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NJL-jet model for quark fragmentation functions

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A description of fragmentation functions which satisfy the momentum and isospin sum rules is presented in an effective quark theory. Concentrating on the pion fragmentation function, we first explain why the elementary (lowest order) fragmentation process $q \rightarrow q\pi$ is completely inadequate to describe the empirical data, although the crossed process $\pi \rightarrow q\bar{q}$ describes the quark distribution functions in the pion reasonably well. Taking into account cascadelike processes in a generalized jet-model approach, we then show that the momentum and isospin sum rules can be satisfied naturally, without the introduction of *ad hoc* parameters. We present results for the Nambu–Jona-Lasinio (NJL) model in the invariant mass regularization scheme and compare them with the empirical parametrizations. We argue that the NJL-jet model, developed herein, provides a useful framework with which to calculate the fragmentation functions in an effective chiral quark theory.

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I. INTRODUCTION

Quark distribution and fragmentation functions are the basic nonperturbative ingredients for a QCD-based analysis of hard scattering processes [1–6]. Distribution functions can be extracted by analyzing inclusive processes [7,8] and their description in terms of effective quark theories of QCD has been quite successful [9,10]. In recent years there has been a significant effort to extract the fragmentation functions by analyzing inclusive hadron production (semi-inclusive) processes in e^+e^- annihilation, deep-inelastic lepton-nucleon scattering and proton-proton collisions [11,12]. Besides being of fundamental interest in their own right, knowledge of fragmentation functions is essential for the extraction of the transversity quark distribution functions [6,13] from data, and to analyze several other interesting effects in semi-inclusive processes [14].

Because of the importance of the fragmentation functions many attempts have been made to describe them using effective quark theories [15]. However, in order to achieve reasonable agreement with the empirical parametrizations it was necessary to introduce new parameters, like normalization constants, which cannot be justified on theoretical grounds. A description of fragmentation functions within effective quark theories, which automatically satisfies the relevant sum rules [3] and describes the empirical data reasonably well—without introducing new parameters into the theory—has hitherto not been achieved.

This failure to describe the fragmentation functions in the same framework which is successful at describing the distribution functions is surprising, because there exists a general relation, the so called Drell-Levy-Yan (DLY) relation [16,17], which suggests a way to compute the fragmentation functions by analytic continuation of the distribution functions into the region of Bjorken $x > 1$. Although the derivation of this relation appears to be very general (as we show in Appendix A), the basic *assumption* that the distributions and fragmentations are essentially one and the same function, defined in different regions of the scaling variable, has not been proven. Moreover, the approximations used to calculate the distribution functions may not be sensible for the fragmentation functions and vice versa. For example, in the fragmentation process of a quark into a pion, $q \rightarrow \pi + n$, where n is a spectator, there is no *a priori* reason to truncate n to a single quark state, as the DLY crossing arguments would suggest for the case of a Bethe-Salpeter type vertex function for $\pi \rightarrow q\bar{q}$. One can actually give a quantitative argument that the lightest component of n is dominant only if the scaling variable z is very close to unity [18].

On the other hand, the phenomenological quark jet-model, as formulated originally by Field and Feynman [19], suggests that the meson observed in a semi-inclusive process is one among many, that is, the spectator state n contains many mesons. This model is based on a *product ansatz* for a chain of elementary fragmentation processes, where in each step a certain fraction of the quark momentum is transferred to a meson, until eventually a very soft quark remains. This final quark is assumed to annihilate with the other remnants of the process without producing

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further observable mesons.[1] In order for all of the quark light-cone momentum to be transferred to the mesons, it is actually necessary to assume an *infinite* number of steps (mesons) in the decay chain, as will be explained in more detail in Sec. IV. In this case, it is possible to satisfy the momentum sum rule for fragmentation functions [3], which is assumed valid in QCD-based fits to the data [11,12]. Clearly, this sum rule cannot be satisfied in a single step elementary fragmentation process.

The purpose of this paper is to apply the method of the quark jet-model to calculate the spin-independent fragmentation functions in an effective chiral quark theory, which has proven to be very successful for the description of quark distribution functions [9,22,23]. We will concentrate on quark fragmentation into pions within the Nambu–Jona-Lasinio (NJL) model [24], however the methods illustrated here can easily be extended to other fragmentation channels and applied within other effective quark theories. In order to reconcile the quark jet-model with our present NJL model description, we will introduce a generalized product ansatz, which allows for the fragmentation of a quark into a finite number of pions according to a certain distribution function, and in the end we take the limit of infinitely many pions. We will show how the momentum and isospin sum rules emerge naturally without introducing any new parameters into the theory. Our numerical results will demonstrate that this NJL-jet model provides a very reasonable framework for describing the fragmentation functions.

This paper is organized as follows: In Sec. II we begin with the operator definitions for the quark distribution and fragmentation functions and move on to discuss the sum rules and the DLY relation. In Sec. III we give the expressions for the elementary fragmentation functions in the NJL model and discuss their physical interpretations and sum rules. In Sec. IV we introduce the generalized product ansatz to describe a chain of elementary fragmentation processes in the spirit of the quark jet-model, derive the integral equation for the fragmentation function and discuss the momentum and isospin sum rules. In Sec. V we explain the model framework for the numerical calculations, present results and compare them with the empirical fragmentation functions. A summary is given in Sec. VI.

II. OPERATOR DEFINITIONS AND SUM RULES

Operator definitions and sum rules for fragmentation functions were first given in Ref. [3] and were further elucidated in Ref. [25]. In this section we summarize the basic relations for the fragmentation functions and for clarity include those for the distribution functions also. The spin-independent distribution function of a quark of flavor q inside a hadron of spin-flavor h (for example $h = p\uparrow$, π^+, etc.) and the spin-independent fragmentation function for $q \to h$ are defined by

$$f_q^h(x) = \frac{1}{2}\int \frac{d\omega^-}{2\pi} e^{ip_-\omega^- x} \hat{\sum}_n \langle p(h)|\bar{\psi}(0)|p_n\rangle \times \gamma^+ \langle p_n|\psi(\omega^-)|p(h)\rangle, \quad (1)$$

$$D_q^h(z) = \frac{z}{12}\int \frac{d\omega^-}{2\pi} e^{ip_-\omega^-/z} \hat{\sum}_n \langle p(h), p_n|\bar{\psi}(0)|0\rangle \times \gamma^+ \langle 0|\psi(\omega^-)|p(h), p_n\rangle. \quad (2)$$

The field operators refer to a quark of flavor q, although it is not indicated explicitly. The symbol $p(h)$ refers to a hadron h with momentum p and p_n labels the spectator state. The light-cone components of a 4-vector are defined as $a^\mu = (a^+, a^-, \boldsymbol{a}_T)$ with $a^\pm = (a^0 \pm a^3)/\sqrt{2}$. Covariant normalization is used throughout this paper and the summation symbol $\hat{\sum}_n$ includes an integration over the on-shell momenta p_n.[2] Both expressions in Eqs. (1) and (2) refer to a frame where $\boldsymbol{p}_T = 0$. The physical content of the functions in Eqs. (1) and (2) is most transparent if we introduce the "good" light-cone quark field ψ_+ [22,26,27], which is defined by $\psi_+ \equiv \Lambda_+\psi$ where $\Lambda_+ = \frac{1}{2}\gamma^-\gamma^+$ and can be expressed as the Fourier decomposition

$$\psi_+(\omega^-) = \int_0^\infty \frac{dk_-}{\sqrt{2k_-}} \int \frac{d^2k_T}{(2\pi)^{3/2}} \sum_\alpha (b_\alpha(k)u_{+\alpha}(k)e^{-ik_-\omega^-} + d_\alpha^\dagger(k)v_{+\alpha}(k)e^{ik_-\omega^-}). \quad (3)$$

The index α denotes the spin-color of a quark with flavor q and the spinors are normalized as $u_{+\alpha'}^\dagger(k)u_{+\alpha}(k) = v_{+\alpha'}^\dagger(k)v_{+\alpha}(k) = \sqrt{2}k_-\delta_{\alpha',\alpha}$. Substituting these expressions into Eqs. (1) and (2) and using the result $\bar{\psi}\gamma^+\psi = \sqrt{2}\psi_+^\dagger\psi_+$, gives the following relations which are independent of the normalization of states:

$$f_q^h(x)dx = dk_- \int d^2k_T \sum_\alpha \frac{\langle p(h)|b_\alpha^\dagger(k)b_\alpha(k)|p(h)\rangle}{\langle p(h)|p(h)\rangle}, \quad (4)$$

$$D_q^h(z)dz = \frac{z^2}{6}dp_- \int d^2k_T \sum_\alpha \frac{\langle k(\alpha)|a_h^\dagger(p)a_h(p)|k(\alpha)\rangle}{\langle k(\alpha)|k(\alpha)\rangle}. \quad (5)$$

[1]This picture of independent fragmentation is appealing because of its simplicity. More elaborate models for hadronization are the string model [20] or the cluster model [21], which are suitable for Monte Carlo analysis.

[2]In this normalization $\langle p'(h')|p(h)\rangle = 2p_-(2\pi)^3\delta^{(3)}(\boldsymbol{p}' - \boldsymbol{p})\delta_{hh'}$ and $|p(h), p_n\rangle = \sqrt{2(2\pi)^3 p_-}a_h^\dagger(p)|p_n\rangle$, with $[a_h(\boldsymbol{p}'), a_h(p)]_\pm = \delta^{(3)}(\boldsymbol{p}' - \boldsymbol{p})$. The summation defined by $\hat{\sum}_n \equiv \sum_n \int \frac{d^4p_n}{(2\pi)^3}\delta(p_n^2 - M_n^2)\Theta(p_{n0})$, where M_n is the invariant mass of n, can also be expressed in terms of light-cone variables.

Here $dx = dk_-/p_-$, that is, $k_- = xp_-$ for some fixed $p_- > 0$ and $dz = dp_-/k_-$, implying $p_- = zk_-$ for some fixed $k_- > 0$. The creation and annihilation operators, $a_h^\dagger$ and a_h, refer to the hadron h (see footnote [2]) and $k(\alpha)$ labels a quark state of flavor q with momentum k and spin-color α.

According to Eq. (4) we can interpret $f_q^h(x)$ as the light-cone momentum distribution of q in h, where a sum over the spin-color of q is understood, while the spin of h is fixed. However, the result is independent of this spin direction, since we will only consider the spin-independent distributions. As mentioned earlier, Eq. (5) refers to the frame where the produced hadron h has $\boldsymbol{p}_T = 0$, but the fragmenting quark has nonzero $\boldsymbol{k}_T$. To interpret this result as a distribution of h in q, it is necessary to make a Lorentz transformation to the frame where $\boldsymbol{k}_\perp = 0$, but h has nonzero $\boldsymbol{p}_\perp$ (note the distinction between the subscripts T and $\perp$). This is discussed in detail in Refs. [3,6], with the result that one can simply substitute

$$\boldsymbol{k}_T = -\frac{\boldsymbol{p}_\perp}{z}, \tag{6}$$

leaving everything else unchanged. We then obtain from Eq. (5) the result

$$D_q^h(z)dz = \frac{1}{6}dp_- \int d^2p_\perp \sum_\alpha \frac{\langle k(\alpha)|a_h^\dagger(p)a_h(p)|k(\alpha)\rangle}{\langle k(\alpha)|k(\alpha)\rangle}, \tag{7}$$

where the fragmenting quark now has $\boldsymbol{k}_\perp = 0$. According to Eq. (7) we can interpret $D_q^h(z)$ as the light-cone momentum distribution of h in q, where the factor $1/6$ indicates an *average* [25] of the spin-color of q, while the spin of h is fixed.[3] In fact, for the elementary distribution and fragmentation functions considered in the next section, the naively expected relation

$$D_q^h(z) = \frac{1}{d_h} f_h^q(z), \tag{8}$$

is valid. Where d_h is the spin degeneracy, or, in the general case, the spin-color degeneracy of h. Generally however, this relation is not necessarily valid, because q is off-shell (its virtuality being determined kinematically by the scaling variable and the transverse momentum) and h is on shell, which breaks the naive symmetry under the interchange $q \leftrightarrow h$.

[3]For the generalized case where h can also be a quark, we summarize the definitions as follows: $f_q^h(x)$ refers to fixed flavors of q and h, while all other quantum numbers of q (spin, color, etc.) are summed over, with those of h are fixed. $D_q^h(z)$ refers to fixed flavors of q and h, with an *average* over the other quantum numbers of q (spin, color, etc.), while those of h are fixed. This definition has the advantage that in a semi-inclusive process, which involves the product $f_q^T(x)D_q^h(z)$, the quark spin-color summation is included in f but not in D, which avoids double counting.

To obtain the momentum sum rule from Eq. (7) we multiply both sides by $z = p_-/k_-$, integrate over z from 0 to 1 and sum over h.[4] Then one notes that the momentum operator, represented in terms of hadron operators, is given by

$$\hat{P}_- \equiv \sum_h \int_0^\infty dp_- \int d^2p_\perp (p_- a_h^\dagger(p)a_h(p)). \tag{9}$$

By assuming that the quark state $|k(\alpha)\rangle$ in Eq. (7) is an eigenstate of this operator with eigenvalue k_-, we obtain the momentum sum rule

$$\sum_h \int_0^1 dz z D_q^h(z) = 1. \tag{10}$$

The physical content of Eq. (10) is that 100% of the initial quark light-cone momentum (k_-) is transferred to the hadrons. The condition which lies at the basis of Eq. (10) is that the initial quark state is an eigenstate of the momentum operator, Eq. (9), expressed solely in terms of hadrons. That is, the quark hadronizes completely in the sense that it gives all of its light-cone momentum to the hadrons.

A similar argument leads to the isospin sum rule [3], namely

$$\sum_h \int_0^1 dz t_h D_q^h(z) = t_q, \tag{11}$$

where t_q and t_h denote the 3-components of the isospins of q and h. The physical content of this sum rule is that all of the isospin of the initial quark is transferred to hadrons, which is possible since the definition in Eq. (2) implies an average over the isospin of the soft quark remainder of a fragmentation chain (see Sec. IV). In general, there is no sum rule for the baryon number or electric charge, because the baryon number or average electric charge of the quark remainder is not zero.[5] If we simply integrate both sides of Eq. (7) over z, we get the hadron multiplicity, which can be interpreted as the *number of mesons per quark*. However, there is no conservation law which leads to a sum rule for the multiplicity.

There is an interesting relation based on charge conjugation and crossing symmetry, between the fragmentation function for physical $z < 1$ and the distribution function for unphysical $x > 1$:

$$D_q^h(z) = (-1)^{2(s_q+s_h)+1} \frac{z}{d_q} f_q^h\left(x = \frac{1}{z}\right), \tag{12}$$

which is called the DLY relation [16,17]. Here s_q and s_h are the spins of q and h respectively, and d_q is the spin-color degeneracy of q. We derive this relation using two

[4]A subtle point here is that in order to get an integral $\int_0^\infty dp_-$ on the right-hand side of Eq. (9), one has to choose $k_- = \infty$. This does not influence the result, which depends only on z.

[5]The average of the electric charge is zero only if SU(3) flavor symmetry is assumed.

independent methods in Appendix A. The first approach, which follows the original arguments [16], compares the hadronic tensors for $eh \to e'X$ (inclusive DIS) and $e^+e^- \to hX$ (inclusive hadron production), and uses crossing relations for matrix elements of the current operator. The second method—which to the best of our knowledge has not been published before—starts directly from the operator definitions in Eqs. (1) and (2) and uses charge conjugation and crossing symmetries for matrix elements of the quark field operator. If one has an effective quark theory to calculate the quark distribution functions, Eq. (12) would suggest a straightforward way to obtain the fragmentation functions. However, as will become clear in the following sections, for the lowest order (elementary) processes such an attempt leads to disastrous results. That is, the fragmentation functions obtained in this way are 1 or 2 orders of magnitude smaller than the empirical functions and the sum rules in Eqs. (10) and (11) are not satisfied.

The reasons why Eq. (12) fails in actual applications are as follows: (i) It is based on the *assumption* that the distribution functions can be continued analytically beyond $x = 1$. However, it is well known that the Q^2 evolution equations lead to singularities at $x = 1$, which are (regularized) infrared singularities arising from the vanishing gluon mass [5,28]. These render an analytic continuation impossible. Someone may still argue that Eq. (12) should be used only at the low energy (model) scale, however it is actually broken there also, because of the cutoff regularization. We will discuss this point in detail in the next section. (ii) Most importantly, approximations which work reasonably well for the distribution functions may not be sensible for the fragmentation functions and vice versa. For example, the assumption that the pion is a $q\bar{q}$ Bethe-Salpeter bound state is very reasonable for the distribution function [9,10], but the DLY crossing arguments then imply the truncation of the spectator state p_n, of Eq. (2), to a single quark state. Although this simple assumption does not lead to any violation of conservation laws, the sum rules in Eqs. (10) and (11) cannot be satisfied in a single step fragmentation process.

For these reasons, we will not rely on Eq. (12) to calculate the fragmentation functions, although we will confirm its formal validity for the lowest order (elementary) functions. We note that the arguments given above do not question the usefulness of Eq. (12) as a means to relate the kernels of the Q^2 evolution equations for the distribution and fragmentation functions (see Ref. [17] and Appendix B). In fact, it is known that at leading order (LO) in α_s this relation between the kernels is valid, although it is violated at next-to-leading order (NLO) [29].

III. ELEMENTARY DISTRIBUTION AND FRAGMENTATION FUNCTIONS

The elementary distribution and fragmentation functions for the pion are represented in Figs. 1 as cut diagrams.

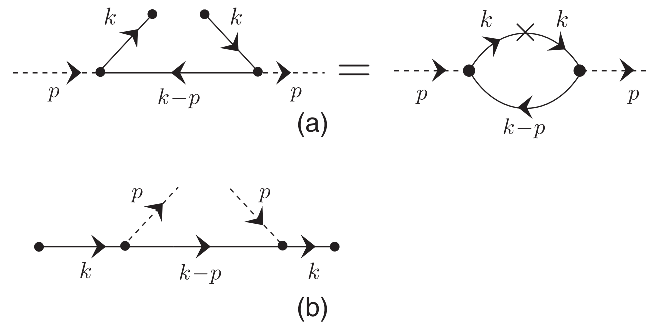

FIG. 1. Figure (a) depicts the cut diagram (left) and Feynman diagram (right) for the distribution function $f_q^\pi(x)$. Solid lines denote the quark and dashed lines the pion. Here $k_- = xp_-$ and the two quark lines with momentum k are connected by a γ^+. Figure (b) depicts the cut diagram for the fragmentation function $d_q^\pi(z)$. Here $p_- = zk_-$ and the two quark lines with momentum k are connected by a γ^+. This diagram refers to a frame where $\boldsymbol{p}_T = 0$ and the substitution given in Eq. (6) is performed in the final transverse momentum integral.

Since the distribution function can also be obtained from a straightforward Feynman diagram calculation [22,30],[6] we also illustrate the Feynman diagram for the distribution function on the right hand side in Fig. 1(a). We denote the elementary fragmentation function by d_q^h in order to distinguish it from the total fragmentation function D_q^h determined in Sec. IV. We obtain the following expressions from the diagrams in Figs. 1[7]:

$$f_q^\pi(x) = \frac{1}{2}(1 + \tau_\pi \tau_q) 3 g_\pi^2 \int \frac{d^4k}{(2\pi)^4} \times \mathrm{Tr}_D[S_F(k)\gamma^+ S_F(k)\gamma_5(\not{k} - \not{p} - M)\gamma_5] \times \delta(k_- - p_- x)\delta((p-k)^2 - M^2) \tag{13}$$

$$= \frac{1}{2}(1 + \tau_\pi \tau_q) 6 g_\pi^2 \int \frac{d^2k_T}{(2\pi)^3} \frac{\boldsymbol{k}_T^2 + M^2}{[\boldsymbol{k}_T^2 + M^2 - m_\pi^2 x(1-x)]^2} \tag{14}$$

$$d_q^\pi(z) = \frac{1}{2}(1 + \tau_\pi \tau_q) g_\pi^2 \frac{z}{2} \int \frac{d^4k}{(2\pi)^4} \times \mathrm{Tr}_D[S_F(k)\gamma^+ S_F(k)\gamma_5(\not{k} - \not{p} - M)\gamma_5] \times \delta(k_- - p_-/z)\delta((p-k)^2 - M^2) \tag{15}$$

$$\left[= \frac{z}{6} f_q^\pi\left(x = \frac{1}{z}\right)\right] \tag{16}$$

[6]This is seen simply by using completeness in Eq. (1) and the identity $\psi_+(0)^\dagger \psi_+(\omega^-) = T(\psi_+(0)^\dagger \psi_+(\omega^-))$ in the limit $\omega^+ \to 0 - \epsilon$, which follows from causality.

[7]The expressions given in this section refer to the NJL model, however they actually have the same form in any effective chiral quark model with pointlike pion-quark vertex functions.

$$= \frac{1}{2}(1+\tau_\pi\tau_q)zg_\pi^2 \int \frac{d^2p_\perp}{(2\pi)^3} \frac{\boldsymbol{p}_\perp^2 + M^2z^2}{[\boldsymbol{p}_\perp^2 + M^2z^2 + (1-z)m_\pi^2]^2} \tag{17}$$

where Tr_D indicates a trace over Dirac indices only. The Feynman propagator of a constituent quark with mass M is denoted by S_F and g_π is the pion-quark coupling constant. In the NJL model g_π is defined via the residue of the $q\bar{q}$ t-matrix at the pion pole, and can be expressed in terms of the $q\bar{q}$ bubble graph by

$$g_\pi^{-2} = -\frac{\partial \Pi_\pi(q^2)}{\partial q^2}\bigg|_{q^2=m_\pi^2}, \tag{18}$$

where $\Pi_\pi(q^2) = 6i \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}_D[\gamma_5 S_F(k)\gamma_5 S_F(k+q)]$.

We use the isospin notations $(\tau_u, \tau_d) = (1, -1)$ and $(\tau_{\pi^+}, \tau_{\pi^0}, \tau_{\pi^-}) = (1, 0, -1)$. For the distribution function in the physical region $(0 < x < 1)$ a factor $\Theta(p_- - k_-) = \Theta(1-x)$ has to be supplied in Eq. (13), which expresses the fact that the intermediate antiquark in Fig. 1(a) has positive energy. Similarly, for the fragmentation function a factor $\Theta(k_- - p_-) = \Theta(1-z)$ has to be supplied in Eq. (15), because the intermediate quark in Fig. 1(b) has positive energy. To obtain Eq. (17) we made the substitution given in Eq. (6).

The DLY relation on this level, indicated in brackets as Eq. (16), shows that Eq. (13) can be considered as a generalized distribution function, which gives the physical distribution function in the region $0 < x < 1$ and the fragmentation function in the region $x = 1/z > 1$. The reason why we indicate this relation only in brackets is that it is violated if the integrals are regularized. For example, if we use a sharp cut-off (Λ) for the transverse quark momentum in Eq. (14), a strict application of the DLY relation would mean that the transverse momentum of the produced pion in Eq. (17) should be cut at $z\Lambda$, which is unacceptable. The more physical procedure is to impose $|\boldsymbol{k}_T| < \Lambda$ on Eq. (14) and $|\boldsymbol{p}_\perp| < \Lambda$ on Eq. (17), which breaks the DLY relation. A similar breakdown of the DLY relation occurs in any other sensible regularization scheme. A noticeable consequence of this is that in the chiral limit the distribution function of Eq. (14) becomes a constant, but the fragmentation function of Eq. (17) is not linear in z, as the DLY relation indicated in Eq. (16) would suggest.

The relations for the distribution function

$$\int_0^1 dx f_q^\pi(x) = \frac{1}{2}(1+\tau_\pi\tau_q), \quad \text{and} \\ \int_0^1 dx x f_q^\pi(x) = \frac{1}{2}(1+\tau_\pi\tau_q)\cdot\frac{1}{2}, \tag{19}$$

lead to the usual number and momentum sum rules. For the elementary fragmentation function the following relation is obtained from Eq. (17):

$$\int_0^1 dz d_q^\pi(z) = \frac{1}{3}(1+\tau_\pi\tau_q)(1-Z_Q) \\ \Rightarrow \int_0^1 dz \sum_{\tau_\pi} d_q^\pi(z) = 1 - Z_Q, \tag{20}$$

where Z_Q is the residue of the quark propagator in the presence of the pion cloud. It is expressed in terms of the renormalized quark self-energy $\Sigma_Q^{(\pi)}(k)$ of Fig. 2 as

$$1 - Z_Q = -\left(\frac{\partial \Sigma_Q^{(\pi)}}{\partial \not{k}}\right)_{\not{k}=M} = -\frac{M}{k_-}\left(\bar{u}_Q(k)\frac{\partial \Sigma_Q^{(\pi)}}{\partial k_+}u_Q(k)\right) \\ = \frac{3}{2}g_\pi^2 \int_0^1 zdz \int \frac{d^2p_\perp}{(2\pi)^3} \\ \times \frac{\boldsymbol{p}_\perp^2 + M^2z^2}{[\boldsymbol{p}_\perp^2 + M^2z^2 + (1-z)m_\pi^2]^2}, \tag{21}$$

where u_Q is the quark spinor $(\bar{u}_Q u_Q = 1)$. Because Z_Q is interpreted as the probability to find a *bare* constituent quark without the pion cloud, Eq. (20) indicates that the elementary fragmentation function is normalized to the *number of pions per quark*. This is expected from our general discussions in Sec. II and will be elucidated further below. Because typical values of Z_Q in models based on constituent quarks are between 0.8 and 0.9, we see from Eq. (20) that the momentum sum rule $\int_0^1 dz z \sum_{\tau_\pi} d_q^\pi(z)$ will be much smaller than typical empirical values. For example, the NLO analysis of Ref. [11] found a momentum sum of $\simeq 0.74$. From this we can anticipate that the elementary fragmentation functions, d_q^π, will be very small compared to the empirical ones (see Sec. V).

In order to confirm that this does not mean that momentum conservation is violated, we also give the expressions for the distribution function of a quark q inside a parent quark Q and for the fragmentation function of $q \to Q$. The operator definitions of these functions $[f_q^Q(x)$ and $D_q^Q(z)]$ are exactly the same as in Eqs. (1) and (2) with the replacement $h \to Q$, where the state $|p(Q)\rangle$ refers to fixed flavor, spin, and color (c.f. the comments in footnote 3). Again we will use the symbol d_q^Q to denote the elementary fragmentation process. The relevant cut diagrams are shown in

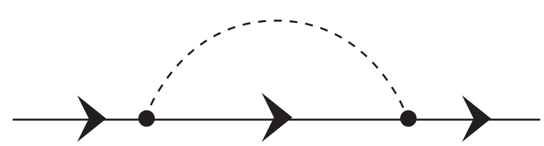

FIG. 2. The quark self-energy, $\Sigma_Q^{(\pi)}(k) = -3ig_\pi^2 \int \frac{d^4p}{(2\pi)^4}\gamma_5 S_F(k-p)\gamma_5 \Delta_F(p)$, where Δ_F is the Feynman propagator of the pion.

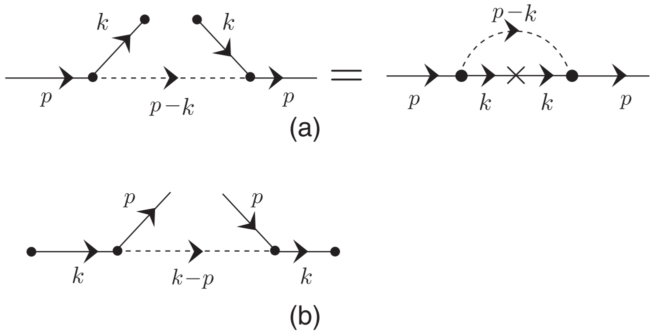

FIG. 3. Figure (a) depicts the cut diagram (left) and Feynman diagram (right) for the loop term in $f_q^Q(x)$ of Eq. (22). Here $k_- = xp_-$ and the two quark lines with momentum k are connected by a γ^+. Figure (b) depicts the cut diagram for the loop term in $d_q^Q(z)$ of Eq. (23). Here $p_- = zk_-$ and the two quark lines with momentum k are connected by a γ^+. This diagram refers to a frame where $\boldsymbol{p}_T = 0$ and the substitution in Eq. (6) is performed in the final transverse momentum integral.

Fig. 3 and a straightforward calculation, following the rules already indicated in Eqs. (13) and (15), gives[8]

$$f_q^Q(x) = Z_Q\delta(x-1)\delta_{q,Q} + \left(\frac{1}{2} - \frac{\tau_q\tau_Q}{6}\right)\frac{3}{2}g_\pi^2(1-x) \times \int\frac{d^2k_T}{(2\pi)^3}\frac{\boldsymbol{k}_T^2 + M^2(1-x)^2}{[\boldsymbol{k}_T^2 + M^2(1-x)^2 + xm_\pi^2]^2}, \tag{22}$$

$$d_q^Q(z) = \frac{1}{6}Z_Q\delta(z-1)\delta_{q,Q} + \frac{1}{6}\left(\frac{1}{2} - \frac{\tau_q\tau_Q}{6}\right)\frac{3}{2}g_\pi^2(1-z) \times \int\frac{d^2p_\perp}{(2\pi)^3}\frac{\boldsymbol{p}_\perp^2 + M^2(1-z)^2}{[\boldsymbol{p}_\perp^2 + M^2(1-z)^2 + zm_\pi^2]^2}. \tag{23}$$

In accordance with Eq. (8) these relations show that

$$d_q^Q(z) = \tfrac{1}{6}f_q^Q(z) = \tfrac{1}{6}f_Q^q(z). \tag{24}$$

Therefore the two quantities in Eqs. (22) and (23) describe essentially the same object, namely, the splitting function of a quark to another quark, which also includes a "non-splitting" term proportional to Z_Q. The normalization is

$$\int_0^1 dz 6\sum_{\tau_Q} d_q^Q(z) = Z_Q + (1 - Z_Q) = 1, \tag{25}$$

where the factor 6 represents the summation over the spin and color of Q. As expected, the second term in Eq. (23) can be obtained from the elementary $q \to \pi$ fragmentation function expressed in Eq. (17), via the substitutions $z \to 1 - z$ and $\tau_\pi \to (\tau_q - \tau_Q)/2$. This directly leads to momentum conservation for the fragmentation of q into either Q or π [see Eq. (29)].

This connection between splitting functions can also be viewed another way: The second term in Eq. (22), which describes the distribution of q inside Q with a pion spectator, suggests that via the substitutions $\tau_q/2 \to \tau_Q/2 - \tau_\pi$ and $x \to 1 - x$ we obtain the distribution function of a pion inside the quark Q, namely

$$f_\pi^Q(x) = \frac{1}{2}(1 + \tau_\pi\tau_Q)g_\pi^2 x\int\frac{d^2k_T}{(2\pi)^3} \times \frac{\boldsymbol{k}_T^2 + M^2x^2}{[\boldsymbol{k}_T^2 + M^2x^2 + (1-x)m_\pi^2]^2}. \tag{26}$$

Comparison with Eq. (17) gives $d_q^\pi(z) = f_\pi^q(z)$, in accordance with Eq. (8). This relation further elucidates the interpretation of the normalization given in Eq. (20) as the number of pions per quark, namely

$$\int_0^1 dz\sum_{\tau_\pi} d_q^\pi(z) = \int_0^1 dz\sum_{\tau_\pi} f_\pi^q(z) = 1 - Z_Q. \tag{27}$$

Finally, we write down the momentum sum rules for the elementary splitting functions. In terms of the distribution functions we have

$$\int_0^1 dxx\left(\sum_{\tau_q} f_q^Q(x) + \sum_{\tau_\pi} f_\pi^Q(x)\right) = Z_Q + \int_0^1 dxx\sum_{\tau_\pi} f_\pi^Q(1-x) + \int_0^1 dxx\sum_{\tau_\pi} f_\pi^Q(x) = 1, \tag{28}$$

where in the second equality we used $x \to 1 - x$ and Eq. (27). In terms of the fragmentation functions Eq. (28) becomes

$$\int_0^1 dzz\left(6\sum_{\tau_Q} d_q^Q(z) + \sum_{\tau_\pi} d_q^\pi(z)\right) = 1. \tag{29}$$

In reference to the form of Eq. (23), we have the following simple interpretation of the momentum sum rule of Eq. (29): Because Z_Q is the probability that the initial quark q does not fragment at all, the fraction Z_Q of the momentum stays with the initial quark. The remaining fraction $(1 - Z_Q)$ is shared among the quark remainder and the produced pion, that is, the first and second terms in Eq. (29).

Although a description of fragmentation functions using only the elementary fragmentation processes does not

[8]The tree level terms proportional to Z_Q in Eqs. (22) and (23) come from the vacuum state in the sum over n in Eqs. (1) and (2), which contributes for the case where $p(h)$ is a quark. Using $\psi = \sqrt{Z_Q}\hat{\psi}$, where $\hat{\psi}$ is the renormalized quark field with unit pole residue of the propagator, gives the Z_Q terms in Eqs. (22) and (23). Note, in the loop terms all factors Z_Q of the propagators cancel. We also note that the loop terms in $f_q^Q(x)$ and $d_q^Q(z)$ formally satisfy the DLY relation, that is $d_{q,\mathrm{loop}}^Q(z) = (-z/6)f_{q,\mathrm{loop}}^Q(x = 1/z)$, however it is violated after regularization.

violate any conservation law, it is completely inadequate for the following reasons: First, there is a large probability (Z_Q) that the initial quark does not fragment. Second, if it does fragment the momentum fraction $1 - Z_Q$ is shared between the quark remainder and the pion. Both points are in contradiction to the usual assumption of complete hadronization, which is expressed by the momentum sum rule of Eq. (10).

IV. GENERALIZED PRODUCT ANSATZ FOR QUARK CASCADES

From the previous section, it is clear that we have to consider the possibility that the fragmenting quark produces a cascade of mesons. A simple model to describe cascades is the quark jet-model of Field and Feynman [19]. However, the product ansatz used in this model assumes from the outset that the probability for fragmentation in each elementary process is 100%, and that the quark produces an infinite number of mesons. Because these assumptions are inconsistent with our present effective quark theory, we will first introduce a generalized product ansatz, then explain its physical significance and its relation to the original quark jet-model.

We assume that the maximum number of mesons which can be produced by the fragmenting quark is N. We then consider a process where the initial quark with light-cone momentum $k_- \equiv W_0$ (which we will simply call the *momentum* in the following) goes through a sequence of momenta $W_0 \geq W_1 \geq W_2 \geq \cdots \geq W_N$, and introduce the momentum ratios

$$\eta_n = \frac{W_n}{W_{n-1}}, \qquad n = 1, \ldots N. \tag{30}$$

Our product ansatz for the fragmentation function, which we will motivate shortly, is

$$\begin{aligned} D_q^\pi(z) = \sum_{m=1}^{N} \int_0^1 d\eta_1 \int_0^1 d\eta_2 \ldots \int_0^1 d\eta_N \\ \times \sum_{Q_N} 6d_q^{Q_1}(\eta_1) \cdot 6d_{Q_1}^{Q_2}(\eta_2) \cdots 6d_{Q_{N-1}}^{Q_N}(\eta_N) \\ \times \delta(z - z_m)\delta(\tau_\pi, (\tau_{Q_{m-1}} - \tau_{Q_m})/2). \end{aligned} \tag{31}$$

Here the functions $d_Q^{Q'}(\eta)$ are our elementary $Q \to Q'$ splitting functions of Eq. (23), which represent the probability that a quark of flavor Q makes a transition to the quark Q', leaving the momentum fraction η to Q'. A sum over repeated flavor indices is implied in Eq. (31); a flavor sum over the quark remainder (Q_N) is included; for the case $N = 1$ we define $Q_0 \equiv q$; and the symbol $\delta(i, j)$ denotes the Kronecker delta. The factor 6 which multiplies each elementary splitting function comes from the sum over spin and color. The delta function in Eq. (31) selects a meson, which is produced in the mth step with momentum fraction z_m of the initial quark:

$$z_m = \frac{W_{m-1} - W_m}{W_0} = \eta_1 \cdot \eta_2 \cdots \eta_{m-1} \cdot (1 - \eta_m), \quad \text{where } m > 1, \quad \text{and} \quad z_1 = 1 - \eta_1. \tag{32}$$

Because the pion has a mass we will exclude the unphysical case of $z = 0$, that is, whenever a pion is produced in the mth step we will assume that $\eta_m \neq 1$ in Eq. (32).

We will write the $q \to Q$ splitting function of Eq. (23), including the spin-color factor 6, in the form

$$6d_q^Q(z) = Z_Q \delta(z - 1)\delta_{q,Q} + F_q^Q(z), \tag{33}$$

where

$$\begin{aligned} F_q^Q(z) &= \left(\frac{1}{2} - \frac{\tau_q \tau_Q}{6}\right) F(z), \quad \text{and} \\ F(z) &= \frac{3}{2} g_\pi^2 (1 - z) \int \frac{d^2 p_\perp}{(2\pi)^3} \frac{\boldsymbol{p}_\perp^2 + M^2(1 - z)^2}{[\boldsymbol{p}_\perp^2 + M^2(1 - z)^2 + z m_\pi^2]^2}. \end{aligned} \tag{34}$$

The function F satisfies the normalization [see Eq. (25)]

$$\sum_Q \int_0^1 dz F_q^Q(z) = \int_0^1 dz F(z) = 1 - Z_Q. \tag{35}$$

For the case $N = 1$ it is easy to see that Eq. (31) reduces to the elementary fragmentation function of Eq. (17), namely

$$\begin{aligned} D_q^\pi(z) \overset{N=1}{\longrightarrow} F_q^Q(1 - z)|_{\tau_Q = \tau_q - 2\tau_\pi} &= \tfrac{1}{3}(1 + \tau_q \tau_\pi) F(1 - z) \\ &= d_q^\pi(z). \end{aligned} \tag{36}$$

In order to illustrate the physical content of the ansatz expressed by Eq. (31) we rewrite it identically as follows: Noting that each factor of the product in Eq. (31) consists of the two terms in Eq. (33), it is easy to see that all products with the same number (call it k) of F's and $(N - k)$ number of Z_Q's make the same contribution to $D_q^\pi(z)$. Therefore, we can introduce an ordering of the η's in Eq. (31). That is, take the first k η's not equal to one ($\eta_1, \eta_2, \ldots \eta_k \neq 1$), and the remaining η's equal to one ($\eta_{k+1}, \eta_{k+2}, \ldots \eta_N = 1$), multiply by the combinatoric factor ${}_N C_k$ and perform a sum over k. For some fixed k, only terms with $m \leq k$ will contribute to the sum in Eq. (31), because z_m of Eq. (32) must be nonzero.[9] Then Eq. (31) is rewritten identically as

[9] As explained earlier, we only consider the case $z > 0$.

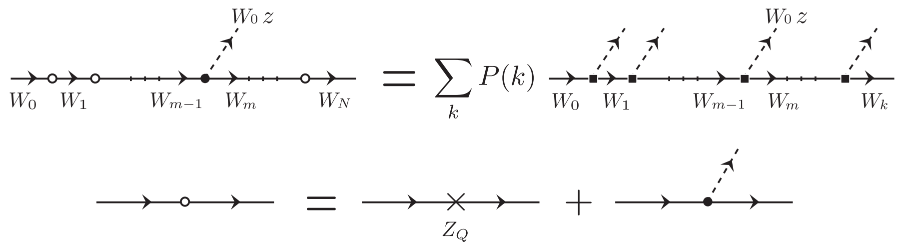

FIG. 4. The left-hand side of the top figure is a graphical representation of Eq. (31) and the right-hand side of this figure represents Eq. (37). The open circles denote the elementary $q \to Q$ fragmentation function of Eq. (33) and the dots represent the second (meson emission) term in Eq. (33). In the mth step, where a meson with momentum zW_0 is selected by the delta function in Eq. (31), only the meson emission term contributes. The term $P(k)$ is the binomial distribution of Eq. (38) and the squares represent the renormalized meson emission term, $\hat{F}_q^Q(z)$, given by Eq. (40). The bottom figure is a graphical representation of Eq. (33).

$$\begin{aligned} D_q^\pi(z) &= \sum_{m=1}^{N}\sum_{k=m}^{N} P(k)\int_0^1 d\eta_1 \int_0^1 d\eta_2 \ldots \int_0^1 d\eta_k \\ &\quad \times \sum_{Q_k} \hat{F}_q^{Q_1}(\eta_1)\hat{F}_{Q_1}^{Q_2}(\eta_2)\ldots \hat{F}_{Q_{k-1}}^{Q_k}(\eta_k)\delta(z - z_m) \\ &\quad \times \delta(\tau_\pi, (\tau_{Q_{m-1}} - \tau_{Q_m})/2), \\ &\equiv \sum_{m=1}^{N} D_{q,(m)}^\pi(z), \end{aligned} \tag{37}$$

which is expressed graphically in Fig. 4. The binomial distribution

$$P(k) = \binom{N}{k} Z_Q^{N-k}(1 - Z_Q)^k, \tag{38}$$

is the probability of producing k mesons out of a maximum of N mesons and satisfies the normalization condition

$$\sum_{k=0}^{N} P(k) = 1. \tag{39}$$

In Eq. (37) we defined the renormalized function $\hat{F}_q^Q \equiv F_q^Q/(1 - Z_Q)$, that is [see Eqs. (34) and (35)]

$$\hat{F}_q^Q(z) = \left(\frac{1}{2} - \frac{\tau_q\tau_Q}{6}\right)\hat{F}(z), \quad \text{where } \hat{F}(z) = \frac{F(z)}{1 - Z_Q}, \tag{40}$$

and

$$\int_0^1 dz \sum_Q \hat{F}_q^Q(z) = \int_0^1 dz \hat{F}(z) = 1. \tag{41}$$

The physical interpretation of Eq. (37) is as follows:

(i) $P(k)$ is the probability that k mesons out of a maximum of N mesons are produced.

(ii) $\hat{F}_Q^{Q'}(\eta)$ is the probability density that, *if* a meson is emitted from the quark Q, the momentum fraction η is left to the remaining quark Q'.

(iii) The product $\hat{F}(\eta_1) \cdot \hat{F}(\eta_2) \ldots \hat{F}(\eta_k)$ is the probability density that, *if* k mesons are produced, each meson carries its momentum fraction z_m ($m = 1, \ldots k$) of the original quark, where z_m is given by Eq. (32).

(iv) $D_{q,(m)}^\pi(z)$ is the probability density that the mth meson has the momentum fraction z of the original quark. This implies that at least m mesons must be produced, which corresponds to the lower limit ($k = m$) of the summation in Eq. (37). The total fragmentation function $D_q^\pi(z)$ is then obtained by summing the probability densities $D_{q,(m)}^\pi(z)$.

We note that the original ansatz of Field and Feynman [19] is an *infinite* product, which formally emerges from Eq. (37) if we take the limit $N \to \infty$ and assume that $P(k)$ is equal to zero for any finite k, that is, the probability of the fragmenting quark to emit a finite number of mesons is zero.

We now proceed with Eq. (37) in order to find the integral equation satisfied by the fragmentation function. For a fixed m, we can integrate over $\eta_{m+1}, \ldots \eta_N$ by using the normalization of $\hat{F}$, that is,

$$\begin{aligned} \int_0^1 d\eta \sum_Q \hat{F}_q^Q(\eta) &= \int_0^1 d\eta \int_0^1 d\eta' \sum_{Q'} \hat{F}_q^Q(\eta)\hat{F}_Q^{Q'}(\eta') = \cdots \\ &= 1. \end{aligned} \tag{42}$$

Then for all $k \geq m$ the integrations over the same variables $\eta_1, \ldots \eta_m$ remain, and the sum over k refers only to the probabilities $P(k)$. Performing the shift $\eta_m \to 1 - \eta_m$ in the integral over η_m, we obtain

$$\begin{aligned} D_{q(m)}^\pi(z) &= \left(\sum_{k=m}^{N} P(k)\right)\int_0^1 d\eta_1 \int_0^1 d\eta_2 \ldots \int_0^1 d\eta_m \\ &\quad \times \hat{F}_q^{Q_1}(\eta_1)\hat{F}_{Q_1}^{Q_2}(\eta_2)\ldots \hat{F}_{Q_{m-2}}^{Q_{m-1}}(\eta_{m-1})\hat{d}_{Q_{m-1}}^\pi(\eta_m) \\ &\quad \times \delta(z - \eta_1\eta_2\ldots\eta_m). \end{aligned} \tag{43}$$

The function $\hat{d}_q^\pi(z) \equiv d_q^\pi(z)/(1 - Z_Q)$ is the renormalized elementary $q \to \pi$ fragmentation function, therefore [see Eq. (36)]

$$\hat{d}_q^{\,\pi}(z) = \hat{F}_q^Q(1-z)|_{\tau_Q=\tau_q-2\tau_\pi} = \tfrac{1}{3}(1+\tau_q\tau_\pi)\hat{F}(1-z). \tag{44}$$

From Eq. (43) it is easy to derive the following recursion relation for $m > 1$:

$$D^\pi_{q(m)}(z) = R_m[\hat{F}_q^Q \otimes D^\pi_{Q(m-1)}](z), \quad \text{where } m > 1, \tag{45}$$

while for $m = 1$ we have

$$D^\pi_{q(1)}(z) = R_1 \hat{d}_q^\pi(z). \tag{46}$$

We have introduced the following ratios:

$$R_n = \frac{\sum_{k=n}^N P(k)}{\sum_{k=n-1}^N P(k)}, \quad \text{where } n = 1, 2, \ldots N, \tag{47}$$

and used the following notation for the convolution of two functions $A(z)$ and $B(z)$:

$$[A \otimes B](z) = \int_0^1 dz_1 \int_0^1 dz_2 \delta(z - z_1 z_2) A(z_1) B(z_2). \tag{48}$$

The total fragmentation function then becomes

$$D_q^\pi(z) = R_1 \hat{d}_q^\pi(z) + \sum_{n=2}^N R_n[\hat{F}_q^Q \otimes D^\pi_{Q(n-1)}](z), \tag{49}$$

where $D^\pi_{q(m)}$ can be obtained from the recursion relation of Eq. (45), with the starting value given by Eq. (46).

It is interesting at this stage to derive the sum rules for the fragmentation function. A simple calculation using Eq. (43) gives the following expressions for the multiplicity, the momentum sum and the isospin sum:

$$\int_0^1 dz \sum_{\tau_\pi} D_q^\pi(z) = \sum_{k=1}^N kP(k) = N(1 - Z_Q), \tag{50}$$

$$\begin{aligned}\int_0^1 dz \sum_{\tau_\pi} z D_q^\pi(z) &= 1 - \sum_{k=0}^N P(k)\langle z\hat{F}\rangle^k \\ &= 1 - (Z_Q + (1-Z_Q)\langle z\hat{F}\rangle)^N,\end{aligned} \tag{51}$$

$$\begin{aligned}\int_0^1 dz \sum_{\tau_\pi} \tau_\pi D_q^\pi(z) &= \frac{\tau_q}{2}\left[1 - \sum_{k=0}^N P(k)\left(-\frac{1}{3}\right)^k\right] \\ &= \frac{\tau_q}{2}\left[1 - \left(Z_Q - \frac{1}{3}(1-Z_Q)\right)^N\right],\end{aligned} \tag{52}$$

where $\langle A\rangle \equiv \int_0^1 dz A(z)$. These expressions can be understood as follows: If k mesons are produced with probability $P(k)$, then Eq. (50) is simply the mean number of mesons; the quantity $P(k)\langle z\hat{F}\rangle^k$ in Eq. (51) is the mean momentum fraction left to the quark remainder; and the quantity $P(k)(-1/3)^k$ in Eq. (52) is the mean isospin fraction left to the quark remainder.

Equations (51) and (52) indicate that, in the present model, it is not possible to transfer the total momentum and isospin of the original quark to the mesons, if the maximum number of mesons is finite. The momentum and isospin sum rules given in Eqs. (10) and (11) are valid only in the limit $N \to \infty$. While this may indicate a conceptual limitation of the jet-model, we note that in general, the QCD-based empirical analysis of fragmentation functions also leads to divergent multiplicities. Therefore, we find it more important to satisfy the momentum and isospin sum rules given in Eqs. (10) and (11) than to have finite multiplicities, and therefore we take the limit $N \to \infty$. The results then become independent of the form of the distribution $P(k)$, if the following condition is satisfied for the ratios in Eq. (47):

$$R_n \overset{N\to\infty}{\longrightarrow} 1, \quad \text{for all } n = 1, 2, \ldots \tag{53}$$

In fact, it is well known that in the limit $N \to \infty$ our binomial distribution of Eq. (38) becomes a normalized Gaussian distribution (normal distribution) $\frac{1}{\sqrt{2\pi c^2}} e^{-((k-k_0)^2/2c^2)}$, with the same mean value $k_0 = N(1 - Z_Q)$ and variance $c^2 = NZ_Q(1 - Z_Q)$ as the original binomial distribution. The validity of Eq. (53) can then easily be confirmed. In fact, *any* distribution which approaches a normal distribution in the limit $N \to \infty$ satisfies the condition given in Eq. (53).[10]

Using Eq. (53), we see from Eq. (49) that our fragmentation function satisfies essentially the same integral equation as in the original quark jet-model [19]:

$$D_q^\pi(z) = \hat{d}_q^\pi(z) + [\hat{F}_q^Q \otimes D_Q^\pi](z), \tag{54}$$

where the driving term is given by Eq. (44) and the integral kernel by Eq. (40). We finally write down the equations which we solve in the next section. Defining two functions $A(z)$ and $B(z)$ by the isospin decomposition

$$D_q^\pi(z) \equiv \tfrac{1}{3}[A(z) + \tau_q\tau_\pi B(z)], \tag{55}$$

and using Eqs. (40) and (44), we find the following integral equations for $A(z)$ and $B(z)$ from Eq. (54):

$$A(z) = \hat{F}(1-z) + \int_z^1 \frac{dy}{y}\hat{F}\left(\frac{z}{y}\right)A(y), \tag{56}$$

[10]The fact that in the limit $N \to \infty$ the binomial distribution becomes a normal distribution is known as the Moivre-Laplace theorem, which can be formulated rigorously in integral form (“weak convergence”). The central limit theorem [31] is an extension of the Moivre-Laplace theorem to general distributions $P(k)$ with mean value proportional to N and variance $c^2 \propto N$. This indicates that Eq. (53) is actually valid for a wide class of distributions. Although our NJL-jet model ansatz of Eq. (31) leads to the binomial distribution, in the limit $N \to \infty$ the results hold for a wide class of distributions.

$$B(z) = \hat{F}(1-z) - \frac{1}{3}\int_z^1 \frac{dy}{y}\hat{F}\left(\frac{z}{y}\right)B(y), \tag{57}$$

where $\hat{F}(z)$ is obtained by renormalizing the function $F(z)$ in Eq. (34) to unity. Using Eq. (55), we obtain the following expressions for the *favored*, *unfavured*, and *neutral* fragmentation functions:

$$D_u^{\pi^+} = D_d^{\pi^-} = D_{\bar{u}}^{\pi^-} = D_{\bar{d}}^{\pi^+} = \tfrac{1}{3}(A+B), \tag{58}$$

$$D_u^{\pi^-} = D_d^{\pi^+} = D_{\bar{u}}^{\pi^+} = D_{\bar{d}}^{\pi^-} = \tfrac{1}{3}(A-B), \tag{59}$$

$$D_u^{\pi^0} = D_d^{\pi^0} = D_{\bar{u}}^{\pi^0} = D_{\bar{d}}^{\pi^0} = \tfrac{1}{3}A. \tag{60}$$

From the form of Eqs. (56) and (57) it is easily seen that $\langle zA\rangle = 1$ and $\langle B\rangle = 3/4$, which leads to the momentum and isospin sum rules of Eqs. (10) and (11). For large z, both functions $A(z)$ and $B(z)$ approach $\hat{F}(1-z)$ and therefore the unfavored fragmentation functions in Eq. (59) are suppressed for large pion momenta.

V. NUMERICAL RESULTS AND DISCUSSIONS

In this section we present the numerical results for the fragmentation function of Eq. (54) in the NJL-jet model. For reference, we also give the results for the elementary distribution function of Eq. (14). Because the application of the NJL model to the calculation of the quark distribution functions in the pion has been explained in detail in Ref. [22], we will not repeat the explanations of the model here. For convenience, we will use the same regularization scheme, namely, the invariant mass, or Lepage-Brodsky (LB) [32] regularization scheme, with the same parameters as in Ref. [22]. The LB scheme is suitable for regularizing integrals in terms of light-cone variables [33] and in terms of the usual variables it is equivalent to the familiar 3-momentum cutoff scheme [22]. That is, if we denote the 3-momentum cutoff by Λ_3, which is fixed in the usual way by reproducing the experimental pion decay constant, a bubble-type loop integral with two intermediate particles of mass M_1 and M_2 is regularized by cutting off their invariant mass M_{12} according to

$$M_{12} \leq \Lambda_{12} \equiv \sqrt{\Lambda_3^2 + M_1^2} + \sqrt{\Lambda_3^2 + M_2^2}. \tag{61}$$

In terms of light-cone variables, if we associate with particle 1 the transverse momentum $\boldsymbol{q}_T$ and the momentum fraction y of the total P_- momentum, and to particle 2 we associate the momentum fraction $(1-y)$ and transverse momentum $-\boldsymbol{q}_T$, then their invariant mass squared is

$$M_{12}^2 = \frac{M_1^2 + \boldsymbol{q}_T^2}{y} + \frac{M_2^2 + \boldsymbol{q}_T^2}{1-y}. \tag{62}$$

The requirement $M_{12} \leq \Lambda_{12}$ then leads to a y-dependent transverse cutoff: $\boldsymbol{q}_T^2 \leq \Lambda_{12}^2 y(1-y) - M_1^2(1-y) - M_2^2 y$. This condition also restricts the values of y from below and above ($0 < y_1 \leq y \leq y_2 < 1$). For example, for the integral in Eq. (17) of the elementary $q \to \pi$ fragmentation function we have $M_1 = m_\pi$ and $M_2 = M$, for the integral in Eq. (23) of the elementary $q \to Q$ fragmentation function we have $M_1 = M$ and $M_2 = m_\pi$ and for the integral in Eq. (14) of the distribution function we have $M_1 = M_2 = M$. We also note that this regularization scheme does not violate the sum rules.

Following Ref. [22] we use a constituent quark mass of $M = 300$ MeV. Then $\Lambda_3 = 670$ MeV and the invariant mass cutoffs for the (π, q) and (q, q) systems are 1.42 GeV and 1.47 GeV, respectively. We did not investigate whether other parameter sets or other regularization schemes lead to a better description of the fragmentation functions.

As usual, we will associate a low energy renormalization scale (Q_0^2) to our NJL results and evolve them in Q^2 by using the QCD evolution equations. For the evolution of the fragmentation functions we limit ourselves to LO. In this case it has been verified [17] that a formal application of the DLY relation, see Eq. (12), leads to the correct connection between the evolution kernels of the distribution and fragmentation functions (see Appendix B). However, the DLY relation is not actually used to relate the distribution and fragmentation functions themselves. We therefore use the Q^2 evolution code of Ref. [34] at LO for the distribution functions, and perform the transformation of the kernels as explained in Appendix B to obtain the LO evolution of the fragmentation functions.[11]

In Fig. 5(a) we recapitulate the results of Fig. 4 of Ref. [22], and show the *minus-type* (valence, $q - \bar{q}$) u-quark distribution in a π^+ and in Fig. 5(b) we give the result for the *plus-type* ($q + \bar{q}$) u-quark distribution in a π^+. The dotted line shows the NJL model result based on Eq. (14), the solid lines illustrate the distribution obtained by associating a low energy scale of $Q_0^2 = 0.18$ GeV^2 to the NJL result and performing the Q^2 evolution at LO and NLO to $Q^2 = 4$ GeV^2. The dashed line shows the empirical NLO parametrizations of Ref. [8]. We see that the LO and NLO results show quantitative differences because of the rather low value assumed for Q_0^2, although the qualitative behaviors are similar.

In Figs. 6 we present the corresponding results for the minus-type and plus-type fragmentation functions for $u \to \pi^+$. The NJL-jet result, given by the dotted line, is the solution of the integral equation in Eq. (54). Therefore the dotted line in Figs. 6(a) and 6(b) show the functions $\frac{2}{3}B(z)$ and $\frac{2}{3}A(z)$, respectively, [see Eqs. (58) and (59)]. In order to see the importance of the cascade processes, we also plot

[11]The DLY based relation between the evolution kernels for distribution and fragmentation functions is violated at NLO [17]. Unfortunately, a NLO evolution code for the fragmentation functions is not yet publicly available. In this paper we do not attempt a quantitative comparison with the empirical functions, therefore we leave the NLO calculation for future work.

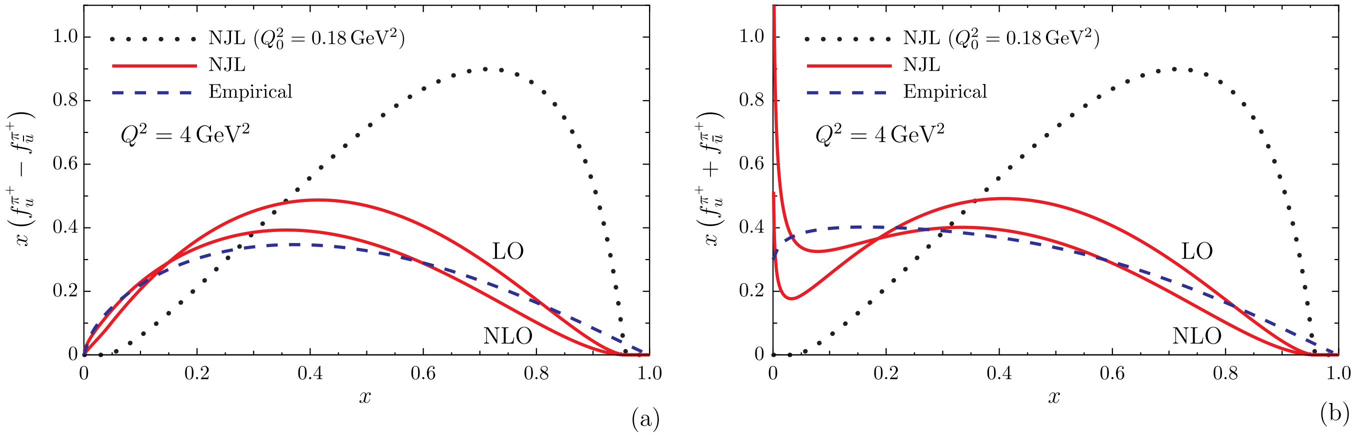

FIG. 5 (color online). Figure (a) depicts the *minus-type* (valence) quark distribution $x(f_u^{\pi^+}(x) - f_{\bar{u}}^{\pi^+}(x))$ and figure (b) illustrates the *plus-type* quark distribution $x(f_u^{\pi^+}(x) + f_{\bar{u}}^{\pi^+}(x))$ of the u-quark in a π^+. The dotted line is the NJL model result, used as input ($Q_0^2 = 0.18$ GeV^2) for the Q^2 evolution. The solid line labeled by LO (NLO) is the result of LO (NLO) evolution to $Q^2 = 4$ GeV^2. The dashed line is the empirical NLO result of Ref. [8] at $Q^2 = 4$ GeV^2.

the driving term of the integral equation, namely $\frac{2}{3}\hat{F}(1 - z)$, as the upper dash-dotted line, which is the renormalized elementary fragmentation function. As the lower dash-dotted line we illustrate the elementary fragmentation function, namely $\frac{2}{3}F(1 - z)$. The result of the evolution of the dotted line ($Q_0^2 = 0.18$ GeV^2) to $Q^2 = 4$ GeV^2 at LO is shown by the solid line and the dashed line shows the empirical NLO result of Ref. [11], evolved to $Q^2 = 4$ GeV^2.

Several important points are illustrated in Figs. 6. First, as anticipated in Sec. III, the elementary fragmentation function (lower dash-dotted line) is very small. Second, Fig. 6(b) shows the tremendous enhancement at intermediate and small z of the plus-type fragmentation function caused by the cascade processes [iterations of the integral equation of Eq. (54)], while for the minus-type fragmentation function of Fig. 6(a) a small reduction is seen. Third, the calculated result shown by the solid line has the correct order of magnitude for intermediate and large z, when compared with the empirical function. This point, which reflects the fact that our model satisfies the momentum sum rule, is very important, because effective quark model calculations completed hitherto only considered the elementary fragmentation functions and introduced some *ad hoc* parameters (like normalization constants) to obtain the correct order of magnitude. Quantitatively, Figs. 6 indicate that our fragmentation functions are too big at large z and too small at smaller z. This is natural for the following

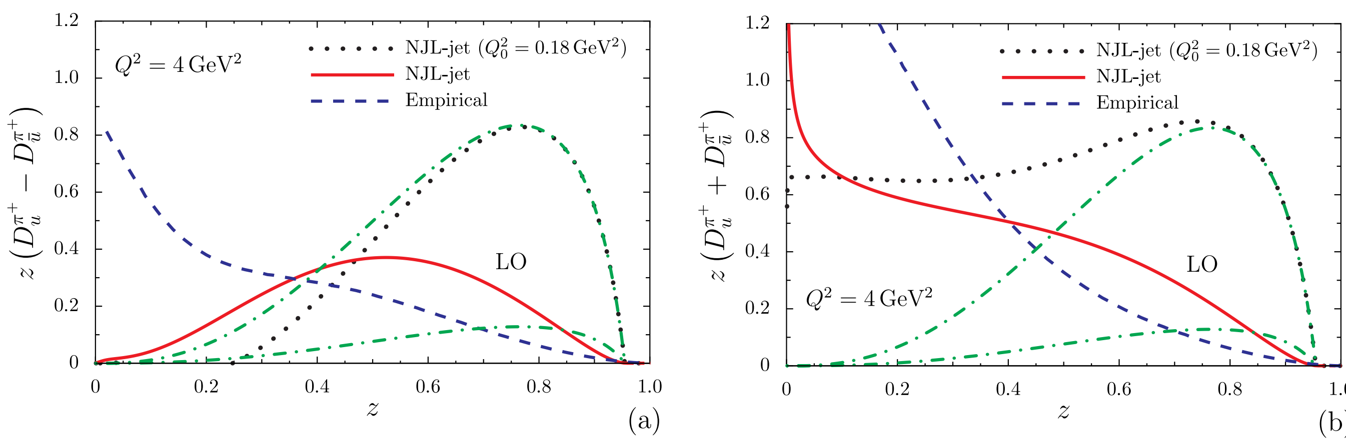

FIG. 6 (color online). Figure (a) depicts the *minus-type* fragmentation function $z(D_u^{\pi^+}(z) - D_{\bar{u}}^{\pi^+}(z))$ and figure (b) illustrates the *plus-type* fragmentation function $z(D_u^{\pi^+}(z) + D_{\bar{u}}^{\pi^+}(z))$ for $u \to \pi^+$. The dotted line is the NJL-jet model result, used as input ($Q_0^2 = 0.18$ GeV^2) for the Q^2 evolution. The lower dash-dotted line is the elementary fragmentation function [d_q^π of Eq. (17)] and the upper dash-dotted line is the renormalized elementary fragmentation function [$\hat{d}_q^\pi$ of Eq. (44)], which is the driving term of the integral equation expressed in Eq. (54). The solid line is the result after LO evolution to $Q^2 = 4$ GeV^2. The dashed line is the empirical NLO result of Ref. [11], evolved to $Q^2 = 4$ GeV^2.

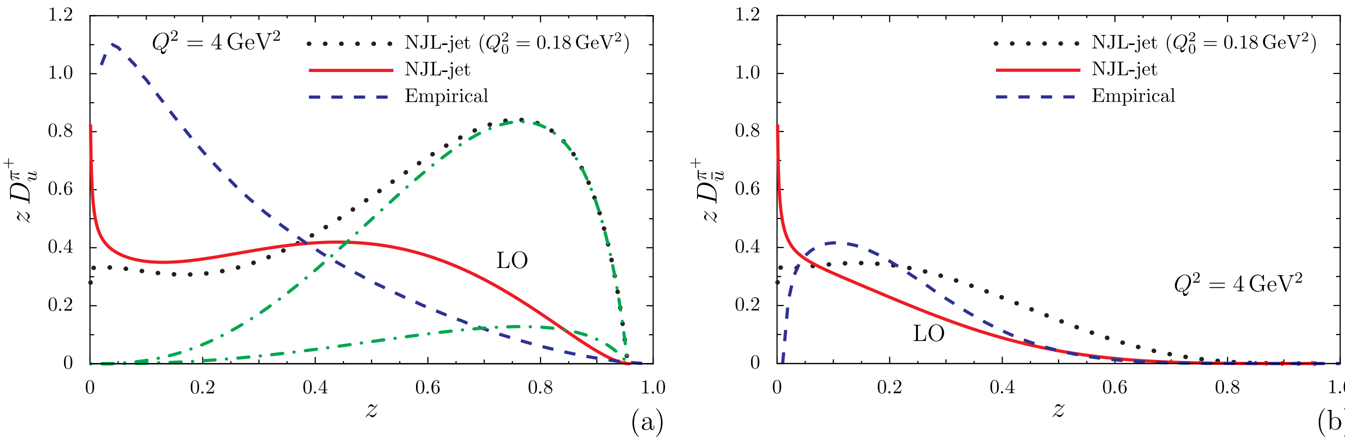

FIG. 7 (color online). Figure (a) depicts the *favored* fragmentation function $zD_u^{\pi^+}(z)$ and the figure (b) illustrates the *unfavored* fragmentation function $zD_{\bar{u}}^{\pi^+}(z)$. In figure (a) the lower dash-dotted line is the elementary fragmentation function (d_q^{π} of Eq. (17)) and the upper dash-dotted line is the renormalized elementary fragmentation function [$\hat{d}_q^{\pi}$ of Eq. (44)], which is the driving term of the integral equation in Eq. (54). Note, these functions are zero for the unfavored case. The solid line is the result after LO evolution to $Q^2 = 4$ GeV^2. The dashed line is the empirical NLO result of Ref. [11], evolved to $Q^2 = 4$ GeV^2.

reasons: First, we can expect that a NLO calculation will lead to a softening of the fragmentation functions. Second, some of the observed pions are secondary ones, which come from the decay of primary ρ and ω mesons. Third, the coupling to other fragmentation channels, in particular, the nucleon, antinucleon and kaon, will transfer some amount of the hard quark momentum to these other hadrons. Also, one should not forget that the empirical fragmentation functions have very large uncertainties, which are not indicated in our figures. Nevertheless, Figs. 6 indicate that the present NJL-jet model provides a reasonable starting point for the description of fragmentation functions.

Figure 7(a) shows the results for the favored fragmentation function of Eq. (58) and Fig. 7(b) shows the unfavored fragmentation function of Eq. (59). Note, these figures correspond to half the sum and half the difference of the curves in Figs. 6. The upper dash-dotted line in Fig. 7(a) shows the driving term, $\frac{2}{3}\hat{F}(1-z)$, of the integral equation in Eq. (54), and the lower dash-dotted line shows the elementary fragmentation function, $\frac{2}{3}F(1-z)$. For the unfavored case these two functions are zero. Both figures demonstrate the importance of cascade processes in the present NJL-jet model.

VI. SUMMARY AND CONCLUSIONS

In this paper we used the NJL model as an effective quark theory to study the simplest fragmentation function, namely, the fragmentation of unpolarized quarks to pions. Our aim was to develop a framework which satisfies the momentum and isospin sum rules in a natural way, without the introduction of *ad hoc* parameters. This framework should also give fragmentation functions that have the correct order of magnitude at intermediate and large z. We explained in detail, that for this purpose, the simplest approximation where a truncation is made to the one-quark spectator state, in the defining relation given by Eq. (2), is completely inadequate. Although this approximation does not violate any conservation law, it gives very small fragmentation functions; because the probability for the elementary fragmentation process is small in effective theories based on constituent quarks and the quark remainder can carry an appreciable amount of momentum.

In order to overcome these difficulties we followed the idea of the quark jet-model and made a generalized product ansatz to describe the cascade processes in the NJL model. We explained that this ansatz corresponds to a binomial distribution for the number of mesons emitted from the quark. However, in the limit that the maximum number of mesons becomes very large the results are independent of the form of this distribution function. Our formulation thus represents an extension of the original quark jet-model, which assumed an infinite number of mesons from the outset. We have shown in detail that this NJL-jet model describes fragmentation processes where 100% of the initial quark light-cone momentum is transferred to mesons. The momentum sum rule of Eq. (10), which is assumed valid in all QCD-based empirical fits, is then satisfied automatically without introducing any new parameters into the theory. We have also shown that the isospin sum rule of Eq. (11) is naturally satisfied in this approach.

The comparison with the empirical fragmentation functions shows that our calculated functions have the correct order of magnitude for intermediate and large z. We highlighted that a straightforward extension to include the NLO terms in the Q^2 evolution and to include the effect of primary ρ and ω mesons, as well as fragmentation to other hadronic channels, will improve the description. Therefore, we can conclude that our NJL-jet model provides a reasonable framework to analyze fragmentation functions in an effective quark theory.

For future work in this direction it is important to derive the jet-model type product ansatz from field theory. The rainbow-ladder scheme for the quark self-energy may provide a suitable framework for this purpose. An attempt can then be made to use this truncation scheme to consistently describe the cascade processes for the fragmentation functions and to include the contribution from the hadron cloud around the quark for the distribution functions. However, it is important to bear in mind that a truncation scheme which works well for fragmentation processes may not be suitable for the distribution functions and vice versa. To establish a scheme which respects the sum rules and which gives a satisfactory description of both types of processes is an important task for future research.

ACKNOWLEDGMENTS

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APPENDIX A: PROOF OF THE DLY RELATION

In this appendix we will prove the DLY relation expressed in Eq. (12) in two independent ways. First, we follow the original derivation of Ref. [16] in terms of the hadronic tensors and second we start from the operator definitions given in Eqs. (1) and (2). In order to illustrate the spinor algebra the formulae in this Appendix refer to the case where h is a proton, however it is trivial to modify the expressions for the case where h is a pion.

1. General crossing relations

We consider the following Green function

$$\bar{M}^{\,a}_{\beta}(p, p_n) = \int d^4x e^{-ip\cdot x}\langle p_n|T(\mathcal{O}^a(0)\bar{\Phi}_\beta(x))|0\rangle, \quad \text{(A1)}$$

where $\Phi_\beta(x)$ is an interpolating field for the nucleon and $\mathcal{O}^a$ is another local field operator. We also define the N-amputated Green function by $\bar{M}^a_\beta(p, p_n) = \bar{\Gamma}^a_\gamma(p, p_n) iG_{N,\gamma\beta}(p)$, where G_N is the nucleon propagator. From the spectral representation of Eq. (A1) or from the familiar reduction formalism, we can derive the relations

$$\langle p_n|\mathcal{O}^a|p\rangle = \bar{\Gamma}^a(p, p_n)\sqrt{2M_N}u_N(\boldsymbol{p}s), \quad \text{(A2)}$$

$$\langle \bar{p}, p_n|\mathcal{O}^a|0\rangle = (\pm)\bar{\Gamma}^a(-p, p_n)\sqrt{2M_N}v_N(\boldsymbol{p}s). \quad \text{(A3)}$$

In Eq. (A3) the sign is (+) if $\mathcal{O}$ is a fermion type operator and (−) if it is a boson type operator. Also, $\bar{p}$ denotes an antinucleon with 4-momentum $p^\mu = (E_N(\boldsymbol{p}), \boldsymbol{p})$. The nucleon spinors are denoted by u_N and v_N. Our covariant normalization implies the following matrix elements of the nucleon field operator: $\langle 0|\Phi(0)|p\rangle = \sqrt{2M_N}u_N(\boldsymbol{p}s)$ and $\langle\bar{p}|\Phi(0)|0\rangle = \sqrt{2M_N}v_N(\boldsymbol{p}s)$. Equations (A2) and (A3) are the basic crossing relations which will be used in the following.

2. Comparison of hadronic tensors

Here we use the above crossing relations to find the connection between the hadronic tensors (spin-independent parts only) for the processes $eh \to e'X$ and $e^+e^- \to hX$, where h denotes a hadron (proton) [4]:

$$\begin{aligned} W^{\mu\nu}_h(p, q) &= \frac{1}{4\pi}\hat{\sum}_n(2\pi)^4\delta^4(q + p - p_n)\langle p|J^\mu|p_n\rangle \\ &\quad \times\langle p_n|J^\nu|p\rangle \\ &= \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right)F^h_1(x, q^2) \\ &\quad + \frac{1}{p\cdot q}\left(p^\mu - \frac{p\cdot q}{q^2}q^\mu\right)\left(p^\nu - \frac{p\cdot q}{q^2}q^\nu\right) \\ &\quad \times F^h_2(x, q^2), \end{aligned} \quad \text{(A4)}$$

$$\begin{aligned} \bar{W}^{\mu\nu}_h(p, q) &= \frac{1}{4\pi}\hat{\sum}_n(2\pi)^4\delta^4(q - p - p_n)\langle 0|J^\mu|p, \bar{p}_n\rangle \\ &\quad \times\langle p, \bar{p}_n|J^\nu|0\rangle \\ &= \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right)\bar{F}^h_1(z, q^2) \\ &\quad + \frac{1}{p\cdot q}\left(p^\mu - \frac{p\cdot q}{q^2}q^\mu\right)\left(p^\nu - \frac{p\cdot q}{q^2}q^\nu\right) \\ &\quad \times \bar{F}^h_2(z, q^2). \end{aligned} \quad \text{(A5)}$$

Here $|p\rangle$ is the state of the hadron h with momentum p and we use $x = \frac{-q^2}{2p\cdot q}$ and $z = \frac{2p\cdot q}{q^2} = -\frac{1}{x}$. We also defined $\hat{\sum}_n = \sum_n \int\frac{d^4p_n}{(2\pi)^3}\delta(p_n^2 - M_n^2)\Theta(p_{n0})$, where M_n is the invariant mass of the state n. Using Eq. (A2) and its complex conjugate for the current operator J^ν:

$$\langle p_n|J^\nu|p\rangle = \sqrt{2M_N}\bar{\Gamma}^\nu(p, p_n)u_N(\boldsymbol{p}s), \quad \text{(A6)}$$

$$\langle p|J^\nu|p_n\rangle = \sqrt{2M_N}\bar{u}_N(\boldsymbol{p}s)\Gamma^\nu(p, p_n), \quad \text{(A7)}$$

where $\Gamma^\nu_\beta = (\gamma_0\bar{\Gamma}^{\dagger\nu})_\beta$, that is, $\bar{\Gamma}^\nu = \Gamma^{\nu\dagger}\gamma^0$. We insert these relations into Eq. (A4). Since we consider the spin-independent part only, we can sum over the nucleon spin s and divide by 2, using $\sum_s u_N(\boldsymbol{p}s)\bar{u}_N(\boldsymbol{p}s) = \frac{\not{p}+M_N}{2M_N}$. This gives

$$4\pi W_h^{\mu\nu}(p,q) = \frac{1}{2}\hat{\sum}_n (2\pi)^4 \delta^4(q+p-p_n) \times \mathrm{Tr}[(\not{p}+M_N)\Gamma^\mu(p,p_n)\bar{\Gamma}^\nu(p,p_n)]. \quad \text{(A8)}$$

For the hadronic tensor in Eq. (A5), we first use charge conjugation and then Eq. (A3) and its complex conjugate for the current operator J^μ:

$$\begin{aligned}\langle 0|J^\mu|p,\bar{p}_n\rangle &= \langle 0|\mathcal{C}^{-1}(\mathcal{C}J^\mu\mathcal{C}^{-1})\mathcal{C}|p,\bar{p}_n\rangle \\ &= \langle 0|(\mathcal{C}J^\mu\mathcal{C}^{-1})|\bar{p},p_n\rangle, = -\langle 0|J^\mu|\bar{p},p_n\rangle \\ &= \sqrt{2M_N}\bar{v}_N(\boldsymbol{p}s)\Gamma^\mu(-p,p_n),\end{aligned} \quad \text{(A9)}$$

$$\begin{aligned}\langle p,\bar{p}_n|J^\mu|0\rangle &= -\langle \bar{p},p_n|J^\mu|0\rangle \\ &= \sqrt{2M_N}\bar{\Gamma}^\mu(-p,p_n)v_N(\boldsymbol{p}s).\end{aligned} \quad \text{(A10)}$$

We insert these relations into Eq. (A5), sum over the nucleon spin s and divide by 2 using $\sum_s v_N(\boldsymbol{p}s)\bar{v}_N(\boldsymbol{p}s) = -\frac{-\not{p}+M_N}{2M_N}$. This gives

$$4\pi \bar{W}_h^{\mu\nu}(p,q) = -\frac{1}{2}\hat{\sum}_n (2\pi)^4\delta^4(q-p-p_n) \times \mathrm{Tr}[(-\not{p}+M_N)\Gamma^\mu(-p,p_n)\bar{\Gamma}^\nu(-p,p_n)]. \quad \text{(A11)}$$

By comparing Eqs. (A8) with (A11) we obtain the DLY crossing relation for the hadronic tensors:

$$\bar{W}_h^{\mu\nu}(p,q) = -W_h^{\mu\nu}(-p,q), \quad \text{where } s_h = \tfrac{1}{2}. \quad \text{(A12)}$$

The minus sign in Eq. (A12) comes from the Dirac algebra, and for a spinless hadron the minus sign is changed to plus. Equation (A12) implies the following relation between the structure functions in Eqs. (A4) and (A5)[12]:

$$\bar{F}_1^h(z,q^2) = -F_1^h(-x,q^2) = -F_1^h\left(\frac{1}{z},q^2\right), \quad \text{(A13)}$$

$$\bar{F}_2^h(z,q^2) = F_2^h(-x,q^2) = F_2^h\left(\frac{1}{z},q^2\right). \quad \text{(A14)}$$

The well-known relation $F_2^h(x) = 2xF_1^h(x)$ becomes, with $x \to -x$ and using the first equalities in Eqs. (A13) and (A14):

$$\bar{F}_2^h(z) = -\frac{2}{z}\bar{F}_1^h(z), \quad \text{(A15)}$$

which also holds for spinless bosons.

[12]By relations like Eq. (A13) we mean the following: Take a particular physical value of z for the (e^+,e^-) process $(0<z<1)$. Then the corresponding (unphysical) value of the Bjorken variable for the (e,e') process is $x=1/z$ and Eq. (A13) gives the connection between the structure functions.

The connection between the structure function $\bar{F}_1^h$ and the fragmentation function $D_q^h(z)$ in the Bjorken limit is as follows: The cross section for the process $e^+e^- \to hX$ is [4][13]

$$\frac{d\sigma^h}{dz} = \frac{2\alpha^2\pi z}{q^2}\left(\bar{F}_1^h(z,q^2) + \frac{z}{6}\bar{F}_2^h(z,q^2)\right) = \frac{4}{3}\frac{\alpha^2\pi z}{q^2}\bar{F}_1^h(z). \quad \text{(A16)}$$

Usually this is divided by the total cross section for $e^+e^- \to$ hadrons

$$\sigma_{\mathrm{tot}} = \frac{4\pi\alpha^2}{q^2}\sum_q e_q^2 \equiv \frac{4\pi\alpha^2}{3q^2}R, \quad \text{(A17)}$$

where $\sum_q$ refers to the quark flavor only. Then we obtain

$$\frac{1}{\sigma_{\mathrm{tot}}}\frac{d\sigma^h}{dz} = \frac{1}{R}z\bar{F}_1^h(z). \quad \text{(A18)}$$

This is compared to the original definition of the fragmentation function [1]:

$$\frac{1}{\sigma_{\mathrm{tot}}}\frac{d\sigma^h}{dz} \equiv \frac{1}{R}3\sum_q e_q^2(D_q^h(z) + D_{\bar{q}}^h(z)) \quad \text{(A19)}$$

to obtain

$$\bar{F}_1^h(z) = \frac{3}{z}\sum_q e_q^2(D_q^h(z) + D_{\bar{q}}^h(z)). \quad \text{(A20)}$$

Because we know how to express $F_1^h(x)$ in the Bjorken limit by the distribution functions $f_q^h(x)$, we obtain from Eq. (A13):

$$\bar{F}_1^h(z) = -F_1^h\left(\frac{1}{z}\right) = -\frac{1}{2}\sum_q e_q^2\left(f_q^h\left(\frac{1}{z}\right) + f_{\bar{q}}^h\left(\frac{1}{z}\right)\right). \quad \text{(A21)}$$

Comparing (A20) and (A21) we obtain

$$D_q^h(z) = -\frac{z}{6}f_q^h\left(\frac{1}{z}\right), \quad \text{where } s_h = \frac{1}{2}, \quad \text{(A22)}$$

and a similar result holds for the antiquarks. Equation (A22) expresses the DLY relation of Eq. (12) between the distribution and fragmentation functions. For the case of a spinless hadron the minus sign in Eq. (A22) becomes a plus sign.

3. Comparing the operator definitions

Starting from the operator definitions given in Eqs. (1) and (2), we obtain

[13]We remind the reader that the symbol h denotes a particular hadron with a specified spin direction, e.g., $p\uparrow$ (although the spin averaged cross section considered here does not depend on the spin direction). Therefore, the cross section measured for the case that the spin of the produced nucleon is not observed has an additional factor of 2, which is not included in Eq. (A16).

$$f_q^h(x) = \frac{1}{2}\hat{\sum}_n \delta(p_- x - p_- + p_{n-})\langle p|\bar{\psi}|p_n\rangle\gamma^+\langle p_n|\psi|p\rangle, \tag{A23}$$

$$D_q^h(z) = \frac{z}{6}\frac{1}{2}\hat{\sum}_n \delta\left(\frac{p_-}{z} - p_- - p_{n-}\right) \times \langle p, \bar{p}_n|\bar{\psi}|0\rangle\gamma^+\langle 0|\psi|p, \bar{p}_n\rangle. \tag{A24}$$

For definiteness we consider again the case where h is a proton. We use $\mathcal{O}^a = \psi_\alpha$ in Eq. (A1), which gives

$$\langle p_n|\psi|p\rangle = \bar{\Gamma}(p, p_n)\sqrt{2M_N}u_N(\boldsymbol{p}s), \tag{A25}$$

$$\langle \bar{p}, p_n|\psi|0\rangle = \bar{\Gamma}(-p, p_n)\sqrt{2M_N}v_N(\boldsymbol{p}s). \tag{A26}$$

We insert Eq. (A25) and its complex conjugate into the operator definition, Eq. (A23), and average over the nucleon spin. This gives

$$f_q^h(x) = \frac{1}{4}\hat{\sum}_n \delta(p_- x - p_- + p_{n-}) \times \mathrm{Tr}[(\not{p} + M_N)\Gamma(p, p_n)\gamma^+\bar{\Gamma}(p, p_n)]. \tag{A27}$$

For the fragmentation function in Eq. (A24), we use the charge conjugation relations of the quark field operators $\mathcal{C}\psi_\alpha\mathcal{C}^{-1} = (C\gamma^0)_{\alpha\beta}\psi_\beta^\dagger$ and $\mathcal{C}\bar{\psi}_\alpha\mathcal{C}^{-1} = \psi_\beta C_{\beta\alpha}$, where $C = i\gamma^2\gamma^0$, to rewrite the matrix elements in Eq. (A24) as follows:

$$\langle 0|\psi_\alpha|p, \bar{p}_n\rangle = (C\gamma_0)_{\alpha\beta}\langle p_n, \bar{p}|\psi_\beta|0\rangle^*, \tag{A28}$$

$$\langle p, \bar{p}_n|\bar{\psi}_\alpha|0\rangle = \langle \bar{p}, p_n|\psi_\beta|0\rangle C_{\beta\alpha}. \tag{A29}$$

Then we use $C\gamma^\mu C = (\gamma^\mu)^T$ and Eq. (A26) to write

$$\langle p, \bar{p}_n|\bar{\psi}_\alpha|0\rangle\gamma^+_{\alpha\beta}\langle 0|\psi_\beta|p, \bar{p}_n\rangle = \bar{v}_N(\boldsymbol{p}s)[\Gamma(-p, p_n)\gamma^+\bar{\Gamma}(-p, p_n)]v_N(\boldsymbol{p}s)\cdot 2M_N. \tag{A30}$$

Averaging over the nucleon spins we finally obtain

$$D_q^h(z) = -\frac{z}{6}\frac{1}{4}\hat{\sum}_n \delta\left(\frac{p_-}{z} - p_- - p_{n-}\right) \times \mathrm{Tr}[(-\not{p} + M_N)\bar{\Gamma}(-p, p_n)\gamma^+\Gamma(-p, p_n)]. \tag{A31}$$

Comparison of Eqs. (A27) and (A31) gives

$$D_q^h(z) = -\frac{z}{6}f_q^h\left(x = \frac{1}{z}\right)\Bigg|_{p\to -p}, \tag{A32}$$

where $p \to -p$ means to reverse all 4 components of p^μ and after this replacement $p^0 = E_N(\boldsymbol{p}) > 0$.

We now consider the property of the distribution function in Eq. (A27) under $p^\mu \to -p^\mu$. Expressing the summation $\hat{\sum}_n$ in terms of light-cone momenta, the distribution in Eq. (A27) can be written in the form

$$f_q^h(x) = \frac{1}{4}\sum_n \int \frac{d^4k}{(2\pi)^3}\frac{\Theta(p_-(1-x))}{2p_-(1-x)}\delta(k_+ - e_N(\boldsymbol{p}) + e_n(\boldsymbol{p}-\boldsymbol{k}))\delta(k_- - p_- x) \times \mathrm{Tr}[(\not{p} + M_N)\Gamma(p, p-k)\gamma^+\bar{\Gamma}(p, p-k)], \tag{A33}$$

where $e_n(\boldsymbol{p}_n) = \frac{\boldsymbol{p}_{n\perp}^2 + M_n^2}{2p_{n-}}$ and $e_N(\boldsymbol{p}) = \frac{\boldsymbol{p}_\perp^2 + M_N^2}{2p_-}$. We then replace $p^\mu \to -p^\mu$ and then $k^\mu \to -k^\mu$ in the integral. This gives

$$f_q^h(x)|_{p\to -p} = -\frac{1}{4}\sum_n \int \frac{d^4k}{(2\pi)^3}\frac{\Theta(p_-(x-1))}{2p_-(1-x)} \times \delta(k_+ - e_N(\boldsymbol{p}) + e_n(\boldsymbol{p}-\boldsymbol{k})) \times \delta(k_- - p_- x) \times \mathrm{Tr}[(-\not{p} + M_N)\Gamma(-p, -p+k)\gamma^+ \times \bar{\Gamma}(-p, -p+k)]. \tag{A34}$$

Because the result of taking the trace in Eq. (A34) must be the plus component of a Lorentz four vector constructed from p^μ and k^μ, we have

$$\mathrm{Tr}[(-\not{p} + M_N)\Gamma(-p, -p+k)\gamma^+\bar{\Gamma}(-p, -p+k)] = -\mathrm{Tr}[(\not{p} + M_N)\Gamma(p, p-k)\gamma^+\bar{\Gamma}(p, p-k)]. \tag{A35}$$

If we use Eq. (A33) to define a function $F(x)$ by $f_q^h(x) = \Theta(1-x)F(x)$, we obtain from Eqs. (A34) and (A35): $f_q^h(x)|_{p\to -p} = \Theta(x-1)F(x)$. From Eq. (A32) we then obtain the connection between the distribution and the fragmentation function as

$$f_q^h(x) = \Theta(1-x)F(x), \tag{A36}$$

$$D_q^h(z) = -\Theta(1-z)\frac{z}{6}F\left(\frac{1}{z}\right). \tag{A37}$$

Note, for spinless bosons there is no minus sign in Eq. (A37). This result agrees with Eq. (A22) and would suggest that f_q^h and D_q^h are essentially one and the same function, defined in different regions of the variable.

APPENDIX B: DLY TRANSFORMATION OF EVOLUTION KERNELS

In this appendix we explain the DLY based relation between the evolution kernels for distribution and fragmentation functions, which is known to be valid at LO [17]. Using Eq. (12), we consider the following transformation of the quark and gluon distribution functions:

$$f_q^h(x) \to \left(\pm \frac{z}{6}\right) f_q^h\left(x = \frac{1}{z}\right), \tag{B1}$$

$$f_g^h(x) \to \left(\mp \frac{z}{16}\right) f_g^h\left(x = \frac{1}{z}\right), \tag{B2}$$

where the upper (lower) sign holds if h is a boson (fermion). Using the well-known evolution equations at LO [5], it is easy to derive the corresponding transformation of the evolution kernels. For the minus-type (flavor nonsinglet) combination $q - \bar{q}$, the kernel (P_{qq}) is unchanged. For the flavor singlet combination, $\sum_{i=1}^{N_f} q_i + \bar{q}_i$, which couples to a gluon, the evolution kernel is transformed as follows:

$$\begin{pmatrix} P_{qq}(x) & P_{qg}(x) \\ P_{gq}(x) & P_{gg}(x) \end{pmatrix} \to \begin{pmatrix} P_{qq}(z) & 2N_f P_{gq}(z) \\ \frac{1}{2N_f} P_{qg}(z) & P_{gg}(z) \end{pmatrix}. \tag{B3}$$

Here $N_f = 3$ is the number of flavors used in the Q^2 evolution equations. For reference, we summarize the forms of the individual kernels below:

$$P_{qq}(x) = \frac{4}{3}\left[\frac{1 + x^2}{(1 - x)_+} + \frac{3}{2}\delta(x - 1)\right], \tag{B4}$$

$$P_{qg}(x) = N_f[x^2 + (1 - x)^2], \tag{B5}$$

$$P_{gq}(x) = \frac{4}{3}\frac{1 + (1 - x)^2}{x}, \tag{B6}$$

$$P_{gg}(x) = 6\left[\frac{x}{(1 - x)_+} + \frac{1 - x}{x} + x(1 - x)\right] + \left(\frac{11}{2} - \frac{N_f}{3}\right)\delta(1 - x). \tag{B7}$$

[1] R. D. Field and R. P. Feynman, Phys. Rev. D **15**, 2590 (1977); F. E. Close, *An Introduction to Quarks and Partons* (Academic Press, London, 1979).
[2] G. Altarelli, R. K. Ellis, G. Martinelli, and S. Y. Pi, Nucl. Phys. **B160**, 301 (1979).
[3] J. C. Collins and D. E. Soper, Nucl. Phys. **B194**, 445 (1982).
[4] R. L. Jaffe, arXiv:hep-ph/9602236.
[5] R. K. Ellis, W. J. Stirling, and B. R. Webber, *QCD and Collider Physics* (Cambridge University Press, Cambridge, England, 1996).
[6] V. Barone, A. Drago, and P. G. Ratcliffe, Phys. Rep. **359**, 1 (2002).
[7] A. D. Martin, R. G. Roberts, W. J. Stirling, and R. S. Thorne, Eur. Phys. J. C **35**, 325 (2004); M. Hirai, S. Kumano, and N. Saito (Asymmetry Analysis Collaboration), Phys. Rev. D **69**, 054021 (2004).
[8] P. J. Sutton, A. D. Martin, R. G. Roberts, and W. J. Stirling, Phys. Rev. D **45**, 2349 (1992).
[9] I. C. Cloët, W. Bentz, and A. W. Thomas, Phys. Lett. B **621**, 246 (2005).
[10] M. Wakamatsu and T. Kubota, Phys. Rev. D **57**, 5755 (1998); M. B. Hecht, C. D. Roberts, and S. M. Schmidt, Phys. Rev. C **63**, 025213 (2001); R. M. Davidson and E. Ruiz Arriola, Phys. Lett. B **348**, 163 (1995).
[11] M. Hirai, S. Kumano, T. H. Nagai, and K. Sudoh, Phys. Rev. D **75**, 094009 (2007).
[12] D. de Florian, R. Sassot, and M. Stratmann, Phys. Rev. D **75**, 114010 (2007); S. Kretzer, E. Leader, and E. Christova, Eur. Phys. J. C **22**, 269 (2001); S. Albino, B. A. Kniehl and G. Kramer, Nucl. Phys. **B803**, 42 (2008).
[13] J. P. Ralston and D. E. Soper, Nucl. Phys. **B152**, 109 (1979).
[14] D. W. Sivers, Phys. Rev. D **41**, 83 (1990); J. C. Collins, Nucl. Phys. **B396**, 161 (1993); D. Boer, P. J. Mulders, and F. Pijlman, Nucl. Phys. **B667**, 201 (2003).
[15] J. T. Londergan, A. Pang, and A. W. Thomas, Phys. Rev. D **54**, 3154 (1996); R. Jakob, P. J. Mulders, and J. Rodrigues, Nucl. Phys. **A626**, 937 (1997); H. Kitagawa and Y. Sakemi, Prog. Theor. Phys. **104**, 421 (2000); J. J. Yang, Phys. Rev. D **65**, 094035 (2002); D. Amrath, A. Bacchetta, and A. Metz, Phys. Rev. D **71**, 114018 (2005); A. Bacchetta, L. P. Gamberg, G. R. Goldstein, and A. Mukherjee, Phys. Lett. B **659**, 234 (2008).
[16] S. D. Drell, D. J. Levy, and T. M. Yan, Phys. Rev. **187**, 2159 (1969); S. D. Drell, D. J. Levy, and T. M. Yan, Phys. Rev. D **1**, 1035 (1970); Phys. Rev. D **1**, 1617 (1970).
[17] J. Blumlein, V. Ravindran, and W. L. van Neerven, Nucl. Phys. **B586**, 349 (2000).
[18] C. Boros, J. T. Londergan, and A. W. Thomas, Phys. Lett. B **473**, 305 (2000).
[19] R. D. Field and R. P. Feynman, Nucl. Phys. **B136**, 1 (1978); T. D. Gottschalk, in 19th Int. School of Elementary Particle Physics, Kupari-Dubrovnik, 1983 (unpublished).
[20] B. Andersson, G. Gustafson, and B. Soderberg, Nucl. Phys. **B264**, 29 (1986).
[21] R. D. Field and S. Wolfram, Nucl. Phys. **B213**, 65 (1983).
[22] W. Bentz, T. Hama, T. Matsuki, and K. Yazaki, Nucl. Phys. **A651**, 143 (1999).
[23] I. C. Cloët, W. Bentz, and A. W. Thomas, Phys. Lett. B **659**, 214 (2008).

[24] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961); **124**, 246 (1961).
[25] J. C. Collins, Nucl. Phys. **B396**, 161 (1993).
[26] R. L. Jaffe and X. D. Ji, Nucl. Phys. **B375**, 527 (1992).
[27] M. Burkardt, Adv. Nucl. Phys. **23**, 1 (1996).
[28] G. Altarelli and G. Parisi, Nucl. Phys. **B126**, 298 (1977).
[29] M. Stratmann and W. Vogelsang, Nucl. Phys. **B496**, 41 (1997).
[30] R. L. Jaffe, Nucl. Phys. **B229**, 205 (1983).
[31] P. J. Huber, *Robust Statistics* (Wiley, New York, 1981).
[32] G. P. Lepage and S. J. Brodsky, Phys. Rev. D **22**, 2157 (1980).
[33] S. Strauss, S. Mattiello, and M. Beyer, J. Phys. G **36**, 085006 (2009).
[34] M. Miyama and S. Kumano, Comput. Phys. Commun. **94**, 185 (1996).