation we have finally arrived to write the jet multiplicity distribution in terms of $\langle n_{\text{jet}}(b, s) \rangle$ and the function $p(\alpha)$, that is unknown, but has fixed normalization and first moment. The expression for the partial jet cross sections given in Eq. (17) that was calculated ignoring the effects of fluctuations in the parton configurations is recovered in the limit where the function $p(\alpha)$ has vanishing width and reduces to the form $p(\alpha) = \delta[\alpha - 1]$.

It is straightforward to see [using Eq. (25)] that one has $\langle k \rangle = \sigma_{\text{jet}}/\sigma_{\text{inel}}$ in agreement with Eq. (7). The second moment of the jet multiplicity distribution can be expressed as

$$\begin{aligned} \langle k^2 \rangle &= \frac{1}{\sigma_{\text{inel}}} \sum_k k^2 \sigma_k^{\text{jet}} \\ &= \frac{\sigma_{\text{jet}}}{\sigma_{\text{inel}}} + \frac{(1+w)}{\sigma_{\text{inel}}} \int d^2b \langle n_{\text{jet}}(b, s) \rangle^2, \end{aligned} \quad (27)$$

where

$$w = \int_0^\infty d\alpha (\alpha^2 - 1) p(\alpha) \quad (28)$$

is the variance of the $p(\alpha)$ distribution. Using the factorization hypothesis (14) one can rewrite Eq. (27) as

$$\langle k^2 \rangle = \frac{\sigma_{\text{jet}}}{\sigma_{\text{inel}}} + (1+w) \frac{\sigma_{\text{jet}}^2}{\sigma_{\text{inel}}} \int d^2b [A(b)]^2. \quad (29)$$

The width of the jet multiplicity distribution is therefore determined by the geometry of the hadronic matter [that is from the shape of the overlap function $A(b)$] and from the variance w of the function $p(\alpha)$. Note that the simple eikonal model corresponds to a vanishing w and therefore to the minimum possible $\langle k^2 \rangle$ [for a fixed overlap function $A(b)$].

The partial cross sections σ_k^{jet} and the quantity $\langle k^2 \rangle$ are (at least in principle) measurable with observations of the jet multiplicity distribution. From these measurements one

can obtain information about the properties of the function $p(\alpha)$.

In fact we have already some (indirect) information about the jet multiplicity distribution, and there are indications that predictions based on the simple eikonal approach are not adequate. Some of the most sophisticated Monte Carlo instruments for the study of high energy hadron collisions, such as PYTHIA [16,17] and HERWIG [18,19] include a treatment of multiparton interactions following the simple eikonal model. The algorithms for multiparton interactions in these Monte Carlo codes compute an inclusive jet cross section σ_{jet} , assume an overlap function $A(b)$, and then generate a number of elementary interactions according to the probability distribution of Eq. (17) or equivalently of Eq. (26) with $p(\alpha) = \delta[\alpha - 1]$. In most cases the presence of multiple interactions cannot be experimentally resolved but is detectable statistically, for example, from fluctuations in the charged particles multiplicity distribution. In order to reproduce the broad fluctuations of the data, it appears necessary to construct some *ad hoc* functional form for the overlap function. For example, PYTHIA [16,17] uses a double Gaussian with a denser core and more extended halo. The overlap function $A(b)$ however (as will be discussed in more detail in the following) in principle also determines the t distribution of elastic scattering, and therefore one does not have the freedom to modify it in an arbitrary way.

The introduction of a nonvanishing variance for the function $p(\alpha)$ allows one to modify the width of the jet multiplicity distribution without modifications of the overlap function $A(b)$.

One open problem in high energy hadron collisions that is likely to be relevant in the interpretation of future data on possible manifestations of new physics in high energy collisions is the problem of the so-called ‘‘underlying event.’’ The observations [20,21] show that in events with the presence of a high p_{\perp} scattering, the ‘‘environment’’ that accompanies the observed jets has an average transverse momentum higher than what is found in minimum bias events. Monte Carlo codes like HERWIG, ISAJET, or PYTHIA at present only partially describe these effects [20,21]. The origin of a higher ‘‘ambient level’’ of p_{\perp} for the underlying event can be related to the presence of additional (softer and unresolved) parton scatterings that accompany the observed jets. The correct description of the underlying event is therefore related to a good theoretical control of the multiplicity distribution of parton interactions. A Monte Carlo implementation of multiparton interactions based on an impact parameter picture for hadronic collisions naturally contains some of the qualitative features observed in underlying events, because events selected with high p_{\perp} jets are more likely to be central collisions, and therefore are more likely to contain additional parton scatterings. A nonvanishing variance of $p(\alpha)$ should however enhance the differences between under-

lying and minimum bias events, and could therefore play a non-negligible role in the description of the data. A quantitative study of this problem with Monte Carlo methods is a goal for future work.

The problem of the fluctuations in the configurations of partons in a hadron is also intimately related to diffractive scattering, as we will illustrate in the following section.

III. TOTAL AND ELASTIC CROSS SECTIONS

A. General formalism

In the following we will need to consider also the elastic and total cross sections in hadronic scattering. The elastic scattering amplitude in the collision of two hadrons of type h_1 and h_2 at c.m. energy \sqrt{s} can be written [22] as a (2-dimensional) integral over impact parameter:

$$F_{\text{el}}(q, s) = i \int \frac{d^2b}{2\pi} e^{i\vec{q}\cdot\vec{b}} \Gamma_{\text{el}}(b, s), \quad (30)$$

where $\Gamma_{\text{el}}(b, s)$ is the profile function that without loss of generality can be written as

$$\Gamma_{\text{el}}(b, s) = 1 - e^{-\chi(b, s)}, \quad (31)$$

with $\chi(b, s)$ the eikonal function. In the notation we are leaving implicit the dependence on the type of hadrons participating in the collision. The elastic scattering amplitude is related to the differential cross section by

$$\frac{d\sigma_{\text{el}}}{dt}(t, s) = \pi \frac{d\sigma_{\text{el}}}{d^2q}(\vec{q}, s) = \pi |F_{\text{el}}(\sqrt{-t}, s)|^2. \quad (32)$$

In this equation $t = (p_{h_1} - p'_{h_1})^2$ is the squared momentum transfer, and we have used the approximation $t = -|\vec{q}|^2$ that is valid for small $-t$ and high energy. Integrating over all t one obtains

$$\sigma_{\text{el}}(s) = \int d^2b |\Gamma_{\text{el}}(b, s)|^2. \quad (33)$$

The total cross section is related to the imaginary part of the forward elastic scattering amplitude by the optical theorem and is given by

$$\sigma_{\text{tot}}(s) = 4\pi \text{Im}[F_{\text{el}}(0, s)] = 2 \int d^2b \text{Re}[\Gamma_{\text{el}}(b, s)]. \quad (34)$$

Combining Eqs. (33) and (34) one obtains an expression for the inelastic cross section as an integral over the impact parameter:

$$\sigma_{\text{inel}}(s) = \int d^2b \{1 - |1 - \Gamma_{\text{el}}(b, s)|^2\}. \quad (35)$$

Equation (32) gives the exact shape of the differential elastic cross section. For small $|t|$ this shape is to a good approximation a simple exponential ($d\sigma_{\text{el}}/dt \propto e^{Bt}$) and it is convenient to define the slope B_{el} of elastic scattering:

$$B_{\text{el}}(s) = \left[\left(\frac{d\sigma_{\text{el}}}{dt} \right)^{-1} \frac{d}{dt} \left(\frac{d\sigma_{\text{el}}}{dt} \right) \right]_{t=0}. \quad (36)$$

It is straightforward to see that if the profile function is real, the slope B_{el} can be calculated as

$$\begin{aligned} B_{\text{el}}(s) &= \left\{ \int d^2b \frac{b^2}{2} \Gamma_{\text{el}}(b, s) \right\} \times \left\{ \int d^2b \Gamma_{\text{el}}(b, s) \right\}^{-1} \\ &= \frac{\langle b^2 \rangle}{2} \end{aligned} \quad (37)$$

and measures the value $\langle b^2 \rangle$ of the profile function.

Using the approximate exponential shape of the differential cross section one finds the relation

$$\sigma_{\text{el}}(s) \simeq \pi \frac{|F_{\text{el}}(0, s)|^2}{B_{\text{el}}(s)} = \pi \frac{(\text{Im}[F_{\text{el}}(0, s)])^2 |i + \rho|^2}{B_{\text{el}}(s)} \quad (38)$$

(where ρ is the ratio of the real to imaginary parts of the forward elastic amplitude). From the optical theorem (34) one obtains the relation:

$$\sigma_{\text{el}} = \frac{\sigma_{\text{tot}}^2 (1 + \rho^2)}{16\pi B_{\text{el}}}. \quad (39)$$

B. The simple eikonal model

The expressions (30)–(35) allow one to compute the total and elastic cross sections for hadron collisions in terms of the profile function $\Gamma_{\text{el}}(b, s)$ or equivalently of the eikonal function $\chi(b, s)$. We remain with the task of constructing these functions. Physical insight on $\Gamma_{\text{el}}(b, s)$ and $\chi(b, s)$ can be obtained using a well-known analogy with the classical treatment of the absorption and scattering of a plane wave of light from an opaque screen. If the ratio between the amplitude “just behind” the screen and the amplitude of the incident plane wave is $\Gamma(\vec{b})$ (where \vec{b} is a 2-dimensional vector spanning the screen), then it is simple to obtain expressions for the total, elastic, and absorption cross sections that are formally identical to Eqs. (30)–(35). In particular the expression (35) for the inelastic cross section suggests to interpret the quantity $1 - |1 - \Gamma(\vec{b})|^2$ as the probability to absorb the wave at the position \vec{b} on the screen.

At high energy the elastic scattering amplitude, in reasonably good approximation, is purely imaginary, and accordingly the profile and eikonal functions are purely real. Neglecting the real part of the scattering amplitude the optical analogy is then sufficient to express the profile function as

$$\Gamma_{\text{el}}(b, s) = 1 - \sqrt{1 - P_{\text{abs}}(b, s)}, \quad (40)$$

with $P_{\text{abs}}(b, s)$ the absorption probability.

In the following we will use the approximation to consider the elastic scattering amplitude as purely imaginary, and the profile and eikonal functions as purely real. The

analyticity of the elastic scattering amplitude that is necessary to respect causality can be imposed, estimating the real part with a dispersion relation.

Equation (40) can be reexpressed in the form:

$$\begin{aligned} \Gamma_{\text{el}}(b, s) &\equiv 1 - e^{-\chi(b, s)} = 1 - \sqrt{P_0(b, s)} \\ &= 1 - \exp\left[-\frac{\langle n(b, s) \rangle}{2}\right]. \end{aligned} \quad (41)$$

In this equation $P_0 = 1 - P_{\text{abs}} = e^{-\langle n(b, s) \rangle}$ is the probability of *no* absorption in a collision at a certain b and s . The notation $P_0(b, s) = e^{-\langle n(b, s) \rangle}$ is motivated by the physical ansatz that “absorption” in a hadron collision corresponds to at least one interaction between the parton constituents of the two hadrons. If $\langle n(b, s) \rangle$ is the expected number of *all* elementary interactions between partons, and if the fluctuations of this average number of interactions are Poissonian, one obtains Eq. (41) for the profile and eikonal function.

The ansatz outlined above connects the profile function $\Gamma_{\text{el}}(b, s)$ to $\langle n(b, s) \rangle$ that is the average number of elementary interactions at impact parameter b and c.m. energy \sqrt{s} . In the following we will call this model the “simple eikonal model.” This ansatz was introduced by Durand and Pi in [1], who proposed to compute the quantity $\langle n(b, s) \rangle$ as

$$\langle n(b, s) \rangle = \sigma_{\text{jet}}(s, p_{\perp}^{\text{min}}) A(b), \quad (42)$$

where $\sigma_{\text{jet}}(s, p_{\perp}^{\text{min}})$ is the quantity discussed in the previous section that is the inclusive cross section for the production of jet pairs above a certain (fixed) p_{\perp} , and the energy independent geometry factor $A(b)$ gives the overlap of hadronic matter. For pp interactions Durand and Pi (see discussion in Appendix A) chose

$$A_{pp}(b) = \frac{b^3}{96\pi R_p^5} K_3\left(\frac{b}{R_p}\right), \quad (43)$$

with $R_p^{-2} = 0.71 \text{ GeV}^2$.

The model in this simplest form (that has in fact a single parameter, the value of p_{\perp}^{min}) soon proved to be inconsistent with the data. Perhaps the main difficulty with the original formulation of the model is that it predicts an incorrect relation between $\sigma_{\text{tot}}(s)$ and $B_{\text{el}}(s)$. This problem in fact is of a general nature and indicates that in the simple eikonal model, the factorization of $\langle n(b, s) \rangle$ as the product of two functions, one of s and the other of b , cannot fit the data.

To falsify the factorization hypothesis (14) it is in fact sufficient to measure both σ_{tot} and the slope of elastic scattering B_{el} [or more in general, because of Eq. (39) and the smallness of ρ , two out of the three quantities σ_{tot} , σ_{el} , and B_{el}] at two different values of \sqrt{s} . It is always possible to find a profile function $A(b)$ and a value of the eikonal function $\sigma_{\text{eik}}(s_1)$ that reproduce the observations of

σ_{tot} and B_{el} at c.m. energy $\sqrt{s_1}$, but then one remains with a single parameter $\sigma_{\text{eik}}(s_2)$ that can be chosen to reproduce $\sigma_{\text{tot}}(s_2)$ or $B_{\text{el}}(s_2)$, however the two values will coincide only if the factorization hypothesis is valid.

Models based on an eikonal model and the factorization hypothesis (14) naturally predict a correlation in the growth of $\sigma_{\text{tot}}(s)$ and $B_{\text{el}}(s)$ with energy; however the growth of $B_{\text{el}}(s)$ in the data is faster than the predictions, as was realized very early by Durand and Pi themselves [23].

This problem is illustrated in Fig. 1 where we show, in the plane (σ_{tot}, B) , experimental data obtained for pp scattering at the ISR [24] and for $p\bar{p}$ scattering at CERN [25] and Fermilab [26–28], and compare with the predictions of the simple eikonal model. The thick curve is the prediction of the simple eikonal model using a factorized form of $\langle n(b, s) \rangle$ and using for $A(b)$ the expression of Eq. (43). This line passes through the ISR data points, but fails at higher energy. The other lines in the figure are calculated using the same functional form for $A(b)$ but replacing the parameter R_p with $r_0 = 1.1R_p$ and $r_0 = 1.2R_p$. It is clear that, in the simple eikonal model the width in impact parameter of the function $\langle n(b, s) \rangle$ cannot be energy independent and must *grow* with s in order to reproduce the data.

An additional problem for the Durand and Pi model, besides the one we have just discussed connected to the relation between σ_{tot} and $B_{\text{el}}(s)$, is also that the energy dependence of $\sigma_{\text{tot}}(s)$ of the model, driven by the growth of

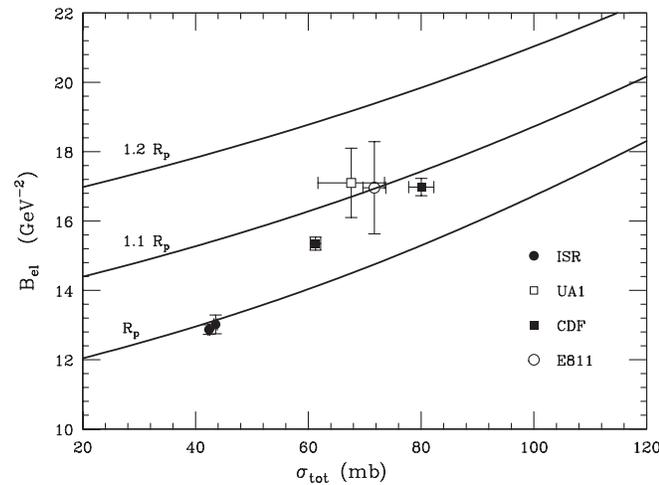


FIG. 1. The points are measurements of the total cross section σ_{tot} and of the forward slope B_{el} of the elastic scattering for pp and $p\bar{p}$ collisions at collider energies. The lines correspond to predictions based on the simple eikonal model using the parametrization of Eq. (45) for $\langle n(b, s) \rangle$. The three lines are computed for three values of the r_0 parameter ($r_0 = R_p$, $1.1R_p$, and $1.2R_p$). The ISR data at $\sqrt{s} = 52.8$ and 62.3 GeV are from [24]; the UA1 data at $\sqrt{s} = 540$ GeV from [25]; the CDF data at $\sqrt{s} = 546$ and 1800 GeV from [27,28]; the E811 data at 1800 GeV from [26].

$\sigma_{\text{jet}}(s, p_{\perp}^{\text{min}})$ with s for p_{\perp}^{min} fixed, is significantly faster than the data.

The ansatz of the simple eikonal model has been considered by several other authors [29–34] that have constructed different models for the function $\langle n(b, s) \rangle$, abandoning the factorization hypothesis. For example, it has been suggested to decompose the function $\langle n(b, s) \rangle$ in the general form:

$$\begin{aligned} \langle n(b, s) \rangle &= \sigma_{\text{eik}}(s)A(b, s) \\ &= \sigma_{\text{soft}}(s)A_{\text{soft}}(b, s) + \sigma_{\text{hard}}(s)A_{\text{hard}}(b, s), \end{aligned} \quad (44)$$

where the two terms describe a hard contribution that can be calculated with perturbative methods, and a soft non-perturbative part. As discussed above, the width in impact parameter of the combined overlap function $A(b, s)$ must increase with s to reproduce the relation between $B_{\text{el}}(s)$ and $\sigma_{\text{tot}}(s)$. Several justifications of this growth have been offered; however in our opinion none is really convincing.

The energy dependence of $\sigma_{\text{eik}}(s)$ and its decomposition in a soft and a hard part have also been the object of considerable discussion. In this work we will not enter in this discussion, because our main purpose here is to discuss the limitations of the simple eikonal model. In the simple eikonal model the quantity $\langle n(b, s) \rangle$ determines completely the profile function, and therefore the total and elastic cross sections. We will also consider the quantity $\langle n(b, s) \rangle$ attributing to it the same physical meaning that it has in the simple eikonal mode, namely, the average number of elementary interactions at impact parameter b and c.m. energy \sqrt{s} ; however in order to describe in a consistent way inelastic diffraction, we will propose a different method to connect $\langle n(b, s) \rangle$ with the profile function and to the observable cross sections.

In the following we will parametrize the function $\langle n(b, s) \rangle$ as

$$\langle n(b, s) \rangle = \sigma_{\text{eik}}(s) \left\{ \frac{b^3}{96\pi[r_0(s)]^5} K_3\left(\frac{b}{r_0(s)}\right) \right\}, \quad (45)$$

with an overlap function that has the same form as in Eq. (43), but with an s dependence obtained substituting for R_p an energy dependent parameter $r_0(s)$:

The simple eikonal model ansatz outlined above does not include a treatment of the inelastic diffraction processes. This is a serious limitation, because the consistent description of these processes is in fact essential. This problem is addressed in the next section.

IV. INELASTIC DIFFRACTION

Inelastic diffraction [35] produces three classes of events: beam, target, and double diffraction. In beam (target) diffraction the beam (target) particle is excited to a higher mass state of the same quantum numbers (with the possible exception of spin) while the other initial state hadron remains unchanged. In double diffraction both

colliding hadrons are excited into higher mass states. The three processes can therefore be described as

$$\begin{aligned} h_1 h_2 &\rightarrow h_1^* h_2 && \text{(beam diffraction),} \\ h_1 h_2 &\rightarrow h_1 h_2^* && \text{(target diffraction),} \\ h_1 h_2 &\rightarrow h_1^* h_2^* && \text{(double diffraction).} \end{aligned} \quad (46)$$

The experimental selection of events that belongs to these three classes is not trivial, especially for the double diffraction ones. These events are characterized by a large ‘‘rapidity gap’’ that separates particles produced in the decay of the two excited states h_1^* and h_2^* . A complete description of beam (target) diffraction is given by the double differential cross sections $d^2\sigma/(dM_1^2 dt)$ [$d^2\sigma/(dM_2^2 dt)$] that gives the probability to produce an event where the beam (target) particle is excited to a state of mass $M_1 > m_1$ ($M_2 > m_2$) with transfer momentum t . The double diffraction cross section is described by the 3-times differential cross section: $d^3\sigma/(dM_1^2 dM_2^2 dt)$. For single diffraction one has a reasonably accurate picture of the M and t dependence of the cross section. The t dependence has the characteristic exponential behavior of all diffractivelike processes, for the mass dependence the cross section grows very quickly above the threshold ($M = m_p + m_\pi$ for proton excitation), oscillates following the structure of the resonances with the quantum numbers of the hadron in question, and then falls at higher masses roughly proportional to M^{-2} . In the following we will indicate as ‘‘diffraction’’ the sum of these three classes of inelastic diffractive events.

Rapidity gaps in the distribution of final state particles can also be present in events with a hard scattering; moreover some events exhibit two or more rapidity gaps. A dynamical description of these classes of events, that account for a much smaller fraction of the total cross sections, is not considered in this work.

We take the point of view that any parton-parton scattering implies an exchange of color (interactions with no color exchange are considered as higher order corrections and neglected) and therefore, in the treatment of the cross section given above, all events with one or more parton-parton interaction are considered as nondiffractive. The total cross section is therefore decomposed into an inelastic nondiffractive part (with at least one parton-parton scattering) and the elastic part, with no room left for diffraction. This lack of inclusion of inelastic diffraction is an important conceptual problem for the simple eikonal model.

A. Good and Walker model

The fundamental idea for the description of inelastic diffraction in hadronic collisions was introduced long ago by Good and Walker [4] as an analogy with the scattering of polarized light on a birefringent absorbing medium. It is well known that if a plane wave of light impinges on an absorbing screen with grayness profile $\Gamma(b)$

one has absorption [$\propto 1 - |1 - \Gamma(b)|^2$] and elastic scattering [$\propto |\Gamma(b)|^2$] with a very forward diffraction pattern that depends on the geometry of the screen.

If the absorption in the screen depends on the polarization of the incident light, a new interesting phenomenon emerges. In general, when the incident beam is in a well-defined polarization state, the absorption is accompanied by the transmission of scattered light with *both* polarization states. The scattering of light with the same polarization of the incident beam and of light in the orthogonal polarization state can be considered as analogous to elastic scattering and inelastic diffraction in hadronic interactions. For a more explicit example, let us consider an incident beam of light in the linear polarization state $|x\rangle$, and an absorbing screen that has the grayness profile $\Gamma_{x'}(b)$ for light with polarization $|x'\rangle$ and the grayness profile $\Gamma_{y'}(b)$ for light with polarization $|y'\rangle$ with

$$\begin{aligned} |x'\rangle &= \cos\varphi|x\rangle + \sin\varphi|y\rangle, \\ |y'\rangle &= -\sin\varphi|x\rangle + \cos\varphi|y\rangle. \end{aligned}$$

It is clear that one will have an absorption cross section according to $1 - |\Gamma_{x'}(b)|^2 - |\Gamma_{y'}(b)|^2$ and after the screen one will find scattered waves with both the $|x\rangle$ and $|y\rangle$ polarizations. The cross sections for elastic scattering and inelastic diffraction turn out to be

$$\sigma_x = \int d^2b |1 - [1 - \Gamma_{x'}(b)]\cos^2\varphi - [1 - \Gamma_{y'}(b)]\sin^2\varphi|^2, \quad (47)$$

$$\sigma_y = \int d^2b \cos^2\varphi \sin^2\varphi |\Gamma_{y'}(b) - \Gamma_{x'}(b)|^2. \quad (48)$$

Note that inelastic diffraction is nonvanishing only when $\Gamma_{x'} \neq \Gamma_{y'}$, and if φ is not a multiple of $\pi/2$, that is if the states $\{|x\rangle, |y\rangle\}$ do not coincide with the states $\{|x'\rangle, |y'\rangle\}$ that are the eigenstates of the transmission across the screen.

It is straightforward [4] to generalize this elementary example. We can consider two complete sets of orthonormal states $\{|\varphi_m\rangle\}$ and $\{|\psi_j\rangle\}$. The states $|\varphi_m\rangle$ are directly observable, with the label m describing the invariant masses of the two final hadronic states (in the forward and backward hemisphere) and their particle content. Without loss of generality we can assume that the state $|\varphi_1\rangle$ with $m = 1$ corresponds to the initial state of the scattering (for example in case of a πp collision to the state $|\pi p\rangle$).

The states $|\psi_j\rangle$ are eigenstates of the scattering matrix. That is, defining as usual the S matrix as $S = I + iT$ one has

$$T|\psi_j\rangle = t_j|\psi_j\rangle. \quad (49)$$

The relation between the two orthonormal bases is given by

$$|\varphi_m\rangle = \sum_j C_{mj} |\psi_j\rangle, \quad (50)$$

$$|\psi_j\rangle = \sum_m C_{mj}^* |\varphi_m\rangle. \quad (51)$$

It is now possible to write the different components of the cross sections as integrals over the impact parameter dependence of the $t_j(b)$. The absorption cross section is

$$\sigma_{\text{abs}} = \int d^2b \left[1 - \sum_j |C_{1j}| \{1 - t_j(b)\} \right]^2. \quad (52)$$

The cross section for diffraction into the state m is

$$\sigma_m = \int d^2b \left| \sum_j C_{mj}^* C_{1j} t_j(b) \right|^2. \quad (53)$$

The elastic cross section corresponds to σ_1 and therefore is

$$\sigma_{\text{el}} = \int d^2b \left| \sum_j |C_{1j}|^2 t_j(b) \right|^2. \quad (54)$$

Summing over all states m and using the orthonormality of the states one obtains

$$\sigma_{\text{diff+el}} = \sum_m \sigma_m = \int d^2b \sum_j |C_{1j}|^2 |t_j(b)|^2. \quad (55)$$

The inelastic diffraction cross section can be obtained subtracting Eq. (54) from (55). The total cross section can be obtained summing the elastic, diffractive, and absorption cross sections obtaining the result:

$$\sigma_{\text{tot}} = \int d^2b \sum_j |C_{1j}|^2 2\text{Re}[t_j(b)], \quad (56)$$

in agreement also with the optical theorem.

The problem with the approach that we have just outlined is that it is only formal and requires the introduction of many, in fact, infinite parameters; moreover the nature of the eigenstates $|\psi_j\rangle$ is not physically obvious.

B. Partonic interpretation of the scattering eigenstates

Miettinen and Pumplin [10] have made the proposal to interpret the states $|\psi_j\rangle$ (the eigenstates of the T matrix) as parton configuration states. In their paper they also include a simple explicit method to construct these partonic models (see [36,37] for a recent discussion).

In this work we will develop the idea that one needs to integrate over the parton configurations of the interacting hadron. The discussion that we have outlined in Sec. II can also be applied to the current problem. Using the notations that we have introduced in Sec. II, the label j of the T matrix eigenstates $|\psi_j\rangle$ corresponds to the direct product of the configurations \mathbb{C}_1 and \mathbb{C}_2 of the colliding hadrons, and the squared coefficient $|C_{1j}|^2$ corresponds to the probability $P_{h_1}(\mathbb{C}_1) \times P_{h_2}(\mathbb{C}_2)$ to find the two hadrons in a certain

configuration. One then has also the correspondence:

$$\sum_j |C_{1j}|^2 \leftrightarrow \int d\mathbb{C}_1 \int d\mathbb{C}_2 P_{h_1}(\mathbb{C}_1) P_{h_2}(\mathbb{C}_2).$$

The transmission eigenvalues $t_j(b)$ have the partonic interpretation as

$$t_j(b) = 1 - \exp\left[-\frac{n_j(b)}{2}\right] = 1 - \exp\left[-\frac{n(b, \mathbb{C}_1, \mathbb{C}_2)}{2}\right], \quad (57)$$

where as in Sec. II $n(b, \mathbb{C}_1, \mathbb{C}_2)$ is the expected number of interactions among partons in a collision with impact parameter b when the colliding hadrons are in configurations \mathbb{C}_1 and \mathbb{C}_2 (the dependence on the c.m. energy has been left implicit). The difference with respect to the discussion made above is that in the previous case one was discussing only a subclass of (hard) interactions, while here one refers to the *total* number of parton interactions.

At this point we can again make the factorization hypothesis, that the fluctuations in the number of interactions at different impact parameters, being related to the distribution of parton configurations is independent from b , and therefore one has

$$\begin{aligned} & \int d\mathbb{C}_1 \int d\mathbb{C}_2 P_{h_1}(\mathbb{C}_1) P_{h_2}(\mathbb{C}_2) \exp\left[-\frac{n(b, \mathbb{C}_1, \mathbb{C}_2)}{2}\right] \\ &= \int_0^\infty d\alpha p(\alpha) \exp\left[-\frac{\langle n(b) \rangle \alpha}{2}\right]. \end{aligned} \quad (58)$$

As before the function $p(\alpha)$ satisfies the two integral relations:

$$\int_0^\infty d\alpha p(\alpha) = 1, \quad \int_0^\infty d\alpha \alpha p(\alpha) = 1.$$

Using the identities (57) and (58) we can now rewrite in manageable form Eqs. (52) and (54)–(56) as

$$\frac{d^2 \sigma_{\text{abs}}}{d^2 b} = 1 - \int_0^\infty d\alpha p(\alpha) e^{-\langle n(b,s) \rangle \alpha}, \quad (59)$$

$$\frac{d^2 \sigma_{\text{el}}}{d^2 b} = \left[\int_0^\infty d\alpha p(\alpha) (1 - e^{-\langle n(b,s) \rangle \alpha / 2}) \right]^2, \quad (60)$$

$$\frac{d^2 \sigma_{\text{diff+el}}}{d^2 b} = \int_0^\infty d\alpha p(\alpha) [1 - e^{-\langle n(b,s) \rangle \alpha / 2}]^2, \quad (61)$$

$$\frac{d^2 \sigma_{\text{tot}}}{d^2 b} = 2 \int_0^\infty d\alpha p(\alpha) (1 - e^{-\langle n(b,s) \rangle \alpha / 2}). \quad (62)$$

The elastic scattering amplitude is given by

$$F_{\text{el}}(q, s) = i \int \frac{d^2 b}{2\pi} e^{i\vec{q} \cdot \vec{b}} \int_0^\infty d\alpha p(\alpha) [1 - e^{-\langle n(b,s) \rangle \alpha / 2}]. \quad (63)$$

It is important to note that in the limit $p(\alpha) \rightarrow \delta[\alpha - 1]$, that is in the limit where one neglects the effects of different parton configurations, one has that Eqs. (60) and (61), that describe the elastic, and elastic plus diffractive cross sections become identical, that is inelastic diffraction vanishes. Moreover, in this case, the expressions for the absorption, elastic, and total cross sections coincide with the expressions of the simple eikonal model that neglects inelastic diffraction.

Equation (61) sums over all diffractive channels, and therefore loses all information about the distributions of the excited masses. It remains however possible to compute the t distribution for elastic scattering and for inelastic diffractive scattering (summing over all possible open channels):

$$\frac{d\sigma_{\text{diff+el}}}{dt} = \sum_m \frac{d\sigma_m}{dt} = \int dM_1 \int dM_2 \frac{d^3\sigma_{\text{diff}}}{dM_1 dM_2 dt}. \quad (64)$$

The differential cross section for elastic scattering $d\sigma_{\text{el}}/dt$ can be calculated from Eqs. (32) and (33):

$$\frac{d\sigma_{\text{el}}}{dt} = \pi \left[\int_0^\infty db b J_0(b\sqrt{|t|}) \times \int_0^\infty d\alpha p(\alpha) (1 - e^{-\langle n(b,s) \rangle \alpha / 2}) \right]^2. \quad (65)$$

Similarly, the differential cross section for diffraction plus elastic scattering can be calculated as

$$\frac{d\sigma_{\text{diff+el}}}{dt} = \pi \int_0^\infty d\alpha p(\alpha) \times \left[\int_0^\infty db b J_0(b\sqrt{|t|}) (1 - e^{-\langle n(b,s) \rangle \alpha / 2}) \right]^2. \quad (66)$$

It is straightforward to see that for $p(\alpha) = \delta[\alpha - 1]$ expressions (65) and (66) become identical and equal to the well-known expression for the simple eikonal model:

$$\left. \frac{d\sigma_{\text{el}}}{dt} \right|_{\text{simple}} = \pi \left[\int db b J_0(b\sqrt{|t|}) (1 - e^{-\langle n(b,s) \rangle / 2}) \right]^2. \quad (67)$$

The slopes at $|t| = 0$ (B_{el} and B_{diff}) can be calculated from the definition (37) (and the analogous for inelastic diffraction). For example, using

$$\lim_{t \rightarrow 0} \frac{d}{dt} [J_0(b\sqrt{|t|})] = -\frac{b^2}{4}, \quad (68)$$

one obtains for B_{el} :

$$B_{\text{el}} = \left[\int_0^\infty db \frac{b^3}{2} \int_0^\infty d\alpha p(\alpha) (1 - e^{-\langle n(b,s) \rangle \alpha / 2}) \right] \times \left[\int_0^\infty db b \int_0^\infty d\alpha p(\alpha) (1 - e^{-\langle n(b,s) \rangle \alpha / 2}) \right]^{-1}. \quad (69)$$

A similar expression for B_{diff} is easily derived.

C. Comparison with multichannel eikonal formalism

A method to apply to the Good-Walker ansatz is to construct explicitly a transition operator as a matrix of $n \times n$ dimensions. In this way the formal indices m and j in Sec. IV A become simply integer indices running from 1 to n . To show the mathematical equivalence between this multichannel method and the use of the effective configuration probability distribution $p(\alpha)$ is straightforward. In the multichannel approach the profile function $\Gamma_{\text{el}}(b, s)$ is replaced by the $(n \times n)$ matrix $\hat{\Gamma}(b, s)$ that can be expressed in terms of the eikonal matrix $\hat{\chi}(b, s)$:

$$\hat{\Gamma}(b, s) = 1 - \exp[-\hat{\chi}(b, s)]. \quad (70)$$

The eigenvalues of $\hat{\Gamma}$ (Γ_j) and $\hat{\chi}$ (χ_j) can be written in the form:

$$\Gamma_j = 1 - e^{-\chi_j} = 1 - \exp\left[-\frac{\langle n(b, s) \rangle}{2} \alpha_j\right]. \quad (71)$$

The interpretation of Γ_j as the effect of absorption suggests that $\langle n(b, s) \rangle$ and all α_j 's are real and positive. In the optical analogy one interprets the quantity:

$$1 - [1 - \Gamma_j]^2 = 1 - e^{-2\chi_j} = 1 - e^{-\langle n(b, s) \rangle \alpha_j},$$

as the absorption probability for the eigenstate $|\psi_j\rangle$ that corresponds to eigenvalue α_j . Assuming a Poisson probability distribution, the quantity $[\langle n(b, s) \rangle \alpha_j]$ can then be interpreted as the average number of elementary interactions for the eigenstate $|\psi_j\rangle$. Without loss of generality, reabsorbing a constant in the definition of $\langle n(b, s) \rangle$, one can impose the constraint

$$\sum_j p_j \alpha_j = 1, \quad (72)$$

where the quantities p_j 's

$$p_j = |\langle \psi_j | \varphi_{\text{initial}} \rangle|^2 \quad (73)$$

measure the probability overlaps between the initial state $|\varphi_{\text{initial}}\rangle$ and the eigenstates $|\psi_j\rangle$ of the transition matrix.

The normalization condition (72) allows one to interpret the quantity $\langle n(b, s) \rangle$ as the average number of elementary interactions (at impact parameter b and c.m. energy \sqrt{s}) for the initial state $|\varphi_{\text{initial}}\rangle$. It is then possible to define the function $p(\alpha)$ as

$$p(\alpha) = \sum_j p_j \delta[\alpha - \alpha_j] \quad (74)$$

and to use this function to express the results for the total, elastic, absorption, and diffractive cross sections as integrals over α according to Eqs. (59)–(62).

It can be instructive to consider an explicit model that implements the multichannel model. The minimum model that includes the 4 distinct processes of elastic scattering together with target, projectile, and double diffraction, must obviously consider a 4-dimensional Hilbert space spanned by 4 physical eigenstates $|\varphi_m\rangle$ (that without loss of generality can symbolically be labeled as $|\pi p\rangle$, $|\pi\Delta\rangle$, $|\rho p\rangle$, and $|\rho\Delta\rangle$).

The structure of this most general Good-Walker model with 4 channels has been presented in [11] and is discussed in detail in Appendix B. The most general 4-channel eikonal (for a fixed \sqrt{s}) is described by the impact parameter multiplicity distribution $\langle n(b, s) \rangle$ and by the 4×4 matrix \hat{M} that in the most general case (where the two initial state hadrons are not identical as in πp scattering) is defined by 4 real parameters (in the case where the two interacting hadrons are identical, as in pp scattering, the parameters reduce to 2).

The eigenvalues and eigenvectors of the most general matrix \hat{M} are easily calculated (as show in Appendix B). Having solved this diagonalization problem, it is simple to obtain the cross sections for the total, elastic, absorption, and diffractive scattering. The cross sections depend on $\langle n(b, s) \rangle$ and on the 4 (or 2 for identical initial particles) parameters of the matrix \hat{M} . The cross sections can be recast in the form (59)–(62) as integrals over α defining $p(\alpha)$ according to Eq. (74) (where the summations over j now run from 1 to 4).

Note that in an n -channel eikonal, the inelastic diffractive cross sections are obtained as an explicit sum of $n - 1$ terms:

$$\sigma_{\text{diff}} = \sum_{m \neq 1} \sigma_m \quad (75)$$

(we are identifying the state $m = 1$ with the initial state). In a realistic discussion the index m should run continuously over all possible diffractively excited states. For the 4-channel model the 3 states can be identified as representing target, projectile, and double diffraction.

It is interesting to study, in the framework of this most general 4-channel model, the relative importance of single versus double diffraction. Considering for simplicity the case of pp collision, the 4-channel Good-Walker model has two free parameters (β_p and ϵ_p). The relative importance of the elastic and diffractive processes can vary significantly with variation of the model parameters, and similarly the ratio between the double and single diffraction cross sections can assume different values; however numerical studies show that to a good approximation (for the scattering of identical particles) one has

$$\frac{\sigma_{\text{TD}}}{\sigma_{\text{el}}} \equiv \frac{\sigma_{\text{BD}}}{\sigma_{\text{el}}} \simeq \frac{\sigma_{\text{DD}}}{\sigma_{\text{TD}}} \equiv \frac{\sigma_{\text{DD}}}{\sigma_{\text{BD}}}, \quad (76)$$

that is the ratios of cross sections single (beam or target) diffraction to the elastic one is approximately equal to the ratio of the double diffraction cross section to the single diffraction one. This result can also be understood from inspection of the structure of the matrix \hat{M} . For negligible ϵ_p for example, the relative importance of the cross sections for elastic, single diffractions and double diffraction scattering is in the ratio $1:\beta_p:\beta_p^2$. This result is compatible with the available data on double diffraction (taking into account the large errors).

The separate calculation of the different components of the diffractive cross sections is a significant merit for the 4-channel model. A limit of such an approach is that it predicts the multiplicity distribution of the elementary interactions as the superposition of 3 (for the scattering of identical particles) or 4 (in the general case) Poissonian distributions of different average values, and such distribution might not be sufficiently smooth for a realistic comparison with data. The approach of using the function of a real positive variable $p(\alpha)$ allows one to consider implicitly an infinity of inelastic channels.

In the next section we will propose a simple parametrization of $p(\alpha)$ that depends on a single parameter.

V. EXPLICIT MODEL

Equations (59)–(61) allow one to compute the total, elastic, diffractive, and absorption cross sections in terms of the impact parameter multiplicity distribution $\langle n(b, s) \rangle$ (the average number of elementary interactions at impact parameter b and c.m. energy \sqrt{s}) and of the function $p(\alpha)$. Unfortunately the shape of the function $p(\alpha)$ is not determined. All we have been able to establish is that the first two moments $\langle \alpha^k \rangle$ with $k = 0, 1$ must be unity. It is however reasonable to expect that the most important property of the function $p(\alpha)$ is its second moment $\langle \alpha^2 \rangle$ or equivalently its width $\sigma_\alpha^2 = \langle \alpha^2 \rangle - 1$.

Given this lack of knowledge about the shape of $p(\alpha)$ we have chosen for it a simple analytic expression that allows easy manipulations:

$$p(\alpha) = \frac{1}{w\Gamma(\frac{1}{w})} \left(\frac{\alpha}{w}\right)^{(1/w)-1} \exp\left[-\frac{\alpha}{w}\right]. \quad (77)$$

The k th moment of the distribution is

$$\langle \alpha^k \rangle = \int_0^\infty d\alpha \alpha^k p(\alpha) = \frac{w^k \Gamma(k + \frac{1}{w})}{\Gamma(\frac{1}{w})}, \quad (78)$$

therefore one finds

$$\langle \alpha^0 \rangle = 1, \quad \langle \alpha \rangle = 1, \quad \langle \alpha^2 \rangle = 1 + w, \quad \sigma_\alpha^2 = w. \quad (79)$$

Therefore the parameter w describes the variance of the α

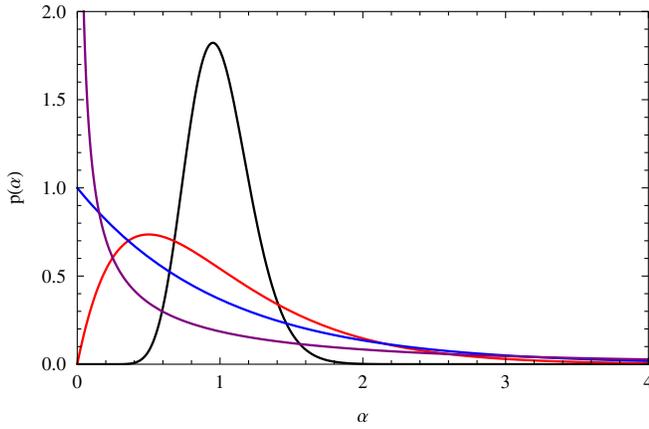


FIG. 2 (color online). Plot of the function $p(\alpha)$ given in Eq. (77) for four values of the parameter w ($w = 0.05, 0.5, 1,$ and 3). For $w \rightarrow 0$ the function takes the form $\delta[\alpha - 1]$.

distribution. The integer value moments for $n \geq 3$ can be written also as

$$\langle \alpha^n \rangle = (1 + w) \cdots (1 + (n - 1)w). \quad (80)$$

Numerical examples of the function $p(\alpha)$ are shown in Fig. 2.

An attractive property of the functional form (77) for $p(\alpha)$ is that it allows one to perform analytically the integrations over α in Eqs. (59)–(62) to obtain the quantities of interest. The profile function $\Gamma_{\text{el}}(b, s)$ becomes

$$\begin{aligned} \Gamma_{\text{el}}(b, s) &= \int_0^\infty d\alpha p(\alpha) [1 - e^{-\langle n(b, s) \rangle \alpha / 2}] \\ &= 1 - \left(1 + \frac{\langle n(b, s) \rangle w}{2} \right)^{-1/w}. \end{aligned} \quad (81)$$

The expressions for the total, elastic, and diffractive cross sections become

$$\frac{d^2 \sigma_{\text{tot}}}{d^2 b} = 2 - 2 \left(1 + \frac{\langle n(b, s) \rangle w}{2} \right)^{-1/w}, \quad (82)$$

$$\frac{d^2 \sigma_{\text{el}}}{d^2 b} = \left(1 - \left(1 + \frac{\langle n(b, s) \rangle w}{2} \right)^{-1/w} \right)^2, \quad (83)$$

$$\frac{d^2 \sigma_{\text{diff}}}{d^2 b} = \left(1 + \frac{\langle n(b, s) \rangle w}{2} \right)^{-1/w} - \left(1 + \frac{\langle n(b, s) \rangle w}{2} \right)^{-2/w}. \quad (84)$$

Using for $\langle n(b, s) \rangle$ the parametrization of Eq. (45) the expressions (82)–(84) allow one to compute the different components of hadron-hadron interactions for any given value of \sqrt{s} in terms of three parameters: $(\sigma_{\text{eik}}, r_0, w)$. It should be noted that the simple eikonal model corresponds

to the case $w \rightarrow 0$ and is therefore included as a limiting case of our model.

As a critical remark we note that the qualitative idea behind our one-parameter modeling of $p(\alpha)$ is that the most important feature of $p(\alpha)$ is its second moment $\langle \alpha^2 \rangle = w + 1$. This however is only true in the first approximation. Functions $p(\alpha)$ that differ only for moments $\langle \alpha^k \rangle$ with $k > 2$ can also produce different cross sections. An example of this behavior is the 4-channel model of Appendix B. For pp interactions this model has two free parameters (β_p and ϵ_p). The variance of $p(\alpha)$ in the model is $(1 + \beta_p)^2 - 1$, uniquely determined by β_p ; however the values of the different cross sections (elastic, diffractive, and absorption) depend on both of the model parameters.

The single parameter description of $p(\alpha)$ of Eq. (77) seems in any case a reasonable form to investigate phenomenologically the consequences of Eqs. (59)–(62).

A. Parameter determination

Equations (82)–(84) allow one to determine (at a certain \sqrt{s}) the set $(\sigma_{\text{eik}}, r_0, w)$ of the three parameters in the model from measurements of $(\sigma_{\text{tot}}, B_{\text{el}}, \sigma_{\text{diff}})$. As an example of this parameter determination we discuss here in some detail the measurements performed at one particular value of the c.m. energy ($\sqrt{s} = 1.8$ TeV) by one detector (CDF at the Fermilab $\bar{p}p$ collider). The CDF experiment [27,28] has measured $\sigma_{\text{tot}}(1 + \rho^2) = 81.83 \pm 2.29$ mb (that estimating $\rho \approx 0.15$ corresponds to $\sigma_{\text{tot}} = 80.03 \pm 2.24$ mb), an elastic cross section $\sigma_{\text{el}} = 19.7 \pm 0.85$ mb, and a slope of the forward elastic cross section $B_{\text{el}} = 16.98 \pm 0.25$ GeV $^{-2}$. In addition the CDF Collaboration has measured [38] the single diffractive cross section $\sigma_{\text{SD}} = 9.46 \pm 0.44$ mb. The three quantities σ_{tot} , σ_{el} , and B_{el} are related by the unitarity relation (39), and therefore only two of them are independent. In the following we will fix our attention on σ_{tot} and B_{el} .

Our formalism allows only the calculation of the total diffractive cross section, summing over single and double diffraction processes. In order to compare the model to the single diffraction measurement of CDF we have therefore to include some estimate of double diffraction. We will use the result (76) that allows one to estimate the complete diffractive cross section σ_{diff} from the measurements of the elastic and single diffractive one as

$$\sigma_{\text{diff}} = \sigma_{\text{SD}} + \sigma_{\text{DD}} \approx \sigma_{\text{SD}} \left(1 + \frac{\sigma_{\text{SD}}}{2\sigma_{\text{el}}} \right). \quad (85)$$

This hypothesis leads us to estimate σ_{diff} at $\sqrt{s} = 1800$ GeV from the CDF data as approximately 11.6 mb.

Neglecting the measurement of the diffractive cross section and considering only the experimental results for σ_{tot} and B_{el} , there is an infinity of sets of parameters $(\sigma_{\text{eik}}, r_0, w)$ that reproduce the data. This infinity of solutions can be parametrized by the value of w , that can take any non-negative value.

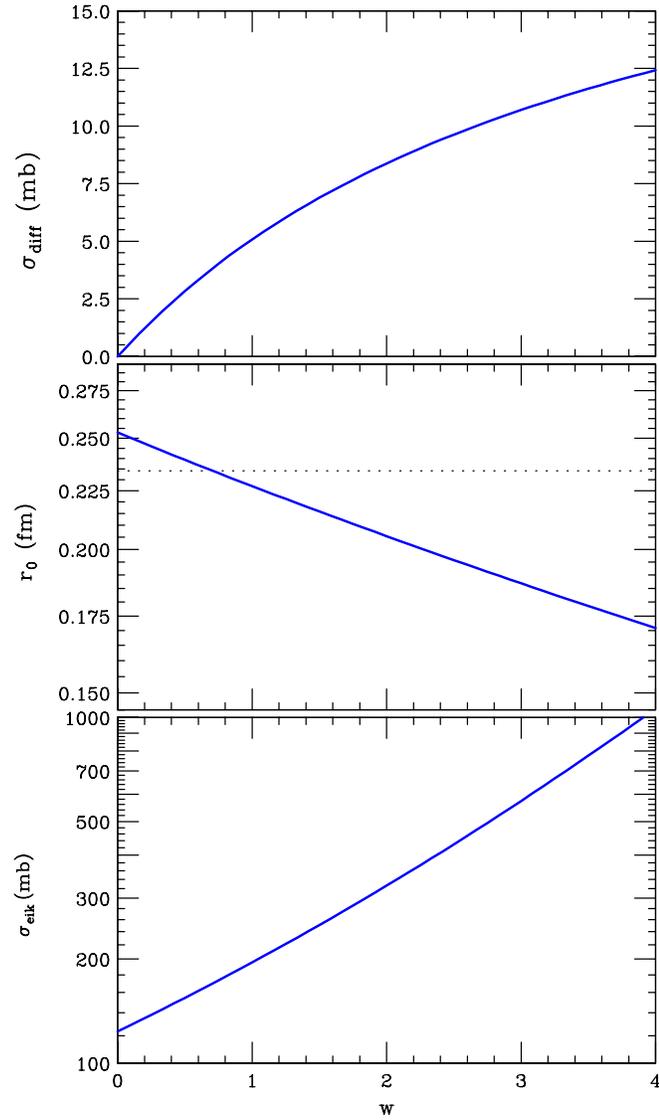


FIG. 3 (color online). The middle and bottom panels show the triplet of parameters (w , $\sigma_{\text{eik}}(w)$, and $r_0(w)$) that reproduce (using Eqs. (82) and (83) with expression (45) for $\langle n(b, s) \rangle$) the measurements of σ_{tot} and B_{el} obtained by CDF [27,28] at $\sqrt{s} = 1.8$ TeV (note the logarithmic scale in the bottom panel for σ_{eik}). The top panel shows the corresponding value of the diffractive cross section.

The limiting case $w = 0$ corresponds to the simple eikonal model. For $w = 0$ the values of the parameters that reproduce the central value of the CDF measurements are $\sigma_{\text{eik}} \simeq 124.1$ mb and $r_0 \simeq 0.2527$ fm ($\simeq 1.08R_p$). To this ($w = 0$) solution corresponds a vanishing diffractive cross section.

The solutions [w , $\sigma_{\text{eik}}(w)$, $r_0(w)$] that reproduce the central value of the CDF measurements for σ_{tot} and B_{el} at $\sqrt{s} = 1800$ GeV are shown in Fig. 3. Increasing w the value of $\sigma_{\text{eik}}(w)$ grows monotonically, while $r_0(w)$ decreases. The triplets $\{w, \sigma_{\text{eik}}(w), r_0(w)\}$ result in identical σ_{tot} and B_{el} , but produce different diffractive cross sec-

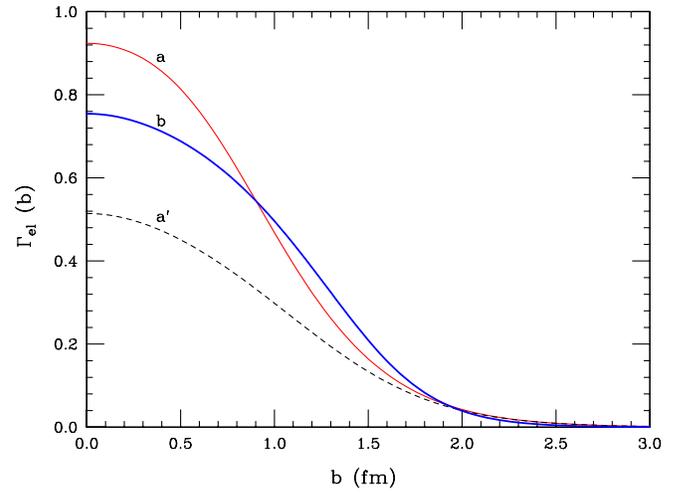


FIG. 4 (color online). Profile function $\Gamma_{\text{el}}(b)$ for pp scattering as a function of the impact parameter b . Curve (a) is calculated in the simple eikonal of Eq. (41), using for the $\langle n(b, s) \rangle$ the parametrization (45) with $\sigma_{\text{eik}} = 124$ mb and $r_0 = 0.253$ fm. The corresponding values of σ_{tot} and B_{el} are $\sigma_{\text{tot}} = 80.3$ mb and $B_{\text{el}} = 16.98$ GeV $^{-2}$. Curve (a') is calculated assuming the same interaction profile $\langle n(b, s) \rangle$ (that is the same parameters σ_{eik} and r_0) as for curve (a) but using the model of Eq. (81) for the profile with the form (77) for $p(a)$ with $w = 3$. The resulting profile function is smaller (implying a smaller σ_{tot}) and broader (implying a larger B_{el}). The profile (b) is calculated using the same model used for curve (a') with the same value $w = 3$; however the parameters that describe $\langle n(b, s) \rangle$ are now $\sigma_{\text{eik}} = 582$ mb and $r_0 = 0.186$ fm. The profile function (b) results in the same σ_{tot} and B_{el} as curve (a).

tions, with $\sigma_{\text{diff}}(w)$ growing monotonically with w as also shown in Fig. 3. The value $\sigma_{\text{diff}} = 11.6$ mb is obtained for $w = 3.48$.

Perhaps the most striking feature of Fig. 3 is the rapid increase of σ_{eik} with w . As an illustration, for $w \simeq 3$ (that results in $\sigma_{\text{diff}} \simeq 10.7$ mb) one needs a smaller r_0 ($r_0 = 0.186$ fm) and $\sigma_{\text{eik}} \simeq 580$ mb, that is almost 5 times larger than the value of σ_{eik} that reproduces the measurements for $w = 0$.

For a qualitative understanding of these results it can be instructive to consider Fig. 4. Curve (a) shows the profile function $\Gamma_{\text{el}}(b)$ that corresponds to the solution with $w = 0$ that we have just discussed. Curve (a') shows the profile that is obtained for the same parameters (σ_{eik} , r_0), that is for the same $\langle n(b, s) \rangle$, of the $w = 0$ solution, but using the value $w = 3$. The resulting profile function is smaller (that is produces a smaller σ_{tot}) and broader (implying a larger B_{el}). These features can be readily understood from inspection of Eq. (81). In order to obtain the desired values of σ_{tot} and B_{el} using the model (81) and $w = 3$ one needs to modify the quantity $\langle n(b, s) \rangle$, choosing both a larger σ_{eik} to increase the area under the profile, and a smaller r_0 to obtain the desired value of $\langle b^2 \rangle \propto B$. The solution is shown in Fig. 4 as curve (b). The profile functions of curves (a) and (b) in Fig. 4 produce identical σ_{tot} , and identical

$d\sigma_{\text{el}}/dt$ for small $|t|$, but differ in the description of elastic scattering at large $|t|$.

To summarize this discussion: in the simple eikonal model the impact parameter multiplicity distributions $\langle n(b, s) \rangle$ uniquely define the profile function. In our model the profile function is determined [see Eq. (81)] also by the function $p(\alpha)$, and different choices for the shape of $p(\alpha)$ produce different profiles, and therefore different values of the total and elastic cross sections. Vice versa, the estimate of $\langle n(b, s) \rangle$ [or in terms of the parametrization (45) the values of σ_{eik} and r_0] that reproduces the measured values of σ_{tot} and σ_{el} (or σ_{tot} and B_{el}) strongly depends on the assumptions made for $p(\alpha)$.

The function $p(\alpha)$, and, in particular, its width, controls the size of the inelastic diffractive cross section; therefore one can obtain information about its properties from the experimental data on the rate of diffractive events. The bottom line is that it is essential to include in a consistent way inelastic diffraction in the theoretical framework that describes hadronic cross sections.

These considerations are the main qualitative results of this work. The consistent introduction of inelastic diffraction in the eikonal formalism results in the following:

- (1) an eikonal cross section σ_{eik} that is several times larger than estimates based on the simple eikonal model that does not consider explicitly diffraction;
- (2) a narrower distribution of hadronic matter. In the case of protons, this distribution is estimated as narrower than the charge distribution inferred by the electromagnetic form factor.

These effects can have important consequences in the prediction of the properties of particle production in high energy hadron interactions, if one takes into account the interpretation of the eikonal as a description of the multiple interaction structure of the collision. In this case the ratio $\sigma_{\text{eik}}/\sigma_{\text{inel}}$ has the physical meaning of the average number of elementary interactions per inelastic event; therefore our results imply that this average number of elementary interactions is several times larger than previous estimates. The precise way to relate this quantity (the average number of elementary interactions per collision) to observable quantities, such as the multiplicity distribution, depends on a number of additional assumptions that have to be made in a Monte Carlo modeling of multiparticle production.

The theoretical framework we are considering predicts not only the average number of interactions in a collision, but also the detailed multiplicity distribution for such interactions in one collision.

In Sec. II we have given in Eq. (26) the multiplicity distribution of the number of hard observable jets per collision, in terms of the quantities $\langle n_{\text{jet}}(b, s) \rangle$ and $p(\alpha)$. The generalization to the multiplicity distribution of the total number of elementary interaction in a collision can be immediately obtained replacing the average number of

hard interactions at a fixed impact parameter and c.m. energy $\langle n_{\text{jet}}(b, s) \rangle$ with the average for the total number of elementary interactions $\langle n(b, s) \rangle$. In addition, the most economical assumption is to assume that the functions $p(\alpha)$ relevant in the two cases are (at least approximately) equal.

For the simple functional form of $p(\alpha)$ given in Eq. (77) the integrals over α in the analogous of Eq. (26) can be performed, with the result:

$$\sigma_k = \frac{w^k}{k!} \Gamma\left(k + \frac{1}{w}\right) \left[\Gamma\left(\frac{1}{w}\right) \right]^{-1} \int d^2b \langle n(b, s) \rangle^k \times \{1 + w \langle n(b, s) \rangle\}^{-k+(1/w)}. \quad (86)$$

This distribution of the number of elementary interactions should be inserted in Monte Carlo implementations to have predictions for the charged particles multiplicity and other observable quantities.

VI. ENERGY DEPENDENCE

The model we have outlined in the previous sections considers the relation between directly observable quantities such as the total and elastic cross sections and on the other hand the eikonal cross section σ_{eik} and the distribution of hadronic matter in the colliding particles (simply parametrized by the quantity r_0), with an additional parameter w that is related to the width of the fluctuations in the parton configurations of the colliding hadrons. Taking into account these fluctuations allows a consistent treatment of inelastic diffraction. The values of σ_{eik} and r_0 that correctly describe the data have a strong dependence on the parameter w , and therefore on the measured values of the cross section for inelastic diffraction.

A calculation of the evolution with energy of the hadronic cross section requires additional theoretical assumptions to predict the energy dependence of the model parameters. To gain insight to this problem, we have taken a phenomenological approach and we have considered a representative subset of the available high energy data. A few high energy experiments have measured both the total and elastic cross sections, together with the forward slope B_{el} . These results can be described in terms of our 3-parameter $\{\sigma_{\text{eik}}, r_0, w\}$ model, using the same approach discussed for the CDF data at $\sqrt{s} = 1800$ GeV. The results are shown in Figs. 5 and 6. In these figures the points give the values of r_0 and σ_{eik} that reproduce the measurements of the pairs (σ_{tot}, B) using two assumptions for the third parameter: $w = 0$ and $w = 3$. The errors on the estimates reflect only the experimental statistical errors. At $\sqrt{s} = 1800$ GeV, one has two independent measurements of the total cross section by the CDF [27] and E710 [39] experiments, both at the Fermilab $p\bar{p}$ collider. The point at $\sqrt{s} = 546$ GeV is also from CDF, while the point at $\sqrt{s} = 62.3$ was obtained for pp scattering at the CERN ISR collider [24].

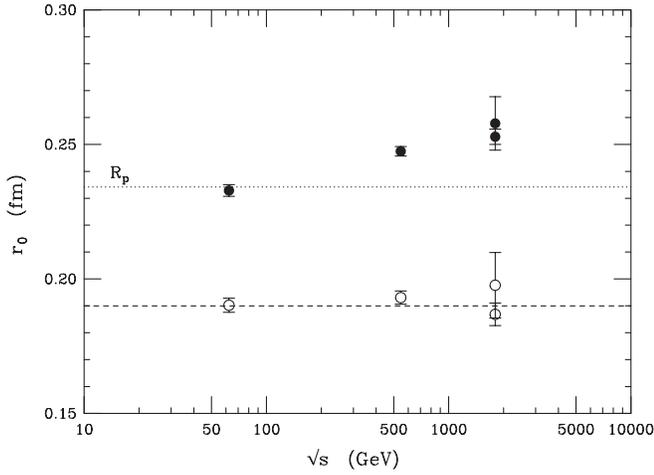


FIG. 5. Values of the r_0 parameter that reproduce the experimental data for σ_{tot} and B_{el} obtained at the ISR pp collider ($\sqrt{s} = 62.3$ GeV), and at the Tevatron $p\bar{p}$ collider ($\sqrt{s} = 546$ GeV by CDF, and $\sqrt{s} = 1800$ GeV by CDF and E710). The solid (empty) points are calculated for $w = 0$ ($w = 3$). The dotted line at $r_0 = 0.234$ fm corresponds to the proton charge radius R_p . The dashed line corresponds to the constant value $r_0 = 0.19$ fm and is a reasonable representation of the results for $w = 3$. The corresponding values of σ_{eik} are shown in Fig. 6.

The calculation with $w = 0$ corresponds to the simple eikonal model, and in the framework to our model leads to a vanishing inelastic diffraction cross section. For each pair of experimental results (σ_{tot} , B) we have performed a scan

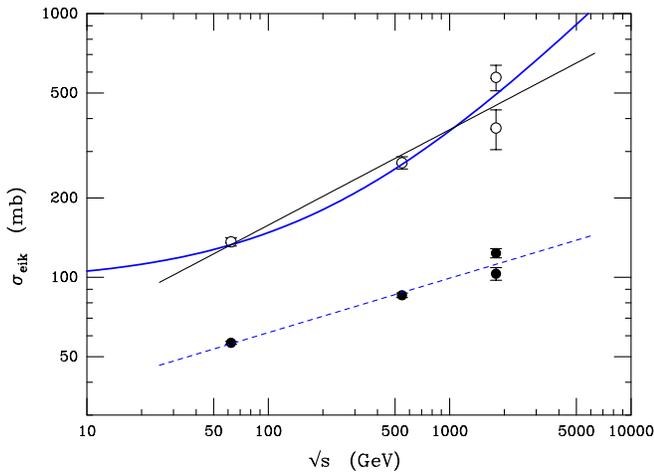


FIG. 6 (color online). Values of the σ_{eik} parameter that reproduce the experimental data for σ_{tot} and B_{el} obtained at the ISR pp collider ($\sqrt{s} = 62.3$ GeV), and at the Tevatron $p\bar{p}$ collider ($\sqrt{s} = 546$ GeV by CDF, and $\sqrt{s} = 1800$ GeV by CDF and E710). The solid (empty) points are calculated for $w = 0$ ($w = 3$). The corresponding values of r_0 are shown in Fig. 5. The dashed line is a power law fit [$\sigma_{\text{eik}}(s) = Ks^\alpha$] to the results for $w = 0$. The thin (black) line is a fit to the results for $w = 3$ with the same power law form. The thick (blue) line is a fit to the same points with the form $\sigma_{\text{eik}}(s) = \sigma_0 + Ks^\alpha$ with $\alpha = 0.35$.

of w similar to the one that we have shown in detail for the CDF point at $\sqrt{s} = 1800$ GeV (see also Fig. 3), calculating the pair $[r_0(w), \sigma_{\text{eik}}(w)]$ that reproduces the experimental results. This also implies a value $\sigma_{\text{diff}}(w)$ obtained from Eq. (84).

At $\sqrt{s} = 1800$ GeV both the CDF [38] and E710 [39] have measured the single diffractive cross sections. Averaging with equal weight (to take into account large systematic uncertainties) and using the ansatz (76) to estimate double diffraction we have estimated $\sigma_{\text{diff}} \approx 10.7$ mb. This value of the diffractive cross section is reproduced in our model (at the corresponding energy $\sqrt{s} = 1800$ GeV) with $w \approx 3$. Including a 20% uncertainty on the estimate of σ_{diff} , w can be estimated at this energy as $w = 3_{-0.9}^{+1.2}$.

The comparison of the calculated diffractive cross section with the data is problematic because the discrepancies between the different experimental results (see Fig. 7) clearly indicate the presence of significant systematic errors; moreover, as we have discussed before, one has the theoretical uncertainty related to the ratio between the single and double diffraction contributions.

Our numerical studies indicate that the choice of an energy independent value $w \approx 3$, that reproduces the diffractive cross section measured at the Fermilab collider at $\sqrt{s} = 1800$ GeV, gives in fact a reasonably good description of the experimental results on diffraction at all energies. The assumption of an energy independent value of w is clearly the simplest one, and in view of the fact that it produces a reasonable agreement with the available data it will be made in the following.

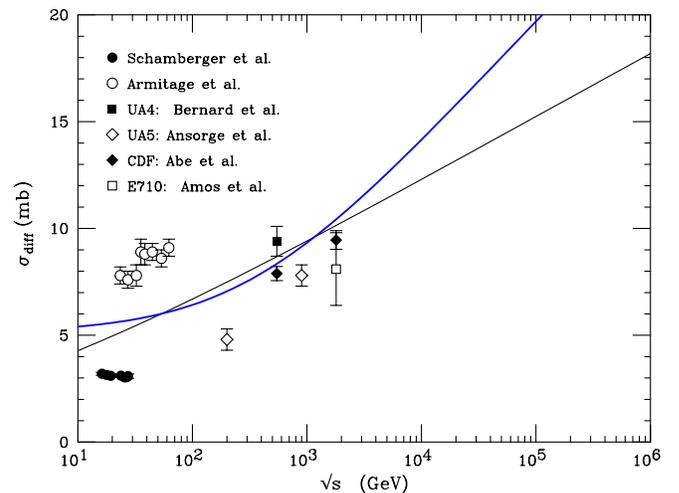


FIG. 7 (color online). Inelastic diffractive cross section calculated according to Eq. (84) using constant values $w = 3$ and $r_0 = 0.19$ fm. For the thin (black) [thick (blue)] curve we have used for $\sigma_{\text{eik}}(s)$ the fit shown with the corresponding lines in Fig. 6. The experimental results are for single diffraction only (Schamberger *et al.* [46], Armitage *et al.* [47], UA4 [48], UA5 [49], CDF [38], and E710 [39]).

It is interesting to note (see Fig. 5) that assuming for w the constant value $w = 3$ the resulting values of the parameter r_0 are also consistent with an energy independent value $r_0 \simeq 0.19$ fm. In the simple eikonal model, as discussed before, in order to reconcile the growth of σ_{tot} and B_{el} , it is necessary to increase the width of the overlap function $A(b, s)$ with s , and therefore (using our parametrization) to increase $r_0(s)$. The necessity of this growth is evident in Fig. 5.

The model we are discussing requires an overlap function that is first of all significantly narrower than previous estimates based on the simple eikonal model; moreover (and in contrast to the simple eikonal model) the overlap function is energy independent. It may appear surprising that the overlap function is narrower than what is estimated on the basis of the proton charge distribution. A possible explanation is that the dominant contribution to the overlap function is the scattering between soft gluons. The narrow $A(b)$ predicted in this model therefore implies that the impact parameter distribution of soft gluons is (i) narrower than the charge distribution (that is presumably controlled by valence quarks), and (ii) independent (for small x) from the x of the gluons. It should soon be possible to test these hypotheses with studies of the impact parameter PDF's.

Figure 6 shows the energy dependence of the third parameter of our model, $\sigma_{\text{eik}}(s)$. There are two remarkable features in this behavior. The first is that (as we have already discussed in Sec. VA) the values of σ_{eik} needed to describe the experimental data in a model that includes diffraction are significantly larger than estimates based on the simple eikonal model, and the second is that the growth of $\sigma_{\text{eik}}(s)$ with c.m. energy is significantly more rapid.

The two straight lines in Fig. 6 are power law fits of the estimated values of form Ks^α . For the simple eikonal model ($w = 0$) the power law fit is $\sigma_{\text{eik}}(s) = 23.8s^{0.10}$ mb (with s measured in GeV^2). For the best fit model ($w = 3$) the power law fit is $\sigma_{\text{eik}}(s) = 29.7s^{0.18}$ mb.

The choice of a power law fit is however clearly not necessary, and the extrapolation of the fit at higher (and lower) energy is therefore very uncertain. Motivated by fits of the $\sigma_{\text{eik}}(s) = \sigma_{\text{soft}} + \sigma_{\text{hard}}(s)$ that include an (approximately) constant soft component and an energy varying hard component, we have fitted the data with the form $\sigma_{\text{eik}}(s) = \sigma_0 + Ks^{0.35}$, obtaining the result $\sigma_{\text{eik}}(s) = 95 + 2.1s^{0.35}$ mb (and s measured in GeV^2).

The motivation for the functional form of this fit (that should however also be considered as purely phenomenological) is that integrating above an energy independent p_\perp^{min} , the jet cross section $\sigma_{\text{jet}}(p_\perp^{\text{min}}, \sqrt{s})$ [see Eq. (3)] has qualitatively the behavior:

$$\sigma_{\text{jet}}(p_\perp^{\text{min}}, \sqrt{s}) \propto \frac{\alpha_s^2}{(p_\perp^{\text{min}})^2} \frac{\tau^{-\epsilon}(-\log \tau)}{\epsilon} \propto \frac{\alpha_s^2}{(p_\perp^{\text{min}})^2} s^\epsilon \log s, \quad (87)$$

where $\tau = 4(p_\perp^{\text{min}})^2/s$, and the quantity ϵ is related to the behavior of the PDF's for $x \rightarrow 0$:

$$\lim_{x \rightarrow 0} f(x) \sim \frac{1}{x^{1+\epsilon}}. \quad (88)$$

The behavior (87) can easily be obtained from the convolution of PDF's with the asymptotic form (88). Recent measurements of the PDF's at HERA [40–43] have shown that their behavior for $x \rightarrow 0$ can be reasonably well represented with the functional form (88) and $\epsilon \simeq 0.3$.

It is interesting to note that it has been argued that the very fast growth of the jet cross section with \sqrt{s} implied by the small x behavior of the PDF's is problematic, and in fact unphysical. Note that in the simple eikonal model, an energy dependence of $\sigma_{\text{eik}}(s)$ of type $s^{0.3}$ (or faster) is not acceptable because it implies that $\sigma_{\text{tot}}(s)$ grows with energy more rapidly than the observations. Following this observation, the rapid growth of $\sigma_{\text{jet}}(s)$ with s has been tamed assuming that the threshold p_\perp^{min} of applicability of perturbation theory also grows with s . This growth has been connected to phenomena of ‘‘saturation,’’ or screening among the partons. In the framework of the model we are considering a fast growth of $\sigma_{\text{eik}}(s)$ is not only acceptable but in fact necessary. A simple model where the growth of $\sigma_{\text{eik}}(s)$ is explained with the dominant contribution of a minijet cross section, calculated perturbatively above an energy independent p_\perp^{min} , can provide a $\sigma_{\text{eik}}(s)$ with the needed properties. For consistency, it is however necessary that the effects of saturation and parton screening are small.

The model we are describing, already at $\sqrt{s} = 1800$ GeV has a ratio $\sigma_{\text{eik}}/\sigma_{\text{inel}} \simeq 10$. This implies that the number of elementary interactions in an inelastic collision at this energy is also approximately 10. This, at first sight, may appear too large. The potential danger is that this large average number of elementary interaction per collision could result in a too large average multiplicity and in a too soft inclusive spectrum of particles in the final state. These questions can (and should) be addressed properly with a detailed Monte Carlo calculation that includes a modeling of particle production in the presence of different numbers of elementary interactions.

Figure 7 shows our calculation of the diffractive cross section $\sigma_{\text{diff}}(s)$ including the extrapolation to high energy. The calculation is performed with Eq. (84), using energy independent values $w = 3$, $r_0 = 0.19$ fm and the two parametrization of $\sigma_{\text{eik}}(s)$ shown in Fig. 6.

Figure 8 shows the result of our model for the total cross section, comparing with the available data and extrapolating at higher energy. In the figures we show two calculations based on Eq. (82) using (as in the previous figure) the constant values $w = 3$ and $r_0 = 0.19$ fm, and the two parametrizations of $\sigma_{\text{eik}}(s)$ (the results for the model are only plotted for $\sqrt{s} > 60$ GeV).

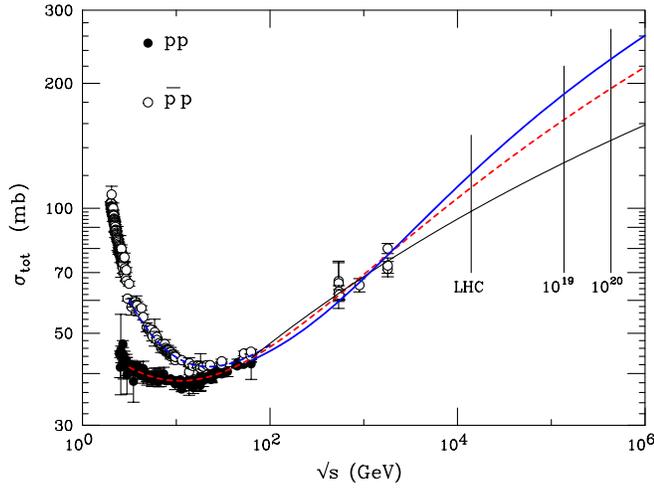


FIG. 8 (color online). The points are measurements of the pp and $p\bar{p}$ total cross sections. The dashed line is the fit of $\sigma_{\text{tot}}(s)$ suggested by the PDG [45]. The other two lines are predictions obtained from Eq. (82) using constant values $w = 3$ and $r_0 = 0.19$ fm. For the thin (black) [thick (blue)] curve we have used for $\sigma_{\text{eik}}(s)$ the fit shown with the corresponding lines in Fig. 6.

Figure 8 also shows the parametrizations for $\sigma_{\text{tot}}^{pp}(s)$ and $\sigma_{\text{tot}}^{p\bar{p}}(s)$ suggested by the PDG [44,45]. The PDG estimate of the extrapolation of the total pp cross sections falls in between our two estimates that mark a range of uncertainty in our prediction.

At the LHC energy ($\sqrt{s} = 14$ TeV) the PDG prediction is $\sigma_{\text{tot}} = 112.2$ mb while our two calculations give $\sigma_{\text{tot}} = 98.1$ and 120.8 mb (at $\sqrt{s} = 10$ TeV the PDG prediction is $\sigma_{\text{tot}} = 105.7$ mb, our calculations give 94.2 and 112.7 mb). The extrapolation to $\sqrt{s} = 4.33 \times 10^5$ GeV (that corresponds to a proton cosmic ray particle with energy $E_{\text{lab}} = 10^{20}$ eV) is $\sigma_{\text{tot}} = 194$ mb for the PDG extrapolations, and 146 and 229 mb for our calculations.

Recently two groups [8,9] discussed predictions of the total pp cross section at high energy in the framework of models that include a treatment of diffraction with a multi-channel eikonal and multipomeron interactions. Both groups arrive at a similar conclusion, namely, that the inclusion of diffraction reduces the estimate of the total cross section at LHC energy. For the two groups the estimate of σ_{tot} at $\sqrt{s} = 14$ TeV is of order 90 mb, approximately 20% smaller than the PDG. Their estimates of the total cross section grow very slowly with energy reaching $\sigma_{\text{tot}} \simeq 108$ mb (for [9]) and $\sigma_{\text{tot}} \simeq 98$ mb (for [8]) at $\sqrt{s} = 10^5$ GeV (that corresponds to $E_{\text{lab}} = 5.33 \times 10^{18}$ eV).

The main point that we want to make here is that it is certainly the case that given a model for $\langle n(b, s) \rangle$ [that in the simple eikonal model is simply equal to twice the eikonal function $\chi(b, s)$], the inclusion of diffraction reduces the cross section. However, the estimate of the total cross section and of its dependence on energy also involves the calculation of the function $\langle n(b, s) \rangle$.

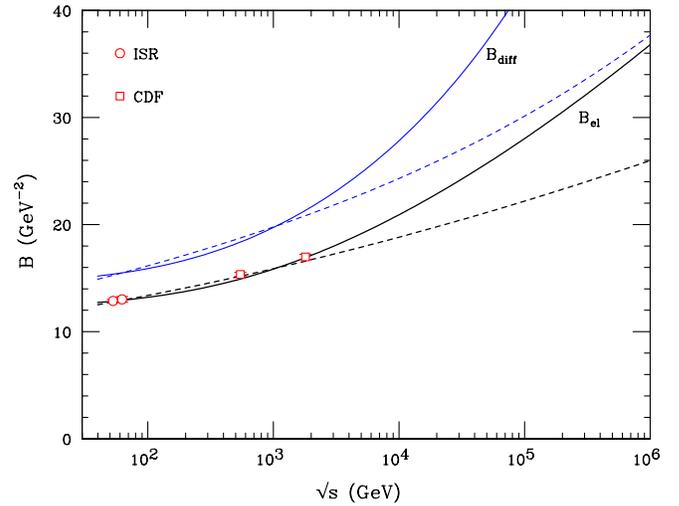


FIG. 9 (color online). Slope at $t = 0$ of the differential cross sections $d\sigma_{\text{el}}/dt$ and $d\sigma_{\text{diff}}/dt$ for elastic and inelastic diffractive events. The predictions are calculated with Eqs. (65) and (66), using the functional form (77) for $p(\alpha)$ with $w = 3$, and the parametrization (45) for $\langle n(b, s) \rangle$ with $r_0 = 0.19$ fm. The solid and dashed curves use the two parametrizations of $\sigma_{\text{eik}}(s)$ shown in Fig. 6.

The conclusion that the cross section at LHC is of order 90 mb obtained by the authors in [8,9] should not be considered as a consequence of the inclusion of diffraction in the theoretical framework, but rather as the consequence of the entire set of theoretical assumptions of their models.

Figure 9 shows, plotted as a function of \sqrt{s} , the predictions of our model for the slopes $B_{\text{el}}(s)$ and $B_{\text{diff}}(s)$ of the differential cross sections $d\sigma_{\text{el}}/dt$ and $d\sigma_{\text{diff}}/dt$. For each slope, the figure shows two curves that differ for the use of the two different parametrizations of $\sigma_{\text{eik}}(s)$ that are shown in Fig. 6, and have already been used in Figs. 7 and 8. Note how $B_{\text{diff}}(s)$ is always larger than $B_{\text{el}}(s)$.

VII. SUMMARY AND OUTLOOK

In this work we have discussed the problem of multiple parton interactions in hadron collisions. If one takes into consideration only the parton scatterings that have sufficiently large momentum transfer, it becomes possible to detect the final state partons as high p_{\perp} jets and determine event by event the number of hard parton interactions that are present. It becomes therefore possible to study experimentally the multiplicity distribution of parton interactions above for example p_{\perp}^{min} . At sufficiently high c.m. energy the probability of having more than one high p_{\perp} parton scattering in a single collision can be appreciable, and the hard scattering multiplicity distribution becomes nontrivial.

The calculation for the inclusive distribution of high p_{\perp} parton scattering is a textbook application of perturbative QCD and can be performed from knowledge of the standard PDF's. A theoretical prediction of the multiplicity

distribution of high p_{\perp} interactions in high energy hadron collisions is however a highly nontrivial problem that requires the introduction of additional theoretical concepts.

The standard PDF's give the inclusive probability density for finding one single parton with fractional longitudinal momentum x . This probability density is obtained integrating over the parton transverse momentum, and integrating over all possible momenta of the other partons in the hadron. If one wants to compute the probability to have exactly n hard interactions in one collision the information contained in the PDF's is clearly insufficient. One needs to know:

- (1) the probability to find a parton of a given x at different impact parameters with respect to the hadron center of mass;
- (2) the *correlated* probabilities for finding different partons at $(x_1, \vec{b}_1), (x_2, \vec{b}_2), (x_3, \vec{b}_3), \dots$

The first problem should be addressed by introducing impact parameter dependent PDF's $F_j^h(x, b, Q^2)$ that give the probability of finding the parton of type j with fractional longitudinal momentum x and impact parameter \vec{b} probing hadron h at the scale Q^2 . This problem has not yet a well-determined solution, and all studies of this problem have made the simplification to assume that the dependences on x and b of the impact parameter PDF's factorize, that is, $F_j^h(x, b, Q^2) = f_j(x, Q^2)\hat{\rho}(b)$, and estimated $\hat{\rho}(b)$ with simple phenomenological considerations. Studies of the generalized PDF's [13–15] should soon be able to shed light on this question.

The problem of obtaining correlated PDF's that give the probability to find simultaneously several partons in different elements of phase space is clearly much more difficult and complex. In this work we suggested to parametrize the effects of our lack of knowledge about the correlated PDF's introducing the effective configuration probability distribution that is one function $p(\alpha)$ of the real, positive variable α . Each one of the configurations of partons in the pair of colliding hadrons has associated the real number α . The physical meaning of α is that the expected number of parton interactions that corresponds to the parton configuration \mathbb{C} is $n(\mathbb{C}) = \langle n \rangle \alpha$, where $\langle n \rangle$ is the average over all configurations. The first two moments of the function $p(\alpha)$ are unity (because of the normalization of a probability density and to reproduce the correct $\langle n \rangle$), and by increasing the 2nd moment of the $p(\alpha)$ distribution the width of the multiplicity of parton interactions grows.

If one considers not only a subset of detectable (high p_{\perp}) parton interactions, but *all* of them, it becomes possible to relate these elementary interactions with the total and elastic cross sections. This general idea has been implemented in many works using an eikonal formalism. A crucial ingredient of these models is the quantity $\langle n(b, s) \rangle$ that gives the average number of elementary interactions for a hadron collision at impact parameter b

and c.m. energy \sqrt{s} . In the simple eikonal model this quantity is related to the elastic scattering profile function by the relation $\Gamma_{\text{el}}(b, s) = 1 - \exp[-\langle n(b, s) \rangle / 2]$. The corresponding inelastic cross section is then

$$\sigma_{\text{inel}}(s) = \int d^2b \{1 - \exp[-\langle n(b, s) \rangle]\}.$$

The physical interpretation is that an inelastic interaction corresponds to absorption and to at least one elementary interaction, assuming Poisson fluctuations in their multiplicity. The same considerations that we have outlined for hard interactions however apply, and it is natural to expect that fluctuations in the number of elementary interactions n are in fact much broader than Poissonian because of fluctuations in the configurations of the colliding hadrons. This effect can again be parametrized with a function $p(\alpha)$. For example the inelastic cross section can be rewritten as

$$\sigma_{\text{inel}}(s) = \int d^2b \int_0^{\infty} d\alpha p(\alpha) \{1 - \exp[-\langle n(b, s) \rangle \alpha]\}.$$

One can see that the parameter α controls the “transparency” of a hadron collision. Different configurations of the colliding hadrons have transparencies that are related to α . Good and Walker [4] proposed that inelastic diffraction originates from the different absorption of the different components of the colliding hadrons. Therefore our formalism can be applied to the calculation of the inelastic diffractive cross section, and in fact unavoidably implies the presence of inelastic diffractive processes.

In other words, the function $p(\alpha)$ allows one to relate the quantity $\langle n(b, s) \rangle$ to the total and elastic cross sections, and at the same time fixes the value of the diffractive cross section. Vice versa, from the data on the total and elastic cross sections, together with the data on inelastic diffraction, it is possible to extract information on $\langle n(b, s) \rangle$ and on the properties of $p(\alpha)$.

We have performed an analysis of the data on pp and $\bar{p}p$ collisions obtained at high energy colliders, and obtained information on $\langle n(b, s) \rangle$ and $p(\alpha)$. For the study of the properties of $p(\alpha)$ we have used as a first approximation a simple analytic form that depends on a single parameter.

To describe the measured diffractive cross section one is forced to have a function $p(\alpha)$ with a large variance. This in turn has very important consequences on the parameters that describe $\langle n(b, s) \rangle$. It is remarkable that we find that (within significant uncertainties) the function $p(\alpha)$ is independent from energy; moreover parametrizing $\langle n(b, s) \rangle$ in the form $\langle n(b, s) \rangle = \sigma_{\text{eik}}(s)A(b, s)$ as the product of an eikonal cross section times a geometrical overlap function, we find that the geometrical factor can be taken as energy independent, in contrast to results obtained in the simple eikonal model that neglects fluctuations. The eikonal cross section $\sigma_{\text{eik}}(s)$ is much larger and grows much faster with energy than in the simple eikonal model. Such a rapid growth can however be readily explainable assuming that

it is controlled by the increase of $\sigma_{\text{jet}}(p_{\perp}^{\text{min}}, s)$ with s assuming a constant p_{\perp}^{min} and negligible screening effects.

We note that, at least in first approximation, the function $p(\alpha)$ that we have extracted from the study of the total, elastic, and diffraction cross sections is also applicable to the study of the multiplicity distribution of high p_{\perp} jets. One can therefore make the prediction that the distribution of the number of hard interactions per event will be broad, with a non-negligible number of events containing several interactions.

The prediction of the total cross section at LHC depends on the energy dependence of $\sigma_{\text{eik}}(s)$. This is a problem we have not discussed in detail here. It seems however natural to expect a result around 110 mb with a significant uncertainty. In the model we are discussing the eikonal cross section $\sigma_{\text{eik}}(s)$ is large and since $\sigma_{\text{eik}}(s)/\sigma_{\text{inel}}(s)$ is equal to the number of elementary interactions in a collision, one is led to expect a large charged particle multiplicity and a soft inclusive spectrum in the final state. These considerations are also relevant for the study of ultrahigh energy cosmic ray showers.

ACKNOWLEDGMENTS

We are grateful to Gianni Salmé for discussions about the GPDF's. P.L. would like to thank Ralph Engel for the introduction to the model of Appendix B, and Tom Gaisser, Todor Stanev, and Eun-Joo Ahn for many discussions about hadron interactions.

APPENDIX A: ELECTROMAGNETIC FORM FACTORS AND OVERLAP FUNCTION $A(b)$

In [1] Durand and Pi estimated the overlap function $A(b)$ for pp collisions from the electromagnetic form factor of the proton. The simple physical idea behind their derivation is that the overlap function is the energy independent geometric overlap of the hadronic matter distributions in the colliding particles. More explicitly, one defines the spatial distribution of matter in the hadron h as $\rho_h(r)$, with the normalization condition:

$$\int d^3r \rho_h(r) = 1. \quad (\text{A1})$$

The density in the transverse plane is then obtained with a simple integration:

$$\hat{\rho}_h(b) = \int_{-\infty}^{+\infty} dz \rho_h(\sqrt{b^2 + z^2}). \quad (\text{A2})$$

The overlap function in the collision between hadrons h_1 and h_2 is then obtained as

$$A(b) = \int d^2b_1 d^2b_2 \hat{\rho}_{h_1}(\vec{b}_1) \hat{\rho}_{h_2}(\vec{b}_2) \delta[\vec{b} - \vec{b}_1 + \vec{b}_2]. \quad (\text{A3})$$

The normalization condition

$$\int d^2b A(b) = 1 \quad (\text{A4})$$

follows automatically from the normalization of $\rho_{h_1}(r)$ and $\rho_{h_2}(r)$ given by (A1).

To estimate the density of $\rho_h(r)$ of hadron h , Durand and Pi make the assumption that it is simply the Fourier transform of its electromagnetic form factor. For the proton one has

$$F_p(q^2) = \frac{1}{(1 + R_p^2 q^2)^2}, \quad (\text{A5})$$

with $R_p = 0.234$ fm (or $R_p^{-2} = 0.71$ GeV²) and correspondingly $\rho_p(r) \propto e^{-r/R_p}$. The geometric convolution for proton-proton collisions is

$$A_{pp}(b) = \frac{b^3}{96\pi R_p^5} K_3\left(\frac{b}{R_p}\right), \quad (\text{A6})$$

where $K_3(x)$ is the modified Bessel function of the third kind.

APPENDIX B: FOUR CHANNELS MODEL

It is instructive to discuss a ‘‘minimum’’ model that implements the Good-Walker ansatz [4] for inelastic diffraction in the collision between two hadrons, where all calculations can be performed explicitly. The minimum model has 4 channels, to describe the 4 possible types of scattering (target, projectile, and double diffraction together with elastic scattering). Without loss of generality we can consider the scattering πp (here π and p are labels to represent arbitrary hadrons). Each of the two colliding hadrons can undergo inelastic diffraction with a transition to an additional state. We will label the excited states for the projectile and target particles as ρ and Δ ; one therefore has to consider the transitions $\pi \rightarrow \pi^* \equiv \rho$ and $p \rightarrow p^* \equiv \Delta$. In the 4-channel model one has to study the 4-dimensional vector space spanned by the orthonormal basis of the 4 physical states $|\varphi_m\rangle$:

$$\{|\varphi_m\rangle\}_{(m=1,4)} = \{|\pi p\rangle, |\pi \Delta\rangle, |\rho p\rangle, \rho \Delta\rangle\}. \quad (\text{B1})$$

One can (in principle) study the 4×4 transitions $\langle \varphi_f | \mathbf{S} | \varphi_i \rangle$. In practice of course one is limited to the study of the transitions $|\pi p\rangle \rightarrow |\varphi_m\rangle$ (that correspond to the processes of elastic scattering, target, projectile, and double diffraction). The scattering amplitude is a 4×4 matrix:

$$\hat{F}(\vec{q}, s) = i \int \frac{d^2b}{2\pi} e^{i\vec{q}\cdot\vec{b}} \hat{\Gamma}(b, s). \quad (\text{B2})$$

The differential cross section for the transition $i \rightarrow f$ is

$$\frac{d\sigma_{i \rightarrow f}}{dt}(t, s) = \pi |[\hat{F}(\vec{q}, s)]_{fi}|^2. \quad (\text{B3})$$

Integrating over all t values one obtains the transition cross

sections

$$\sigma_{i \rightarrow f}(s) = \int d^2b |[\hat{\Gamma}(b, s)]_{fi}|^2. \quad (\text{B4})$$

The profile matrix $\hat{\Gamma}$ can be written in terms of the eikonal matrix $\hat{\chi}(b, s)$:

$$\hat{\Gamma}(b, s) = 1 - \exp[-\hat{\chi}(b, s)]. \quad (\text{B5})$$

Using the Good and Walker ansatz, the eikonal matrix $\hat{\chi}(b, s)$ takes the form:

$$\hat{\chi}(b, s) = \frac{\langle n(b, s) \rangle}{2} \hat{M}, \quad (\text{B6})$$

where $\langle n(b, s) \rangle$ has the usual meaning of the average number of parton interactions in a πp collision, and we have introduced the 4×4 matrix \hat{M} . This matrix must be real and have 4 real and positive eigenvalues α_j ; moreover one must have $\hat{M}_{11} = 1$.

This is a consequence of the fact that one can define (as in the previous section) the states $|\psi_j\rangle$ as the eigenstates of

$$\hat{M} = \begin{pmatrix} 1 & \beta_\pi \\ \beta_\pi & 1 - 2\epsilon_\pi \end{pmatrix} \otimes \begin{pmatrix} 1 & \beta_p \\ \beta_p & 1 - 2\epsilon_p \end{pmatrix} = \begin{pmatrix} 1 & \beta_\pi & \beta_p & \beta_\pi \beta_p \\ \beta_\pi & 1 - 2\epsilon_\pi & \beta_\pi \beta_p & \beta_p(1 - 2\epsilon_\pi) \\ \beta_p & \beta_\pi \beta_p & 1 - 2\epsilon_p & \beta_\pi(1 - 2\epsilon_p) \\ \beta_\pi \beta_p & \beta_p(1 - 2\epsilon_\pi) & \beta_\pi(1 - 2\epsilon_p) & (1 - 2\epsilon_\pi)(1 - 2\epsilon_p) \end{pmatrix}. \quad (\text{B8})$$

The eigenvalues and eigenvectors of the matrix \hat{M} are easily calculable, noting that each of the 2×2 matrices of form

$$\begin{pmatrix} 1 & \beta \\ \beta & 1 - 2\epsilon \end{pmatrix}$$

has eigenvalues:

$$\lambda_{1,2} = 1 \pm \gamma - \epsilon, \quad (\text{B9})$$

where

$$\gamma = \sqrt{\beta^2 + \epsilon^2}, \quad (\text{B10})$$

and the corresponding eigenvectors are

$$C_{mj} = \frac{1}{2} \begin{pmatrix} \sqrt{1+r_\pi}\sqrt{1+r_p} & \sqrt{1-r_\pi}\sqrt{1+r_p} & \sqrt{1+r_\pi}\sqrt{1-r_p} & \sqrt{1-r_\pi}\sqrt{1-r_p} \\ -\sqrt{1-r_\pi}\sqrt{1+r_p} & \sqrt{1+r_\pi}\sqrt{1+r_p} & -\sqrt{1-r_\pi}\sqrt{1-r_p} & \sqrt{1+r_\pi}\sqrt{1-r_p} \\ -\sqrt{1+r_\pi}\sqrt{1-r_p} & -\sqrt{1-r_\pi}\sqrt{1-r_p} & \sqrt{1+r_\pi}\sqrt{1+r_p} & \sqrt{1-r_\pi}\sqrt{1+r_p} \\ \sqrt{1-r_\pi}\sqrt{1-r_p} & -\sqrt{1+r_\pi}\sqrt{1-r_p} & -\sqrt{1-r_\pi}\sqrt{1+r_p} & \sqrt{1+r_\pi}\sqrt{1+r_p} \end{pmatrix}. \quad (\text{B14})$$

The profile functions for the different scattering processes can now be given explicitly as

$$\Gamma_{m_f m_i} = \sum_j C_{m_f j} C_{m_i j} \left[1 - \exp\left(-\frac{\langle n(b, s) \rangle}{2} \alpha_j\right) \right]. \quad (\text{B15})$$

The model outlined above requires an estimate of the

the \hat{M} matrix. These states undergo only absorption or elastic scattering, and each has a transparency $P_0 = e^{-\langle n(b, s) \rangle \alpha_j}$. Therefore $\langle n(b, s) \rangle \alpha_j$ can be interpreted as the average number of interactions for the state $|\psi_j\rangle$ and this quantity (and α_j) must be positive. Moreover, to have the correct average multiplicity of elementary interactions for the initial state $|\phi_1\rangle = |\pi p\rangle$ one must have

$$\sum_j |\langle \psi_j | \phi_1 \rangle|^2 \alpha_j = 1 \quad (\text{B7})$$

that implies $\hat{M}_{1,1} = 1$.

The matrix \hat{M} can be constructed explicitly making the additional hypothesis that the (4-dimensional) space of the physical states is the direct product of two (2-dimensional) spaces for the beam and target particle, and moreover that one has time reversal symmetry, and the amplitudes for the transitions $\pi \rightarrow \rho$ ($p \rightarrow \Delta$) and $\rho \rightarrow \pi$ ($\Delta \rightarrow p$) are equal. With these assumptions the most general form for the matrix \hat{M} is

$$\vec{v}_{1,2} = \frac{1}{\sqrt{2}} \{\pm \sqrt{1 \pm r}, \sqrt{1 \mp r}\}, \quad (\text{B11})$$

with $r = \epsilon/\gamma = \epsilon/\sqrt{\beta^2 + \epsilon^2}$.

The eigenvalues of the 4×4 matrix \hat{M} are then

$$\alpha_j = (1 \pm \gamma_\pi - \epsilon_\pi)(1 \pm \gamma_p - \epsilon_p) \quad (\text{B12})$$

(with $j \in \{1, 2, 3, 4\}$). The condition that the eigenvalues are non-negative gives

$$\epsilon_{\pi,p} \leq 1/2, \quad \beta_{\pi,p}^2 \leq 1 - 2\epsilon_{\pi,p}. \quad (\text{B13})$$

The rotation matrix C_{mj} that connects the scattering eigenstates $|\psi_j\rangle$ to the physical states $|\phi_m\rangle$ is

function $\langle n(b, s) \rangle$ that can be interpreted as the average number of ‘‘elementary’’ interactions for a hadron crossing at impact parameter b and c.m. energy \sqrt{s} . In the general case of the collision of two different hadrons (such as in $\pi^\pm p$ scattering) the model has 4 additional parameters ($\beta_\pi, \epsilon_\pi, \beta_p, \epsilon_p$) that describe the matrix structure of the

eikonal function. Obviously for pp scattering the model has only two parameters (β_p, ϵ_p).

With the labeling of the physical states that we have been using (namely, $|\varphi_1\rangle = |\pi p\rangle$, $|\varphi_2\rangle = |\pi\Delta\rangle$, $|\varphi_3\rangle = |\rho p\rangle$, and $|\varphi_4\rangle = |\rho\Delta\rangle$), the integration over all impact parameters b of $|\Gamma_{11}(b)|^2$ yields the elastic cross section, the integral of $|\Gamma_{21}(b)|^2$ ($|\Gamma_{31}(b)|^2$) gives the target (projectile) single diffraction cross section, and finally the integral of $|\Gamma_{41}(b)|^2$ gives the double diffractive cross section.

To connect this analysis to the discussion performed in the main text, we note that we can define the 4 quantities p_j that are the probabilities $|\langle\varphi_1|\psi_j\rangle|^2$ to find the initial state $|\varphi_1\rangle \equiv |\pi p\rangle$ in the scattering eigenstates $|\psi_j\rangle$. The p_j are given by

$$\begin{aligned} p_j &= |\langle\psi_j|\pi p\rangle|^2 = \frac{(\gamma_\pi \pm \epsilon_\pi)(\gamma_p \pm \epsilon_p)}{4\gamma_\pi\gamma_p} \\ &= \frac{1}{4}(1 \pm r_\pi)(1 \pm r_p). \end{aligned} \quad (\text{B16})$$

It is straightforward to verify that

$$\sum_j p_j = \sum_j p_j \alpha_j = 1. \quad (\text{B17})$$

One can now define the function $p(\alpha)$:

$$p(\alpha) = \sum_j p_j \delta[\alpha - \alpha_j]. \quad (\text{B18})$$

This function, as a consequence of Eq. (B17) satisfies the conditions:

$$\int_0^\infty d\alpha p(\alpha) = 1, \quad \int_0^\infty d\alpha \alpha p(\alpha) = 1.$$

It is now straightforward to see that one can recast the expressions for the total, elastic, absorption, and diffractive (that is the sum of the target, projectile, and double diffraction) cross sections as integrals over α identical to the expressions (59)–(62) in Sec. IV B.

It can be interesting to note that the 2nd moment of the $p(\alpha)$ distribution is given by

$$\int d\alpha \alpha^2 p(\alpha) = \sum_j p_j \alpha_j^2 = (1 + \beta_p)(1 + \beta_\pi). \quad (\text{B19})$$

-
- [1] L. Durand and H. Pi, Phys. Rev. D **38**, 78 (1988).
 [2] T. K. Gaisser and F. Halzen, Phys. Rev. Lett. **54**, 1754 (1985).
 [3] G. Pancheri and Y. Srivastava, Phys. Lett. **159B**, 69 (1985).
 [4] M. L. Good and W. D. Walker, Phys. Rev. **120**, 1857 (1960).
 [5] A. B. Kaidalov, Phys. Rep. **50**, 157 (1979).
 [6] R. S. Fletcher, Phys. Rev. D **46**, 187 (1992).
 [7] S. Roesler, R. Engel, and J. Ranft, Z. Phys. C **59**, 481 (1993).
 [8] M. G. Ryskin, A. D. Martin, and V. A. Khoze, Eur. Phys. J. C **54**, 199 (2008).
 [9] E. Gotsman, E. Levin, U. Maor, and J. S. Miller, Eur. Phys. J. C **57**, 689 (2008).
 [10] H. I. Miettinen and J. Pumplin, Phys. Rev. D **18**, 1696 (1978).
 [11] E. J. Ahn, R. Engel, T. K. Gaisser, P. Lipari, and T. Stanev, arXiv:0906.4113.
 [12] F. Abe *et al.* (CDF Collaboration), Phys. Rev. D **47**, 4857 (1993).
 [13] M. Burkardt, Int. J. Mod. Phys. A **18**, 173 (2003).
 [14] M. Diehl, Phys. Rep. **388**, 41 (2003).
 [15] A. V. Belitsky and A. V. Radyushkin, Phys. Rep. **418**, 1 (2005).
 [16] T. Sjostrand and M. van Zijl, Phys. Rev. D **36**, 2019 (1987).
 [17] T. Sjostrand, S. Mrenna, and P. Skands, J. High Energy Phys. **05** (2006) 026.
 [18] M. Bahr, S. Gieseke, and M. H. Seymour, J. High Energy Phys. **07** (2008) 076.
 [19] M. Bahr, J. M. Butterworth, and M. H. Seymour, J. High Energy Phys. **01** (2009) 065.
 [20] D. E. Acosta *et al.* (CDF Collaboration), Phys. Rev. D **70**, 072002 (2004).
 [21] A. A. Affolder *et al.* (CDF Collaboration), Phys. Rev. D **65**, 092002 (2002).
 [22] M. M. Block and R. N. Cahn, Rev. Mod. Phys. **57**, 563 (1985).
 [23] L. Durand and H. Pi, Phys. Rev. D **40**, 1436 (1989).
 [24] N. A. Amos *et al.*, Nucl. Phys. **B262**, 689 (1985).
 [25] G. Arnison *et al.* (UA1 Collaboration), Phys. Lett. **128B**, 336 (1983).
 [26] C. Avila *et al.* (E811 Collaboration), Phys. Lett. B **445**, 419 (1999).
 [27] F. Abe *et al.* (CDF Collaboration), Phys. Rev. D **50**, 5550 (1994).
 [28] F. Abe *et al.* (CDF Collaboration), Phys. Rev. D **50**, 5518 (1994).
 [29] J. Dias de Deus and J. Kwiecinski, Phys. Lett. B **196**, 537 (1987).
 [30] B. Margolis, P. Valin, M. M. Block, F. Halzen, and R. S. Fletcher, Phys. Lett. B **213**, 221 (1988).
 [31] A. Capella, U. Sukhatme, C. I. Tan, and J. Tran Thanh Van, Phys. Rep. **236**, 225 (1994).
 [32] K. Werner, Phys. Rep. **232**, 87 (1993).

- [33] R. S. Fletcher, T. K. Gaisser, P. Lipari, and T. Stanev, *Phys. Rev. D* **50**, 5710 (1994).
- [34] R. M. Godbole, A. Grau, G. Pancheri, and Y. N. Srivastava, *Phys. Rev. D* **72**, 076001 (2005).
- [35] U. Amaldi, M. Jacob, and G. Matthiae, *Annu. Rev. Nucl. Sci.* **26**, 385 (1976).
- [36] S. Sapeta, *Phys. Lett. B* **597**, 352 (2004).
- [37] S. Sapeta and K. J. Golec-Biernat, *Phys. Lett. B* **613**, 154 (2005).
- [38] F. Abe *et al.* (CDF Collaboration), *Phys. Rev. D* **50**, 5535 (1994).
- [39] N. A. Amos *et al.* (E710 Collaboration), *Phys. Lett. B* **301**, 313 (1993).
- [40] M. Gluck, E. Reya, and A. Vogt, *Z. Phys. C* **67**, 433 (1995).
- [41] P. M. Nadolsky *et al.*, *Phys. Rev. D* **78**, 013004 (2008).
- [42] A. D. Martin, R. G. Roberts, W. J. Stirling, and R. S. Thorne, *Eur. Phys. J. C* **4**, 463 (1998).
- [43] A. D. Martin, W. J. Stirling, R. S. Thorne, and G. Watt, arXiv:0901.0002.
- [44] J. R. Cudell *et al.* (COMPETE Collaboration), *Phys. Rev. Lett.* **89**, 201801 (2002).
- [45] W. M. Yao *et al.* (Particle Data Group), *J. Phys. G* **33**, 1 (2006).
- [46] R. D. Schamberger, J. Lee-Franzini, R. McCarthy, S. Childress, and P. Franzini, *Phys. Rev. Lett.* **34**, 1121 (1975).
- [47] J. C. M. Armitage *et al.*, *Nucl. Phys.* **B194**, 365 (1982).
- [48] D. Bernard *et al.* (UA4 Collaboration), *Phys. Lett. B* **186**, 227 (1987).
- [49] R. E. Ansorge *et al.* (UA5 Collaboration), *Z. Phys. C* **33**, 175 (1986).