Dynamical compactification and inflation in Einstein-Yang-Mills theory with higher derivative coupling

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We study cosmology of the Einstein-Yang-Mills theory in ten dimensions with a quartic term in the Yang-Mills field strength. We obtain analytically a class of cosmological solutions in which the extra dimensions are static and the scale factor of the four-dimensional Friedmann-Lemaitre-Robertson-Walker metric is an exponential function of time. This means that the model can explain inflation. Then we look for solutions that describe dynamical compactification of the extra dimensions. The effective cosmological constant λ_1 in the four-dimensional universe is determined from the gravitational coupling, ten-dimensional cosmological constant, gauge coupling, and higher derivative coupling. By numerical integration, the solution with $\lambda_1 = 0$ is found to behave as a matter-dominated universe which asymptotically approaches flat space-time, while the solution with a nonvanishing λ_1 approaches de Sitter space-time in the asymptotic future.

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I. INTRODUCTION

There have been many attempts to consider extra dimensions in addition to our world of four-dimensional space-time, even though they have not been observed. The original idea dates back to Nordström [1], Weyl [2], Kaluza [3], and Klein [4], who considered extra dimensions in order to unify gravity and electromagnetic force in five space-time dimensions. Now the most promising unified theory, describing all fundamental forces including two types of nuclear forces, is considered to exist in ten, eleven, or twelve dimensions after string theory, *M* theory [5], and *F* theory [6] have appeared. Superstring theory is consistent in ten-dimensional space-time. The extra six dimensions should be compactified. Some people require supersymmetry in four-dimensional space-time and the extra-dimensional space was assumed as a Calabi-Yau manifold. After the discovery of *D*-branes, *D*-branes or more generally ‘‘branes’’ offer the possibility of large extra dimensions or the brane world scenario [7].

There were many efforts to describe cosmological solutions in the framework of higher-dimensional theories. Especially, the realization of de Sitter-like expansion of a four-dimensional part has attracted much attention in connection with the inflationary scenario or the current accelerated expansion of our Universe. One of those attempts is the flux compactification, which has received a lot of attention in recent years [8]. One of the most important and basic features of the flux compactification is to stabilize the size of a compactified space by certain configurations of high-rank differential form fields. There were also efforts to realize dynamical compactifications by using various kinds of scalar fields [9].

Before string theory was discovered, Cremmer and Scherk studied an attractive possibility of compactification with the size of a compactified space being stabilized [10]. In order to achieve it, they placed a nontrivial topological solution (soliton) of a gauge field on the compactified space, for instance a monopole on the sphere S^2 or a Yang-Mills instanton on the four-dimensional sphere S^4. In these cases, the compactified space is stabilized at a finite radius rather than decompactified to an infinite radius. So they called it ‘‘spontaneous compactification.’’

In this paper we would like to study if such a compactification can occur dynamically or not. In general, in order to stabilize a topological configuration of a Yang-Mills field in dimensions greater than four, we need higher order terms of the gauge field strength [11]. Some years ago Tchrakian introduced such a term, which we call the Tchrakian term, in order to generalize ’t Hooft-Polyakov monopoles and Yang-Mills instantons to those analogues in dimensions greater than four [12]. The term is not renormalizable, but still quadratic in the time derivative. Recently some of the present authors have numerically studied a monopolelike solution in six-dimensional Minkowski space by adding the Tchrakian term [13].[1]

[1]This was originally motivated by the computation of non-Abelian Berry’s phases in *T*-dualized unitary simplectic matrix model [14].

One of the authors has further studied an asymptotic solution of five-dimensional Tchrakian monopole, the generalization of Tchrakian monopole [15]. In the case of a six-dimensional sphere, an exact solution to a generalized self-duality relation has been constructed for SO(6) Yang-Mills fields with the Tchrakian term [16].[2] Then this relation has been successfully embedded in the Einstein-Yang-Mills theory with the Tchrakian term in the geometry of the direct product of the four-dimensional Minkowski space (anti–de Sitter space AdS_4) and S^6 of a constant radius, with (without) a ten-dimensional cosmological constant [18]. In this solution the gauge field distributes on S^6 homogeneously, so it is a natural generalization of Cremmer and Scherk [10]. At least for the Yang-Mills part, the configuration attains the minimum of the Bogomol'nyi bound when the radius of S^6 satisfies a certain relation with the gauge coupling constant and the coupling strength of the Tchrakian term. Therefore we expect that if we turn on the time variation of the space-time, we obtain a solution which describes the process of dynamical compactification.

In this paper we consider cosmological solutions with a time-dependent scale factor of the three dimensions as well as with a time-dependent radius of S^6, and study if there exist solutions with the radius of S^6 tending to a finite value, as a possible model of dynamical compactification.

This paper is organized as follows. In Sec. II, we describe our theory; that is, the Einstein-Yang-Mills theory with the Tchrakian term in ten dimensions. We review the discussion of Bogomol'nyi completion. In Sec. III, we introduce an ansatz on the ten-dimensional metric, namely, the direct product of the four-dimensional Friedmann-Lemaitre-Robertson-Walker metric and S^6 with the radius as a function of time. Then we specify a gauge configuration which satisfies the self-duality relation and solves the Yang-Mills equation with the Tchrakian term. In Sec. IV, simple analytical solutions with a fixed radius of S^6 are given. The four-dimensional part of the solutions is either Minkowski or de Sitter, depending on the choice of the model parameters. In Sec. V, we consider solutions that describe the process of dynamical compactification. We investigate the behavior of the solutions both analytically and numerically. In general, the four-dimensional part behaves as de Sitter plus small oscillations, while the S^6 radius undergoes damped oscillations toward a finite value. For a particular choice of the model parameters that gives the product of a flat space-time times S^6 with a fixed radius, we find the four-dimensional part behaves as a dust-dominated universe, that is, with the scale factor proportional to $t^{2/3}$. Sec. VI is devoted to conclusion and discussions.

[2]Generalization of instantons on the complex projective space $\mathbb{C}\mathbf{P}^3$ has been also given [17].

II. MODEL SETTING AND BOGOMOL'NYI EQUATION

Let us start from the following action in ten-dimensional space-time:

$$\begin{aligned} S_{\rm tot} &:= S_{\rm EH} + S_{\rm YMT}, \\ S_{\rm EH} &:= \frac{1}{16\pi G}\int dv \mathcal{R}, \\ S_{\rm YMT} &:= \frac{1}{16}\int \mathrm{Tr}\{-F\wedge *F + \alpha^2 (F\wedge F)\wedge *(F\wedge F) \\ &\quad - V_0 dv\}. \end{aligned} \tag{1}$$

Here dv is the invariant volume form, $\mathcal{R}$ is the scalar curvature with respect to the metric g_{MN}, and F is the field strength twoform which takes values in the Lie algebra SO(6). The star ($*$) denotes the Hodge dual operator acting on differential forms in ten dimensions. Our notation is summarized in Appendix A. For more details, see [19].

We consider the case where the space-time is locally a product space of $\mathcal{M}$ and $\mathcal{N}$. $\mathcal{M}$ is a four-dimensional curved space-time and $\mathcal{N}$ is a compact space. Let us denote the total space $\mathcal{T}$. The metric on this space is

$$\begin{aligned} ds^2 &= g_{\mu\nu}(x)dx^\mu dx^\nu + g_{IJ}(x,y)dy^I dy^J = ds^2_{\mathcal{M}} + ds^2_{\mathcal{N}}, \\ \mu,\nu &= 0,1,2,3, \qquad I,J = 4,5,\cdots,9. \end{aligned} \tag{2}$$

For the case where the field strength has only components along the compact directions, we can manipulate the Yang-Mills action as [18]

$$\begin{aligned} &\frac{1}{16}\int_{\mathcal{T}} \mathrm{Tr}\{-F\wedge *F + \alpha^2(F\wedge F)\wedge *(F\wedge F)\} \\ &= \frac{1}{16}\int_{\mathcal{M}} dv^{(4)} \int_{\mathcal{N}} \mathrm{Tr}[(F \mp \boldsymbol{i}\alpha\gamma_7 *_6 (F\wedge F)) \\ &\quad \wedge *_6 (F \mp \boldsymbol{i}\alpha\gamma_7 *_6 (F\wedge F))] \\ &\quad \pm \frac{1}{16}\int_{\mathcal{M}} dv^{(4)} \int_{\mathcal{N}} \mathrm{Tr} 2\boldsymbol{i}\alpha\gamma_7 F\wedge F\wedge F, \end{aligned} \tag{3}$$

where $*_6$ represents the Hodge dual operator along the compact direction $\mathcal{N}$. We call this procedure Bogomol'nyi completion. The term $Q := \int_{\mathcal{N}} \mathrm{Tr}\gamma_7 F^3$ is a surface term and it gives the bound on the energy density. Then the Bogomol'nyi equation is

$$F \mp \boldsymbol{i}\alpha\gamma_7 *_6 (F\wedge F) = 0. \tag{4}$$

If either of these equations is satisfied, the energy attains the minimum given by Q irrespective of the sign $\pm$.

Suppose that $A^{(0)}$ is a solution of equation of motion and $F^{(0)}$ is the corresponding field strength. We denote the fluctuations around this solution δA, $A = A^{(0)} + \delta A$. Let us expand the left-hand side of Eq. (4) in terms of the following fluctuations:

$$F - \boldsymbol{i}\alpha\gamma_7 *_6 F\wedge F = \mathcal{B}_0 + \mathcal{B}_1(\delta A) + \mathcal{B}_2(\delta A), \tag{5}$$

where

$$\mathcal{B}_0 := F^{(0)} - i\alpha\gamma_7 *_6 F^{(0)} \wedge F^{(0)},$$
$$\mathcal{B}_1(\delta A) := D_0\delta A - i\alpha\gamma_7 *_6 (D_0\delta A \wedge F_0 + F_0 \wedge D_0\delta A),$$
$$\mathcal{B}_2(\delta A) := q\delta A \wedge \delta A - i\alpha\gamma_7 *_6 (q\delta A \wedge \delta A \wedge F_0 + F_0 \wedge q\delta A \wedge \delta A) - i\alpha\gamma_7 *_6 [(D_0\delta A + q\delta A \wedge \delta A) \wedge (D_0\delta A + q\delta A \wedge \delta A)]. \quad (6)$$

Here $\mathcal{B}_0$ is the 0th order term with respect to δA. The term $\mathcal{B}_1(\delta A)$ is linear in δA. The remaining $\mathcal{B}_2(\delta A)$ includes higher order terms. By substituting this to Eq. (3), we obtain

$$\begin{aligned} &-\frac{1}{16}\int \mathrm{Tr}\{-F\wedge *F + \alpha^2(F\wedge F)\wedge *(F\wedge F)\} \\ &= -\frac{1}{16}\int dv^{(4)}\,\mathrm{Tr}\{\mathcal{B}_0\wedge *_6\mathcal{B}_0 + 2\mathcal{B}_0\wedge *_6\mathcal{B}_2(\delta A)\} \\ &\quad -\frac{1}{16}\int dv^{(4)}\,\mathrm{Tr}\{\mathcal{B}_1(\delta A)\wedge *_6\mathcal{B}_1(\delta A)\} + O(\delta A^3) \\ &\quad + \int(\text{total derivative}). \end{aligned} \quad (7)$$

Here the term $\mathcal{B}_0 \wedge *_6\mathcal{B}_1(\delta A)$ is a total derivative term because $A^{(0)}$ is a solution of the equation of motion. The term $2\mathcal{B}_0 \wedge *_6\mathcal{B}_2(\delta A)$ includes the indefinite quadratic form of δA, which might yield a tachyonic mass term. When $A^{(0)}$ is a solution of $\mathcal{B}_0 = 0$ which is one of Eq. (4), no tachyonic mass term appears in gauge sector. We mention that this does not necessarily mean the stability of the system under the presence of fluctuations of both the metric and the gauge field. This is an issue to be studied in the future.

III. ANSATZ FOR THE METRIC AND GAUGE FIELDS

In this section we consider time-dependent solutions in the sense of Freund [20]. Namely, the metric is assumed to be in the form,

$$\begin{aligned} ds^2 &= ds_4^2 + ds_6^2; \\ ds_4^2 &= -dt^2 + L_0^2 e^{2\phi_1}\frac{|d\sigma|^2}{(1+\kappa|\sigma|^2/4)^2}, \\ ds_6^2 &= L_0^2 e^{2\phi_2}\frac{|dy|^2}{(1+|y|^2/4)^2}, \end{aligned} \quad (8)$$

where the coordinates $\sigma^i = (\sigma^1, \sigma^2, \sigma^3)$ span the three-dimensional space and $y^I = (y^4, y^5, \cdots, y^9)$ span S^6. $|\sigma|^2 := (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2$ and $|y|^2 := (y^4)^2 + (y^5)^2 + \cdots + (y^9)^2$. The parameter κ is ± 1 or 0. ϕ_1 and ϕ_2 are functions of time t. L_0 is a constant with dimension of length. The radius of S^6 is given by $R = L_0 e^{\phi_2}$. These types of metrics were considered in various contexts, for instance in [21,22].

The SO(6) gauge field configuration, represented in terms of differential forms, is assumed to be in the form

$$A = \frac{1}{4qL_0 e^{\phi_2}}\gamma_{ab} y^{a+3} V^{b+3}, \quad (9)$$

where $a, b = 1, 2, \cdots, 6$ are the indices of the Lie algebra of SO(6), $\gamma_{ab} := (1/2)[\gamma_a, \gamma_b]$ are the infinitesimal generators represented by spinor, and V^I is the vielbein of the six-dimensional metric ds_6^2,

$$V^I := L_0 e^{\phi_2}\frac{dy^I}{(1+|y|^2/4)}. \quad (10)$$

q is the gauge coupling constant. In the configuration of the gauge field A, the internal indices $a, b, \cdots$ and the spatial indices $I, J, \cdots$ are identified by an embedding of the spin connection of the six-dimensional sphere into the gauge group.

The exterior derivatives of the vielbeins are

$$\begin{aligned} dV^I &= -L_0 e^{\phi_2}\frac{\delta_{JK} y^J dy^K \wedge dy^I}{2(1+|y|^2/4)^2} + L_0\frac{\dot\phi_2 e^{\phi_2} dt\wedge dy^I}{(1+|y|^2/4)} \\ &= -\frac{\delta_{JK} y^J V^K \wedge V^I}{2L_0 e^{\phi_2}} + \dot\phi_2 dt \wedge V^I. \end{aligned} \quad (11)$$

Then the Ricci tensor components are given by

$$\begin{aligned} \mathcal{R}_{tt} &= -3(\ddot\phi_1 + \dot\phi_1^2) - 6(\ddot\phi_2 + \dot\phi_2^2), \\ \mathcal{R}_{ij} &= g_{ij}\left(\ddot\phi_1 + 2\frac{\kappa}{L_0^2 e^{2\phi_1}} + 3\dot\phi_1^2 + 6\dot\phi_1\dot\phi_2\right), \\ \mathcal{R}_{IJ} &= g_{IJ}\left(\ddot\phi_2 + 5\frac{1}{L_0^2 e^{2\phi_2}} + 6\dot\phi_2^2 + 3\dot\phi_1\dot\phi_2\right). \end{aligned} \quad (12)$$

The scalar curvature is

$$\mathcal{R} = 6\ddot\phi_1 + 12\ddot\phi_2 + 12\dot\phi_1^2 + 42\dot\phi_2^2 + 36\dot\phi_1\dot\phi_2 + \frac{6}{L_0^2}(\kappa e^{-2\phi_1} + 5e^{-2\phi_2}). \quad (13)$$

Thus the Einstein tensor components are given by

$$\begin{aligned} \mathcal{G}_{tt} &= 3\dot\phi_1^2 + 15\dot\phi_2^2 + 18\dot\phi_1\dot\phi_2 + \frac{3}{L_0^2}(\kappa e^{-2\phi_1} + 5e^{-2\phi_2}), \\ \mathcal{G}_{ij} &= -g_{ij}\left(2\ddot\phi_1 + 6\ddot\phi_2 + 3\dot\phi_1^2 + 21\dot\phi_2^2 + 12\dot\phi_1\dot\phi_2 + \frac{1}{L_0^2}(\kappa e^{-2\phi_1} + 15e^{-2\phi_2})\right), \\ \mathcal{G}_{IJ} &= -g_{IJ}\left(3\ddot\phi_1 + 5\ddot\phi_2 + 6\dot\phi_1^2 + 15\dot\phi_2^2 + 15\dot\phi_1\dot\phi_2 + \frac{1}{L_0^2}(3\kappa e^{-2\phi_1} + 10e^{-2\phi_2})\right). \end{aligned} \quad (14)$$

As for the gauge field, its field strength is given by

$$F = \frac{1}{4qL_0^2 e^{2\phi_2}}\gamma_{ab} V^{a+3} \wedge V^{b+3}. \quad (15)$$

This satisfies the following duality relations:

$$*F = -\boldsymbol{i}\beta\gamma_7 dv^{(4)} \wedge F \wedge F,$$
$$*(F \wedge F) = \frac{\boldsymbol{i}}{\beta} dv^{(4)} \wedge \gamma_7 F, \tag{16}$$

where

$$\beta := \frac{\boldsymbol{q}L_0^2}{3} e^{2\phi_2}. \tag{17}$$

The self-duality relation Eq. (16) becomes the Bogomol'nyi Eq. (4) if $\beta = \alpha$. In this case, there are no tachyonic modes, at least in the gauge sector. This determines a particular radius $R = L_c$ of the extra dimensions in terms of the gauge coupling constants $\boldsymbol{q}$ and α,

$$L_c := \sqrt{\frac{3\alpha}{\boldsymbol{q}}}. \tag{18}$$

Because ϕ_2 depends only on the time coordinate, the exterior derivative of $\beta d^{(4)} v$ vanishes,

$$d(\beta d^{(4)} v) = 0. \tag{19}$$

This means that the configuration satisfies the equation of motion,

$$D(*F) - \alpha^2 D\{*(F \wedge F) \wedge F + F \wedge *(F \wedge F)\} = 0. \tag{20}$$

The energy momentum tensor of the gauge field is given by

$$\mathcal{T}_{MN} = \frac{1}{8}\,\mathrm{Tr}\Big(-F_{MP}F_N{}^P + \frac{\alpha^2}{3!}H_{MPQS}H_N^{PQS}\Big) - \frac{1}{2}g_{MN}\chi,$$
$$\chi := \frac{1}{8}\,\mathrm{Tr}\Big(-\frac{1}{2}F_{MN}F^{MN} + \frac{\alpha^2}{4!}H_{MNPQ}H^{MNPQ} + V_0\Big). \tag{21}$$

Here H_{IJKL} are the components of $F \wedge F$ introduced in Appendix A. For our gauge configuration we have

$$\mathcal{T}_{tt} = \frac{1}{2}\chi,$$
$$\mathcal{T}_{ij} = -\frac{1}{2}g_{ij}\chi, \tag{22}$$
$$\mathcal{T}_{IJ} = -\frac{5}{8\boldsymbol{q}^2L_0^4}e^{-4\phi_2}\Big(1 - \frac{3^2\alpha^2}{\boldsymbol{q}^2L_0^4}e^{-4\phi_2}\Big)g_{IJ} - \frac{1}{2}g_{IJ}V_0,$$

where

$$\chi \equiv \chi(\phi_2) = \frac{15}{4\boldsymbol{q}^2L_0^4}e^{-4\phi_2}\Big(1 + \frac{3^2\alpha^2}{\boldsymbol{q}^2L_0^4}e^{-4\phi_2}\Big) + V_0. \tag{23}$$

In this gauge configuration, the Einstein field equations are

$$\frac{8\pi G}{2}\chi = 3\dot\phi_1^2 + 15\dot\phi_2^2 + 18\dot\phi_1\dot\phi_2 + \frac{3}{L_0^2}(\kappa e^{-2\phi_1} + 5e^{-2\phi_2}), \tag{24}$$

$$\frac{8\pi G}{2}\chi = \Big(2\ddot\phi_1 + 6\ddot\phi_2 + 3\dot\phi_1^2 + 21\dot\phi_2^2 + 12\dot\phi_1\dot\phi_2 + \frac{1}{L_0^2}(\kappa e^{-2\phi_1} + 15e^{-2\phi_2})\Big), \tag{25}$$

$$8\pi G\Big[-\frac{5}{8\boldsymbol{q}^2L_0^4}e^{-4\phi_2}\Big(1 - \frac{3^2\alpha^2}{\boldsymbol{q}^2L_0^4}e^{-4\phi_2}\Big) - \frac{1}{2}V_0\Big] = -\Big(3\ddot\phi_1 + 5\ddot\phi_2 + 6\dot\phi_1^2 + 15\dot\phi_2^2 + 15\dot\phi_1\dot\phi_2 + \frac{1}{L_0^2}(3\kappa e^{-2\phi_1} + 10e^{-2\phi_2})\Big), \tag{26}$$

where the first equation is a constraint on the field and its derivatives, the Hamiltonian constraint equation, determining the three-dimensional hypersurface in the four-dimensional phase space. We note that the kinetic term in the Hamiltonian constraint is quadratic in the field velocities, and it has one positive and one negative eigenvalues. The above system of differential equations is invariant under the time translation and time reversal transformation. If $\kappa = 0$, there is in addition an invariance under the shift of ϕ_1. The time evolution of the fields ϕ_1 and ϕ_2 is determined by Eqs. (25) and (26), describing the trajectory on the three-dimensional hypersurface defined by the constraint equation.

It is convenient to express the field equations in terms of a rescaled time coordinate $\tau = t/L_0$, and introduce the following dimensionless parameters,

$$a = \frac{8\pi G}{\boldsymbol{q}^2L_0^2}, \tag{27}$$

$$b = \frac{\alpha^2}{\boldsymbol{q}^2L_0^4}, \tag{28}$$

$$c = 4\pi G V_0 L_0^2, \tag{29}$$

where c is related to the ten-dimensional cosmological constant Λ by $c = \Lambda L_0^2$. In what follows, the τ derivative of a function $h(\tau)$ will be denoted by h'. For $L_0 = L_c$ the parameter b is fixed to the value $b = 1/9$, leaving only two free parameters in the field equations.

By manipulating the field equations we can reduce them to the following convenient set of two differential equations:

$$V_1 = (\phi_1')^2 + 5(\phi_2')^2 + 6(\phi_1')(\phi_2'), \tag{30}$$

$$V_2 = \phi_2'' + 6(\phi_2')^2 + 3(\phi_1')(\phi_2'), \tag{31}$$

where V_1 and V_2 are defined by

$$\begin{aligned} V_1(\phi_1, \phi_2) &:= \frac{4\pi G L_0^2}{3}\chi(\phi_2) - 5e^{-2\phi_2} - \kappa e^{-2\phi_1} \\ &= \frac{5}{8}ae^{-4\phi_2}(1 + 9be^{-4\phi_2}) + \frac{c}{3} - 5e^{-2\phi_2} \\ &\quad - \kappa e^{-2\phi_1}, \end{aligned} \tag{32}$$

$$V_2(\phi_2) := \frac{5a}{32}e^{-4\phi_2}(5 + 63be^{-4\phi_2}) + \frac{c}{4} - 5e^{-2\phi_2}. \tag{33}$$

We can solve Eq. (30) for ϕ_1' to obtain

$$\phi_1' = -3\phi_2' \pm \sqrt{V_1 + 4(\phi_2')^2}. \tag{34}$$

By using this equation, we can eliminate ϕ_1' from Eq. (31). Then the Einstein field equations are reduced to a system of coupled differential equations given by

$$\phi_1' + 3\phi_2' - \sqrt{V_1 + 4(\phi_2')^2} = 0, \tag{35}$$

$$\phi_2'' - 3(\phi_2')^2 + 3\phi_2'\sqrt{V_1 + 4(\phi_2')^2} - V_2 = 0, \tag{36}$$

where we have chosen the positive value of the square root in Eq. (35). In the next section, we look for a solution in which the extra-dimensional part of the metric is static; that is, a solution with $\phi_2' = 0$. In this case, ϕ_1 grows with time for the above choice of the square root sign, ensuring that the four-dimensional part of the metric describes an expanding universe.

IV. SOLUTIONS WITH STATIC EXTRA DIMENSIONS

In this section we consider solutions in which the metric of the extra-dimensional space, S^6, is static; that is when $\phi_2 = \text{const}$. In this case, Eq. (35) becomes integrable with respect to ϕ_1, and Eq. (36) becomes an algebraic equation for $e^{-2\phi_2}$. We note that we do *not* require our solution to satisfy the Bogomol'nyi Eq. (4). Hence for those solutions whose extra-dimensional radius is different from L_c given by Eq. (18), the absence of tachyon modes is not guaranteed. Therefore we simply assume that there is a sufficiently wide range of parameters in which there appears no harmful tachyons. This issue is left for a future study.

Below we first consider general solutions. As we will see shortly, there is a particular solution given by $e^{\phi_1} = \sqrt{-\kappa\tau + C}$ for $\kappa = -1, 0$. Since this is somehow special, we treat it separately.

A. The general case

Static solutions of Eq. (36) are determined by the roots of $V_2(\phi_2) = 0$. Let us set $Z := ae^{-2\phi_2}$. We note that $Z \propto (L_0e^{\phi_2})^{-2}$, where $L_0e^{\phi_2}$ is the linear scale of the extra dimensions. The equation $V_2 = 0$ becomes

$$f(Z) \equiv Z^4 + 5\nu_1 Z^2 - 32\nu_1 Z + \frac{8\nu_2\nu_1}{5} = 0, \tag{37}$$

where

$$\nu_1 := \frac{a^2}{63b} = \frac{64\pi^2G^2}{63\alpha^2\boldsymbol{q}^2}, \qquad \nu_2 := ac = \frac{32\pi^2G^2V_0}{\boldsymbol{q}^2}. \tag{38}$$

Note that ν_1 and ν_2 are independent of L_0 and ϕ_2. As demonstrated in Appendix B, the equation $f(Z) = 0$ has one or two real solutions Z_1 and Z_2 (we assume $Z_1 \geq Z_2$) provided that ν_1 and ν_2 satisfy a certain inequality.

Let us first consider the solution Z_1. The relation between the original variables and Z_1 can be written as

$$L_0^2\exp(2\phi_2) = L_1^2 := \frac{8\pi G}{\boldsymbol{q}^2}\frac{1}{Z_1}, \tag{39}$$

where L_1 represents the size of the extra dimensions. Thus the size of the extra dimensions is completely fixed by the coupling constants.

As easily seen, Eqs. (35) and (36) are invariant under the rescaling,

$$\begin{gathered} L_0 \to CL_0, \qquad \exp(\phi_2) \to C^{-1}\exp(\phi_2), \\ \exp(\phi_1) \to C^{-1}\exp(\phi_1). \end{gathered} \tag{40}$$

Using this degree of freedom, we fix the length scale L_0 to be the size of the extra dimensions L_1, or equivalently, we set $\phi_2 = \phi_2^{(1)} = 0$ for this solution. Then we have $Z_1 = a$. Therefore a must be a solution of Eq. (37)

$$f(a) = 0 \Leftrightarrow c = 20 - \frac{5a}{8}(5 + 63b). \tag{41}$$

With this normalization, we find a positive real Z_2 for $c > 0$. The condition $Z_2 \leq Z_1$ and $c \geq 0$ give the following inequalities:

$$\frac{32}{5 + 63b} \geq a \geq \frac{16}{5 + 126b}. \tag{42}$$

In terms of ϕ_2 these two solutions are given by

$$\phi_2^{(1)} := -\frac{1}{2}\log(Z_1/a) = 0, \qquad \phi_2^{(2)} := -\frac{1}{2}\log(Z_2/a). \tag{43}$$

Those points are critical points or equilibrium solutions of the differential Eq. (36). The value of $\phi_2^{(2)}$ is depicted as a function of a for each value of b in Fig. 1. The discussions in the rest of this subsection are valid for both solutions.

Now we turn to Eq. (35). Setting $\phi_2' = 0$, we have

$$\phi_1' = \sqrt{\lambda_i^2 - \kappa e^{-2\phi_1}} \Leftrightarrow (e^{\phi_1})' = \sqrt{\lambda_i^2 e^{2\phi_1} - \kappa}, \tag{44}$$

where λ_i $(i = 1, 2)$ is defined by

$$\lambda_i^2 := \frac{4}{3}\pi G L_0^2\chi(\phi_2^{(i)}) - 5e^{-2\phi_2^{(i)}}. \tag{45}$$

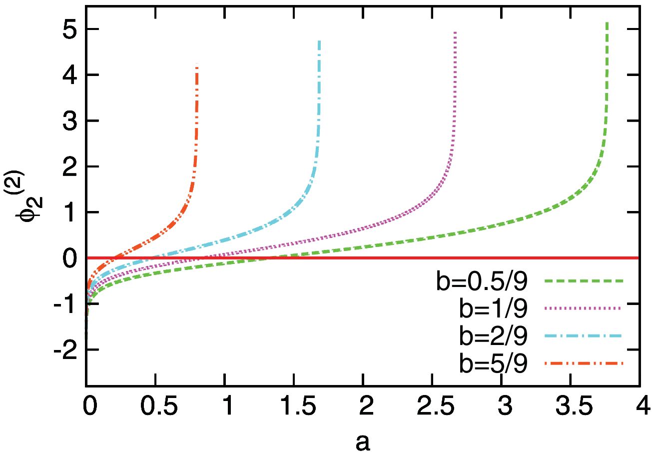

FIG. 1 (color). The plot of $\phi_2^{(2)}$ as a function of a for $b = 0.5/9$, $1/9$, $2/9$, and $5/9$. Because $\phi_2^{(2)} \geq \phi_2^{(1)} = 0$, only the part of the curves above the line $\phi_2^{(2)} = 0$ is meaningful. The value of a is bounded as given by Eq. (42).

We assume λ_i^2 is positive. $\lambda_1^2 \geq 0$ gives an additional condition on the parameter a,

$$\frac{4}{1+18b} \geq a \geq \frac{16}{5+126b}. \tag{46}$$

If this inequality is satisfied, $\lambda_2^2 \geq 0$, because $\frac{4}{3}\pi G L_0^2 \chi(\phi_2) - 5e^{-2\phi_2}$ is concave downward as a function of $e^{-2\phi_2}$ and its derivative at $e^{-2\phi_2} = 1$ is negative. The allowed region of a and b given by Eq. (46) is depicted in Fig. 2.

The Eq. (44) can be integrated to give

$$e^{\phi_1} = \frac{1}{2\lambda_i}(e^{\lambda_i \tau} + \kappa e^{-\lambda_i \tau}), \tag{47}$$

where the origin of the time coordinate has been chosen to make the expression simple. Four-dimensional parts of these solutions are the same as those of Ishihara [21].

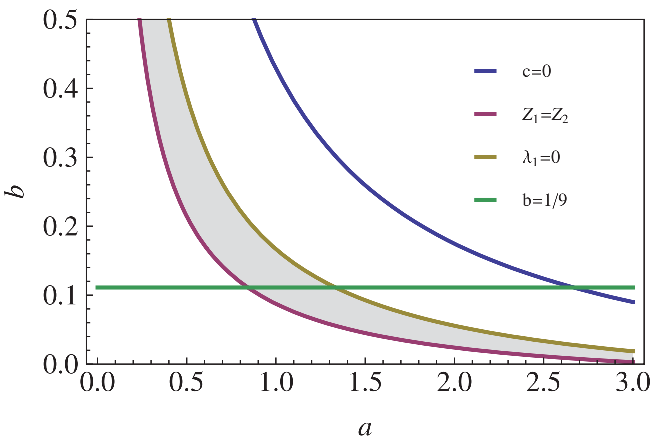

FIG. 2 (color). The allowed region of a and b. The filled region is the allowed region which is bounded by the lines $\lambda_1^2 = 0$ and $c = 0$.

For large τ, the term proportional to κ can be neglected and the scale factor of the four-dimensional space-time approaches $R(t) = L_0 e^{\lambda_i \tau}$, which describes a universe with accelerated expansion. Thus, although we do not claim that our model can give a realistic model of the universe, depending on the value of the constant λ_i, it can reproduce a period of inflation in the very early universe or the present universe dominated by a very small cosmological constant.

B. The case $\lambda_i = 0$

When $\lambda_i = 0$ the solution (47) is no longer valid as it is, and we need a special treatment. In this case Eq. (44) implies that κ must be either -1 or 0. In either case, $(e^{\phi_1})' = \sqrt{-\kappa}$, and the solution is

$$e^{\phi_1} = \sqrt{-\kappa}\tau + C \qquad (\kappa = -1, 0), \tag{48}$$

where C is an integration constant.

The four-dimensional part of the solution for $\kappa = 0$ is flat. It was obtained in [18], which is almost the same as the one obtained by Cremmer-Scherk [10], but with the radius of S^6 and the value of ten-dimensional cosmological constant modified by the presence of the Tchrakian term.

The solution for $\kappa = -1$ is also flat. The four-dimensional line element is

$$ds^2 = L_0^2\left(-d\tau^2 + \tau^2 \frac{d\sigma^2}{(1 - \frac{|\sigma|^2}{4})^2}\right). \tag{49}$$

This metric covers the inside of either the future light cone or the past light cone of the flat space-time.

V. DYNAMICAL COMPACTIFICATION

In this section, we switch on the time dependence of ϕ_2 in order to see if our model has the possibility to describe the process of dynamical compactification. For this purpose, we analyze the stability of the solution $\phi_2 = \phi_2^{(1)}$ $(=0)$ and $\phi_2 = \phi_2^{(2)}$ in the second order nonlinear differential Eq. (36) in the case of $\kappa = 0$ both analytically and numerically.

We first analyze the stability of the critical points analytically. For this purpose, we linearize the system of differential equations (see e.g. [23]). We find, however, that for $\lambda_1 = 0$ this method is not sufficient to establish the stability of the critical point $\phi_2' = \phi_2 = 0$. Therefore we will try a different approach in this case.

We first consider the critical point $(\phi_2, \phi_2') = (0, 0)$, which is a stationary or equilibrium solution of the differential Eq. (36). Denoting $X := \phi_2$ and $Y := \phi_2'$ and keeping only terms linear in X and Y, Eq. (36) is written as the following system of first order differential equations:

$$\mathbf{X}' := \begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} Y \\ \frac{dV_2}{d\phi_2}|_{\phi_2=0} X - 3\lambda_1 Y \end{pmatrix} = \mathbf{A}\mathbf{X}, \tag{50}$$

where the matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ \frac{dV_2}{d\phi_2}\big|_{\phi_2=0} & -3\lambda_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\Omega^2 + \frac{21}{2}\lambda_1^2 & -3\lambda_1 \end{pmatrix} \tag{51}$$

with

$$\lambda_1^2 = \frac{5}{3} - \frac{5}{12}a(1 + 18b), \tag{52}$$

$$\Omega^2 := \frac{5}{4}(6 - a). \tag{53}$$

Note that λ_1 in the above is equal to the one defined by Eq. (45) with the normalization condition (41). The solution $(X, Y) = (0, 0)$ is asymptotically stable if both of the two eigenvalues of the matrix $\mathbf{A}$,

$$\Upsilon_{1,\pm} = -\frac{3}{2}\left[\lambda_1 \pm \sqrt{\lambda_1^2 + \frac{4}{9}\frac{dV_2}{d\phi_2}\bigg|_{\phi_2=0}}\right], \tag{54}$$

have negative real part.

To analyze the stability of the second critical point, $(\phi_2, \phi_2') = (\phi_2^{(2)}, 0)$, we simply replace ϕ_2 by $\phi_2 - \phi_2^{(2)}$ when linearizing Eq. (36). Then the eigenvalues are

$$\Upsilon_{2,\pm} = -\frac{3}{2}\left[\lambda_2 \pm \sqrt{\lambda_2^2 + \frac{4}{9}\frac{dV_2}{d\phi_2}\bigg|_{\phi_2=\phi_2^{(2)}}}\right]. \tag{55}$$

For $\lambda_1 = 0$ the real part of the two eigenvalues is zero, and the linear system corresponds to the harmonic oscillator for $\Omega^2 > 0$. In this case we can not apply Poincare-Lyapunov's theorem above and additional information is required to establish the character of the critical point for the full nonlinear equation. We therefore treat this case separately.

A. The case $\lambda_1 > 0$

As in Sec. IV, we are interested in the solutions with λ_1 real and positive. Then the real part of the eigenvalues $\Upsilon_{1,\pm}$ is negative if

$$-\frac{dV_2}{d\phi_2}\bigg|_{\phi_2=0} = \left(\Omega^2 - \frac{21}{2}\lambda_1^2\right) > 0. \tag{56}$$

This condition coincides with the condition $Z_2 < Z_1$, which is satisfied when the parameters satisfy Eq. (46). Thus the critical point $(\phi_2, \phi_2') = (0, 0)$ is stable.

The system shows two different kinds of behavior in the neighborhood of the critical point $(\phi_2, \phi_2') = (0, 0)$. When $\lambda_1^2 - \frac{4}{9}(\Omega^2 - \frac{21}{2}\lambda_1^2) < 0$, the system undergoes damped oscillations with the amplitude decreasing as $e^{-3/2\lambda_1\tau}$. Otherwise the system is over-damped, showing simple exponential damping toward the critical point.

For the second critical point $(\phi_2, \phi_2') = (\phi_2^{(2)}, 0)$, it can be shown that

$$\frac{dV_2}{d\phi_2}\bigg|_{\phi_2^{(2)}} = -\frac{5}{16}\frac{63b}{a^3}Z_2\frac{df}{dZ}(Z_2) \geq 0, \tag{57}$$

where $f(Z)$ is the function introduced in Eq. (37) and Z_2 is the solution of $f(Z) = 0$ corresponding to the second critical point. This inequality follows from the fact that df/dZ is a monotonically increasing function of Z with the unique $df/dZ = 0$ at $Z = Z_0$ and $Z_2 \leq Z_0$, which is proved in Appendix B. If $df/dZ(Z_2) < 0$, there are two real eigenvalues with opposite signs. Hence the critical point is an unstable saddle-point. In the special case when $df/dZ(Z_2) = 0$, the first and second critical points become degenerate, and $\phi_2 = 0$ becomes the only equilibrium solution of the system. Note that we have $\Omega^2 = 21/2\lambda_1^2$ in this case.

To confirm the above stability analysis, we have performed numerical integration of Eq. (36). Our numerical results indicate that the linear analysis around the first critical point is accurate. In Fig. 3, we show the phase space orbits of the solutions of Eq. (36) with $\lambda_1 > 0$. In this case, the first critical point is stable and the other critical point along the $\phi_2' = 0$ axis is an unstable saddle point. The location of the saddle point depends on the values of the parameters a and b as well, and it roughly defines an effective stability radius for orbits near the solution (0, 0). The time evolution of ϕ_2 for an asymptotically stable solution is shown in Fig. 4, where e^{ϕ_2} oscillates with a decreasing amplitude until ϕ_2 reaches zero.

We have also integrated Eq. (35) for ϕ_1. The time evolution of the three-dimensional cosmic scale factor $R \propto e^{\phi_1}$ is shown in Fig. 5. Initially when the oscillatory energy of ϕ_2 is non-negligible, the scale factor behaves as the one in a matter-dominated universe, $R(\tau) \propto \tau^{2/3}$. For sufficiently large τ, after the amplitude of ϕ_2 has decayed

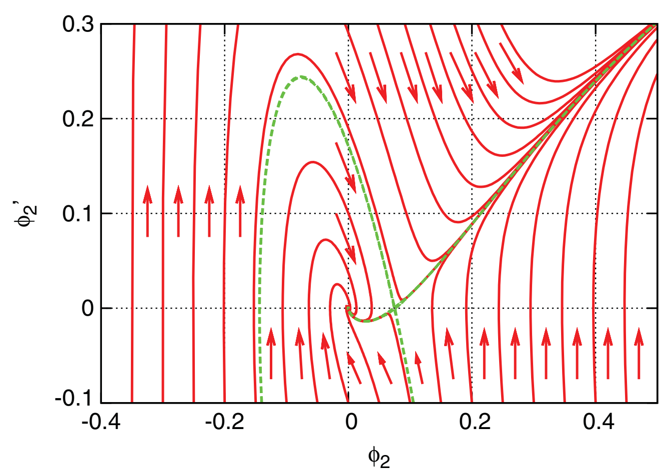

FIG. 3 (color). The phase space diagram (ϕ_2, ϕ_2') for $\lambda_1^2 = 5/12$ and $\Omega^2 = 25/4$. These are equivalent to take $a = 1$ and $b = 1/9$. The figure shows the two critical points of the system. The point (0, 0) is stable while the second critical point is a saddle point with unstable orbits to its right.

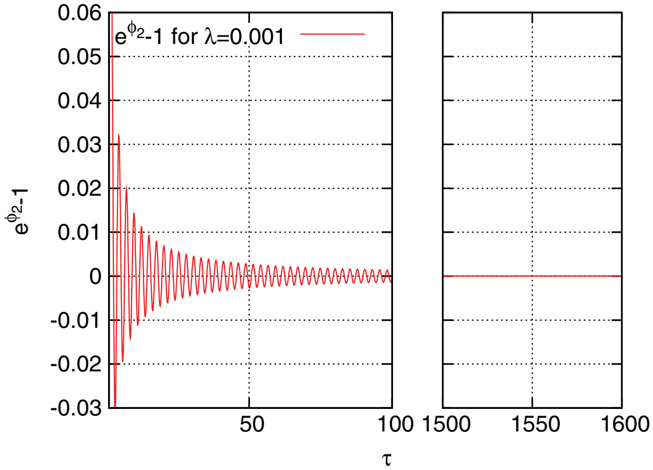

FIG. 4 (color). This figure shows the damped oscillations of the radius of the extra dimensions with time $\tau = tL_0^{-1}$.

exponentially, the universe eventually enters a stage of accelerated expansion, $R(\tau) \propto e^{H\tau}$, with the (dimensionless) Hubble parameter $H = \lambda_1$.

B. The case $\lambda_1 = 0$

In the case $\lambda_1 = 0$ the real part of $\Upsilon_{1,\pm}$ is zero, rendering the linear analysis insufficient to determine the stability of the solution. Therefore we have to take into account the second order terms.

To second order in X and Y, Eq. (36) gives the equations,

$$X' = Y, \tag{58}$$

$$Y' = -\Omega^2 X + G(X, Y). \tag{59}$$

where $G(X, Y)$ is a quadratic function given by

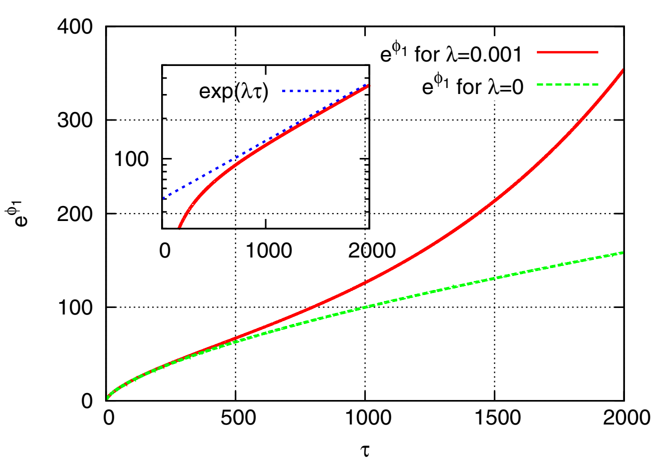

FIG. 5 (color online). This figure shows the time evolution of the scale factor $R(\tau)$ for $\Omega^2 = 25/4$ and $\lambda_1 = 0.001$.

$$G(X, Y) = \left(9\Omega^2 - \frac{15}{2}\right)X^2 + 3Y^2 - 3Y\sqrt{4\Omega^2 X^2 + 4Y^2}. \tag{60}$$

Let us solve Eqs. (58) and (59) perturbatively. We assume $\Omega^2 > 0$. To first order in X and Y, the system describes a harmonic oscillator. Namely, we have

$$X(\tau) = r\cos(\Omega\tau + \psi), \tag{61}$$

$$Y(\tau) = -\Omega r \sin(\Omega\tau + \psi) \tag{62}$$

as a solution of the first order equations, where r and ψ are arbitrary constants. Then the orbits in phase space are ellipses about the critical point (0, 0).

Now we consider the effect of the second order terms. Here we just apply the so-called Krylov-Bogoliubov method of averaging [23] to study the behavior of the solutions.[3]

First, we introduce varying constants in the harmonic oscillator solution as

$$X(\tau) = r(\tau)\cos(\Omega\tau + \psi(\tau)), \tag{63}$$

$$Y(\tau) = -\Omega r(\tau)\sin(\Omega\tau + \psi(\tau)). \tag{64}$$

Then the system of differential equations may be expressed as

$$r' = f_r(\tau, r, \psi), \tag{65}$$

$$\psi' = f_\psi(\tau, r, \psi), \tag{66}$$

where

$$f_r(\tau, r, \psi) = -\frac{1}{\Omega}\sin(\Omega\tau + \psi) \times G(r\cos(\Omega\tau + \psi), -\Omega r\sin(\Omega\tau + \psi)), \tag{67}$$

$$f_\psi(\tau, r, \psi) = -\frac{1}{\Omega r}\cos(\Omega\tau + \psi) \times G(r\cos(\Omega\tau + \psi), -\Omega r\sin(\Omega\tau + \psi)). \tag{68}$$

Note that the right-hand sides of Eqs. (65) and (66) are periodic in τ with the period $2\pi\Omega^{-1}$. Then instead of these equations, applying the Krylov-Bogoliubov method of averaging we consider the following time-averaged equations:

$$\bar{r}' = \frac{\Omega}{2\pi}\int_0^{2\pi/\Omega} f_r(s, \bar{r}, \bar{\psi})ds = -3\Omega\bar{r}^2, \tag{69}$$

[3]The detailed calculation is shown in Appendix C.

$$\bar{\psi}' = \frac{\Omega}{2\pi}\int_0^{2\pi/\Omega} f_\psi(s, \bar{r}, \bar{\psi}) ds = 0 \tag{70}$$

for $\bar{r}$ and $\bar{\psi}$. The solution is given by

$$\bar{r} = \frac{1}{3\Omega\tau + \text{const}}, \tag{71}$$

$$\bar{\psi} = \text{const}. \tag{72}$$

These give approximate behavior of r and ϕ at sufficiently large τ. From Eq. (63), approximations to ϕ_2 and ϕ_2' for large τ are given by

$$\phi_2(\tau) \sim \frac{1}{3\Omega\tau}\cos\Omega\tau, \tag{73}$$

$$\phi_2'(\tau) \sim -\frac{1}{3\tau}\sin\Omega\tau. \tag{74}$$

Then, for large τ, Eq. (35) gives

$$\phi_1' \sim -\frac{\sin\Omega\tau}{\tau} + \frac{2}{3\tau}. \tag{75}$$

We can now read off an approximate solution for the field ϕ_1,

$$\phi_1 \sim \frac{2}{3}\log\tau - \text{Si}(\Omega\tau). \tag{76}$$

Thus the scale factor behaves as

$$R(\tau) = L_0 e^{\phi_1(\tau)} \sim L_0 \tau^{2/3} e^{-\text{Si}(\Omega\tau)}. \tag{77}$$

Apart from the small oscillations, this describes a matter-dominated universe.

In Figs. 6 and 7, we show numerical solutions of the full nonlinear system for $\lambda_1 = 0$. The numerical results are in good agreement with our analytical estimations. The time evolution of e^{ϕ_1} in Fig. 7 clearly exhibits oscillations around its central value $\tau^{2/3}$ as we have shown analytically.

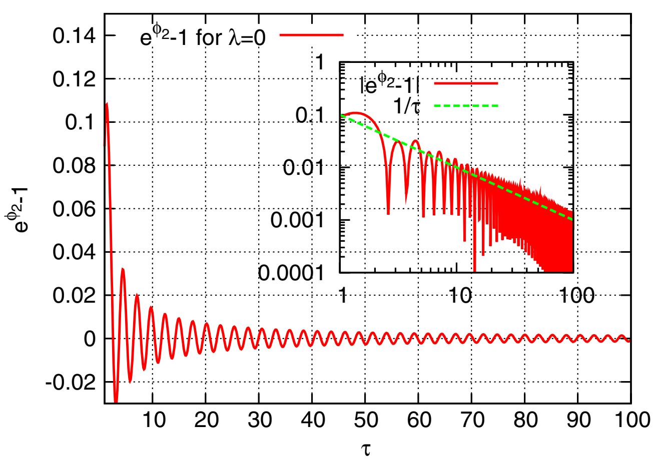

FIG. 6 (color online). The time evolution of $(e^{\phi_2} - 1)/L_0$ with time $\tau = t/L_0$. We have chosen $\Omega = 25/4$, corresponding to the choice of $a = 1$. The oscillations in the proximity of the equilibrium solution $\phi_2 = 0$ are rapidly damped out as $1/\tau$.

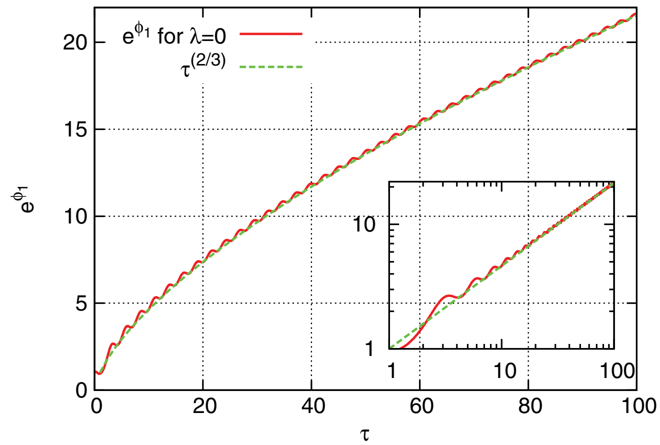

FIG. 7 (color). The time evolution of e^{ϕ_1}/L_0 for $\lambda = 0$. In this plot $\Omega = 25/4$, which corresponds to $a = 1$. The time-averaged scale factor $a(t) \propto \langle e^{\phi_1}\rangle$ describes a matter-dominated universe.

VI. CONCLUSION AND DISCUSSION

In this article, we studied time-dependent solutions of the ten-dimensional Einstein-Yang-Mills theory with the Tchrakian term. We obtained a class of simple analytic solutions in which the extra dimensions are static and the scale factor of the four-dimensional Friedmann-Lemaitre-Robertson-Walker metric behaves exponentially in time with the rate of expansion given by constants denoted by λ_i ($i = 1, 2$). Thus our model admits solutions describing inflation.

We then considered a possible dynamical compactification of the extra dimensions by allowing them to be time dependent. In the case $\lambda_1 > 0$, we found solutions in which the scale factor of the extra dimensions undergoes damped oscillations and approaches a constant value, while the four-dimensional scale factor approaches $e^{\lambda_1\tau}$. In the case of $\lambda_1 = 0$, we found numerically that the scale factor behaves as a matter-dominated universe $R \propto \tau^{2/3}$.

Our model includes four dimensionful constants $(G, V_0, \boldsymbol{q}, \alpha)$. They define four typical length scales in our model. Or if we fix the Planck scale or the gravitational constant, G, we are left with three dimensionless parameters. In addition, if we require the Bogomol'nyi equation to be satisfied, the linear size of the extra dimensions is fixed to be $L_c = \sqrt{3\alpha/\boldsymbol{q}}$, and there remains only two dimensionless parameters.

As is shown in Sec. II, when the radius of the compact direction is equal to L_c, there are no tachyonic modes in the gauge sector. However, for a set of model parameters that gives a radius substantially different from L_c, a tachyonic mode may appear. To investigate when a tachyon appears and how it affects our model is certainly an important issue. Also for a complete analysis, in addition to fluctuations of the gauge field, it is necessary to include fluctuations of the metric and cross terms between them. These are left for future work.

We also note that all the discussions given in this paper apply equally to the gauge group SU(4) in place of SO(6), because the matrices γ_{ab} are block diagonalizable. Namely, if we project those matrices on the four-dimensional eigenspace with respect to the eigenvalue $+1$ of γ_7, we obtain self-duality relation of SU(4) without γ_7. Thus all cosmological solutions obtained in this paper are also valid for models with SU(4) gauge theory. Furthermore, since SU(4) is a subgroup of SU(N), our cosmological solutions can be embedded into the Einstein-Yang-Mills theory with the Tchrakian term with the SU(N) gauge group. Generalization to other gauge groups like E_8 or SO(N) with $N \geq 8$ remains as a future issue [24].

Recently some of us (H.K. and M.N.) considered the Bogomol'nyi equation on $\mathbb{C}\mathbf{P}^n$ [17]. By using the gauge configuration on $\mathbb{C}\mathbf{P}^3$, we expect that we will be able to obtain similar cosmological solutions for $\mathbb{C}\mathbf{P}^3$ compactification instead of S^6 studied in this paper. Also, it is interesting to see if similar cosmological solutions can be obtained for other types of compactification such as the compactification in terms of the Casimir energy [25]. These are also issues to be investigated in the future.

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APPENDIX A: NOTATION

1. Definitions and properties of tensors

Here we explain our notation. The Einstein tensor and the energy momentum tensor are defined as

$$\mathcal{G}_{MN} := \mathcal{R}_{MN} - \frac{1}{2} g_{MN}\mathcal{R}, \qquad \mathcal{T}_{MN} := -\frac{2}{\sqrt{-g}}\frac{\delta S_{\mathrm{YMT}}}{\delta g^{MN}}. \tag{A1}$$

In terms of these tensors the Einstein equation is

$$\mathcal{G}_{MN} = 8\pi G \mathcal{T}_{MN}. \tag{A2}$$

The Einstein tensor is obtained by the differentiation of the Einstein-Hilbert action S_{EH} with respect to the metric g^{MN}. The corresponding Levi-Civita connection Γ^M_{NP} is defined as

$$\Gamma^M_{NP} := \tfrac{1}{2} g^{MQ}(\partial_N g_{QP} + \partial_P g_{QN} - \partial_Q g_{NP}). \tag{A3}$$

The Riemannian curvature $\mathcal{R}^M_{NPQ}$ is defined as

$$\mathcal{R}^{\ M}_{NPQ} := \partial_P \Gamma^M_{NQ} - \partial_Q \Gamma^M_{NP} + \Gamma^M_{PA}\Gamma^A_{NQ} - \Gamma^M_{QA}\Gamma^A_{NP}. \tag{A4}$$

The Ricci tensor $\mathcal{R}_{MN}$ and scalar curvature $\mathcal{R}$ are

$$\mathcal{R}_{MN} := \mathcal{R}^Q_{MQN}, \qquad \mathcal{R} := g^{MN}\mathcal{R}_{MN}. \tag{A5}$$

2. Differential forms

The tangent vector space of a point is spanned by ∂_M. The basis dx^M of the cotangent space is the dual vector, $dx^M(\partial_N) = \delta^M_N$. For vector space V the Grassmann algebra $\Lambda^*(V)$ is defined as $T(V)/I$ where $T(V)$ is the tensor algebra $T(V) := \oplus_{p=0}^{\infty} V^{\otimes p}$ and I is the two-sided ideal generated by $v \otimes v$, $v \in V$. We can define a linear operation which is called the Hodge dual. Let us fix p and $q := D - p$. The Hodge dual operator $*$ is defined as

$$* \, dX^{M_1\cdots M_p} := \frac{1}{q!\sqrt{-g}} \epsilon^{M_1\cdots M_p}{}_{N_1\cdots N_q} dX^{N_1\cdots N_q}. \tag{A6}$$

By using the Hodge dual operation the metric on the differential suppose that ω is a p form,

$$\omega := \frac{1}{p!}\omega_{M_1\cdots M_p} dX^{M_1\cdots M_p}. \tag{A7}$$

The inner product is given by $(\omega, \omega) := \omega \wedge *\omega$. Let us show the metric in terms of the component,

$$\begin{aligned}\omega \wedge *\omega &= \frac{1}{(p!)^2}\omega_{M_1\cdots M_p}\omega_{K_1\cdots K_p} dX^{M_1\cdots M_p} \wedge \frac{1}{q!\sqrt{-g}}\epsilon^{K_1\cdots K_p}{}_{N_1\cdots N_q} dX^{N_1\cdots N_q} \\ &= \frac{1}{(p!)^2 q!\sqrt{-g}}\omega_{M_1\cdots M_p}\omega_{K_1\cdots K_p}\epsilon^{K_1\cdots K_p}{}_{N_1\cdots N_q} dX^{M_1\cdots M_p N_1\cdots N_q} \\ &= -\frac{1}{g(p!)^2 q!}\omega_{M_1\cdots M_p}\omega_{K_1\cdots K_p}\epsilon^{K_1\cdots K_p}{}_{N_1\cdots N_q}\epsilon^{M_1\cdots M_p N_1\cdots N_q} dv = -\frac{1}{p!}\omega_{M_1\cdots M_p}\omega_{K_1\cdots K_p}\Delta^{K_1\cdots K_p, M_1\cdots M_p},\end{aligned} \tag{A8}$$

where the metric $\Delta^{K_1\cdots K_p,M_1\cdots M_p}$ is defined as follows:

$$\Delta^{K_1\cdots K_p,M_1\cdots M_p} := \frac{1}{p!}\sum_{\sigma\in\mathfrak{S}_p}\mathrm{sign}(\sigma)\prod_{i=1}^{p} g^{K_i M_{\sigma(i)}}. \quad \text{(A9)}$$

Here $\mathfrak{S}_p$ is the pth symmetric group consisting of all permutations of p characters. Finally we obtain

$$\omega\wedge *\omega = -\frac{1}{p!}\omega_{M_1\cdots M_p}\omega^{M_1\cdots M_p}. \quad \text{(A10)}$$

The minus sign is from the fact that the signature of the metric g_{MN} is Lorentzian.

3. Clifford algebra

We will use the Clifford algebra with respect to the six-dimensional Euclidean metric in order to represent the algebra SO(6). Indices $a, b = 1, 2, \cdots, 6$ refer to the inner space. The Clifford algebra is generated by γ_a which satisfy

$$\{\gamma_a, \gamma_b\} = 2\delta_{ab}, \qquad \gamma_{ab} := \tfrac{1}{2}[\gamma_a, \gamma_b]. \quad \text{(A11)}$$

These generators are represented as 8×8 matrices. γ_{ab} satisfy the commutation relation of the Lie algebra SO(6). Anticommutation relation of γ_{ab} is

$$\{\gamma_{ab}, \gamma_{cd}\} = 2\gamma_{abcd} - 4\delta^{ab}_{[cd]}. \quad \text{(A12)}$$

Here γ_{abcd} is an antisymmetric product of four generators defined as

$$\gamma_{a_1 a_2\cdots a_p} := \frac{1}{p!}\sum_{\sigma\in\mathfrak{S}_p}\mathrm{sgn}\sigma\,\gamma_{a_{\sigma(1)}\cdots a_{\sigma(p)}}. \quad \text{(A13)}$$

The chirality operator γ_7 is defined as

$$\gamma_7 = -\boldsymbol{i}\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6, \qquad \gamma_7^2 = 1, \qquad \gamma_7^\dagger = \gamma_7. \quad \text{(A14)}$$

By using this matrix, γ_{abcd} is written as a sum of products of γ_7 and γ_{ab},

$$\gamma_{abcd} = -\frac{\boldsymbol{i}}{2!}\epsilon_{abcdef}\gamma_7\gamma_{ef}. \quad \text{(A15)}$$

4. Notation for gauge fields

The degree of freedom of a gauge boson is represented by the Lie algebra-valued oneform A,

$$A := \tfrac{1}{2}A^{ab}_M\gamma_{ab}dX^M, \qquad F = dA + \boldsymbol{q}A\wedge A, \quad \text{(A16)}$$

where F is the corresponding gauge field strength twoform and $\boldsymbol{q}$ is the gauge coupling constant. Let us rewrite the action in terms of the components,

$$\begin{aligned}\frac{1}{16}\mathrm{Tr}(-F\wedge *F) &= \frac{1}{32}\mathrm{Tr}(F_{MN}F^{MN})dv\\ &= \frac{1}{4\cdot 32}F^{ab}_{MN}F^{cd,MN}\mathrm{Tr}\gamma_{ab}\gamma_{cd}dv\\ &= \frac{1}{4\cdot 32}F^{ab}_{MN}F^{cd,MN}dv\,8(\delta_{bc}\delta_{ad} - \delta_{bd}\delta_{ac})\\ &= -\frac{1}{4\cdot 2}F^{ab}_{MN}F^{ab,MN}dv. \end{aligned} \quad \text{(A17)}$$

For notational simplicity, we introduce the composite four form operator H,

$$H := F\wedge F = \frac{1}{4!}H_{MNPQ}dX^{MNPQ}. \quad \text{(A18)}$$

The energy momentum tensor is

$$\begin{aligned}\mathcal{T}_{MN} &= \frac{1}{2}F^{ab}_{MP}F^{abP}_N + \frac{\alpha^2}{8\cdot 3!}\mathrm{Tr}H_{MPQS}H_N{}^{PQS} - \frac{1}{2}g_{MN}\chi\\ &= \frac{1}{8}\mathrm{Tr}\Big(-F_{MP}F_N{}^P + \frac{\alpha^2}{3!}H_{MPQS}H_N{}^{PQS}\Big) - \frac{1}{2}g_{MN}\chi,\end{aligned} \quad \text{(A19)}$$

where

$$\chi := \mathrm{Tr}\Big(-\frac{1}{16}F_{MN}F^{MN} + \frac{\alpha^2}{4!}H_{MNPQ}H^{MNPQ} + V_0\Big). \quad \text{(A20)}$$

APPENDIX B: ϕ_2 = constant SOLUTIONS

In this Appendix, we derive an inequality which gives the condition for Eq. (37) to have two real solutions.

Because $d^2f(Z)/dZ^2 = 12Z^2 + 10\nu_1 > 0$ for arbitrary Z, the polynomial $f(Z)$ has a unique minimum. This means that the number of real solutions of $f(Z) = 0$, Eq. (37), is at most 2. Let the value of Z at the minimum be Z_0. Then $f(Z_0)$ must be nonpositive for a real solution to exist

$$f(Z_0) = Z_0^4 + 5\nu_1 Z_0^2 - 32\nu_1 Z_0 + \frac{8}{5}\nu_1\nu_2 \le 0. \quad \text{(B1)}$$

Also Z_0 must be the unique real solution of the equation,

$$\frac{df(Z)}{dZ} = 4Z^3 + 10\nu_1 Z - 32\nu_1 = 0. \quad \text{(B2)}$$

Because $-32\nu_1$ is negative, Z_0 must be positive. In fact, by using Cardano's formula, we obtain

$$\begin{aligned}Z_0 &= \Big(4\nu_1 + 4\nu_1\sqrt{1 + \frac{5^3}{3^3\cdot 2^7}\nu_1}\Big)^{1/3}\\ &\quad - \Big(-4\nu_1 + 4\nu_1\sqrt{1 + \frac{5^3}{3^3\cdot 2^7}\nu_1}\Big)^{1/3},\end{aligned} \quad \text{(B3)}$$

which is manifestly positive definite.

Now using $df(Z_0)/dZ = 0$, the condition (B1) reduces to

$$\frac{5}{2}Z_0^2 - 24Z_0 \le -\frac{8\nu_2}{5}. \tag{B4}$$

Thus when the couplings $(G, V_0, \boldsymbol{q}, \alpha)$ satisfy the condition (B4), there are one or two real solutions Z_1 and Z_2, $(Z_1 \ge Z_2)$. Because Z_0 is positive, and we have the relation $Z_1 \ge Z_0 \ge Z_2$, Z_1 is always positive if it exists. When the equality in Eq. (B4) is satisfied, we have $Z_1 = Z_2 (= Z_0)$.

We assume that the parameters satisfy Eq. (B4). Then for $\nu_2 \ge 0$ or equivalently $c \ge 0$, we have $f(0) \ge 0$, hence both solutions are non-negative, $Z_1 \ge Z_2 \ge 0$. The solutions are given by the Ferrari's formula,

$$\begin{aligned} Z &= \epsilon_1 \frac{\sqrt{u}}{2} + \epsilon_2 \sqrt{D}; \\ D &:= -\frac{1}{4}(10\nu_1 + u) + 16\epsilon_1 \frac{\nu_1}{\sqrt{u}}. \end{aligned} \tag{B5}$$

Here ϵ_1 and ϵ_2 are ± 1, and u is

$$\begin{aligned} u = -\frac{10\nu_1}{3} &+ \left(-\frac{J}{2} - \sqrt{\frac{J^2}{4} - \frac{H^3}{27}}\right)^{1/3} \\ &+ \left(-\frac{J}{2} + \sqrt{\frac{J^2}{4} - \frac{H^3}{27}}\right)^{1/3}, \end{aligned} \tag{B6}$$

where

$$\begin{aligned} H &= \frac{25\nu_1^2}{3} + \frac{32\nu_2\nu_1}{5}, \\ J &= \frac{64\nu_1^2\nu_2}{3} - 2^{10}\nu_1^2 - \frac{250\nu_1^3}{3^3}. \end{aligned} \tag{B7}$$

If ν_1 and ν_2 satisfy Eq. (B4), $J^2/4 - H^3/27 > 0$. This means that u is positive. The equation must have only one or two real solutions. This implies $\epsilon_1 = 1$ because $D < 0$ if $\epsilon_1 = -1$. Thus the two real solutions are

$$\begin{aligned} Z_1 &= \frac{\sqrt{u}}{2} + \sqrt{-\frac{1}{4}(10\nu_1 + u) + 16\frac{\nu_1}{\sqrt{u}}}, \\ Z_2 &= \frac{\sqrt{u}}{2} - \sqrt{-\frac{1}{4}(10\nu_1 + u) + 16\frac{\nu_1}{\sqrt{u}}}, \end{aligned} \tag{B8}$$

where $Z_1 \ge Z_2$.

The above expressions for the solutions Z_1 and Z_2 are quite complicated as they are. However, using the scaling freedom of L_0, it is possible to simplify the expressions. For this purpose, let us first recapitulate Eq. (39) where the length L_1 representing the linear extension of the extra dimensions was introduced,

$$L_1^2 = \frac{8\pi G}{\mathbf{q}^2}\frac{1}{Z_1}. \tag{B9}$$

Then we set $L_0 = L_1$, which implies $Z_1 = a$.

Also for Z_2, we may also simplify the expression in terms of a, b, c with the normalization $L_0 = L_1$. In this case, since $Z = Z_1 = a$ is a solution of $f(Z) = 0$, we have Eq. (41),

$$f(a) = 0 \Leftrightarrow c = 20 - \frac{5a}{8}(5 + 63b), \tag{B10}$$

and $f(Z)$ can be divided by $(Z - a)$. The quotient is

$$\frac{63ab}{8}((Z/a)^3 + (Z/a)^2) + \left(4 - \frac{c}{5}\right)(Z/a) - \frac{c}{5} = 0. \tag{B11}$$

In order to use the Cardano's formula, let us change the equation into the normal form,

$$(Z/a + 1/3)^3 + A(Z/a + 1/3) + B = 0, \tag{B12}$$

where

$$A = \frac{5 + 42b}{63b} > 0, \qquad B = -\frac{2(48 - 5a(1 + 14b))}{3 \cdot 63ab}. \tag{B13}$$

This equation has only one real positive solution Z_2. Therefore the solution is

$$\begin{aligned} Z_2/a = -\frac{1}{3} &- \left\{\frac{1}{2}\left(B + \sqrt{B^2 + \frac{4A^3}{27}}\right)\right\}^{1/3} \\ &+ \left\{\frac{1}{2}\left(-B + \sqrt{B^2 + \frac{4A^3}{27}}\right)\right\}^{1/3}. \end{aligned} \tag{B14}$$

Finally let us derive the bounds on the parameters a and b. We assume $c \ge 0$. From Eq. (B10), this gives a bound on a and b,

$$32 - a(5 + 63b) > 0. \tag{B15}$$

In addition, since Eq. (B11) has only one real positive solution Z_2 which is equal to or smaller than Z_1, the left-hand side of it is non-negative at $Z = a$,

$$\frac{63ab}{4} + \left(4 - \frac{c}{5}\right) - \frac{c}{5} = \frac{1}{4}(a(5 + 126b) - 16) \ge 0. \tag{B16}$$

Therefore the conditions that $Z_2 \le Z_1$ and $c \ge 0$ yield the bounds on the parameters a and b as

$$\frac{32}{5 + 63b} \ge a \ge \frac{16}{5 + 126b}. \tag{B17}$$

APPENDIX C: ASYMPTOTIC BEHAVIOR IN THE CASE OF $\lambda_1 = 0$

Here we derive the asymptotic behavior of the solution of the system given by Eqs. (58) and (59). Eqs. (65) and (66) can be written as

$$\begin{aligned} &\frac{\psi'}{r}\Omega\cos(\Omega\tau + \psi) + \left(\frac{r'}{r^2} + 6\Omega\right)\Omega\sin(\Omega\tau + \psi) \\ &\quad = -3\Omega^2 - \left(6\Omega^2 - \frac{15}{2}\right)\cos^2(\Omega\tau + \psi), \end{aligned} \tag{C1}$$

$$\frac{\psi'}{r} = \frac{r'}{r^2}\tan^{-1}(\Omega\tau + \psi). \tag{C2}$$

By eliminating the term ψ'/r from these equations, we obtain

$$\frac{r'}{r^2} = -3\Omega + \mathcal{F}, \tag{C3}$$

where

$$\mathcal{F} = -\left(\frac{9}{2}\Omega - \frac{15}{8\Omega}\right)\sin(\Omega\tau + \psi) + 3\Omega\cos(2\Omega\tau + 2\psi) - \left(\frac{3}{2}\Omega - \frac{15}{8\Omega}\right)\cos(3\Omega\tau + 3\psi). \tag{C4}$$

We can integrate this to obtain an expression for r,

$$\frac{1}{r} = 3\Omega\tau - \int^{\tau} d\tau\mathcal{F}. \tag{C5}$$

As for the angle ψ, from Eqs. (C2) and (C3), it satisfies

$$\psi' = -r\Omega\cos(\Omega\tau + \psi)\left(3 + 6\sin(\Omega\tau + \psi) + \left(6 - \frac{15}{2\Omega^2}\right)\cos^2(\Omega\tau + \psi)\right). \tag{C6}$$

As clear from this equation, ψ tends to a constant for $r \to 0$. Then $\mathcal{F}$ will be a function oscillating around zero. This implies that the integral of $\mathcal{F}$ in Eq. (C3) cannot be large. Thus in the region where τ is large enough, r damps out in time as $1/\tau$,

$$r = \frac{1}{3\Omega\tau - \int d\tau\mathcal{F}} \sim \frac{1}{3\Omega\tau}. \tag{C7}$$

This is consistent with our anticipation that ψ tends to a constant. Therefore ignoring an irrelevant integration constant, the asymptotic behaviors of ϕ_1 and ϕ_2 at large τ are given by

$$\phi_2 \sim \frac{1}{3\Omega\tau}\sin\Omega\tau, \qquad \phi_1 \sim \frac{2}{3}\log\tau - \mathrm{Si}(\Omega\tau). \tag{C8}$$

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