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Invariants for particle propagation in non-Abelian fields

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Characterizing the propagation of particles in an external non-Abelian field only in terms of invariants constructed from its field tensor is not always sufficient, especially, in many analytically tractable and phenomenologically interesting cases.

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The concept of external fields has many uses, from theoretical tools to phenomenological motivations. In the latter case, the motivations range from computational feasibility to the fact that the vacuum without external field is not the correct expansion point [1]. The investigation of quantum electrodynamics in external fields leads, e.g., to the seminal and as yet untested prediction of particle creation in (originally constant electric) external fields [2]. This and other effects are about to be tested, e.g., with ultrastrong light sources [3]. The generalization to quantum chromodynamics is of interest in the context of high energy collisions. A concept that is inseparably linked to external fields is that of effective actions [2,4]. For a covariantly constant background [see Eq. (8)], the corresponding computations proceed in close analogy to the Abelian case [5].

Observables in gauge field theories are by definition gauge invariant. In the presence of backgrounds this means that the results may only depend on said background in a gauge invariant way. A way to make the gauge invariance manifest is to identify gauge invariant combinations of the background field tensor [6] and express the observables in terms of these. Backgrounds allowing for analytically tractable calculations, due to technical limitations, have typically only a small number of nonzero Lorentz and color components. Therefore, they are subject to the Wu-Yang ambiguity [7]. It states that in non-Abelian field theories there exist field tensors that have realizations in terms of different gauge field configurations that are *not* gauge equivalent. To see that these different gauge fields do indeed lead to different physics consider the constant non-Abelian field tensor,

$$E_3^a = F_{03}^a = \partial_0 A_3^a - \partial_3 A_0^a + f^{abc} A_0^b A_3^c, \tag{1}$$

and all other components equal to zero. f^{abc} stands for the antisymmetric structure constant of the gauge group $\mathcal{G}$. This field tensor can be realized by the gauge field,

$$A_3^a = +E_3^a x^0, \tag{2}$$

and zero otherwise. The gauge transformation $U = \mathrm{e}^{-iE_3 x^3 x^0}$, where $E_3 = E_3^a T^a$ and T^a represent the generators of $\mathcal{G}$, turns it into $A_0^a = -E_3^a x^3$, while leaving the field tensor invariant. Now regard,

$$A_0^a = a_0^a, \quad \text{and} \quad A_3^a = a_3^a, \tag{3}$$

where a_0^a and a_3^a are constant such that $f^{abc} a_0^b a_3^c = E_3^a$. (All of the above field configurations satisfy Lorenz as well as Coulomb gauge.) The gauge transformation that removes a_3^a reads $U = \mathrm{e}^{-ia_3 x^3}$. It turns a_0 into

$$U a_0 U^\dagger = \mathrm{e}^{-ia_3 x^3} a_0 \mathrm{e}^{+ia_3 x^3} = a_0 \mathrm{e}^{+2ia_3 x^3} \neq a_0, \tag{4}$$

where we assumed $\{a_0, a_3\} = 0$ for simplicity. This gauge transformation also does not leave the field tensor invariant: Assuming $\{a_3, E_3\} = 0$,

$$U E_3 U^\dagger = E_3 \mathrm{e}^{+2ia_3 x^3}. \tag{5}$$

Another way of seeing that this last configuration is not gauge equivalent to the first is computing gauge invariant Wilson loops. Take the rectangular path $\mathcal{C}(x^0, x^3)$: $(0,0) \to (y^0, 0) \to (y^0, y^3) \to (0, y^3) \to (0,0)$. For configurations (2) and (3) this yields

$$W = \mathrm{tr}\,\mathrm{e}^{i \oint_{\mathcal{C}} dx \cdot A} = \mathrm{tr}\,\mathrm{e}^{iE_3 y^0 y^3}, \tag{6}$$

and

$$W = \mathrm{tr}\,\mathrm{e}^{-ia_3 y^3} \mathrm{e}^{-ia_0 y^0} \mathrm{e}^{ia_3 y^3} \mathrm{e}^{ia_0 y^0}, \tag{7}$$

respectively, which do not coincide.

In 4 dimensions, a necessary condition for the presence of this ambiguity is $\det \mathbb{F} = 0$, where $\mathbb{F}_{\mu\nu}^{ab} = \frac{1}{2} \epsilon_{\mu\nu\kappa\lambda} F^{c\kappa\lambda} f^{abc}$ [8]. (Accordingly, such a determinant also appears as part of the Jacobian when translating path integrals from the gauge field to a field tensor formulation [9].) $\mathbb{F}$ is in the adjoint representation. Therefore, each submatrix of a single Lorentz component has zero eigenvalues. The corresponding eigenvectors of different submatrices must be misaligned to have $\det \mathbb{F} \neq 0$.

Further, configuration (2) is covariantly constant (a gauge invariant statement),

$$D_\lambda F_{\mu\nu} = 0 \quad \forall\ \lambda, \mu, \nu, \tag{8}$$

as there $D_\lambda F_{\mu\nu} = \partial_\lambda F_{\mu\nu} = 0 \quad \forall\ \lambda, \mu, \nu$. For configuration (3) we have $\partial_\lambda F_{\mu\nu} = 0 \quad \forall\ \lambda, \mu, \nu$, and thus, $D_\lambda^{cd} F_{\mu\nu}^d = f^{abc} A_\lambda^a F_{\mu\nu}^b \neq 0$ for λ, μ, and $\nu \in \{0; 3\}$.

Thus, here $D_\lambda F_{\mu\nu}$ are the gauge covariant quantities that allow one to distinguish between the gauge-inequivalent settings. They cannot be expressed in terms of $F_{\mu\nu}^a$ alone.

They can serve to construct gauge invariant quantities, which can also be contracted into Lorentz scalars. In particular, the current in the Yang-Mills equation, $D^{ab}_\mu F^{b\mu\nu} = J^{a\nu}$, can be used in $J^a_\mu J^{a\mu}$. After all, covariant conservation is a sufficient albeit not necessary condition for a vanishing current. Hence, for covariantly constant fields all the invariants involving J^a_μ are zero. A related invariant is $(D^{ab}_\kappa F^b_{\mu\nu})(D^{ac\kappa}F^{c\mu\nu})$ [10].

In fact, the covariant derivative and not the field tensor is the elementary building block for invariants, in the sense that it carries more information than the latter. Odd powers of the covariant derivative cannot be contracted into Lorentz scalars. Order 2 does not have nontrivial contributions. Order 4 has $F_{\mu\nu}F^{\mu\nu}$ and $F_{\mu\nu}\tilde{F}^{\mu\nu}$. Order 6 contains the aforementioned $J^a_\mu J^{a\mu}$.

A rescaling $a_0 \to a_0 c$, $a_3 \to a_3/c$ leaves the field tensor invariant [10]. This rescaling cannot be generated by a unitary global gauge transformation and hence, the parameter c characterizes a continuous class of gauge-inequivalent gauge field configuration belonging to the same field tensor. (There are no additional classes of gauge-inequivalent representations, for a constant field tensor; covariantly constant and static configurations exhaust all possibilities [10].) Fixing as reference $J^a_\mu J^{a\mu} = 0$ for $c = 1$, we obtain $J^a_\mu J^{a\mu} = (c^{-2} - c^2)|E_3|^3$. The $c = 1$ case can be told apart from $J^a_\mu = 0$ by means of $J^a_\mu J^{b\mu} J^a_\nu J^{b\nu} = (c^{-4} + c^4)|E_3|^6$, where for simplicity we assumed $J^a_0 J^a_3 = 0$. After inclusion of a third gauge field component, such that all components are noncommuting, which leads to 3 nonzero components for the field tensor, this continuous scaling symmetry breaks down to a simultaneous overall sign change. The Wilson loop (7) is also c dependent. In comparison, Klein-Gordon and Dirac propagators have additional structure [11]. To illustrate more how much the situations with equal field tensor, but different gauge-inequivalent gauge fields differ, we study these propagators in the presence of the 2 different configurations (2) and (3).

A *more general nonstatic configuration* $E_3 = E_3(x^0) = [Q, E(x^0)]$, where $Q = Q^a T^a = \text{const}$, can also be realized either as a derivative of

$$A_3 = A_3(x^0) = [Q, \int dx^0 E(x^0)], \qquad (9)$$

or as a commutator of a constant

$$A_3 = QC \quad \text{with} \quad A_0 = E(x^0)/C, \qquad (10)$$

where C also accounts for the correct mass dimension of the vector potential. Clearly, $\det\mathbb{F} = 0$. These field configurations still satisfy Coulomb, but not always Lorenz gauge because of $A_0 = A_0(x^0)$, which, however, could be rotated away. For a covariantly constant electric field, the first realization leads only to $J_3 \neq 0$, while the second has also $J_0 \neq 0$, i.e., a net charge density. For the second configuration, the field tensor can again not distinguish between gauge fields rescaled by a constant c as described above, which here is equivalent to dividing C by c. When it comes to gauge transformations, $A_3 = QC$ can be removed by $U = e^{-iQCx^3}$. This leads to $A_0 C \to e^{-iQCx^3} E e^{iQCx^3}$, which is x^3 dependent. Choosing $x^0 = 0$ as the lower integration bound in the expression for A_3, we find for the Wilson loops the 2 different results,

$$W = \text{tr} e^{i[Q, \int_0^{y^0} dt E(t)]y^3} \quad \text{and}$$

$$W = \text{tr} e^{-iQCy^3} e^{i\int_0^{y^0} dt E(t)/C} e^{iQCy^3} e^{i\int_0^{y^0} dt E(t)/C}.$$

In mixed representation, in a purely time-dependent background, the equation of motion for the *Klein-Gordon* propagator $\tilde{S} = \tilde{S}(x_0, y_0, \vec{p})$ reads

$$(\partial_0^2 - i\dot{A}_0 - 2iA_0\partial^0 + 2A_j p^j - A\cdot A + \omega^2)\tilde{S} = \delta^{(1)}, \qquad (11)$$

where $\delta^{(1)} = \delta(x_0 - y_0)$, $j \in \{1;2;3\}$, and $\omega^2 = |\vec{p}|^2 + m^2$. For configuration (2) this becomes

$$[\partial_0^2 + (p_3 - E_3 x^0)^2 + m_\perp^2]\tilde{S} = \delta^{(1)}, \qquad (12)$$

where $m_\perp^2 = |\vec{p}_\perp|^2 + m^2$. A decomposition into eigenvectors of E_3 leads to

$$[\partial_0^2 + (p_3 - e_n x^0)^2 + m_\perp^2]\tilde{S}_n = P_n \delta^{(1)}, \qquad (13)$$

where $E_3|n\rangle = e_n|n\rangle$, $\langle n|m\rangle = \delta_{nm}$, $P_n = |n\rangle\langle n|$, and $\tilde{S}_n = P_n\tilde{S}$. The homogeneous solutions to this differential equation are

$$M_l(x^0) = t^{-1/2} M_{-im_\perp^2/(4e_n),-(-1)^l/4}(ie_n t^2), \qquad (14)$$

where $l \in \{1;2\}$, $t = x^0 - p_3/e_n$, and $M_{\kappa,\mu}(z)$ is a Whittaker function. [See Eqs. (13.1.31) and (13.1.32) in [12].] With the boundary conditions $\tilde{S} = 0$ and $\dot{\tilde{S}} = 1$ at $x_0 = y_0$, we find for the retarded solution,

$$\tilde{S} = \sum_n P_n \frac{M_1(x^0)M_2(y^0) - (1 \leftrightarrow 2)}{\dot{M}_1(y^0)M_2(y^0) - (1 \leftrightarrow 2)} \theta^{(1)}, \qquad (15)$$

where $\theta^{(1)} = \theta(x_0 - y_0)$ stands for the Heaviside step function. The denominator contains a known Wronskian and evaluates to i. [See Eqs. (13.1.34), (13.1.32), (13.1.33), and (13.1.22) in [12].] In the limit of large t, Eq. (14) becomes [see Eqs. (13.1.32) and (13.5.1) in [12]]

$$M_l \to i^{c_l/2}\Gamma(c_l) e^{-\pi m_\perp^2/(8e_n)} t^{-1/2} \times \left[\frac{(e_n t^2)^{-im_\perp^2/(4e_n)} e^{-(i/2)e_n t^2} i^{c_l/2}}{\Gamma(\frac{c_l}{2} - \frac{im_\perp^2}{4e_n})} + \text{c.c.}\right], \qquad (16)$$

where $c_1 = 3/2$ and $c_2 = 1/2$. In the previous expression, we can already see the typical exponential $m_\perp$ behavior of the pair production rate.

For configuration (3), Eq. (11) becomes

$$[\partial_0^2 - 2ia_0\partial^0 - (a_0)^2 + (p_3 - a_3)^2 + m_\perp^2]\underline{\tilde{S}} = \delta^{(1)}. \qquad (17)$$

Let us continue with $SU(2)$ [at least an $SU(2)$ subgroup], the generalization to higher gauge groups being straightforward. Define

$$[\partial_0^2 + 2ia_0\partial^0 - (a_0)^2 + (p_3 + a_3)^2 + m_\perp^2]\underline{\tilde{s}} = \underline{\tilde{S}}.$$

Then, from Eq. (17), assuming $\{a_0, a_3\} = 0$,

$$\{[\partial_0^2 - (a_0)^2 + (a_3)^2 + \omega^2]^2 + 4(a_0)^2\partial_0^2 - 4(p_3)^2(a_3)^2\}\underline{\tilde{s}} = \delta^{(1)}. \quad (18)$$

The exponential ansatz $\underline{\tilde{s}} \sim e^{\lambda x^0}$ yields the 4 values,

$$\lambda_\pm^2 = -[\omega^2 + (a_0)^2 + (a_3)^2] \pm 2\sqrt{(a_0)^2(a_3)^2 + (a_0)^2\omega^2 + (p_3)^2(a_3)^2}.$$

For $[\omega^2 - (a_0)^2 + (a_3)^2]^2 < 4(p_3)^2(a_3)^2$, this corresponds to 2 oscillatory, 1 exponentially decaying, and 1 exponentially growing mode; otherwise, the behavior is purely oscillatory. For comparison, repeating the same steps for a magnetic field $F_{12}^a = f^{abc}a_1^b a_2^c$ yields

$$\lambda_\pm^2 = -[(a_1)^2 + (a_2)^2 + \omega^2] \pm 2\sqrt{(a_1)^2(p_1)^2 + (a_2)^2(p_2)^2},$$

implying always purely oscillatory solutions.

In mixed representation the *Dirac* equation is given by

$$(i\gamma^0\partial_0 - \gamma^j p_j + \not{A} - m)\tilde{G} = \delta^{(1)}, \quad (19)$$

where $\tilde{G} = \tilde{G}(x_0, y_0, \vec{p})$. With the help of

$$-(i\gamma^0\partial_0 - \gamma^j p_j + \not{A} + m)\tilde{g} = \tilde{G}, \quad (20)$$

we obtain the squared Dirac equation,

$$[\partial_0^2 - 2iA^0\partial_0 + 2p_jA^j - i\gamma^0\dot{\not{A}} - \not{A}\not{A} + \omega^2]\tilde{g} = \delta^{(1)}. \quad (21)$$

For configuration (2) this becomes

$$[\partial_0^2 + (p_3 - E_3x^0)^2 + m_\perp^2 - i\gamma^0\gamma^3E_3]\tilde{g} = \delta^{(1)}. \quad (22)$$

We continue by carrying out a decomposition with the projectors $P_\pm = (1 \pm \gamma^0\gamma^3)/2$ and P_n,

$$[\partial_0^2 + (p_3 - e_nx^0)^2 + m_\perp^2 \mp ie_n]\tilde{g}_\pm^n = P_nP_\pm\delta^{(1)}, \quad (23)$$

where $\tilde{g}_\pm^n = P_nP_\pm\tilde{g}$. Up to the substitutions $m_\perp^2 \to m_\perp^2 \mp ie_n$, $\tilde{g}_\pm^n$ are the same as Eq. (15),

$$i\tilde{g} = \sum_{n,\pm} P_\pm P_n[M_1^\pm(x_0)M_2^\pm(y_0) - (1 \leftrightarrow 2)]\theta^{(1)}. \quad (24)$$

In the limit of large t, $M_l^\pm$ become [see Eqs. (13.1.32) and (13.5.1) in [12]],

$$M_l^\pm \to i^{c_l/2\mp 1/4}\Gamma(c_l)e^{-\pi m_\perp^2/(8e_n)}(e_nt^2)^{-1/4} \times\left[\frac{(e_nt^2)^{\pm 1/4 - im_\perp^2/(4e_n)}e^{-(i/2)e_nt^2}i^{c_l/2}}{\Gamma(\frac{c_l}{2} \pm \frac{1}{4} - \frac{im_\perp^2}{4e_n})} + \begin{pmatrix}\text{c.c.\&}\\ \pm \leftrightarrow \mp\end{pmatrix}\right]. \quad (25)$$

At the end, the Dirac propagator is obtained by putting Eq. (24) into Eq. (20). At late times, the Dirac operator in Eq. (20) is dominated by the gauge field term, which grows linearly and the derivative term, which, when acting on the Gaussian in time in the previous equation also generates an extra factor of time. Hence, the dominant components of $\tilde{G}$ are growing approximately like the square root of time. If we use $\tilde{G}$ to construct the fermion current $\bar{\psi}\gamma^\mu\psi$ this factor appears twice and the current grows linearly in time. Therefore, one talks of a constant pair production rate in this field configuration.

For configuration (3), Eq. (21) becomes

$$[\partial_0^2 - 2ia^0\partial_0 - (a_0)^2 + (p_3 - a_3)^2 + m_\perp^2 - i\gamma^0\gamma^3E_3]\underline{\tilde{g}} = \delta^{(1)}. \quad (26)$$

We carry out the same decomposition with the projectors $P_\pm$ as before for Eq. (22),

$$[\partial_0^2 - 2ia^0\partial_0 - (a_0)^2 + (p_3 - a_3)^2 + m_\perp^2 \mp iE_3]\underline{\tilde{g}}_\pm = P_\pm\delta^{(1)}, \quad (27)$$

where $\underline{\tilde{g}}_\pm = P_\pm\underline{\tilde{g}}$. Define

$$[\partial_0^2 + 2ia^0\partial_0 - (a_0)^2 + (p_3 + a_3)^2 + m_\perp^2 \pm iE_3]\underline{\tilde{\Gamma}}_\pm = \underline{\tilde{g}}_\pm. \quad (28)$$

Then, with pairwise anticommuting a_0, a_3, and E_3,

$$\{[\partial_0^2 - (a_0)^2 + (a_3)^2 + \omega^2]^2 + 4(a_0)^2\partial_0^2 - 4(p_3)^2(a_3)^2 + (E_3)^2\}\underline{\tilde{\Gamma}}_\pm = P_\pm\delta^{(1)}. \quad (29)$$

From the exponential ansatz $\underline{\tilde{\Gamma}}_\pm \sim e^{\mu x^0}$ we get

$$\mu_\pm^2 = -[\omega^2 + (a_0)^2 + (a_3)^2] \pm\sqrt{4[(a_0)^2(a_3)^2 + (a_0)^2\omega^2 + (p_3)^2(a_3)^2] - (E_3)^2},$$

which leads to a purely oscillatory behavior, as does the analogous result for a magnetic B_3 field,

$$\mu_\pm^2 = -[(a_1)^2 + (a_2)^2 + \omega^2] \pm\sqrt{4[(a_1)^2(p_1)^2 + (a_2)^2(p_2)^2] + (B_3)^2}. \quad (30)$$

In conclusion, there exist non-Abelian field tensors that can be realized by different gauge field configurations that are not linked by gauge transformations, i.e., that are not gauge equivalent. Under these circumstances the covariant derivative carries more information than its commutator, the field tensor. In most of the gauge-inequivalent configurations leading to the same field tensor there exist observables (gauge invariant quantities) that cannot be expressed exclusively in terms of the field tensor. Here, we have demonstrated this explicitly for various field tensors that allow for gauge-inequivalent gauge field realizations. As examples we have picked static field tensors, electric or magnetic, and purely time-dependent configurations. Concretely, we showed that a direct gauge transformation of different gauge field configurations into each other cannot be found despite the fact that they yield the same field tensor; further, that for these different configurations the corresponding Wilson loops and Yang-Mills currents differ, as do the Klein-Gordon and Dirac propagators. For example, while the induced fermion current in the Abelian-like realization for a static electric field exhibits asymptotically linear growth with time, which leads to the rate interpretation of the result, the propagators in the genuinely non-Abelian realization possess only purely oscillatory modes. In the latter realization, the scalar propagator can also feature exponentially growing and decaying modes in the presence of an electric field. A particular quantity that cannot be expressed in terms of a Wu-Yang ambiguous field tensor is the Yang-Mills current. It is exactly the covariantly constant case, where this current vanishes, which explains why the effective actions for scalars or fermions in this configuration can be expressed in terms of the field tensor alone. For $\det\mathbb{F} \neq 0$ there is no Wu-Yang ambiguity and A^a_μ can be expressed in terms of $F^a_{\mu\nu}$ and therefore, all invariants and observables.

In the worldline approach [13] to effective actions all these differences discussed above reflect in the precession of the color as described by Wong's equation [14].

The above facts may have consequences for the flux-tube picture [15] for ultrarelativistic collisions, which features static chromoelectric fields. Depending on how the latter is realized, by configurations (2) or (3) [or if, e.g., a decaying field is assumed by configurations (9) or (10)], the particle yields differ. J_0 can serve to distinguish between the realizations.

As mentioned in [16], also Coulomb fields have $\det\mathbb{F} = 0$. Boosted onto the light cone, i.e., as Weizsäcker-Williams fields, they are used in the color glass condensate framework [17] to model the initial conditions of ultrarelativistic collisions.

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