

Use of dispersion relations in hard exclusive processes and the partonic interpretation of deeply virtual Compton scattering

Gary R. Goldstein^{1,*} and Simonetta Liuti^{2,†}

¹*Department of Physics and Astronomy, Tufts University, Medford, Massachusetts 02155, USA*

²*Department of Physics, University of Virginia, Charlottesville, Virginia 22901, USA*

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Recently dispersion relations have been applied to hard exclusive processes such as deeply virtual Compton scattering, and a holographic principle was proposed that maps out the generalized parton distributions entering the soft matrix elements for the processes from their values on a given kinematical ridge. We examine possible pitfalls associated with the implicit, direct identification in this approach of the physical hadronic states with colored partons, and suggest an improved treatment of this assumption.

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A number of studies have recently advocated using dispersion relations (DRs) both to facilitate the extraction from deeply virtual exclusive experiments, such as deeply virtual Compton scattering (DVCS), of the soft matrix elements for hard exclusive processes, the generalized parton distributions (GPDs), and to determine their model parameters [1–3]. In this paper we are going to show that there are important limitations to the use of DRs for processes described by GPDs.

DRs have a long history in hadronic physics. For a general exclusive, two body hadronic reaction, invariant amplitudes can be written in terms of energy and angle variables, such as the Mandelstam variables s and t , or $\nu = (s - u)/4M$ and t . When the energy variable is continued into the complex plane, the amplitudes become holomorphic functions, i.e. analytic functions over regions of the complex plane. Unitarity of the amplitudes determines the physical intermediate states that in turn determine branch cuts in the complex energy plane. Each physical state has a kinematic threshold that fixes the branch point.

DRs were derived for *inclusive* deep inelastic scattering (DIS) as well, when viewed as forward virtual Compton scattering [4]. ν in this case translates into the virtual photon's energy in the laboratory system ($\nu \equiv \nu_{\text{Lab}}$), in turn connected to the variable $\omega = 1/x_{Bj}$, where $x_{Bj} = Q^2/2M\nu$. DIS can be considered as a special case of elastic scattering where unitarity relates the imaginary part of the forward amplitude to the total cross section, the inclusive sum over all physical final states allowed by the energy. As discussed thus far, DRs do not necessarily include the partonic structure of the target. Partonic degrees of freedom are integrated over. In fact, all remaining kinematical variables, including x_{Bj} , can be considered to be fixed by the kinematic conditions “external” to any partonic loop or QCD elaboration.

A connection with the partonic structure, through the operator product expansion (OPE), therefore QCD, can be

established e.g. by following the derivation in Ref. [4], where the important assumption is made that the support for both the integrals defining the Mellin moments of the operators and the final amplitude is $x_{Bj} \in [-1, 1]$, in the asymptotic limit, $Q^2 \rightarrow \infty$. It is also assumed that the intermediate states that are summed over in a factorized amplitude are physical. This leads to the identification of the (twist two) quark distribution, $H(x_{Bj})$ [4] with the measured structure function, $F_2(x_{Bj})$, or the imaginary part of the forward amplitude. More specifically, two steps are taken in establishing DRs for DIS: (i) the identification of the physical threshold for the scattering process, ν_{th} , with the continuum threshold, $\nu_C = M + m_\pi$, or $x_C = 1$, appearing in the integral definition of the scattering amplitude; and (ii) the identification of x_{Bj} with the partonic variable present in the factorized amplitude. The question of the applicability of dispersion relations to processes interpreted through the parton and QCD description is a long-standing theoretical concern. For example, it has been noted that difficulties arise in treating the pion form factor dispersion relation in terms of partons in the medium Q^2 region where data are available (see Refs. [5,6] and references contained therein to earlier work).

In this paper we argue that these assumptions cannot be carried straightforwardly to the off-forward case described e.g. in DVCS. In fact, as explained later, one has a mismatch between the supports for the scattering amplitude and for the corresponding DR, namely, $\nu_C \neq \nu_{\text{th}}$. This mismatch is a straightforward consequence of t -dependent physical thresholds, not present in the DIS forward/elastic case, that are long known to hinder the useful and practical applications of DRs. The mismatch exists for both ν and ζ . This point does not touch upon the partonic aspects of the process. However, in the factorized form of DVCS, described by a handbag picture, x_{Bj} in (ii) is replaced by two longitudinal fractions, X and ζ , where $X \equiv (kq)/(Pq)$ and the external variable, the skewness, $\zeta = (q\Delta)/(qP) \approx Q^2/(2M\nu_{\text{Lab}})$, with Δ being the momentum transfer for the two body scattering process, $\Delta^2 = t$ [we will use either the set (X, ζ, t) or the alternative

*gary.goldstein@tufts.edu

†sl4y@virginia.edu

variables, ($x = \frac{x-\xi/2}{1-\xi/2}$, $\xi = \frac{\xi}{2-\xi}$, t) throughout the paper—see Refs. [7,8] for reviews on DVCS]. The expression for the DVCS amplitude at leading order in QCD factorization is

$$T^{\mu\nu}(\nu, Q^2, t) = -\frac{1}{2} g_T^{\mu\nu} \bar{u}(p') \hat{n} u(p) \sum_{\text{flavors}} e_f^2 \mathcal{H}_f(\xi, t), \quad (1)$$

where the analog of the Compton form factor is

$$\mathcal{H}_f^+(\xi, t) = \int_{-1}^{+1} dx \frac{H_f^+(x, \xi, t)}{x - \xi + i\epsilon}. \quad (2)$$

The GPD $H(x, \xi, t)$ is convoluted with the hard part, $1/(x - \xi + i\epsilon)$, and integrated over x in the range $[-1, 1]$. Crossing symmetry is implemented by

$$H_f^{(\pm)}(x, \xi, t) = H_f(x, \xi, t) \mp H_f(-x, \xi, t), \quad (3)$$

recalling that for parton distribution functions, $q(-x) = -\bar{q}(x)$ relates negative x to positive x antiquark probability.

It follows straightforwardly from Eq. (2) that $\text{Im}\mathcal{H}(\xi, t) = H(\xi, \xi, t)$. To relate this to the discontinuity across the physical branch cut of a holomorphic function, unitarity is invoked through the insertion of a complete set of intermediate states,

$$\begin{aligned} \text{Im}\mathcal{H}(\xi, t) &= \int dX [\delta(X - \xi) + \delta(X)] \\ &\times \sum_N \langle P' | \bar{\psi}^+(k') | N \rangle \langle N | \psi^+(k) | P \rangle \\ &\times (2\pi) \delta(XP^+ + p_N^+ - P^+). \end{aligned} \quad (4)$$

The resulting analytic structure allows the DR to be written

$$\begin{aligned} \text{Re}\mathcal{H}^{(\pm)}(\xi, t) &= \frac{1}{\pi} \left[P.V. \int_{-1}^{\xi_{\text{th}}} dx \frac{H^{(\pm)}(x, x, t)}{x - \xi} \right. \\ &\left. + \int_{\xi_{\text{th}}}^{+1} dx \frac{H_{\text{unphys}}^{(\pm)}(x, x, t)}{x - \xi} \right], \end{aligned} \quad (5)$$

where the subscript ‘‘unphys’’ emphasizes that the integration should be over the whole range, but because the integration variable is now interpreted as the skewness, external to the quark loop, a threshold mismatch ensues due to the inelasticity of the two body process for nonzero t . In fact $\xi_{\text{th}} = [-t + (t^2 - 4M^2t)^{1/2}]/2M$ for $Q^2 \gg t$, the physical threshold for the two body, $\gamma^* p \rightarrow \gamma p'$ scattering process, originates from the limiting values for the angles defining the invariant $t = (q - q')^2$. One obtains in the limit $Q^2 \gg t$, $t_{\text{min}} = Q^4/4s - (q^{CM} - q'^{CM})^2 = -M^2 \xi^2/(1 - \xi)$. Notice that for DIS, the physical and continuum thresholds coincide because the final photon has the same Q^2 as the initial one, $t_{\text{min}} = 0$ and $\xi_{\text{th}} = x_{\text{th}} = 1 = \xi_C$.

In DVCS the region $x \in [\xi_{\text{th}}, 1]$ is unphysical and the second term in Eq. (5) cannot be obtained from experiment. The physical meaning of this discrepancy is illus-

trated in Fig. 1 where both the continuum and physical thresholds for several variables describing DVCS, s , ν , and ξ , are plotted as a function of t . For s , as Q^2 increases, only higher and higher invariant mass states are sampled. Under $s \rightarrow u$ crossing, there are corresponding branch cuts for negative ν . So it is not clear how the dispersion integral can be written in the partonic variables. Although the mismatch between physical and continuum thresholds addresses the issue of the physical interpretation of GPDs, it was a well-known problem for two body scattering processes [9], where it was dealt with by either constructing models for the analytic continuation, or developing some other prescription. The threshold mismatch seen in these fixed t DRs in ξ could be reduced by introducing new variables, a method used in hadronic processes. We will show the consequences of introducing a jet mass in the factorized picture [10].

To illustrate the different physics involved in forward and off-forward processes, respectively, we discuss the proof of the DR given in Ref. [1]. This was obtained similarly to the DIS case (see [4], for example). The hadronic tensor related to the forward Compton amplitude $T^{\mu\nu}(x, Q^2)$ can be given a partonic interpretation when the operators are expressed via interacting quark fields and subjected to the OPE. The coefficients of the leading twist terms in that expansion are the Mellin moments of the quark distribution functions $H(x)$. Summing this geometric series (for $|x| > 1$) leads to the form of the DR for $T^{\mu\nu}(x, Q^2)$ with integrand $H(\alpha)/(x \pm \alpha)$. Because the

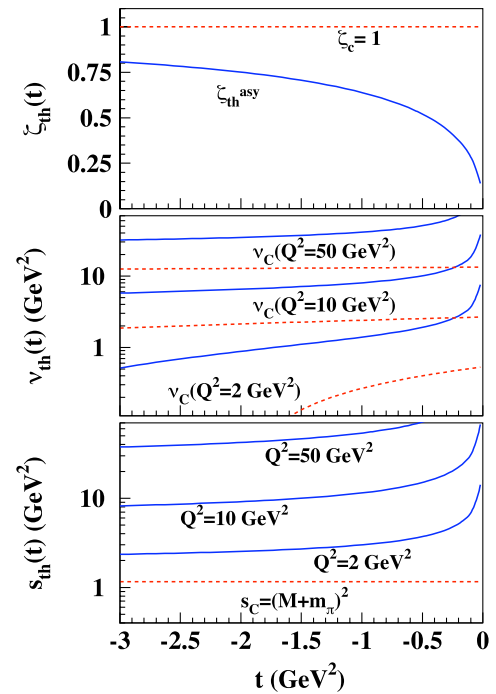


FIG. 1 (color online). Thresholds for the variables ξ , ν , and s plotted vs t . The dashed lines are the continuum thresholds whereas the full lines represent the physical thresholds (see text).

Compton form factor is known to satisfy analyticity as a forward elastic amplitude, that analyticity allows the continuation of the integration to the complex x plane and the DR follows. In Ref. [1] the GPD, $H(x, \xi, t)$, enters observables through integration over x as in Eq. (2). The denominator in the integrand, which arises from the light cone limit of the struck quark's propagator, can be written as a geometric series in $\frac{x}{\xi}$. Because the GPD must satisfy polynomiality in ξ (the x moments are polynomials in ξ with t -dependent coefficients), based on the underlying covariance, the resulting series must converge for large $|\xi| > 1$. So in the complex ξ plane the $\mathcal{H}_f(\xi, t)$ will be analytic for the unphysical $|\xi| > 1$.

Then, by analogy with the hadronic DR, it is assumed that there is a *physical branch cut* from -1 to $+1$ on the real ξ axis and no other poles or cuts. For this interpretation however, the intermediate states, the \hat{s} -channel cuts, have to be determined, given nonzero t and Q^2 . But for these kinematic constraints the support is limited, as Eq. (5) indicates. A separate consideration is that intermediate states carry bare color, so there is no analog of unitarity for factorized DVCS. In DIS this distinction is irrelevant, but here however, the absence of intermediate hadronic states means that the GPD cannot have the proper physical branch cuts. Figure 1 shows that the gap remains even at high Q^2 .

The suggestion [2] that experimental analyses provide information *only on the kinematical domain on a ridge at $x = \xi$ and fixed t and Q^2* therefore depends on whether one can disregard or treat otherwise the unphysical term in Eq. (5) (note that in next-to-leading order analyses the domain is smeared beyond the ridge [2]). It is this point about the sufficiency of the “ridge” that we are examining with care, by assuming DRs are satisfied in various model GPDs.

To illustrate these crucial questions we consider two examples of models that should satisfy DRs, namely, an asymptotic Regge model and a covariant spectator model. The Regge pole model contributes to the scattering amplitude $T(\nu, t, Q^2)$ for a single Regge trajectory $\alpha(t)$ in the simple form

$$T^R(\nu, Q^2, t) = \beta(t, Q^2)(1 - e^{i\pi\alpha(t)})\left(\frac{\nu}{\nu_0}\right)^{\alpha(t)}. \quad (6)$$

So the DR should be

$$\text{Re } T^R(\nu, Q^2, t) = \frac{2\nu}{\pi} \int_{\nu_{\text{th}}}^{\infty} d\nu' \frac{\text{Im } T^R(\nu', Q^2, t)}{\nu'^2 - \nu^2}, \quad (7)$$

providing that the integral converges. For a low lying trajectory or large enough t so that $\alpha(t) < 0$, this will converge. But this relation is exact only for $\nu_{\text{th}} = 0$. The actual threshold for $Q^2 = 0$ is at $-t/4M$ and further for nonzero Q^2 . So the DR is satisfied asymptotically, for $\nu \gg \nu_{\text{th}}$. This is illustrated for several cases in Fig. 2(a) where the real and imaginary parts are plotted against ν

against ζ . The directly calculated real part and the dispersion relation result for the real part in this unsubtracted dispersion relation are quite separated for low ν or high ζ . Note that for current typical JLab kinematics (Hall B) $Q^2 \leq 4.5 \text{ GeV}^2$, $|t| < 2.0 \text{ GeV}^2$, and $0.09 < x_{Bj} < 0.6$, so the nonasymptotic values of Q^2 and t are quite relevant.

We next consider a quark-diquark model with spinless partons (for simplicity, as in Ref. [3]). Because this is a covariant model it satisfies the polynomiality condition thus allowing the GPD to be continued into the large ξ or small x/ξ region in which the analyticity requirements apply [1]. The subtraction, $\Delta(\xi, t)$, the difference between the evaluation of Eqs. (2) and (5) for the symmetric case, is presented in Fig. 2(b), which clearly displays non-negligible ξ and t variations of $\Delta(\xi, t)$, thus demonstrating that $\Delta(\xi, t)$ cannot be identified with a dispersion subtraction constant. In this case, since all x , ξ , and t dependences are part of the model in a nontrivial way, the threshold ξ_{max} is necessarily the physical one. Given that our subtraction “constant” $\Delta(\zeta, t)$ is actually a function of ζ , due to the threshold dependence, we cannot see a direct relation to either the so-called D-term [1] or the $J = 0$ fixed pole [11], although at high $|t|$ and Q^2 , there is a flattening out. We have not addressed the nature of the states yet. For GPDs some kind of a duality model needs to be introduced that makes the colored quark and remnant jets look like hadrons (see the recent study on this subject [12]), in addition to providing a prescription for analytically continuing to the appropriate threshold. The prescription we suggest as an alternative to analytic continuation aims at reducing the kinematical threshold mismatch by replacing the variables

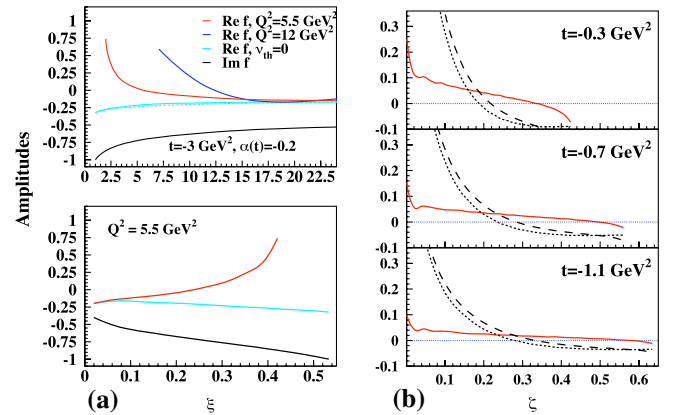


FIG. 2 (color online). Threshold induced violations of dispersion relations for both the Regge model (a), and the covariant quark-diquark model (b), described in the text. For the Regge model we show the gap between the calculated real parts using the physical threshold at $t = 0.3 \text{ GeV}^2$ and different values of Q^2 , and the analytic continuation labeled as $\nu_{\text{th}} = 0$. The same gap is plotted vs ξ in the lower panel of (a). For the quark-diquark model calculation the direct (dotted lines), the dispersion relation (dashed lines), and their difference (solid lines) are plotted vs ζ for different t values.

used in Figs. 1 and 2 with variables including a mass, m_J , for the hard partonic jet. Our prescription is in line with Ref. [10], where it was exactly pointed out that kinematical threshold mismatches might arise in the collinear factorization approach if the mass of the hard partonic jet is disregarded. Although considering jets with mass is not equivalent to hadronization, it might get us closer to what a hadronic intermediate state is. Following [10,13] we replace the hard propagators for the struck quark in the hard part of the handbag with a variable jet mass

$$\frac{1}{\zeta - X + i\epsilon} \quad \text{with} \quad \frac{1}{\zeta(1 + \frac{m_J^2}{Q^2}) - X + i\epsilon}. \quad (8)$$

The dispersion relation becomes

$$\text{Re } H = PV \int dX \int dm_J^2 \rho(m_J^2) \frac{H(X, (1 + \frac{m_J^2}{Q^2})X, t)}{\zeta - X}, \quad (9)$$

where $\rho(m_J^2)$ is a jet mass distribution. The results shown in Fig. 3 demonstrate that the gap obtained as a result of having two different thresholds in the massless calculation (Fig. 2) is softened, due to the new set of variables that better account for the correct range of integration over the partons' virtuality and transverse momentum (see also discussion in [10]).

In conclusion, we have shown the limitations of applying DRs to deeply virtual exclusive processes, and have given insight into the partonic nature of GPDs by examining the role of variables external and internal, respectively, to the quark loop that appears in the leading order factorization formulation. In particular, we show that it could lead

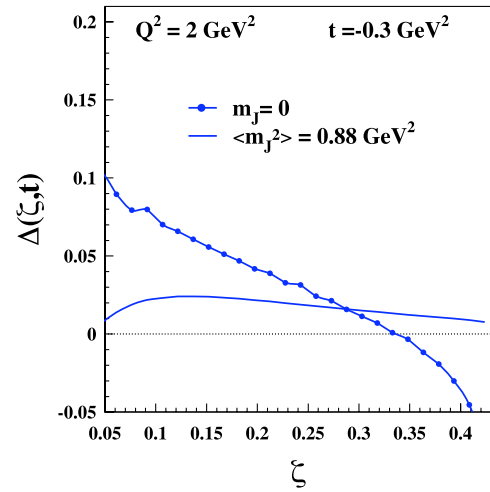


FIG. 3 (color online). The difference between the dispersion relation and direct calculation in a scalar quark-diquark model including the hard jet hadronic mass as from Eqs. (8) and (9).

to misleading results to base global parametrizations on DRs as recently done in [14]. To pin down GPDs we advocate comprehensive measurements of both the real and imaginary components through various asymmetries and cross section components in a wide range of all kinematical variables ζ , t , and Q^2 .

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